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MULTI-DIMENSIONAL RANDOM WALKS AND INTEGRABLE PHASE MODELS

Abstract. We consider random multi-dimensional lattice walks bounded by a hyperplane, calling them walks over multi-dimensional simplicial lattices. We demonstrate that generating functions of these walks are dynamical correlation functions of a certain type of exactly solvable quantum phase models describing strongly correlated bosons on a chain. Walks over oriented lattices are related to the phase model with a non-Hermitian Hamiltonian, while walks over disoriented ones are related to the model with a Hermitian Hamiltonian. The calculation of the generating functions is based on the algebraic Bethe Ansatz approach to the solution of integrable models. The answers are expressed through symmetric functions. Continuous-time quantum walks bounded by a one-dimensional lattice of finite length are also studied.

§1. Introduction

The theory of random walks, being a classical part of enumerative combinatorics [1], appears in different fields of mathematics [2–4], physics [5,6], and information theory [7–9]. Certain quantum integrable models solvable by the quantum inverse scattering method [10, 11] demonstrate a close relationship [12–16] with different objects of combinatorics [17] and the theory of symmetric functions [18]. In the present paper, we consider random multi-dimensional lattice walks bounded by a hyperplane. We call them walks over *multi-dimensional simplicial lattices*. Our approach is based on the analysis of dynamical correlation functions of integrable phase models. The obtained correlation functions have the meaning of the generating functions of random walks.

Quantum walks [19–21] are of considerable recent interest due to their role in quantum computations and quantum information processing [22–24]. Walks on multi-dimensional lattices were studied by numerous authors [25–29].

Key words and phrases: multi-dimensional random walk, quantum walk, phase model, correlation function, symmetric functions.

The dynamical variables of exactly solvable phase models are the so-called *phase operators* introduced in quantum optics in connection with the quantum phase problem [30]. We apply a model defined by a non-Hermitian Hamiltonian [31] to the description of walks on an oriented M-dimensional simplicial lattice (see [32] for the theory of equipped graded graphs and projective limits of simplices). A model with a Hermitian Hamiltonian [33,34] is used in the description of walks on an disoriented lattice. Various boundary conditions will be studied. It should be mentioned that the model with the non-Hermitian Hamiltonian finds an application in the theory of low-dimensional non-equilibrium statistical mechanics, and the model with the Hermitian Hamiltonian describes strongly correlated bosons on a chain.

The behavior of continuous-time quantum walks under the influence of an external environment is of interest. The analysis of one-dimensional continuous-time quantum walks at different values of the bias potential will be given.

The paper is organized as follows. Section 1 is introductory. In Sec. 2, random walks over an M-dimensional simplicial lattice with general and retaining boundary conditions are introduced. The integrable phase models are introduced in Sec. 3. The solution of the generalized phase model is given in Sec. 4. In Sec. 5, the calculation of generating functions is presented. The phase model is studied in Sec. 6, and continuous-time quantum walks are studied in Sec. 7.

§2. Multi-dimensional lattice walks bounded by a hyperplane

Starting from the $(M+1)$-dimensional hypercubical lattice with unit spacing $\mathbb{Z}^{M+1} \ni \mathbf{m} \equiv (m_0, m_1, \ldots, m_M)$, let us define the nonnegative orthant $\mathbb{N}_0^{M+1} \equiv \{\mathbf{m} \mid 0 \leqslant m_i,\ i \in \mathcal{M}\}$ as a subspace of $\mathbb{Z}^{M+1}$ (hereafter, $\mathcal{M} \equiv \{0, 1, \ldots, M\}$). Consider the subset of $\mathbb{N}_0^{M+1}$ consisting of the sites with coordinates constrained by the requirement $m_0 + m_1 + \ldots + m_M = N$:

$$\mathrm{Simp}_{(N)}(\mathbb{Z}^{M+1}) \equiv \Big\{\mathbf{m} \in \mathbb{N}_0^{M+1} \,\Big|\, \sum_{i \in \mathcal{M}} m_i = N\Big\}\,. \tag{1}$$

Then $\mathrm{Simp}_{(N)}(\mathbb{Z}^{M+1})$ is a compact M-dimensional set, and we will call it a *simplicial lattice*. A two-dimensional triangular simplicial lattice is presented in Fig. 1.

