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DUAL MULTIPARAMETER SCHUR Q-FUNCTIONS

Abstract. For the Schur Q-functions there is a Cauchy identity, which shows a duality between the Schur P- and Q-functions. We will be interested in the multiparameter Schur Q-functions, which were introduced by V. N. Ivanov, and we will give dual analogs of the multiparameter Schur Q(P)-functions, with a corresponding multiparameter Cauchy identity.

§1. Introduction

The Schur Q-functions were introduced by Schur and are useful to describe the projective representations of the symmetric group. These supersymmetric polynomials are enumerated by the strict partitions, and differ by a scalar factor from the Hall–Littlewood polynomials with $t = -1$. The latter are sometimes called the Schur P-polynomials (for strict partitions). As described in [2, Chap. 3], we have the following identity, which is called the Cauchy identity:

$$\sum_{\lambda} P_\lambda(x_1, \dots, x_N) Q_\lambda(y_1, \dots, y_K) = \prod_{\substack{i \leqslant N \\ j \leqslant K}} \frac{1 + x_i y_j}{1 - x_i y_j}. \tag{1}$$

It shows the duality between the Schur Q- and P-functions.

Similarly to other series of symmetric functions, the Schur Q(P)-functions have interpolation analogs introduced by V. N. Ivanov in [1] and called the multiparameter Schur Q-functions. For an infinite sequence $(a_0 = 0, a_1, a_2, \dots)$ and a strict partition λ of length l, we have

$$Q_\lambda(x_1, \dots, x_N | a) = \frac{2^l}{(N-l)!} \sum_{\omega \in S(N)} \prod_{i=1}^{l} (x_{\omega(i)} | a)^{\lambda_i} \prod_{i \leqslant l, i \leqslant j \leqslant N} \frac{x_{\omega(i)} + x_{\omega(j)}}{x_{\omega(i)} - x_{\omega(j)}},$$

where by $(x|a)^k$ we denote the generalized power $(x - a_0) \dots (x - a_{k-1})$. These functions have a lot of properties that can be regarded as multiparameter versions of corresponding properties of the Schur Q-functions,

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and at the same time they have an interpolation property that introduces another method of working with these functions and proving some combinatorial identities.

We give dual analogs of the multiparameter Schur Q-functions, which we call the dual multiparameter Schur Q-functions and denote by $\widehat{Q}_\lambda$. For a strict partition λ of length l, we define

$$\widehat{Q}_\lambda(t_1,\dots,t_K|a) = \frac{2^l}{(K-l)!}\sum_{\omega\in S(K)}\prod_{i=1}^{l}\frac{1}{(t_{\omega(i)}|\tau a)^{\lambda_i}}\prod_{i\leqslant l, i\leqslant j\leqslant K}\frac{t_{\omega(j)}+t_{\omega(i)}}{t_{\omega(j)}-t_{\omega(i)}},$$

where τa is the shifted sequence $(a_1,a_2,\dots)$. We will prove a multiparameter analog of the Cauchy identity, which holds in $\mathbb{C}[x_1,\dots,x_N][[t_1^{-1},\dots,t_K^{-1}]]$:

$$\sum_\lambda P_\lambda(x_1,\dots,x_N|a)\widehat{Q}_\lambda(t_1,\dots,t_K|a) = \prod_{\substack{i\leqslant N\\ j\leqslant K}}\frac{t_j+x_i}{t_j-x_i}. \tag{2}$$

For $K\leqslant 2$, an equivalent identity was proved in [1, Sec. 8] with the usage of an interpolation property. We will follow another method, proving the parameter independence of the left-hand side of (2).

The results of this work can be regarded as a projective analog of some results from Molev's paper [3], where multiparameter analogs of the ordinary Schur functions are considered. Even earlier, in the case of the shifted Schur functions, dual Schur functions and a corresponding Cauchy identity appeared in [4].

