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N. Gogin, M. Hirvensalo

ON THE GENERATING FUNCTION OF DISCRETE CHEBYSHEV POLYNOMIALS

Abstract. We give a closed form for the generating function of the discrete Chebyshev polynomials. The closed form consists of the MacWilliams transform of Jacobi polynomials together with a binomial multiplicative factor. It turns out that the desired closed form is a solution to a special case of the Heun differential equation, and that the closed form implies combinatorial identities that appear quite challenging to prove directly.

§1. Introduction

The discrete Chebyshev polynomials belong to the rich family of *orthogonal polynomials* (see [9] for a general treatise on orthogonal polynomials and [2] for a previous work of the authors). The inner product associated to the discrete Chebyshev polynomials is defined with a discrete weight function, and hence the vector space $\mathcal{P}_N$ of polynomials having degree at most N forms a natural reference for the orthogonal polynomials discussed in this article.

The sum and the scalar product in $\mathcal{P}_N$ are defined pointwise, the scalar product being defined as

$$\langle p, q\rangle_w = \sum_{l=0}^{N} w_l p(l) q(l). \tag{1.1}$$

The *Krawtchouk polynomials* (see [6]) $K_0^{(N)}, K_1^{(N)}, \ldots, K_N^{(N)}$ (of order N) are orthogonal with respect to the weight function $w_l = \binom{N}{l}$, and the discrete Chebyshev polynomials $D_0^{(N)}, D_1^{(N)}, \ldots, D_N^{(N)}$ of order N, with respect to the weight function given by $w_l = 1$ for every l. In addition to orthogonality, we have $\deg(K_k^{(N)}) = \deg(D_k^{(N)}) = k$ for every $k \in \{0, 1, \ldots, N\}$.

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As (orthogonal) polynomials with increasing degrees, the discrete Chebyshev polynomials form a basis of $\mathcal{P}_N$, and hence any polynomial p of degree at most N can be uniquely represented as

$$p = d_0 D_0^{(N)} + d_1 D_1^{(N)} + \ldots + d_N D_N^{(N)}, \tag{1.2}$$

where $d_l \in \mathbb{C}$. The coefficients d_l in (1.2) are called the *discrete Chebyshev coefficients* of p. Since the discrete Chebyshev polynomials are orthogonal with respect to a constant weight function, they have the following property important in approximation theory: with respect to the norm $||p-q||^2 = \sum_{l=0}^{N} (p(l)-q(l))^2$, the best approximation of p in $\mathcal{P}_M$ can be found by simply taking $M+1$ first summands of (1.2) (see [4], for instance).

§2. Preliminaries

2.1. The discrete Chebyshev polynomials. There are various ways to construct polynomials orthogonal with respect to the scalar product (1.1) with the weight function $w_l = 1$ so that $\deg(D_k^{(N)}) = k$.

We choose a construction analogous to that of the *Legendre polynomials* [9]. We first define the difference operator Δ by $\Delta f(x) = f(x+1) - f(x)$, the binomial coefficient $\binom{x}{k} = \frac{1}{k!} x(x-1)\ldots(x-k+1)$, and, finally,

$$D_k^{(N)}(x) = (-1)^k \Delta^k \left(\binom{x}{k} \binom{x-N-1}{k} \right). \tag{2.1}$$

It is straightforward to see that the polynomials D_k (hereafter, we omit the superscript N if there is no danger of confusion) defined above form a basis of $\mathcal{P}_N$ orthogonal with respect to the scalar product (1.1) with $w_l = 1$. Moreover, clearly, $\deg(D_k) = k$, since one application of Δ decreases the degree of a polynomial by one, see [3].