A sequence of $K+1$ points in $\mathbb{Z}^{M+1}$ is called a *random walk* with K steps, see [4]. Random walks over sites of $\mathrm{Simp}_{(N)}(\mathbb{Z}^{M+1})$ are determined

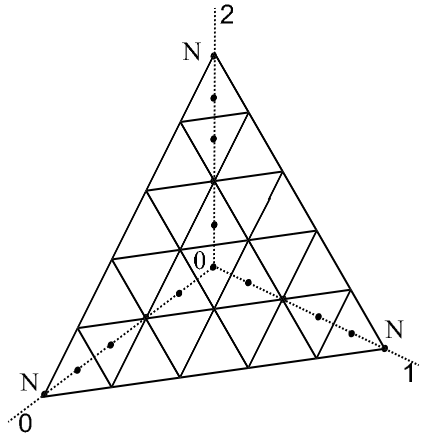

Fig. 1. A two-dimensional triangular simplicial lattice.

by a set of admissible steps (*step set*) Ω_M such that at each step the ith coordinate m_i increases by 1 while the nearest neighboring one decreases by 1. Namely, each element of Ω_M is a sequence $(e_0, e_2, \dots, e_M)$ such that $e_i = \pm 1$, $e_{i+1} = \mp 1$ for all pairs $(i, i+1)$ with $0 \leqslant i \leqslant M$ and $M + 1 = 0 \pmod 2$, and $e_j = 0$ for all $j \in \mathcal{M}$ and $j \neq i, i+1$. The step set $\Omega_M \equiv \Omega_M(\mathbf{m}_0)$ ensures that the trajectory of the random walk determined by the starting point $\mathbf{m}_0$ lies in the M-dimensional set $\mathrm{Simp}_{(N)}(\mathbb{Z}^{M+1})$. An example of step processes in the one-dimensional lattice is given in Fig. 2.

Directed random walks on the M-dimensional *oriented simplicial lattice* are determined by a step set $\Gamma_M = (k_0, k_1, \dots, k_M)$ such that $k_i = -1$, $k_{i+1} = 1$ for all pairs $(i, i+1)$ with $i \in \mathcal{M}$ and $M + 1 = 0 \pmod 2$, and $k_j = 0$ for $j \in \mathcal{M} \backslash \{i, i+1\}$. An oriented two-dimensional simplicial lattice is presented in Fig. 3.

It might occur that some points on the boundary of the simplicial lattice also belong to a random walk trajectory. Therefore, the walker's movements should be supplied with appropriate boundary conditions. The *general boundary conditions* are defined by the requirement that admissible steps along the boundary of $\mathrm{Simp}_{(N)}(\mathbb{Z}^{M+1})$ are taken from the step sets Ω_M or Γ_M.

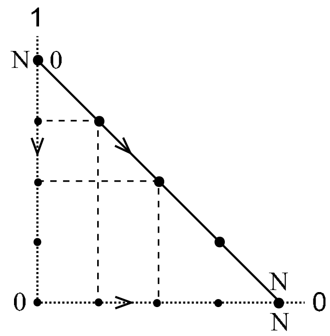

Fig. 2. Step processes for the one-dimensional nearest-neighbor random walk on a segment [0, N].

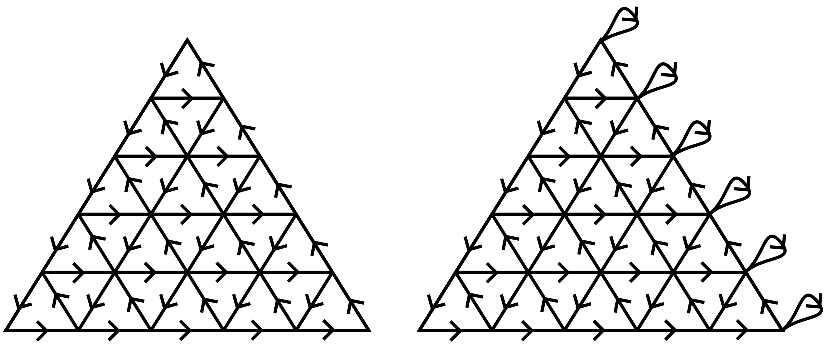

Fig. 3. The oriented two-dimensional simplicial lattice determined by the step set Γ_2 with (a) general and (b) retaining ($g_0 = 1$, $g_1 = g_2 = 0$) boundary conditions. Steps are allowed only in the direction of arrows.

Under the *retaining boundary conditions*, the walker comes to a node of the boundary and either continues to move in accordance with the elements of Ω_M (or Γ_M) or keeps staying in the node. The boundary of the simplicial lattice consists of $M+1$ faces of highest dimension $M-1$. To each node of the sth component of the boundary, $s \in \mathcal{M}$, a weight g_s is assigned which is 0 or 1. When all g_s are zero, the model corresponds to the general boundary conditions.

