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THE MALLOWS MEASURES ON THE HYPEROCTAHEDRAL GROUP

Abstract. The Mallows measures on the symmetric group S_n form a deformation of the uniform distribution. These measures are commonly used in mathematical statistics, and in recent years they found applications in other areas of mathematics as well.

As shown by Gnedin and Olshanski, there exists an analog of the Mallows measure on the infinite symmetric group. These new measures are diffuse, and they are quasi-invariant with respect to the two-sided action of a countable dense subgroup.

The purpose of the present note is to extend the Gnedin–Olshanski construction to the infinite hyperoctahedral group. Along the way, we obtain some results for the Mallows measures on finite hyperoctahedral groups, which may be of independent interest.

§1. Introduction

For positive q, the Mallows measure on a finite symmetric group is defined as the probability measure whose value at a permutation σ is proportional to $q^{\mathrm{inv}(\sigma)}$, where $\mathrm{inv}(\sigma)$ denotes the number of inversions of σ. This measure was introduced by Mallows in [1], and it can be extended to a probability measure on the infinite symmetric group, as described in [2].

The Mallows measures can also be defined on other Weyl groups. We analyze the Mallows measure on the finite hyperoctahedral group – the Weyl group of the root systems B_n and C_n, and extend it to a probability measure on the infinite hyperoctahedral group.

§2. Finite hyperoctahedral groups

The hyperoctahedral group $\mathcal{H}_n$ is the semidirect product $S_n \ltimes (\mathbb{Z}/2\mathbb{Z})^n$, where S_n denotes the group of permutations of $\{1, \dots, n\}$.

Let $\widetilde{S}_{2n}$ denote the group of permutations of $\{-n, \dots, -1, 1, \dots, n\}$. We will regard $\mathcal{H}_n$ as a subgroup of $\widetilde{S}_{2n}$, formed by the permutations

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commuting with the involution $i \mapsto -i$; such permutations will be called symmetric.

The finite hyperoctahedral group can also be regarded as the Weyl group of the root systems B_n and C_n, which are dual to each other. If $\{e_1, \ldots, e_n\}$ is the canonical orthonormal basis of $\mathbb{R}^n$, then B_n consists of the vectors $\pm e_i$ (of length 1) and the vectors $\pm(e_i \pm e_j)$ for $i \neq j$ (of length $\sqrt{2}$). The vectors $e_1, e_2 - e_1, \ldots, e_n - e_{n-1}$ form a base Δ for this system, and the corresponding fundamental Weyl chamber $\mathfrak{C}(\Delta)$ is formed by the points $(x_1, \ldots, x_n) \in \mathbb{R}^n$ with $0 < x_1 < x_2 < \cdots < x_n$. The Weyl group is generated by the transformations $x_i \longleftrightarrow x_{i+1}$ $(i = 1, \ldots, n-1)$ and $x_1 \mapsto -x_1$, which are the reflections in the hyperplanes orthogonal to the vectors $e_{i+1} - e_i$ and e_1, respectively. This gives an isomorphism between the Weyl group and the hyperoctahedral group $\mathcal{H}_n$. The elements of $\mathcal{H}_n$ that correspond to these transformations are denoted by $\sigma_{i,i+1}$ and ε_1, respectively.

The action of the Weyl group of B_n on $\mathbb{R}^n$ gives us a realization of $\mathcal{H}$ as the group of linear operators on $\mathbb{R}^n$ preserving the set $\{\pm e_1, \ldots, \pm e_n\}$. In the basis $\{e_1, \ldots, e_n\}$, these operators are given by the $n \times n$ matrices with exactly one nonzero number in each row and column, this nonzero number being ± 1. Given a symmetric permutation σ, the corresponding operator on $\mathbb{R}^n$, also denoted by σ, is given by

$$\sigma e_i = \operatorname{sign}(\sigma(i)) e_{|\sigma(i)|}.$$

Let us consider the action of $\mathcal{H}_n$ on Weyl chambers of B_n induced by the above isomorphism. Two Weyl chambers are said to be *separated by the hyperplane* P_α orthogonal to a root α if these chambers lie in different connected components of $\mathbb{R}^n \setminus P_\alpha$.

Definition 1. *An inversion of the first type* of $\sigma \in \mathcal{H}_n$ is a hyperplane orthogonal to a root of length $\sqrt{2}$ and separating $\mathfrak{C}(\Delta)$ and $\sigma.\mathfrak{C}(\Delta)$. *An inversion of the second type* of $\sigma \in \mathcal{H}_n$ is a hyperplane orthogonal to a root of length 1 and separating $\mathfrak{C}(\Delta)$ and $\sigma.\mathfrak{C}(\Delta)$.

Let $\operatorname{inv}_1(\sigma)$ denote the number of inversions of σ of the first type, and $\operatorname{inv}_2(\sigma)$ denote the number of inversions of σ of the second type.

Proposition 2. *Let* $\sigma \in \mathcal{H}_n$. *Then* $\operatorname{inv}_1(\sigma) = \operatorname{inv}_1(\sigma^{-1})$ *and* $\operatorname{inv}_2(\sigma) = \operatorname{inv}_2(\sigma^{-1})$.