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§2. Dual multiparameter Schur Q-functions

Let $a=(a_0,a_1,\dots)$ be an infinite sequence of parameters, with $a_0=0$. Let $(x|a)^k$ denote the generalized powers $(x-a_0)\dots(x-a_{k-1})$. Following Ivanov, we define functions $P_\alpha(x_1,\dots,x_N|a)$ and $Q_\alpha(x_1,\dots,x_N|a)$ for an arbitrary sequence of positive integers of length $l\leqslant N$.

Definition 1. Let $\alpha=(\alpha_1,\dots,\alpha_l)$ be a sequence of positive integers and $N\geqslant l$. Then the *multiparameter Schur P- and Q-functions* are defined by

$$P_\alpha(x_1,\dots,x_N|a) = \frac{1}{(N-l)!}\sum_{\sigma\in S(N)}\sigma\Bigg(\prod_{i=1}^{l}(x_i|a)^{\alpha_i}\prod_{\substack{i\leqslant l,\\ i<j\leqslant N}}\frac{x_i+x_j}{x_i-x_j}\Bigg),$$

$$Q_\alpha(x_1,\dots,x_N|a) = 2^l P_\alpha(x_1,\dots,x_N|a),$$

where a permutation σ acts by permuting the variables x_i. If $l=0$, we set $P_\alpha = P_\varnothing = 1$. For $N<l$, set $P_\alpha(x_1,\dots,x_N|a) = Q_\alpha(x_1,\dots,x_N|a) = 0$.

For partitions of length l, this definition coincides with Ivanov's definition. Moreover, for strict partitions of length l and $a=0$, this definition gives the Schur P- and Q-functions.

Let τ denote the shift operator acting on the space of infinite sequences by $\tau a = (a_1, a_2, \dots)$.

Definition 2. Let $\alpha = (\alpha_1,\dots,\alpha_l)$ be a sequence of positive integers and $K \geqslant l$. The *dual multiparameter Schur P- and Q-functions with index* α are defined by

$$\widehat{P}_\alpha(t_1,\dots,t_K|a) = \frac{1}{(K-l)!}\sum_{\sigma\in S(K)} \sigma\biggl(\prod_{i=1}^{l}\frac{1}{(t_i|\tau a)^{\alpha_i}} \prod_{\substack{i\leqslant l,\\ i<j\leqslant K}} \frac{t_j+t_i}{t_j-t_i}\biggr),$$

$$\widehat{Q}_\alpha(t_1,\dots,t_K|a) = 2^l\,\widehat{P}_\alpha(t_1,\dots,t_K|a),$$

where σ acts by permuting the variables t_i. If $l=0$, we define $\widehat{P}_\alpha = \widehat{P}_\varnothing = 1$. For $K<l$, set $\widehat{P}_\alpha(t_1,\dots,t_K|a) = \widehat{Q}_\alpha(t_1,\dots,t_K|a) = 0$.

For $a=0$ and a strict partition λ of length l, we have $\widehat{P}_\lambda(t_1,\dots,t_K|a) = P_\lambda(t_1^{-1},\dots,t_K^{-1})$, where $P_\lambda(y_1,\dots,y_K)$ denote the Schur P-function.

Proposition 3. *For $\omega \in S(l)$ and a sequence α of length l, one has*

$$P_{\omega\alpha}(x_1,\dots,x_N|a) = \varepsilon(\omega)P_\alpha(x_1,\dots,x_N|a),$$
$$\widehat{P}_{\omega\alpha}(t_1,\dots,t_K|a) = \varepsilon(\omega)\widehat{P}_\alpha(t_1,\dots,t_K|a),$$

where $\varepsilon(\omega)$ is the sign of ω and $\omega\alpha = (\alpha_{\omega^{-1}(1)},\dots,\alpha_{\omega^{-1}(l)})$.