As an example, walks determined by the step set $\Omega_1 = \{(1,-1),(-1,1)\}$ go over sites of the one-dimensional set $\mathrm{Simp}_{(N)}(\mathbb{Z}^2)$. The step set

$$\Omega_2 = \{(-1,1,0),(1,-1,0),(0,-1,1),(0,1,-1),(1,0,-1),(-1,0,1)\}$$

for random walks, or $\Omega_2 = \{(-1,1,0),(0,-1,1),(1,0,-1)\}$ for directed walks, ensures that the trajectories belong to $\mathrm{Simp}_{(N)}(\mathbb{Z}^3)$.

Consider the case of the retaining boundary conditions. The exponential generating function of lattice walks is defined as the formal series

$$F^{(N)}(\mathbf{l},\mathbf{j}|t) = \sum_{k=0}^{\infty} \frac{t^k}{k!} G_k^{(N)}(\mathbf{l},\mathbf{j})\,, \tag{2}$$

where the coefficients $G_k^{(N)}(\mathbf{l},\mathbf{j})$ characterize the k-step walks with the end nodes $\mathbf{l},\mathbf{j} \in \mathrm{Simp}_{(N)}(\mathbb{Z}^{M+1})$. The generating function (2) satisfies the master equation

$$\begin{aligned}\partial_t F^{(N)}(\mathbf{l},\mathbf{j}|t) = & \sum_{s=0}^{M} F^{(N)}(\mathbf{l}, j_0, j_1, \dots, j_s - 1, j_{s+1} + 1, \dots j_M) \\ & + \sum_{s=0}^{M} F^{(N)}(\mathbf{l}, j_0, j_1, \dots, j_s + 1, j_{s+1} - 1, \dots j_M) \\ + \sum_{s=0}^{M} g_s F^{(N)}&(\mathbf{l}, j_0, j_1, \dots, j_{s-1}, 0, j_{s+1}, \dots j_M)\, \delta(N, \sideset{}{'}\sum_{0\leqslant k\leqslant M} j_k)\,,\end{aligned} \tag{3}$$

where $\delta(n,m)$ is the Kronecker symbol and $\sum'$ implies that the term with $k = s$ is omitted. The equation for directed walks is obtained by removing the second sum in the right-hand side of (3). Substituting (2) into (3), we obtain a system of equations for $G_k^{(N)}(\mathbf{l},\mathbf{j})$:

$$\begin{aligned}G_k^{(N)}(\mathbf{l},\mathbf{j}) = & \sum_{s=0}^{M} G_{k-1}^{(N)}(\mathbf{l}, j_0, j_1, \dots, j_s - 1, j_{s+1} + 1, \dots j_M) \\ & + \sum_{s=0}^{M} G_{k-1}^{(N)}(\mathbf{l}, j_0, j_1, \dots, j_s + 1, j_{s+1} - 1, \dots j_M) \\ + \sum_{s=0}^{M} g_s G_{k-1}^{(N)}&(\mathbf{l}, j_0, j_1, \dots, j_{s-1}, 0, j_{s+1}, \dots j_M)\, \delta(N, \sideset{}{'}\sum_{0\leqslant k\leqslant M} j_k)\,,\end{aligned} \tag{4}$$

where $k \geqslant 1$, while it is natural to set $G_0^{(N)}(\mathbf{l},\mathbf{j}) = \prod\limits_{l=0}^{M} \delta_{l_l j_l}$.

§3. The integrable phase models

To give the problem a quantum flavor, we will interpret the coordinates n_j of a particle $\mathbf{n} = (n_0, n_1, \dots, n_M) \in \mathbb{Z}^{M+1}$ as the occupation numbers of the $(M+1)$-component Fock space and describe the dynamics of a particle with the help of the Fock state vectors $|\mathbf{n}\rangle \equiv |n_0, n_1, \dots, n_M\rangle$. Let us introduce a description in terms of N bosonic particles on a cyclic chain consisting of $M+1$ nodes. The number of particles at any site is arbitrary, and each configuration of particles is characterized by the collection of occupation numbers $\big\{n_0, n_1, \dots, n_M \big| \sum\limits_{l\in\mathcal{M}} n_l = N\big\}$. Each particle hops to one of the nearest sites.

To describe the hops of the particles, let us introduce the so-called *phase operators* $\phi_n, \phi_n^\dagger$ (see [33]) which satisfy the commutation relations

$$[\widehat{N}_i, \phi_j] = -\phi_i \delta_{ij}\,, \quad [\widehat{N}_i, \phi_j^\dagger] = \phi_i^\dagger \delta_{ij}\,, \quad [\phi_i, \phi_j^\dagger] = \pi_i \delta_{ij}\,, \tag{5}$$

where $\widehat{N}_j$ is the number operator and $\pi_i = 1 - \phi_i^\dagger \phi_i$ is the vacuum projection: $\phi_j \pi_j = \pi_j \phi_j^\dagger = 0$.