Proof. The action on $\mathbb{R}^n$ by σ^{-1} gives a one-to-one correspondence between the hyperplanes separating $\mathfrak{C}(\Delta)$ and $\sigma.\mathfrak{C}(\Delta)$ and the hyperplanes separating $\sigma^{-1}.\mathfrak{C}(\Delta)$ and $\mathfrak{C}(\Delta)$. Moreover, this action preserves the root system and the scalar product, hence the images of hyperplanes orthogonal to roots of length l are again hyperplanes orthogonal to roots of length l. So this gives a one-to-one correspondence between the inversions of σ and inversions of σ^{-1} which preserves the type of inversions. $\square$

Proposition 3. *Let σ be an element of $\mathcal{H}_n$. There is a one-to-one correspondence between the inversions of σ of the first type and the pairs (i,j) such that $-n \leqslant i < j \leqslant n$, $|i| < |j|$, and $\sigma(i) > \sigma(j)$. Also, there is a one-to-one correspondence between tne inversions of σ of the second type and the pairs (i,j) such that $-n \leqslant i < j \leqslant n$, $|i| = |j|$, and $\sigma(i) > \sigma(j)$.*

Proof. Proposition 2 yields a bijection between the inversions of σ and σ^{-1}. Then the desired correspondence is provided by the action of $\mathcal{H}_n$ on $\mathbb{R}^n$.

The image of $e_1 + 2e_2 + \cdots + ne_n \in \mathfrak{C}(\Delta)$ under the action by σ^{-1} is $\sigma(1)e_1 + \cdots + \sigma(n)e_n \in \sigma^{-1}.\mathfrak{C}(\Delta)$. Thus, the hyperplane orthogonal to e_i is an inversion of the second type if and only if $\sigma(i) < 0$, and this gives the second correspondence. As to the first correspondence, the pairs (i,j) such that $-n \leqslant i < j \leqslant n$, $|i| < |j|$, and $\sigma(i) > \sigma(j)$ are either pairs (i,j) such that $0 < i < j$ and $\sigma(i) > \sigma(j)$ or pairs $(-i,j)$ such that $0 < i < j$ and $\sigma(i) + \sigma(j) < 0$. Then a pair from the first group gives an inversion of the first type corresponding to the reflection in $P_{e_i - e_j}$, and a pair from the second group gives an inversion of the first type corresponding to the reflection in $P_{e_i + e_j}$, and all inversions of the first type are given by this correspondence. $\square$

§3. A description of $\mathcal{H}_n$ using Young diagrams

Let Λ^n_{sym} denote the set of symmetric Young diagrams of length $\leqslant n$. We will construct a bijection $\mathcal{H}_n \to S_n \times \Lambda^n_{\mathrm{sym}}$ using the construction described in [3].

According to [3], there is a bijection $\widetilde{S}_{2n} \to S_n \times \Lambda^n \times S_n$, where Λ^n denotes the set of partitions λ of length $\leqslant n$ with $\lambda_1 \leqslant n$. It maps a permutation $\sigma \in \widetilde{S}_{2n}$ to a triple $(\sigma_+, \lambda(\sigma), \sigma_-)$, where the permutations σ_+ and σ_- are obtained by taking the subsequences of positive and negative terms in $(\sigma(-n), \dots, \sigma(n))$. As for the diagram $\lambda(\sigma)$, the length of its ith row is, by definition, the number of negative terms in $(\sigma(-n), \dots, \sigma(n))$ that are to the right of the ith leftmost positive number.

Proposition 4. *The correspondence $\sigma \mapsto (\sigma_+, \lambda(\sigma))$ defined above determines a bijection between $\mathcal{H}_n$ and $S_n \times \Lambda^n_{\mathrm{sym}}$.*

Proof. The embedding $\mathcal{H}_n \subset \widetilde{S}_{2n}$ gives rise to an embedding

$$\mathcal{H}_n \to S_n \times \Lambda^n \times S_n$$

which maps σ to the triple $(\sigma_+, \lambda(\sigma), \sigma_-)$. Since σ is a symmetric permutation, we have $\sigma_+ = \sigma_-$. Moreover, according to [3, Part 3], we have $\lambda(\sigma) = \lambda(\sigma)'$, so λ is symmetric. On the other hand, if $\sigma_- = \sigma_+$ and $\lambda(\sigma)$ is symmetric, then σ is a symmetric permutation.

Thus, the image of $\mathcal{H}_n$ in $S_n \times \Lambda^n \times S_n$ is the set of triples $(\omega_1, \lambda, \omega_2)$ such that $\omega_1 = \omega_2$ and $\lambda = \lambda'$. The projection onto the first two components, which maps $(\omega_1, \lambda, \omega_2)$ to (ω_1, λ), provides a one-to-one correspondence between these triples and $S_n \times \Lambda^n_{\mathrm{sym}}$. Hence, the composition

$$\sigma \mapsto (\sigma_+, \lambda(\sigma), \sigma_-) \mapsto (\sigma_+, \lambda(\sigma))$$

gives a one-to-one correspondence from $\mathcal{H}_n$ to $S_n \times \Lambda^n_{\mathrm{sym}}$. $\square$

This description gives a way to compute the number of inversions of $\sigma \in \mathcal{H}_n$.