Proof. We will prove the assertion for P_α, the proof for $\widehat{P}_\alpha$ is similar. If $l>N$, both sides are 0, so we assume that $l\leqslant N$. Since any permutation from $S(N)$ can be written in the form $\sigma\omega^{-1}$, we have

$$\begin{aligned} &P_{\omega\alpha}(x_1,\dots,x_N|a)\\ &\quad= \frac{1}{(N-l)!}\sum_{\sigma\in S(N)} \sigma\omega^{-1}\biggl(\prod_{i=1}^{l}(x_i|a)^{\alpha_{\omega^{-1}(i)}} \prod_{\substack{i\leqslant l,\\ i<j\leqslant N}} \frac{x_i+x_j}{x_i-x_j}\biggr)\\ &\quad= \frac{1}{(N-l)!}\sum_{\sigma\in S(N)} \sigma\biggl(\prod_{i=1}^{l}(x_{\omega^{-1}(i)}|a)^{\alpha_{\omega^{-1}(i)}} \prod_{\substack{i\leqslant l,\\ i<j\leqslant N}} \frac{x_{\omega^{-1}(i)}+x_{\omega^{-1}(j)}}{x_{\omega^{-1}(i)}-x_{\omega^{-1}(j)}}\biggr). \end{aligned}$$

Since ω^{-1} permutes only the first l variables, the first product is the same as in the definition of P_α. As to the second product, the numerator is also the same, but the denominator behaves as the Vandermonde determinant, so the action of ω^{-1} on it produces the desired factor $\varepsilon(\omega^{-1}) = \varepsilon(\omega)$. □

Corollary 4. *If for $i \neq j$ we have $\alpha_i = \alpha_j$, then $P_\alpha = Q_\alpha = \widehat{P}_\alpha = \widehat{Q}_\alpha = 0$.*

Proposition 5.

(1) *$\widehat{P}_\lambda(t_1, \dots, t_K|a)$ is an element of $\mathbb{C}[[t_1^{-1}, \dots, t_K^{-1}]]$ of degree $-|\lambda|$.*

(2) *The $(-|\lambda|)$th homogenous component of $\widehat{P}_\lambda(t_1, \dots, t_K|a)$ is*

$$P_\lambda(t_1^{-1}, \dots, t_K^{-1}).$$

Proof. Rewrite the definition of $\widehat{P}_\lambda$ as

$$\widehat{P}_\lambda(t_1, \dots, t_K|a) = \frac{1}{(K-l)!} \sum_{\sigma \in S(K)} \sigma \left(\prod_{i=1}^{l} \frac{1}{(t_i|\tau a)^{\lambda_i}} \prod_{\substack{i \leqslant l, \\ i<j \leqslant K}} \frac{t_i^{-1} + t_j^{-1}}{t_i^{-1} - t_j^{-1}} \right). \tag{3}$$

Then $A = V(t_1^{-1}, \dots, t_K^{-1}) \widehat{P}_\lambda(t_1, \dots, t_K|a)$ is an element of $\mathbb{C}[[t_1^{-1}, \dots, t_K^{-1}]]$, where $V(x_1, \dots, x_k)$ denote the Vandermonde determinant. At the same time, A is skew-symmetric in $t_1^{-1}, \dots, t_K^{-1}$, so $\widehat{P}_\lambda(t_1, \dots, t_K|a)$ is also an element of $\mathbb{C}[[t_1^{-1}, \dots, t_K^{-1}]]$. The degree with respect to the variables $t_1, \dots, t_K$ of each summand in (3) is $-|\lambda|$, so the degree of $\widehat{P}_\lambda(t_1, \dots, t_K|a)$ is also $-|\lambda|$. Moreover, the highest degree terms of each summand give the definition of $P_\lambda(t_1^{-1}, \dots, t_K^{-1})$, thus the second claim follows. □

Proposition 6 (stability condition)**.** *For $K > l$ and α of length l, we have*

$$\widehat{P}_\alpha(t_1, \dots, t_{K-1}, \infty|a) = \widehat{P}_\alpha(t_1, \dots, t_{K-1}|a).$$