Consider the Fock state vectors $|n_l\rangle_l = (\phi_l^\dagger)^{n_l} |0\rangle_l$, where $|0\rangle_l$ is the vacuum state $|0\rangle$ at the lth site defined by the relation $\phi_l |0\rangle = 0$, $l \in \mathcal{M}$. A representation of the phase algebra (5) is given by the relations

$$\phi_l^\dagger \, |n_l\rangle_l = |n_l + 1\rangle_l\,, \qquad \phi_l \, |n_l\rangle_l = |n_l - 1\rangle_l\,, \qquad \widehat{N}_l \, |n_l\rangle_l = n_l \, |n_l\rangle_l\,. \tag{6}$$

We introduce the $(M+1)$-dimensional vacuum vector

$$|\mathbf{0}\rangle \equiv \bigotimes_{l=0}^{M} |0\rangle_l \tag{7}$$

and define, taking into account (6), an appropriate state vector $|\mathbf{n}\rangle$:

$$|\mathbf{n}\rangle \equiv \bigotimes_{l=0}^{M} |n_l\rangle_l\,, \tag{8}$$

where $\mathbf{n} \in \mathrm{Simp}_{(N)}(\mathbb{Z}^{M+1})$. The Fock state vectors $|\mathbf{n}\rangle$ are generated from the vacuum state $|\mathbf{0}\rangle$ by the action of the raising operators $\phi_j^\dagger$:

$$|\mathbf{n}\rangle = \Bigl(\prod_{j=0}^{M} (\phi_j^\dagger)^{n_j}\Bigr)|\mathbf{0}\rangle\,. \tag{9}$$

This algebra has a representation in the Fock space:

$$\begin{aligned}
\phi_j \mid n_0, \ldots, 0_j, \ldots, n_M\rangle &= 0,\\
\phi_j^\dagger \mid n_0, \ldots, n_j, \ldots, n_M\rangle &= \mid n_0, \ldots, n_j + 1, \ldots, n_M\rangle,\\
\phi_j \mid n_0, \ldots, n_j, \ldots, n_M\rangle &= |n_0, \ldots, n_j - 1, \ldots, n_M\rangle,\\
\widehat{N}_j \mid n_0, \ldots, n_j, \ldots, n_M\rangle &= n_j|n_0, \ldots, n_j, \ldots, n_M\rangle,\\
\pi_j \mid n_0, \ldots, 0_j, \ldots, n_M\rangle &= \mid n_0, \ldots, 0_j, \ldots, n_M\rangle.
\end{aligned} \tag{10}$$

The states $|n_0, \ldots, n_M\rangle$ are orthogonal, $\langle p_0, \ldots, p_M|n_0, \ldots, n_M\rangle = \prod\limits_{l=0}^{M} \delta_{p_l n_l}$.

The generator of undirected walks over sites of $\mathrm{Simp}_{(N)}(\mathbb{Z}^{M+1})$ is determined by the Hermitian phase model [33] with the Hamiltonian

$$H_{\mathrm{ph}} = \sum_{m=0}^{M} \bigl(\phi_m \phi_{m+1}^\dagger + \phi_m^\dagger \phi_{m+1}\bigr)\,. \tag{11}$$

As the generator of directed walks with the retaining boundary conditions, we can consider the non-Hermitian Hamiltonian of the generalized phase model [31, 35]:

$$H_{\mathrm{gph}} = \sum_{m=0}^{M} \bigl(\phi_m \phi_{m+1}^\dagger + g_m \pi_m\bigr)\,, \tag{12}$$

where $M + 1 = 0 \pmod 2$ and the periodic boundary conditions are imposed. The Hamiltonians (11) and (12) commute with the number operator:

$$[H_{\mathrm{ph}}, \widehat{N}] = [H_{\mathrm{gph}}, \widehat{N}] = 0\,, \qquad \widehat{N} = \sum_{j=0}^{M} \widehat{N}_j\,. \tag{13}$$

This ensures that the walks themselves belong to the symplicial lattices $\mathrm{Simp}_{(N)}(\mathbb{Z}^{M+1})$.