Proposition 5. *Let $\sigma \in \mathcal{H}_n$. Then we have*

$$\mathrm{inv}_1(\sigma) = \mathrm{inv}(\sigma_+) + \frac{|\lambda(\sigma)| - d(\lambda(\sigma))}{2}, \tag{1}$$

$$\mathrm{inv}_2(\sigma) = d(\lambda(\sigma)), \tag{2}$$

where $\mathrm{inv}(\sigma_+)$ *denotes the number of inversions of σ_+ and $d(\lambda)$ denotes the number of diagonal squares of the Young diagram of λ.*

Proof. We begin with (2). According to the definition of $\lambda(\sigma)$, we see that $\lambda(\sigma)_i$ is the number of negative integers to the right of the ith positive integer in the sequence $(\sigma(-n), \ldots, \sigma(n))$. Thus, to the right of the ith positive number there are $n - i + \lambda(\sigma)_i$ numbers. It follows that the ith positive number is in the left half of the sequence if and only if $n - i + \lambda(\sigma)_i \geqslant n$. The number of inversions of the second type is equal to the number of positive numbers in the left half of the sequence, which is equal to the number of i such that $\lambda(\sigma)_i \geqslant i$, which is equal to $d(\lambda(\sigma))$.

Let us prove (1). Given $\sigma \in \mathcal{H}_n$, we consider its ordinary inversions, i.e., pairs $i < j$ such that $\sigma(i) > \sigma(j)$ (where $i, j \in \{-n, \ldots, -1, 1, \ldots, n\}$). Let $\mathrm{Inv}(\sigma)$ denote the number of ordinary inversions of σ. We count this number in two ways.

On the one hand, by Proposition 3, each inversion (i,j) of σ of the first type determines two ordinary inversions, (i,j) and $(-i,-j)$, while each inversion of the second type determines a single ordinary inversion of the form $(i,-i)$. Moreover, all ordinary inversions are obtained in this way, which gives the equality

$$\mathrm{Inv}(\sigma) = 2\mathrm{inv}_1(\sigma) + \mathrm{inv}_2(\sigma).$$

On the other hand, each ordinary inversion (i,j) is either positive in the sense that $\sigma(i) > \sigma(j) > 0$, or negative in the sense that $0 > \sigma(i) > \sigma(j)$, or else sign-changing, meaning that $\sigma(i) > 0 > \sigma(j)$. Both the number of positive and the number of negative inversions are equal to the number of inversions of σ_+, while the number of sign-changing inversions is equal to $|\lambda(\sigma)|$ by the definition of $\lambda(\sigma)$. So we have

$$\mathrm{Inv}(\sigma) = 2\mathrm{inv}(\sigma_+) + |\lambda(\sigma)| = 2\mathrm{inv}_1(\sigma) + \mathrm{inv}_2(\sigma).$$

Since $\mathrm{inv}_2(\sigma) = d(\lambda(\sigma))$ by (2), we get (1). $\square$

§4. The Mallows measures on $\mathcal{H}_n$

In the remainder of the paper we fix two real numbers q_1 and q_2 such that $0 < q_1, q_2 < 1$. We will also use the standard q-notation:

$$[n]_q = 1 + q + \dots + q^{n-1}, \qquad [n]_q! = [1]_q[2]_q \dots [n]_q,$$

$$(x;q)_n = (1-x)(1-xq)\dots(1-xq^{n-1}).$$

Definition 6. *The Mallows measure* on $\mathcal{H}_n$ is the probability measure μ_n such that for any $\sigma \in \mathcal{H}_n$

$$\mu_n(\sigma) = \gamma_n^{-1} q_1^{\mathrm{inv}_1(\sigma)} q_2^{\mathrm{inv}_2(\sigma)},$$

where γ_n is a normalization constant.

Proposition 7. *The constant γ_n is given by*

$$\gamma_n = [n]_{q_1}!(-q_2;q_1)_n.$$

Proof. Since μ_n is a probability measure on $\mathcal{H}_n$, we have

$$\gamma_n = \sum_{\sigma \in \mathcal{H}_n} q_1^{\mathrm{inv}_1(\sigma)} q_2^{\mathrm{inv}_2(\sigma)}.$$

Using the description via Young diagrams, we have

$$\begin{aligned}\gamma_n &= \sum_{\sigma\in\mathcal{H}_n} q_1^{\mathrm{inv}(\sigma_+)+\frac{|\lambda(\sigma)|-d(\lambda(\sigma))}{2}} q_2^{d(\lambda(\sigma))} \\ &= \sum_{\omega\in S_n} q_1^{\mathrm{inv}(\omega)} \cdot \sum_{\lambda\in\Lambda^n_{\mathrm{sym}}} q_1^{\frac{|\lambda|-d(\lambda)}{2}} q_2^{d(\lambda)}.\end{aligned}$$