In this article, *we regard* (2.1) *as the definition of the discrete Chebyshev polynomials*, but it is also easy to see that the following explicit expressions hold (see [3]):

$$\begin{aligned} D_k^{(N)}(x) &= \sum_{l=0}^{k} (-1)^l \binom{k}{l} \binom{N-x}{k-l} \binom{x}{l} \\ &= \sum_{l=0}^{k} (-1)^l \binom{k+l}{k} \binom{N-l}{k-l} \binom{x}{l}. \end{aligned} \tag{2.2}$$

Also, it is rather easy to verify that the discrete Chebyshev polynomials satisfy the following recurrence relation:

$$k^2 D_k = (2k-1)D_1 D_{k-1} - (N+k)(N-k+2)D_{k-2}, \qquad (2.3)$$

$D_0 = 1$, $D_1 = N - 2x$ (see [3]). The recurrence (2.3) also extends the definition of D_k to $k > N$.

The method of using *generating functions* is among the cornerstones of various areas of mathematics, and does not need any further introduction. We merely focus on the very simple form of the generating function of the Krawtchouk polynomials (see [6]):

$$(1+t)^{N-x}(1-t)^x = \sum_{k=0}^{\infty} K_k^{(N)}(x) t^k. \qquad (2.4)$$

In fact, when studying the binomial distributions, it is quite natural to *define* the Krawtchouk polynomials via (2.4).

On the other hand, the quest for the generating function of the discrete Chebyshev polynomials seems to be a more complicated task. In what follows, we give a closed form for the generating function

$$\sum_{k=0}^{\infty} D_k^{(N)}(x) t^k. \qquad (2.5)$$

It should be noticed, however, that some useful closed-form expressions carrying information about the discrete Chebyshev polynomials have been found before. For instance, in [5], the expression

$$(1+t)^k (1+s)^{N-x} (1-st)^x \qquad (2.6)$$

having the property that the coefficient of $s^k t^k$ equals $D_k^{(N)}(x)$ is given.

2.2. A differential equation for Jacobi polynomials. For a nonnegative integer n, the *Jacobi polynomial* $P_n^{(\alpha,\beta)}(x)$ is, up to a constant factor, the unique entire rational solution to the *differential equation* (*for Jacobi polynomials*)

$$(1-x^2)y'' + \big(\beta - \alpha - (\alpha+\beta+2)x\big)y' + n(n+\alpha+\beta+1)y = 0 \qquad (2.7)$$

(see [1]).

In this article, we are interested in the Jacobi polynomials with parameters $\alpha = 0, \beta = -(N+1)$, where $N > 0$ is a fixed integer. We also substitute x for n and t for x in Eq. (2.7), and denote $J_x^{(N+1)}(t) = P_x^{(0,-N-1)}(t)$. We usually omit the superscript $N+1$ and denote $J_x(t) = J_x^{(N+1)}(t)$. Then $J_x(t)$ satisfies the differential equation

$$(1-t^2)J_x''(t) - (N+1-(N-1)t)J_x'(t) + x(x-N)J_x(t) = 0. \tag{2.8}$$

Recall that in this context, x is a fixed nonnegative integer. The polynomial $J_x(t)$ can be expressed as

$$J_x(t) = \frac{1}{2^x}\sum_{k=0}^{x}\binom{x}{k}\binom{x-N-1}{k}(t-1)^k(t+1)^{x-k} \tag{2.9}$$

(see [1]). Since (2.8) is clearly invariant under the substitution $x \leftarrow N-x$, we have the symmetry

$$J_{N-x}(t) = J_x(t) \tag{2.10}$$

(see [1]).

2.3. The MacWilliams transform. The *MacWilliams transform* of order x for a polynomial P is defined as

$$\widehat{P}_x(t) = (1+t)^x P\left(\frac{1-t}{1+t}\right). \tag{2.11}$$

As definition (2.11) shows, the MacWilliams transform is a special case of the Möbius transformation together with the factor $(1+t)^x$. If the subscript x is clear by the context, we may omit it. It is also straightforward to see that if x is an integer such that $\deg(P) \leq x$, then $\widehat{P}$ is again a polynomial. In this article, we will, however, face situations with noninteger values of x, and it is worth noticing already here that (2.11) shows that if $t > -1$, then $\widehat{P}_x(t)$ is a uniquely defined differentiable function of the real variable x.