The exponential generating function (2) of random walks can be expressed as the dynamical correlation function:

$$F^{(N)}(\mathbf{l}, \mathbf{j}|t) = \langle \mathbf{l} \mid e^{tH} \mid \mathbf{j}\rangle\,, \tag{14}$$

where H is the Hamiltonian (11) or (12) and the coordinates of a particle moving over the sites of $\mathrm{Simp}_{(N)}(\mathbb{Z}^{M+1})$ coincide with the occupation numbers of the Fock states. Differentiating (14) by t and taking into account (10), we obtain Eq. (3) for undirected or directed walks, respectively. Expanding the correlator in powers of t, we obtain an expression for the coefficients $G_k^{(N)}(\mathbf{l},\mathbf{j})$ characterizing the k-step lattice walks between the nodes $\mathbf{j}$ and $\mathbf{l}$:

$$G_k^{(N)}(\mathbf{l},\mathbf{j}) = \langle \mathbf{l} \mid H^k \mid \mathbf{j}\rangle \,. \tag{15}$$

To find analytic answers for $F^{(N)}$ and $G_k^{(N)}$, we will apply the quantum inverse scattering method [10, 11].

§4. Solution of the generalized phase model

To apply the scheme of the quantum inverse scattering method to the solution of the Hamiltonian (12), we introduce the L-operator [31], which is a 2×2 matrix with operator-valued entries acting on the Fock states according to (10):

$$L(n|u) \equiv \begin{pmatrix} u^{-1} + u g_n \pi_n & \phi_n^\dagger \\ \phi_n & u \end{pmatrix} , \tag{16}$$

where $u \in \mathbb{C}$ is a parameter and $g_n \in \mathbb{R}$. This L-operator satisfies the intertwining relation

$$R(u,v)\,(L(n|u) \otimes L(n|v)) = (L(n|v) \otimes L(n|u))\,R(u,v)\,, \tag{17}$$

in which $R(u,v)$ is the R-matrix

$$R(u,v) = \begin{pmatrix} f(v,u) & 0 & 0 & 0 \\ 0 & g(v,u) & 1 & 0 \\ 0 & 0 & g(v,u) & 0 \\ 0 & 0 & 0 & f(v,u) \end{pmatrix} , \tag{18}$$

where

$$f(v,u) = \frac{u^2}{u^2 - v^2}, \qquad g(v,u) = \frac{uv}{u^2 - v^2}\,, \qquad u, v \in \mathbb{C}\,. \tag{19}$$

The monodromy matrix is the matrix product of L-operators:

$$T(u) = L(M|u)L(M-1|u)\cdots L(0|u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix} . \tag{20}$$

The commutation relations for the matrix elements of the monodromy matrix are given by the same R-matrix (18):

$$R(u,v)\,(T(u)\otimes T(v)) = (T(v)\otimes T(u))\,R(u,v)\,. \tag{21}$$

The transfer matrix $\tau(u)$ is the trace of the monodromy matrix in the auxiliary space:

$$\tau(u) = \operatorname{tr} T(u) = A(u) + D(u)\,. \tag{22}$$

Relation (21) means that $[\tau(u), \tau(v)] = 0$ for arbitrary $u, v \in \mathbb{C}$.

Using the definition of the L-operator (16) in order to calculate the monodromy matrix $T(u)$, one obtains the corresponding entries in the form of relations polynomial in powers of u^2:

$$\begin{aligned} u^{M+1}A(u) &= 1 + u^2\Big(\sum_{m=0}^{M-1}\phi_m\phi_{m+1}^\dagger + \sum_{m=0}^{M} g_m\pi_m\Big) + \dots \\ &\quad + u^{2(M+1)}\prod_{m=0}^{M} g_m\pi_m\,, \\ u^{M+1}D(u) &= u^2\phi_0^\dagger\phi_M + \dots + u^{2(M+1)}\,, \end{aligned} \tag{23}$$

and

$$u^M B(u) = \phi_0^\dagger + \dots + u^{2M}\mathcal{P}_R \equiv \widetilde{B}(u)\,, \tag{24}$$

$$u^M C(u) = \phi_M + \dots + u^{2M}\mathcal{P}_L \equiv \widetilde{C}(u)\,, \tag{25}$$

where

$$\mathcal{P}_R = \phi_M^\dagger + \sum_{k=0}^{M-1}\phi_k^\dagger g_{k+1}\pi_{k+1}\dots g_M\pi_M\,,$$

$$\mathcal{P}_L = \phi_0 + \sum_{k=1}^{M} g_0\pi_0\dots g_{k-1}\pi_{k-1}\phi_k\,.$$

The representation (23) allows one to express the Hamiltonian (12) through the transfer matrix (22):

$$H_{\mathrm{gph}} = \frac{\partial}{\partial u^2}\,u^{M+1}\tau(u)\Big|_{u=0} = \frac{\partial}{\partial u^2}u^{M+1}(A(u)+D(u))\Big|_{u=0}. \tag{26}$$