This is the product of two sums. The first one can be computed by induction, and the result is $[n]_{q_1}!$. The second sum can be rewritten using the Frobenius notation $(\alpha|\alpha)$ for a symmetric partition λ:

$$\sum_{\lambda\in\Lambda^n_{\mathrm{sym}}} q_1^{\frac{|\lambda|-d(\lambda)}{2}} q_2^{d(\lambda)} = \sum_{(\alpha|\alpha)\in\Lambda^n_{\mathrm{sym}}} q_1^{|\alpha|} q_2^{l(\alpha)} = \sum_{(\alpha|\alpha)\in\Lambda^n_{\mathrm{sym}}} \prod_{i=1}^{l(\alpha)} q_1^{\alpha_i} q_2.$$

For elements from Λ^n_{sym}, the numbers α_i are distinct and less than n. So we have

$$\sum_{(\alpha|\alpha)\in\Lambda^n_{\mathrm{sym}}} \prod_{i=1}^{l(\alpha)} q_1^{\alpha_i} q_2 = \prod_{i=0}^{n-1}(1+q_1^i q_2).$$

Hence $\gamma_n = [n]_{q_1}! \prod_{i=0}^{n-1}(1+q_1^i q_2) = [n]_{q_1}!(-q_2;q_1)_n$. □

The description via Young diagrams also gives a way to write the Mallows measure on $\mathcal{H}_n$ as the product of the Mallows measure on S_n and the measure on Λ^n_{sym} with the value $q_1^{\frac{|\lambda|-d(\lambda)}{2}} q_2^{d(\lambda)}(-q_2;q_1)_n^{-1}$ at a partition λ.

Proposition 8. *The Mallows measure is invariant under the inversion map $g \to g^{-1}$.*

Proof. This follows from Proposition 2. □

Given $k < n$ and a k-tuple $\alpha = (\alpha_1,\dots,\alpha_k)$ of numbers from the set $\{-n,\dots,-1,1,\dots,n\}$ such that $|\alpha_i| \neq |\alpha_j|$ for $i \neq j$, we define

$$C^n_k(\alpha) = C^n_k(\alpha_1,\dots,\alpha_k) = \{\sigma\in\mathcal{H}_n | \sigma(i) = \alpha_i\}.$$

Subsets of this form will be called *elementary cylinders*.

Theorem 9. *The Mallows measure μ_n on $\mathcal{H}_n$ is uniquely determined by the following properties*:

(a) *μ_n is a probability measure*;

(b) *for any elementary cylinder* $C_k^n(\alpha_1,\dots,\alpha_k)$ *and any* $i \leqslant k-1$ *such that* $\alpha_i < \alpha_{i+1}$*, one has*

$$\mu_n(C_k^n(\alpha_1,\dots,\alpha_i,\alpha_{i+1},\dots,\alpha_k)) = q_1^{-1}\mu_n(C_k^n(\alpha_1,\dots,\alpha_{i+1},\alpha_i,\dots,\alpha_k));$$

(c) *for any elementary cylinder* $C_k^n(\alpha_1,\dots,\alpha_k)$ *with* $\alpha_1 > 0$*, one has*

$$\mu_n(C_k^n(\alpha_1,\dots,\alpha_k)) = q_2^{-1}\mu_n(C_k^n(-\alpha_1,\dots,\alpha_k)).$$

Proof. Let ν be a probability measure with properties (b) and (c). For $\sigma \in \mathcal{H}_n$ we have

$$C_n^n((\sigma(1),\dots,\sigma(n))) = \{\sigma\}.$$

The right multiplication by the elementary transposition $\sigma_{i,i+1}$ swaps $\sigma(i)$ and $\sigma(i+1)$, while the right multiplication by ε_1 changes the sign of $\sigma(1)$.

Take $\sigma \in \mathcal{H}_n$. If $\sigma(i) < \sigma(i+1)$ for some i, we have

$$\mathrm{inv}_1(\sigma\sigma_{i,i+1}) = \mathrm{inv}_1(\sigma) + 1, \qquad \mathrm{inv}_2(\sigma\sigma_{i,i+1}) = \mathrm{inv}_2(\sigma); \tag{3}$$

and if $\sigma(1) > 0$, we have

$$\mathrm{inv}_1(\sigma\varepsilon_1) = \mathrm{inv}_1(\sigma), \qquad \mathrm{inv}_2(\sigma\varepsilon_1) = \mathrm{inv}_2(\sigma) + 1. \tag{4}$$

All these identities can be proved using Proposition 3. Then, using properties (a)–(c), we obtain that

$$\nu(\sigma\sigma') = \nu(\sigma) q_1^{\mathrm{inv}_1(\sigma\sigma')-\mathrm{inv}_1(\sigma)} q_2^{\mathrm{inv}_2(\sigma\sigma')-\mathrm{inv}_1(\sigma)}, \tag{5}$$

where σ' is either $\sigma_{i,i+1}$ or ε_1. Since these elements generate $\mathcal{H}_n$, we can deduce by induction that (5) holds for any $\sigma' \in \mathcal{H}_n$. Taking $\sigma = e$, we get

$$\nu(\sigma') = \nu(e) q_1^{\mathrm{inv}_1(\sigma')} q_2^{\mathrm{inv}_2(\sigma')}.$$

It follows that $\nu = \mu_n$.