In what follows, $\widehat{J}_x(t)$ stands for the MacWilliams transform of J_x of order x. It is then straightforward to uncover a representation for $\widehat{J}_x(t)$:

$$\widehat{J}_x(t) = \widehat{(J_x)}_x(t) = \sum_{k=0}^{x}(-1)^k\binom{x}{k}\binom{x-N-1}{k}t^k. \tag{2.12}$$

The symmetry (2.10) implies straightforwardly that

$$\begin{aligned}\widehat{J}_{N-x}(t) &= \widehat{(J_{N-x})}_{N-x}(t) = (1+t)^{N-x}J_{N-x}\left(\frac{1-t}{1+t}\right)\\ &= (1+t)^{N-2x}(1+t)^x J_x\left(\frac{1-t}{1+t}\right) = (1+t)^{N-2x}\widehat{J}_x(t).\end{aligned}$$

The equality

$$\widehat{J}_{N-x}(t) = (1+t)^{N-2x}\widehat{J}_x(t) \tag{2.13}$$

thus obtained will be important in understanding the alternative representation of the generating function introduced in Sec. 5.

§3. The Heun equation

A differential equation for the MacWilliams transform of $J_x(t)$ can be found easily. For short, we denote $J(t) = J_x(t)$ and $\widehat{J}(t) = \widehat{J}_x(t)$ in the following lemmas.

Lemma 1. *The function $\widehat{J}(t)$ satisfies the differential equation*

$$t(1+t)\widehat{J}''(t) + (Nt + 1 - 2t(x-1))\widehat{J}'(t) + x(x-N-1)\widehat{J}(t) = 0. \tag{3.1}$$

Proof. By computing the derivatives of $\widehat{J}(t) = (1+t)^x J(\frac{1-t}{1+t})$, we can represent $\widehat{J}(t)$, $\widehat{J}'(t)$, and $\widehat{J}''(t)$ in terms of $J(\frac{1-t}{1+t})$, $J'(\frac{1-t}{1+t})$, and $J''(\frac{1-t}{1+t})$. A direct calculation allows us also to reverse the representations to get

$$\begin{aligned} J(\frac{1-t}{1+t}) &= (1+t)^{-x}\widehat{J}(t), & (3.2)\\ J'(\frac{1-t}{1+t}) &= \frac{1}{2}x(1+t)^{-x+1}\widehat{J}(t) - \frac{1}{2}(1+t)^{-x+2}\widehat{J}'(t), \text{ and} & (3.3)\\ J''(\frac{1-t}{1+t}) &= \frac{1}{4}x(x-1)(1+t)^{-x+2}\widehat{J}(t) & \\ &- \frac{1}{2}(x-1)(1+t)^{-x+3}\widehat{J}'(t) + \frac{1}{4}(1+t)^{-x+4}\widehat{J}''(t). & (3.4) \end{aligned}$$

Replacing t with $\frac{1-t}{1+t}$ in (2.8) and substituting (3.2)–(3.4) into (2.8) gives us the claim.

Another way to prove the lemma is to use (2.12) and verify by direct calculations that the differential equation (3.1) is satisfied. □

Lemma 2. *Let $T(t)$ be defined as $T(t) = (1+t)^{N-2x}\widehat{J}(-t^2)$. Then $T(t)$ satisfies the differential equation*

$$\begin{aligned} (t^3 - t)T''(t) &+ \big(2t(N-2x) + 3t^2 - 1\big)T'(t) \\ &+ \big(N - 2x - tN(N+2)\big)T(t) = 0. \end{aligned} \tag{3.5}$$