Proof. From the definition we have

$$\widehat{P}_\alpha(t_1, \dots, t_K|a) = \frac{1}{(K-l)!} \sum_{\sigma \in S(K)} \sigma \left(\prod_{i=1}^{l} \frac{1}{(t_i|\tau a)^{\alpha_i}} \prod_{\substack{i \leqslant l, \\ i<j \leqslant K}} \frac{t_j + t_i}{t_j - t_i} \right).$$

As t_K tends to infinity, the only surviving terms on the right-hand side will be those that correspond to permutations $\sigma \in S(K)$ such that $\sigma^{-1}(K) > l$.

Any such permutation can be written in the form $\omega\sigma_0$ where $\sigma_0 = (i, K)$ for $i > l$ and $\omega \in S(K-1)$ permutes $\{1, \dots, K-1\}$. Hence we have

$$\begin{aligned}
&\widehat{P}_\alpha(t_1, \dots, t_{K-1}, \infty|a) \\
&= \frac{1}{(K-l)!} \sum_{\sigma \in S(K)} \omega\sigma_0 \left(\prod_{i=1}^{l} \frac{1}{(t_i|\tau a)^{\alpha_i}} \prod_{\substack{i \leqslant l, \\ i<j \leqslant K-1}} \frac{t_j + t_i}{t_j - t_i} \right) \\
&= \frac{1}{(K-l-1)!} \sum_{\omega \in S(K-1)} \omega \left(\prod_{i=1}^{l} \frac{1}{(t_i|\tau a)^{\alpha_i}} \prod_{\substack{i \leqslant l, \\ i<j \leqslant K-1}} \frac{t_j + t_i}{t_j - t_i} \right) \\
&= \widehat{P}_\alpha(t_1, \dots, t_{K-1}|a).
\end{aligned}$$

□

The next formula can be thought of as a multiparameter analog of formula (2.14) in [2, Chap. 3] for the Schur P-functions.

Proposition 7. *For $N \geqslant l \geqslant 1$ and α of length l, one has*

$$P_\alpha(x|a) = \sum_{i=1}^{N} (x_i|a)^{\alpha_1} g_i(x) P_\beta(x^{(i)}|a), \tag{4}$$

where $x = (x_1, \dots, x_N)$, $x^{(i)} = (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_N)$, $g_i(x) = \prod_{j \neq i}^{N} \frac{x_i + x_j}{x_i - x_j}$, and $\beta = (\alpha_2, \dots, \alpha_l)$,

Proof. Each element $\sigma \in S(N)$ can be uniquely written as $\sigma_0\omega$, where $\sigma_0 = (1, i)$ and $\omega \in S(N-1)$ acts on $2, \dots, N$. Then one has

$$\begin{aligned}
P_\alpha(x|a) &= \frac{1}{(N-l)!} \sum_{\sigma \in S(N)} \sigma_0\omega \left(\prod_{i=1}^{l} (x_i|a)^{\alpha_i} \prod_{\substack{i \leqslant l, \\ i<j \leqslant N}} \frac{x_i + x_j}{x_i - x_j} \right) \\
&= \frac{1}{((N-1)-(l-1))!} \sum_{\sigma \in S(N)} \sigma_0 \left((x_1|a)^{\alpha_1} g_1 \omega \left(\prod_{i=2}^{l} (x_i|a)^{\alpha_i} \prod_{\substack{2 \leqslant i \leqslant l, \\ i<j \leqslant N}} \frac{x_i + x_j}{x_i - x_j} \right) \right) \\
&= \sum_{i}^{N} (1, i) \left((x_1|a)^{\alpha_1} g_1 P_{\alpha_2, \dots, \alpha_l}(x_2, \dots, x_N|a) \right),
\end{aligned}$$

which is equivalent to (4). □

§3. Independence of the parameters and the multiparameter Cauchy identity

We will say that an expression $f(x,t,a)$ is independent of the parameters, or independent of a, if for every pair of infinite sequences a and a' beginning with 0 we have $f(x,t,a) = f(x,t,a')$. A basic and important example of an expression independent of a is

$$\sum_{i=1}^{\infty} \frac{(x|a)^i}{(t|\tau a)^i} = \frac{x-a_0}{t-x} = \frac{x}{t-x}. \tag{5}$$

In the particular case $a = (0,1,2,\dots)$ this identity reduces to formula (12.5) in [5], and in the general case it can be checked in exactly the same way.