By construction, this Hamiltonian commutes with the transfer matrix:

$$[H_{\mathrm{gph}}, \tau(u)] = 0.$$

Since the Hamiltonian (12) is non-Hermitian, we must distinguish between its right and left eigenvectors. The N-particle right state vectors are taken in the form

$$|\Psi_N(\mathbf{u})\rangle = \Big(\prod_{j=1}^{N} \widetilde{B}(u_j)\Big)|\mathbf{0}\rangle, \tag{27}$$

where $\widetilde{B}(u)$ is defined in (24) and $\mathbf{u}$ is a collection of arbitrary complex parameters $u_j \in \mathbb{C}$: $\mathbf{u} = (u_0, u_1, \ldots, u_N)$. The left state vectors are

$$\langle\Psi_N(\mathbf{u})| = \langle\mathbf{0}|\Big(\prod_{j=1}^{N} \widetilde{C}(u_j)\Big)\,, \tag{28}$$

where $\widetilde{C}(u)$ is given by (25). The vacuum state $|\,\mathbf{0}\rangle$ (see (7)) is an eigenvector of $A(u)$ and $D(u)$,

$$A(u)|\mathbf{0}\rangle = \alpha(u)|\mathbf{0}\rangle, \qquad D(u)|\mathbf{0}\rangle = \delta(u)|\mathbf{0}\rangle, \tag{29}$$

with the eigenvalues

$$\alpha(u) = \prod_{j=0}^{M}(u^{-1} + g_j u), \qquad \delta(u) = u^{M+1}. \tag{30}$$

The state vectors (27) and (28) are eigenvectors both of the Hamiltonian (12) and of the transfer matrix $\tau(u)$ if and only if the variables u_j satisfy the Bethe equations:

$$\frac{\alpha(u_n)}{\delta(u_n)} = \prod_{m\neq n}^{N} \frac{f(u_m,u_n)}{f(u_n,u_m)}\,, \tag{31}$$

or, in an explicit form,

$$u_n^{-2N}\prod_{j=0}^{M}\big(g_j + u_n^{-2}\big) = \frac{(-1)^{N-1}}{U^2}\,, \qquad U^2 \equiv \prod_{j=1}^{N} u_j^2\,. \tag{32}$$

The eigenvalues $\Theta_N(v,\mathbf{u})$ of the transfer matrix (22) in general form are equal to

$$\Theta_N(v;\mathbf{u}) = \alpha(v)\prod_{j=1}^{N} f(v,u_j) + \delta(v)\prod_{j=1}^{N} f(u_j,v)\,. \tag{33}$$

For the model under consideration,

$$v^{M+1}\Theta_N^{\mathrm{gph}}(v;\mathbf{u}) = \prod_{j=0}^{M}(1+g_jv^2)\prod_{m=1}^{N}\frac{u_m^2}{u_m^2-v^2} + v^{2(M+1)}\prod_{m=1}^{N}\frac{v^2}{v^2-u_m^2} \tag{34}$$

$$= \Big(\prod_{j=0}^{M}(1+g_jv^2) + (-1)^N v^{2(M+N+1)}U^{-2}\Big)H(v^2;\mathbf{u}^{-2})\,. \tag{35}$$

In (35), we have introduced the generating function of complete symmetric functions $h_l(\mathbf{u}^{-2}) \equiv h_l(u_1^{-2}, u_2^{-2}, \ldots, u_N^{-2})$, see [18]:

$$H(v^2;\mathbf{u}^{-2}) \equiv \prod_{m=1}^{N}\frac{1}{1-v^2/u_m^2} = \sum_{l\geqslant 0} h_l(\mathbf{u}^{-2})v^{2l}\,. \tag{36}$$

Equation (26) allows one to obtain the spectrum of the Hamiltonian (12). The N-particle eigenenergies are equal to

$$E_N^{\mathrm{gph}}(\mathbf{u}) = \frac{\partial}{\partial v^2}v^{M+1}\Theta_N^{\mathrm{gph}}(v;\mathbf{u})\Big|_{v=0} \tag{37}$$

$$= \sum_{m=0}^{M} g_m + \sum_{k=1}^{N} u_k^{-2} = h_1(\mathbf{u}^{-2}) + \sum_{m=0}^{M} g_m\,. \tag{38}$$