Proof. As in the previous lemma, we can express $T(t)$, $T'(t)$, and $T''(t)$ in terms of $\widehat{J}(-t^2)$, $\widehat{J}'(-t^2)$, and $\widehat{J}''(-t^2)$, and then reverse the representations to get

$$\widehat{J}(-t^2) = (1+t)^{2x-N}T(t), \tag{3.6}$$

$$\widehat{J}'(-t^2) = \frac{1}{2t}(N-2x)(1+t)^{2x-N-1}T(t) - \frac{1}{2t}(1+t)^{2x-N}T'(t), \tag{3.7}$$

$$\begin{aligned}\widehat{J}''(-t^2) = {} & \frac{1}{4t^3}(N-2x)(1+t)^{2x-N-2}(t(N-2x+2)+1)T(t) \\ & - \frac{1}{4t^3}(2t(N-2x)+t+1)(1+t)^{2x-N-1}T'(t) \\ & + \frac{1}{4t^2}(1+t)^{2x-N}T''(t)\end{aligned} \tag{3.8}$$

by direct calculations. By substituting $-t^2$ for t in (3.1) and using (3.6)–(3.8), we get the differential equation (3.5) after some direct calculations. □

The differential equation (3.5) can be easily rewritten in *standard natural form for the Heun differential equation*

$$\begin{aligned} t(t-1)(t-q)y''(t) + \big(c(t-1)(t-q) + d\cdot t(t-q) \\ + (a+b+1-c-d)t(t-1)\big)y'(t) + (abt-\lambda)y(t) = 0\end{aligned}$$

(see [8]) by taking $q=-1$, $a=-N$, $b=N+2$, $c=1$, $d=N-2x+1$, and $\lambda = 2x-N$. So, the *generalized Riemann scheme* (see [8]), describing the local characteristic properties of this equation, is as follows:

$$\begin{pmatrix} 1 & 1 & 1 & 1 & \\ 0 & 1 & -1 & \infty & ;t \\ 0 & 0 & 0 & -N & ;2x-N \\ 0 & 2x-N & N-2x & N+2 & \end{pmatrix}.$$

§4. The generating function

By (2.12), the function $T(t) = (1+t)^{N-2x}\widehat{J}_x(-t^2)$ can be represented as

$$T(t) = (1+t)^{N-2x}\sum_{k=0}^{x}\binom{x}{k}\binom{x-N-1}{k}t^{2k}. \tag{4.1}$$

If $t\in(-1,1)$, we should keep in mind that $\widehat{J}_x(-t^2) = (1+t^2)^x J(\frac{1+t^2}{1-t^2})$ can be straightforwardly defined for any real values of x. Hence for $t\in(-1,1)$, also $T(t) = (1+t)^{N-2x}\widehat{J}_x(-t^2)$ can be defined for an arbitrary real value of x, even though (4.1) is meaningful only for integer values of x (because of the summation upper bound). Another way of generalizing (4.1) even to complex values of x is to expand (4.1) straightforwardly to see that if we write

$$T(t) = \sum_{k=0}^{\infty} \tau_k(x) t^k, \tag{4.2}$$

then

$$\tau_k(x) = \sum_{0 \le l \le k/2} \binom{N-2x}{k-2l} \binom{x}{l} \binom{x-N-1}{l} \tag{4.3}$$

is a polynomial of degree k. For any fixed x, the function $T(t)$ is an analytic function of t in the disc $|t| < 1$ (we can use the principal branch of the logarithm to define the power), and hence it has a unique Maclaurin expansion (4.2) convergent when $|t| < 1$.

That (4.2) converges for $|t| < 1$ can also be verified by using the ratio test, but estimating $|\tau_{k+1}(x)/\tau_k(x)|$ as k tends to infinity is not very straightforward. On the other hand, the recurrence from the next lemma reveals that $\lim\limits_{k\to\infty} |\tau_{k+1}(x)/\tau_k(x)| = 1$.

Remark 1. The polynomials $\tau_k(x)$ for small values of k are easy to find by using formula (4.3). For instance, $\tau_0(x) = 1$, $\tau_1(x) = N - 2x$, and $\tau_2(x) = 3x^2 - 3Nx + \frac{1}{2}N(N-1)$.