If an expression f is independent of a, then $f(x,t,a) = f(x,t,0)$. We will use this idea to prove a multiparameter Cauchy identity.

Let $\mathbb{Y}$ denote the set of all partitions, $\mathbb{Y}_l$ denote the set of partitions of length l, and $\mathbb{Y}'_l$ denote the set of strict partitions of length l.

Theorem 8 (multiparameter Cauchy identity). *For $N, K \geqslant 1$ one has*

$$\sum_{\lambda\in\mathbb{Y}} 2^{-l(\lambda)} Q_\lambda(x|a)\widehat{Q}_\lambda(t|a) = \prod_{i=1}^{N}\prod_{j=1}^{K} \frac{t_j+x_i}{t_j-x_i}. \tag{6}$$

Proof. We will prove that the left-hand side of (6) is independent of a. For that we will need several lemmas. For a sequence $\alpha = (\alpha_1,\dots,\alpha_l)$, let $(t|a)^\alpha$ denote $(t_1|a)^{\alpha_1}\dots(t_l|a)^{\alpha_l}$.

Lemma 8.1. *For arbitrary integers $N \geqslant l > 0$, the expression*

$$\sum_{\alpha_1,\dots,\alpha_l\geqslant 1}^{\infty} \frac{P_\alpha(x_1,\dots,x_N|a)}{(t|\tau a)^\alpha} \tag{7}$$

is independent of a.

Proof. We prove this by induction on l. For $l=1$ we have

$$P_{(i)}(x_1,\dots,x_N|a) = \sum_{j=1}^{N} (x_j|a)^i g_j.$$

Hence, using (5), we can rewrite (7) as

$$\sum_{j=1}^{N}\sum_{i=1}^{\infty} \frac{(x_j|a)^i}{(t|\tau a)^i} g_j = \sum_{j=1}^{N} \frac{x_j}{t-x_j} g_j,$$

which is independent of a.

Assume that we have already proved the lemma for l. Then for $N \geqslant l+1$ we have from (4)

$$\sum_{\substack{\alpha_j \geqslant 1\\ j\in[1,l+1]}}^{\infty} \frac{P_\alpha(x_1,\dots,x_N|a)}{(t|\tau a)^\alpha} = \sum_{i=1}^{N}\sum_{\alpha_j\geqslant 1}^{\infty} g_i \frac{(x_i|a)^{\alpha_1}}{(t_1|\tau a)^{\alpha_1}} \frac{P_{(\alpha_2,\dots,\alpha_{l+1})}(x^{(i)}|a)}{(t_2|\tau a)^{\alpha_2}\dots(t_{l+1}|\tau a)^{\alpha_{l+1}}}$$
$$= \sum_{i=1}^{N} g_i \left(\sum_{\alpha_1\geqslant 1}^{\infty} \frac{(x_i|a)^{\alpha_1}}{(t_1|\tau a)^{\alpha_1}}\right)\left(\sum_{\alpha_2,\dots,\alpha_{l+1}\geqslant 1}^{\infty} \frac{P_{(\alpha_2,\dots,\alpha_{l+1})}(x^{(i)}|a)}{(t_2|\tau a)^{\alpha_2}\dots(t_{l+1}|\tau a)^{\alpha_{l+1}}}\right).$$

Both sums in the parentheses are independent of a, the first one is (5) and the second one is (7) for l, both independent of a. Then the whole expression is independent of a. □

Lemma 8.2. *For arbitrary integers $N, K \geqslant l \geqslant 1$, the expression*

$$\sum_{\lambda\in\mathbb{Y}_l} P_\lambda(x_1,\dots,x_N|a)\widehat{P}_\lambda(t_1,\dots,t_K|a) \tag{8}$$

is independent of a.