Also, from (3) and (4) we see that the properties from the statement of the theorem hold for μ_n. $\square$

For elements of the hyperoctahedral group there is one more description. For every $\sigma \in \mathcal{H}_n$, the sequence $(\sigma(-n),\dots,\sigma(n))$ determines a linear ordering of the set $\{-n,\dots,-1,1,\dots,n\}$ (namely, $a \succ b$ if a is to the right of b). This ordering is symmetric in the sense that $a \succ b$ entails $-a \prec -b$. Let $\mathcal{O}_n^{\mathrm{sym}}$ denote the set of such orderings of $\{-n,\dots,-1,1,\dots,n\}$. There is a natural projection $\tilde{\pi}_n : \mathcal{O}_n^{\mathrm{sym}} \to \mathcal{O}_{n-1}^{\mathrm{sym}}$, defined by restricting the ordering to the subset $\{-n+1,\dots,n-1\}$. We denote by π_n the corresponding projection $\pi_n : \mathcal{H}_n \to \mathcal{H}_{n-1}$.

Proposition 10. *The pushforward of the Mallows measure μ_n under π_n is μ_{n-1}.*

Proof. This can be checked by a simple computation. For $\sigma \in \mathcal{H}_{n-1}$, consider the sequence $\alpha = (\sigma(1), \dots, \sigma(n-1))$. Then the corresponding sequences for elements in the preimage of σ should have α as a subsequence, and all absolute values should be distinct. So we should insert in α either n or $-n$. If we insert n in the ith position, then we add $n-i$ inversions of the first type to the inversions of σ. If we insert $-n$ in the ith position, then we add $n-1+i-1$ inversions of the first type and one inversion of the second type. It follows that

$$\mu_n \pi_n^{-1}(\sigma) = \left(\sum_{i=1}^{n} q_1^{n-i} + \sum_{i=1}^{n} q_1^{n+i-2} q_2\right) \frac{[n-1]_{q_1}!(-q_2;q_1)_{n-1}}{[n]_{q_1}!(-q_2;q_1)_n} \mu_{n-1}(\sigma)$$
$$= \left([n]_{q_1} + [n]_{q_1} q_1^{n-1} q_2\right) \frac{1}{[n]_{q_1}(1+q_2 q_1^{n-1})} \mu_{n-1}(\sigma) = \mu_{n-1}(\sigma). \square$$

§5. The Mallows measure on $\mathcal{H}$

Now we focus on the infinite hyperoctahedral group $\mathcal{H} = S \ltimes (\mathbb{Z}/2\mathbb{Z})^\infty$, where S denotes the group of all permutations of $\mathbb{N}$. As in the finite case, we regard $\mathcal{H}$ as the group of symmetric permutations of $\mathbb{Z}\backslash\{0\}$.

Next, we introduce two subgroups of $\mathcal{H}$. One is

$$\mathcal{H}' = S \ltimes (\mathbb{Z}/2\mathbb{Z})^\infty_{\mathrm{fin}},$$

where $(\mathbb{Z}/2\mathbb{Z})^\infty_{\mathrm{fin}}$ denotes the subset of $(\mathbb{Z}/2\mathbb{Z})^\infty$ consisting of the elements $(g_1, g_2, \dots)$ with finitely many nonidentity elements. In other words, $\mathcal{H}'$ is the subgroup of symmetric permutations with finitely many changes of sign, that is, finitely many i's such that $\mathrm{sign}(i) \neq \mathrm{sign}(\sigma(i))$. The second, smaller, subgroup is

$$\mathcal{H}_{\mathrm{fin}} = S_{\mathrm{fin}} \ltimes (\mathbb{Z}/2\mathbb{Z})^\infty_{\mathrm{fin}},$$

where $S_{\mathrm{fin}} \subset S$ denotes the set of finitary permutations. In other words, $\mathcal{H}_{\mathrm{fin}}$ is the set of finitary symmetric permutations.

The description via Young diagrams can be extended to the subgroup $\mathcal{H}'$, giving a bijection $\mathcal{H}' \to S \times \Lambda_{\mathrm{sym}}$. For $\omega \in S$ and $\lambda \in \Lambda_{\mathrm{sym}}$, let $\sigma(\omega, \lambda)$ denote the preimage of (ω, λ).

For a k-tuple $(\alpha_1, \dots, \alpha_k)$ of nonzero integers with distinct absolute values, consider the elementary cylinder

$$C_k(\alpha_1, \dots, \alpha_k) = \{\sigma \in \mathcal{H} | \sigma(i) = \alpha_i\}.$$

It can be checked that such cylinders form a base for a topology on $\mathcal{H}$. In this topology, elementary cylinders are open and closed. In what follows, all measures on $\mathcal{H}$ are assumed to be Borel measures.

Recall that $\sigma_{i,i+1} \in \mathcal{H}$ is the elementary transposition swapping i and $i+1$, and ε_1 is the elementary transposition swapping -1 and 1.

Definition 11. A measure μ on $\mathcal{H}$ is said to be *right* (q_1, q_2)*-exchangeable* if the following properties hold:

(a) For every $i \geqslant 1$, the pushforward $\mu_{i,i+1}$ of the measure μ under the transformation $\sigma \mapsto \sigma\sigma_{i,i+1}$ is equivalent to μ, and the value of the Radon–Nikodym derivative $\frac{d\mu_{i,i+1}}{d\mu}$ at $\sigma \in \mathcal{H}$ is equal to $q_1^{\operatorname{sign}(\sigma(i+1)-\sigma(i))}$.