Lemma 3. *For $k \ge 2$, the polynomials $\tau_k(x)$ satisfy the recurrence relation*

$$k^2\tau_k(x) = (2k-1)(N-2x)\tau_{k-1}(x) - (N+k)(N-k+2)\tau_{k-2}(x). \tag{4.4}$$

Proof. This is a general property for a generic solution to the Heun equation, see [8]. Recurrence (4.4) can also be obtained by differentiating and substituting (4.2) to Eq. (3.5). □

Remark 2. From (4.4) it follows that

$$\frac{\tau_k(x)}{\tau_{k-1}(x)} = \frac{(2k-1)(N-2x)}{k^2} - \frac{(N+k)(N-k+2)}{k^2}\frac{\tau_{k-2}(x)}{\tau_{k-1}(x)},$$

which shows that the relation $\lim_{k\to\infty} |\tau_{k+1}(x)/\tau_k(x)| = \infty$ cannot hold. Since, clearly, $\tau_k(x)$ is a rational expression in k, the limit exists and is finite. Now the equation

$$\frac{\tau_k(x)}{\tau_{k-1}(x)} \cdot \frac{\tau_{k-1}(x)}{\tau_{k-2}(x)} = \frac{(2k-1)(N-2x)}{k^2} \cdot \frac{\tau_{k-1}(x)}{\tau_{k-2}(x)} - \frac{(N+k)(N-k+2)}{k^2}$$

shows that $\lim_{k\to\infty} |\tau_{k+1}(x)/\tau_k(x)| = 1$.

We are now ready to state the main result.

Theorem 1. *The function*

$$T_{N,x}(t) = (1+t)^{N-2x}\widehat{J}_x(-t^2) \tag{4.5}$$

is the generating function of the discrete Chebyshev polynomials, that is, $\tau_k(x) = D_k(x)$ for every $k \geq 0$.

Proof. By (2.3), the discrete Chebyshev polynomials satisfy the same recurrence relation (4.4) as the polynomials $\tau_k(x)$. Since the initial conditions $\tau_0(x) = D_0(x)$ and $\tau_1(x) = D_1(x)$ hold by Remark 1, we have the equality $\tau_k(x) = D_k(x)$ for every k. □

Remark 3. It may be useful to compare formulas (4.5) and (2.4). Let $I_x(u) = 1 = u^0$. Then, by (2.11),

$$\widehat{I}_x(t) = (1+t)^x\left(\frac{1-t}{1+t}\right)^0 = (1+t)^x,$$

so the generating function of the Krawtchouk polynomials can be written as

$$(1+t)^{N-x}(1-t)^x = (1+t)^{N-2x}(1-t^2)^x = (1+t)^{N-2x}\widehat{I}_x(-t^2),$$

whereas the generating function of the discrete Chebyshev polynomials is

$$(1+t)^{N-2x}\widehat{J}_x(-t^2),$$

where

$$J_x(u) = \frac{1}{2^x}\sum_{k=0}^{x}\binom{x}{k}\binom{x-N-1}{k}(u-1)^k(u+1)^k.$$

Note also that formula (2.13) is (trivially) valid for I as well as for J.

Moreover,

$$I_x(u) = 1 = \frac{1}{2^x}((1-u)+(1+u))^x = \frac{1}{2^x}\sum_{k=0}^{x}\binom{x}{k}(1-u)^k(1+u)^{x-k},$$

whereas

$$\begin{aligned} J_x(u) &= \frac{1}{2^x}\sum_{k=0}^{x}(-1)^k\binom{x-N-1}{k}\left(\binom{x}{k}(1-u)^k(1+u)^{x-k}\right) \\ &= \frac{1}{2^x}\sum_{k=0}^{x}\binom{N+k-x}{k}\binom{x}{k}(1-u)^k(1+u)^{x-k}. \end{aligned}$$

In addition, the sum of the coefficients is equal to

$$\sum_{k=0}^{x} \binom{N+k-x}{k} = \sum_{k=0}^{x} \binom{(N-x)+k}{(N-x)} = \binom{N+1}{x}.$$