Proof. By the definition of $\widehat{P}$ and Corollary 4, we have that (8) is equal to

$$\sum_{\lambda\in\mathbb{Y}'_l} \frac{1}{(K-l)!} P_\lambda(x_1,\dots,x_N|a) \sum_{\sigma_t\in S(K)} \sigma_t\left(\prod_{i=1}^{l}\frac{1}{(t_i|\tau a)^{\lambda_i}} \prod_{\substack{i\leqslant l,\\ i<j\leqslant K}} \frac{t_j+t_i}{t_j-t_i}\right),$$

where σ_t acts on $t_1,\dots,t_K$. Consider the quotient $S(K)/S(l)$, and for each $\omega_t \in S(K)/S(l)$ choose an arbitrary element $\omega_t^0 \in \omega_t$. Then any permutation $\sigma_t \in S(K)$ can be uniquely written in the form $\omega_t^0\phi_t$, where $\phi_t \in S(l)$ permutes only $\{1,\dots,l\}$ and $\omega_t \in S(K)/S(l)$. As in the proof of Lemma 3, acting by ϕ_t on $\prod_{i\leqslant l, i<j\leqslant K} \frac{t_j+t_i}{t_j-t_i}$ gives us the multiplication by $\varepsilon(\phi_t)$, so we

have that (8) is equal to

$$\sum_{\omega_t}\sum_{\phi_t}\sum_{\lambda\in\mathbb{Y}'_l}\frac{1}{(K-l)!}\,\omega_t^0\biggl(\varepsilon(\phi_t)P_\lambda(x_1,\dots,x_N|a)\prod_{i=1}^{l}\frac{1}{(t_{\phi_t(i)}|\tau a)^{\lambda_i}}\prod_{\substack{i\leqslant l,\\ i<j\leqslant K}}\frac{t_j+t_i}{t_j-t_i}\biggr)$$

$$=\frac{1}{(K-l)!}\sum_{\omega_t}\omega_t^0\biggl(\sum_{\phi_t\in S(l)}\sum_{\lambda\in\mathbb{Y}'_l}\varepsilon(\phi_t)P_\lambda(x|a)\prod_{i=1}^{l}\frac{1}{(t_i|\tau a)^{\lambda_{\phi_t^{-1}(i)}}}\prod_{\substack{i\leqslant l,\\ i<j\leqslant K}}\frac{t_j+t_i}{t_j-t_i}\biggr)$$

$$=\frac{1}{(K-l)!}\sum_{\omega_t}\omega_t^0\biggl(\sum_{\phi_t\in S(l)}\sum_{\lambda\in\mathbb{Y}'_l}\frac{P_{\phi_t\lambda}(x|a)}{(t|\tau a)^{\phi_t\lambda}}\prod_{\substack{i\leqslant l,\\ i<j\leqslant K}}\frac{t_j+t_i}{t_j-t_i}\biggr)$$

$$=\frac{1}{(K-l)!}\sum_{\omega_t}\omega_t^0\biggl(\Bigl(\sum_{\alpha_i\geqslant 1,\alpha_i\neq\alpha_j}\frac{P_\alpha(x|a)}{(t|\tau a)^\alpha}\Bigr)\prod_{\substack{i\leqslant l,\\ i<j\leqslant K}}\frac{t_j+t_i}{t_j-t_i}\biggr),$$

which is independent of a by Lemma 8.1 (the sums coincide by Corollary 4). So (8) is independent of a. □

From the previous lemma we derive that the left-hand side of (6) is independent of a. Hence it suffices to prove (6) for $a = 0$. But in this case the desired identity is the Cauchy identity for the Schur Q-polynomials with $y_i = t_i^{-1}$. □

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