(b) The pushforward $\mu_{-1,1}$ of the measure μ under the transformation $\sigma \mapsto \sigma\varepsilon_1$ is equivalent to μ, and the value of the Radon–Nikodym derivative $\frac{d\mu_{-1,1}}{d\mu}$ at $\sigma \in \mathcal{H}$ is equal to $q_2^{\operatorname{sign}(\sigma(1))}$.

We can similarly define the *left* (q_1, q_2)*-exchangeability* by replacing right shifts by left shifts and imposing the relations

$$\frac{d\mu_{i,i+1}}{d\mu} = q_1^{\operatorname{sign}(\sigma^{-1}(i+1)-\sigma^{-1}(i))}, \qquad \frac{d\mu_{-1,1}}{d\mu} = q_2^{\operatorname{sign}(\sigma^{-1}(1))}.$$

Remark 12. The property of right (respectively, left) (q_1, q_2)-exchangeability is equivalent to that of quasi-invariance with respect to the right (respectively, left) action of the dense countable subgroup $\mathcal{H}_{\text{fin}} \subset \mathcal{H}$ with a certain special 1-cocycle.

The next proposition is an obvious reformulation of the right (q_1, q_2)-exchangeability in terms of elementary cylinders.

Proposition 13. *A probability measure μ on $\mathcal{H}$ is right (q_1, q_2)-exchangeable if and only if the following properties hold*:

(1) *For a sequence of nonzero integers $(\alpha_1, \dots, \alpha_k)$ such that $\alpha_i < \alpha_{i+1}$ for some i, we have*

$$\mu(C_k(\alpha_1, \dots, \alpha_i, \alpha_{i+1}, \dots, \alpha_k)) = q_1^{-1}\mu(C_k(\alpha_1, \dots, \alpha_{i+1}, \alpha_i, \dots, \alpha_k)).$$

(2) *For a sequence of nonzero integers $(\alpha_1, \dots, \alpha_k)$ such that $\alpha_1 > 0$, we have*

$$\mu(C_k(\alpha_1, \dots, \alpha_k)) = q_2^{-1}\mu(C_k(-\alpha_1, \dots, \alpha_k)).$$

Theorem 14. *There exists a unique right (q_1, q_2)-exchangeable probability measure on $\mathcal{H}$, which we call the Mallows measure. Moreover, this measure is left (q_1, q_2)-exchangeable, it is supported by $\mathcal{H}'$, and it is invariant under the inversion map $\sigma \to \sigma^{-1}$.*

We begin with constructing the required measure μ. Let μ_{sym} be the q_1-exchangeable measure on S (as described in [2]) and μ_Λ be the measure on Λ_{sym} defined by

$$\mu_\Lambda(\lambda) = (-q_2; q_1)_\infty^{-1} q_1^{\frac{|\lambda| - d(\lambda)}{2}} q_2^{d(\lambda)}.$$

Set $\mu' := \mu_{\mathrm{sym}} \otimes \mu_\Lambda$ (this is a measure on $S \times \Lambda_{\mathrm{sym}} = \mathcal{H}'$) and define μ as the pushforward of μ' under the embedding $\mathcal{H}' \to \mathcal{H}$. Thus, μ is concentrated on $\mathcal{H}' \subset \mathcal{H}$.

Lemma 14.1. *The measure μ is right (q_1, q_2)-exchangeable.*

Proof. We use Proposition 13. Take α such that $\alpha_i < \alpha_{i+1}$ for some i. If α_i and α_{i+1} have the same sign, then their transposition does not change the partition, while $\mu_{\mathrm{sym}}(\sigma_+)$ is multiplied by q_1. If they have different signs, then σ_+ is unchanged and the new partition has the same length of the diagonal but two more squares. Then the required factor q_1 results from the transformation of μ_Λ.

To prove the second property, we observe that the change of the sign of $\sigma(1)$ does not affect σ_+, while $\lambda(\sigma)$ acquires one diagonal box, which results in the multiplication by q_2, as desired. $\square$

Let Seq be the set of finite sequences of nonzero integers with distinct absolute values.

Lemma 14.2. *The measure μ is left (q_1, q_2)-exchangeable.*

Proof. We begin with the left shift by $\sigma_{i,i+1}$. Take $(\alpha_1, \dots, \alpha_k) \in \mathrm{Seq}$ such that $i = |\alpha_j|$ and $i + 1 = |\alpha_{j+s}|$ for some positive j and s. Let β denote the subsequence of α consisting of the integers between α_j and α_{j+s}, while γ_l denote the subsequence to the left of α_j and γ_r denote the subsequence to the right. Note that the absolute values of integers in β, γ_l, and γ_r are not equal to i or $i+1$. We examine four cases depending on the signs of α_j and α_{j+s}.