§5. Concluding remarks

Example 1. Expression (4.5) shows that if x is an integer not exceeding $N/2$, then $T_{N,x}(t)$ is a polynomial in t of degree $N-2x+2x=N$. Thus we can find expressions

$$T_{N,x}(t) = \sum_{n=0}^{N} D_n^{(N)}(x) t^n$$

by simply evaluating $D_n(N)(x)$ for $n \in \{0,1,\ldots,N\}$ by via (2.3) or (4.3). For example, $N=6$ gives

$$T_{6,0}(t) = 1+6t+15t^2+20t^3+15t^4+6t^5+t^6 = (1+t)^6,$$
$$T_{6,1}(t) = 1+4t+0\cdot t^2-20t^3-35t^4-24t^5-6t^6 = (1+t)^4(1-6t^2),$$
$$T_{6,2}(t) = 1+2t-9t^2-20t^3+5t^4+30t^5+15t^6 = (1+t)^2(1-10t^2+15t^4),$$
$$T_{6,3}(t) = 1-12t^2+30t^4-20t^6,$$

which is in full accordance with (4.5) and (2.12). For $x \in \{4,5,6\}$, the power $6-2x$ of $1+t$ in (4.5) is no longer positive, so it is not clear anymore whether $T_{6,x}(t)$ would be a polynomial. But if $T_{6,x}$ were not a polynomial for $x \in \{4,5,6\}$, then there would be a rather mysterious asymmetry between $x \le 3$ and $x > 3$. Fortunately, it is easy to show that $T_{N,x}(t)$ is indeed a polynomial for each $x \in \{0,1,\ldots,N\}$ and the asymmetry actually vanishes via the trivial equality $1-t^2 = (1+t)(1-t)$.

Theorem 2. *The generating function $T_{N,x}(t)$ can also be represented as*

$$T_{N,x}(t) = (1-t)^{2x-N} \widehat{J}_{N-x}(-t^2). \tag{5.1}$$

Proof. Equality (2.13) implies

$$\widehat{J}_{N-x}(-t^2) = (1-t^2)^{N-2x} \widehat{J}_x(-t^2) = (1-t)^{N-2x}(1+t)^{N-2x} \widehat{J}_x(-t^2),$$

and the claim follows immediately. □

Example 2 (Example 1 continued)**.** Since, by Theorem 2, the expressions $T_{6,x}(t)$ are polynomials in t of degree 6, we can evaluate their values for $x \in \{4,5,6\}$ as

$$T_{6,4}(t) = 1 - 2t - 9t^2 + 20t^3 + 5t^4 - 30t^5 + 15t^6 = (1-t)^2(1 - 10t^2 + 15t^4),$$
$$T_{6,5}(t) = 1 - 4t + 0 \cdot t^2 + 20t^3 - 35t^4 + 24t^5 - 6t^6 = (1-t)^4(1-6t^2),$$
$$T_{6,6}(t) = 1 - 6t + 15t^2 - 20t^3 + 15t^4 - 6t^5 + t^6 = (1-t)^6.$$

This is again in full accordance with (5.1) and (2.12).

Combining Theorems 1 and 2 into a single presentation is straightforward.

Theorem 3 (explicit polynomial form for $x \in \{0,1,\dots,N\}$)**.** *The generating function $T_{N,x}(t)$ can be presented as a polynomial in t of degree N:*

$$T_{N,x}(t) = (1 + t \cdot \operatorname{sign}(N-2x))^{|N-2x|} \hat{\mathcal{J}}^{(N)}_{\min\{x,N-x\}}(-t^2).$$

Remark 4. Theorem 1 implies that (2.2) and (4.3) are equal, i.e.,

$$\sum_{0 \le l \le k/2} \binom{N-2x}{k-2l}\binom{x}{l}\binom{x-N-1}{l} = \sum_{l=0}^{k} (-1)^l \binom{k}{l}\binom{N-x}{k-l}\binom{x}{l}. \qquad (5.2)$$

A direct combinatorial proof of (5.2) appears to be challenging, for instance, the techniques of [7] seem to be powerless in this case. Theorem 2 implies an identity similar to (5.2).

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Department of Mathematics and Statistics,
University of Turku,
FI-20014 Turku, Finland
E-mail: ngiri@list.ru
E-mail: mikhirve@utu.fi

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