§5. Calculation of generating functions

For the models associated with the R-matrix (18), the scalar product of the state vectors (27) and (28) is given by the formula (see [36])

$$\langle\Psi_N(\mathbf{v})|\Psi_N(\mathbf{u})\rangle = \frac{\det Q}{\mathcal{V}_N(\mathbf{v}^2)\mathcal{V}_N(\mathbf{u}^{-2})}\prod_{j=1}^{N}\Big(\frac{v_j}{u_j}\Big)^{M+N-1}, \tag{39}$$

where $\mathcal{V}_N(\mathbf{x})$ is the Vandermonde determinant,

$$\mathcal{V}_N(\mathbf{x}) \equiv \mathcal{V}_N(x_1, x_2, \ldots, x_N) = \prod_{1\leqslant i<k\leqslant N}(x_k - x_i)\,, \tag{40}$$

and the matrix Q is characterized by the entries Q_{jk}, $1 \leqslant j,k \leqslant N$:

$$Q_{jk} = \frac{\alpha(v_j)\delta(u_k)\Big(\dfrac{u_k}{v_j}\Big)^{N-1} - \alpha(u_k)\delta(v_j)\Big(\dfrac{v_j}{u_k}\Big)^{N-1}}{\dfrac{u_k}{v_j} - \dfrac{v_j}{u_k}}\,, \tag{41}$$

with $\alpha(u)$ and $\delta(u)$ given by (30).

The norm of the state vector, $\mathcal{N}^2(\mathbf{u}) \equiv \langle \Psi_N(\mathbf{u}) | \Psi_N(\mathbf{u}) \rangle$, is defined by the scalar product (39) when the arguments $\mathbf{v}$ and $\mathbf{u}$ satisfy the Bethe equations (32). In the present case of the generalized phase model, we substitute $v_k = u_k$ for all k, with u_k respecting the Bethe equations (32), into the entries of the matrix Q. The resulting matrix is denoted by $\widetilde{Q}$, and its entries at $j \neq k$ are equal to

$$\widetilde{Q}_{jk} = \frac{(-1)^N (u_k u_j)^{N+M+1}}{U^2}\,, \tag{42}$$

where U^2 is given by (32). L'Hôspital's rule gives the diagonal entries of $\widetilde{Q}$:

$$\begin{aligned} \widetilde{Q}_{jj} \quad &= (N-1)\alpha(u_j)\delta(u_j) \;+\; \big(\alpha(u_j)\delta'(u_j) - \alpha'(u_j)\delta(u_j)\big)\frac{u_j}{2} \\ &= \Big(1 - N - \mathfrak{G}_j\Big)\frac{(-1)^N u_j^{2(N+M+1)}}{U^2}\,, \end{aligned} \tag{43}$$

where

$$\mathfrak{G}_j \equiv \sum_{s=0}^{M} \frac{1}{g_s u_j^2 + 1}\,. \tag{44}$$

As a result, the squared norm $\mathcal{N}^2(\mathbf{u})$ on the Bethe solution takes the form

$$\mathcal{N}^2(\mathbf{u}) = \frac{\det \widetilde{Q}}{\mathcal{V}_N(\mathbf{u}^2)\mathcal{V}_N(\mathbf{u}^{-2})}\,, \tag{45}$$

$$\det \widetilde{Q} \;=\; U^{2(M+1)}\Big(1 - \sum_{l=1}^{N} \frac{1}{N + \mathfrak{G}_l}\Big)\prod_{j=1}^{N}(N + \mathfrak{G}_j)\,. \tag{46}$$

There are $\frac{(N+M)!}{N!M!}$ sets of solutions of the Bethe equations (32), and state vectors belonging to different sets are orthogonal. The eigenvectors (27) and (28) provide a resolution of the identity operator:

$$I = \sum_{\{\mathbf{u}\}} \frac{|\Psi_N(\mathbf{u})\rangle\langle\Psi_N(\mathbf{u})|}{\mathcal{N}^2(\mathbf{u})}, \tag{47}$$

where the summation $\sum\limits_{\{\mathbf{u}\}}$ is over all independent solutions of the Bethe equations (32).

The general answer for the generating function (14) of directed random walks is

$$F^{(N)}(\mathbf{l},\mathbf{j}|t) = \sum_{\{\mathbf{u}\}} e^{tE_N^{\mathrm{gph}}} \frac{\langle \mathbf{l}|\Psi_N(\mathbf{u})\rangle\langle\Psi_N(\mathbf{u})|\mathbf{j}\rangle}{\mathcal{N}^2(\mathbf{u})}\,. \tag{48}$$