In the first case, we have $\alpha_j, \alpha_{j+s} > 0$. Let a denote the number of elements of β greater than $i+1$, and b denote the number of elements less than i. Then, using the right (q_1, q_2)-exchangeability, we have

$$\mu(C_k(\gamma_l, i, \beta, i+1, \gamma_r)) = q_1^{a-b}\mu(C_k(\gamma_l, i, i+1, \beta, \gamma_r))$$
$$= q_1^{a-b-1}\mu(C_k(\gamma_l, i+1, i, \beta, \gamma_r)) = q_1^{-1}\mu(C_k(\gamma_l, i+1, \beta, i, \gamma_r)).$$

In the second case, we have $\alpha_j, \alpha_{j+s} < 0$. Similarly, let a denote the number of elements of β greater than $-i$, and b denote the number of elements less than $-i-1$. Then we have

$$\mu(C_k(\gamma_l, -i, \beta, -i-1, \gamma_r)) = q_1^{b-a}\mu(C_k(\gamma_l, -i, -i-1, \beta, \gamma_r))$$
$$= q_1^{b-a+1}\mu(C_k(\gamma_l, -i-1, -i, \beta, \gamma_r)) = q_1\mu(C_k(\gamma_l, -i-1, \beta, -i, \gamma_r)).$$

In the third case, we have $\alpha_j < 0$ and $\alpha_{j+s} > 0$. Let c denote the number of elements of γ_l greater than $i+1$, and d denote the number of elements of γ_l with absolute value less than $i+1$, while e denote the number of elements of γ_l less than $-i-1$. We have (using the result of the first case)

$$\begin{aligned}
&\mu(C_k(\gamma_l, -i, \beta, i+1, \gamma_r)) = q_1^{c+d-e}\mu(C_k(-i, \gamma_l, \beta, i+1, \gamma_r))\\
&= q_1^{c+d-e}q_2\mu(C_k(i, \gamma_l, \beta, i+1\gamma_r)) = q_1^{2d}q_2\mu(C_k(\gamma_l, i, \beta, i+1, \gamma_r))\\
&= q_1^{2d-1}q_2\mu(C_k(\gamma_l, i+1, \beta, i, \gamma_r)) = q_1^{c+d-e-1}q_2\mu(C_k(i+1, \gamma_l, \beta, i, \gamma_r))\\
&= q_1^{c+d-e-1}\mu(C_k(-i-1, \gamma_l, \beta, i, \gamma_r)) = q_1^{-1}\mu(C_k(\gamma_l, -i-1, \beta, i, \gamma_r)).
\end{aligned}$$

In the fourth case, we have $\alpha_j > 0$ and $\alpha_{j+s} < 0$. Using the same notation as in the third case and applying the result of the second case, we have

$$\begin{aligned}
&\mu(C_k(\gamma_l, i, \beta, -i-1, \gamma_r)) = q_1^{c-d-e}\mu(C_k(i, \gamma_l, \beta, -i-1, \gamma_r))\\
&= q_1^{c-d-e}q_2^{-1}\mu(C_k(-i, \gamma_l, \beta, -i-1\gamma_r))\\
&= q_1^{-2d}q_2^{-1}\mu(C_k(\gamma_l, -i, \beta, -i-1, \gamma_r))\\
&= q_1^{-2d+1}q_2^{-1}\mu(C_k(\gamma_l, -i-1, \beta, -i, \gamma_r))\\
&= q_1^{c-d-e+1}q_2^{-1}\mu(C_k(-i-1, \gamma_l, \beta, -i, \gamma_r))\\
&= q_1^{c-d-e+1}\mu(C_k(i+1, \gamma_l, \beta, -i, \gamma_r))\\
&= q_1\mu(C_k(\gamma_l, i+1, \beta, -i, \gamma_r)).
\end{aligned}$$

These arguments imply that μ behaves as desired under the left shift by $\sigma_{i,i+1}$.

Consider the left shift by ε_1. The image of $C_k(\gamma_l, 1, \gamma_r)$ under the left shift by ε_1 is $C_k(\gamma_l, -1, \gamma_r)$. Let a denote the number of positive elements of γ_l, and b denote the number of negative elements of γ_l. We have

$$\mu(C_k(\gamma_l,1,\gamma_r)) = q_1^{a-b}\mu(C_k(1,\gamma_l,\gamma_r))$$
$$= q_1^{a-b}q_2^{-1}\mu(C_k(-1,\gamma_l,\gamma_r)) = q_2^{-1}\mu(C_k(\gamma_l,-1,\gamma_r)).$$

This completes the proof. □

Now we turn to the uniqueness. As in the finite case, we can define the set of symmetric linear orderings on $\mathbb{Z}\backslash\{0\}$, which is denoted by $\mathcal{O}_\infty^{\mathrm{sym}}$. For each element of $\mathcal{H}$ there is a symmetric ordering, defined in the same way as in the finite case. Thus we obtain an embedding $\tau : \mathcal{H} \hookrightarrow \mathcal{O}_\infty^{\mathrm{sym}}$ (however, it is not surjective). For each n there is a natural projection $\widetilde{\psi}_n : \mathcal{O}_\infty^{\mathrm{sym}} \to \mathcal{O}_n^{\mathrm{sym}}$, which is defined as the restriction to $\{-n,\dots,n\}$. Let ψ_n denote the corresponding projection $\mathcal{H} \to \mathcal{H}_n$.

Lemma 14.3. *Assume that ν is a right (q_1,q_2)-exchangeable probability measure on $\mathcal{H}$, and denote by ν_n its pushforward under the projection $\psi_n : \mathcal{H} \to \mathcal{H}_n$. Then ν_n coincides with the Mallows measure.*

Proof. We introduce partial orderings on Seq as follows: $\alpha \preceq \beta$ if α is a subsequence of β, and $\alpha \preceq_n \beta$ if $\alpha \preceq \beta$ and any integer in β that does not belong to α has absolute value greater than n. We prove that ν_n is the Mallows measure using the defining properties from Proposition 9.

Take $\alpha \in \mathrm{Seq}$ of length k, and let N be a positive integer greater than the absolute values of all integers in α. For any $n \geqslant N$ we have

$$\nu_n(C_k^n(\alpha)) = \nu(\psi_n^{-1}(C_k^n(\alpha))).$$

If $\sigma \in \psi_n^{-1}(C_k^n(\alpha))$, then the subsequence of $(\sigma(1),\sigma(2),\dots)$ consisting of the elements with absolute value not greater than n begins with α. Then we can take l such that $h = (\sigma(1),\dots,\sigma(l)) \succeq_n \alpha$. On the other hand, if for some $\sigma \in \mathcal{H}$ the sequence $(\sigma(1),\sigma(2),\dots)$ begins with $h \succeq_n \alpha$, then $\sigma \in \psi_n^{-1}(C_k^n(\alpha))$. Hence

$$\psi_n^{-1}(C_k^n(\alpha)) = \bigcup_{h \succeq_n \alpha} C_{l(h)}(h).$$

Moreover, if for any σ we take the minimum l such that $h \succeq_n \alpha$, then we have $\sigma(l) = h_l = \alpha_k$, and thus

$$\psi_n^{-1}(C_k^n(\alpha)) = \bigsqcup_{\substack{h \succeq_n \alpha \\ h_{l(h)}=\alpha_k}} C_{l(h)}(h),$$

$$\nu_n(C_k^n(\alpha)) = \sum_{\substack{h \succeq_n \alpha \\ h_{l(h)}=\alpha_k}} \nu(C_{l(h)}(h)).$$

Assume that for some i we have $\alpha_i < \alpha_{i+1}$. To prove that

$$\nu_n((C_k^n(\alpha)) = q_1^{-1}\nu_n((C_k^n((i,i+1).\alpha)),$$

it is enough to prove that for any $h \succeq_n \alpha$ we have

$$\nu(C_{l(h)}(h)) = q_1^{-1}\nu(C_{l(h)}(\tilde{h})),$$

where $\tilde{h}$ denotes the sequence h with the transposed integers α_i and α_{i+1}. Take $h \succeq_n \alpha$, and let $(\beta_1,\ldots,\beta_t)$ be the subsequence of integers between α_i and α_{i+1}, while γ_l and γ_r be the subsequences to the left of α_i and to the right of α_{i+1}. Then $|\beta_j| > n$, so either $\beta_j > \alpha_i, \alpha_{i+1}$ or $\beta_j < \alpha_i, \alpha_{i+1}$. Let a be the number of positive β_j's and b be the number of negative β_j's. Then

$$\begin{aligned}
\nu(C_{l(h)}(h)) &= \nu(C_{l(h)}(\gamma_l, \alpha_i, \beta_1, \ldots, \beta_t, \alpha_{i+1}, \gamma_r)) \\
&= q_1^{b-a}\nu(C_{l(h)}(\gamma_l, \beta, \alpha_i, \alpha_{i+1}, \gamma_r)) \\
&= q_1^{b-a-1}\nu(C_{l(h)}(\gamma_l, \beta, \alpha_{i+1}, \alpha_i, \gamma_r)) \\
&= q_1^{-1}\nu(C_{l(h)}(\gamma_l, \alpha_{i+1}, \beta, \alpha_i, \gamma_r)) = q_1^{-1}\nu(C_{l(h)}(\tilde{h})).
\end{aligned}$$

The second property can be proved similarly. □

The uniqueness of a right (q_1,q_2)-exchangeable measure on $\mathcal{H}$ follows from its embedding into $\mathcal{O}_\infty^{\mathrm{sym}}$, which is the projective limit of the sets $\mathcal{O}_n^{\mathrm{sym}}$. Using the lemma above and the Kolmogorov extension theorem, we see that this measure should be the limit of the Mallows measures on $\mathcal{H}_n$.

Lemma 14.4. *The measure μ is invariant under the inversion map.*

Proof. This follows from the uniqueness of a right (q_1,q_2)-exchangeable probability measure and the left (q_1,q_2)-exchangeability of μ. Note that the pushforward image of a left (q_1,q_2)-exchangeable measure is right (q_1,q_2)-exchangeable. So the image of μ is right (q_1,q_2)-exchangeable, and thus it is equal to μ. □

Thus μ satisfies all the properties from the theorem.

It is worth noting that varying q_2 with fixed q_1 gives a family of equivalent measures on $\mathcal{H}'$.

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