For simplicity, let us consider a walker on $\mathrm{Simp}_{(N)}(\mathbb{Z}^{M+1})$ under the condition that the starting point is located at the node $(N, 0, \ldots, 0)$ and the walk terminates at $(0, 0, \ldots, N)$. The generating function (14) of these walks is specified as follows:

$$F^{(N)}(t) \equiv \langle 0, 0, \ldots, N \mid e^{tH_{\mathrm{gph}}} \mid N, 0, \ldots, 0 \rangle = \langle \mathbf{0} \mid (\phi_M)^N e^{tH_{\mathrm{gph}}} (\phi_M^\dagger)^N \mid \mathbf{0} \rangle \,, \tag{49}$$

where (7) and (9) have been used. Inserting the resolution of the identity operator (47) into (49), we obtain

$$F^{(N)}(t) = \sum_{\{\mathbf{u}\}} \frac{e^{tE_N^{\mathrm{gph}}(\mathbf{u})}}{\mathcal{N}^2(\mathbf{u})} \langle \mathbf{0} \mid (\phi_M)^N \mid \Psi_N(\mathbf{u}) \rangle \langle \Psi_N(\mathbf{u}) \mid (\phi_M^\dagger)^N \mid \mathbf{0} \rangle \,, \tag{50}$$

where the summation is over all independent solutions of (32).

The decomposition (25) for $B(u)$ and $C(u)$ gives

$$\langle \mathbf{0} \mid (\phi_M)^N \mid \Psi_N(\mathbf{u}) \rangle = \lim_{\mathbf{v} \to 0} \langle \Psi_N(\mathbf{v}) | \Psi_N(\mathbf{u}) \rangle = 1 \,, \tag{51}$$

$$\langle \Psi_N(\mathbf{u}) \mid (\phi_M^\dagger)^N \mid \mathbf{0} \rangle = \lim_{\mathbf{v} \to 0} \langle \Psi_N(\mathbf{u}) | \Psi_N(\mathbf{v}) \rangle = 1 \,, \tag{52}$$

and eventually we obtain

$$F^{(N)}(t) = \sum_{\{\mathbf{u}\}} \frac{e^{t\left(\sum\limits_{k=0}^{M} g_s + \sum\limits_{k=1}^{N} u_k^{-2}\right)}}{\mathcal{N}^2(\mathbf{u})} \,, \tag{53}$$

where $\mathcal{N}^2(\mathbf{u})$ is given by (45), (46).

When $g_s = 0$ for all s, the model described by the non-Hermitian Hamiltonian (12) is still applicable for the description of directed walks over the simplicial lattice (1). The Hamiltonian (12) at $g_s = 0$ and the Hamiltonian of the phase model (11) commute and are diagonalized by the same set of eigenfunctions. Therefore, the expression for the scalar product $\langle \Psi_N(\mathbf{v}) | \Psi_N(\mathbf{u}) \rangle$ in the case where $g_s = 0$ for all s and M, N are arbitrary should be the same as in the case of the phase model. The corresponding formula was derived in [15] and will be given in the next section.

§6. The phase model

In this section, we will consider the phase model [33] described by the Hermitian Hamiltonian (11):

$$H_{\rm ph} = \sum_{m=0}^{M} \left(\phi_m \phi_{m+1}^\dagger + \phi_m^\dagger \phi_{m+1}\right).$$

The Hamiltonian $H_{\rm ph}$ may be regarded as the generator of random walks on $\mathrm{Simp}_{(N)}(\mathbb{Z}^{M+1})$ that are characterized by the step set Ω_M. The L-operator of the phase model is given by (16) with $g_i = 0$.

The state vectors of the phase model have the form (see [36, 37])

$$|\Psi_N(\mathbf{u})\rangle = \sum_{\boldsymbol{\lambda} \subseteq \{M^N\}} S_{\boldsymbol{\lambda}}(\mathbf{u}^2) \left(\prod_{l=0}^{M} (\phi_l^\dagger)^{n_l}\right) |\mathbf{0}\rangle . \tag{54}$$

The coefficients of the expansion (54) are the Schur functions $S_{\boldsymbol{\lambda}}(\mathbf{u}^2)$ given by the Jacobi–Trudi identity [18]:

$$S_{\boldsymbol{\lambda}}(\mathbf{x}) \equiv \frac{\det(x_j^{\lambda_k+N-k})_{1\leqslant j,k\leqslant N}}{\mathcal{V}_N(\mathbf{x})} . \tag{55}$$

Here $\boldsymbol{\lambda}$ denotes a partition $(\lambda_1, \dots, \lambda_N)$, which is a sequence of nonincreasing nonnegative integers,

$$M \geqslant \lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_N \geqslant 0,$$

and $\mathcal{V}_N(\mathbf{x})$ is the Vandermonde determinant (40). The summation in Eq. (54) is over all partitions $\boldsymbol{\lambda}$ into at most N parts with $N \leqslant M$.

There is a one-to-one correspondence between the sequences of occupation numbers $(n_M, \dots, n_1, n_0)$, $\sum\limits_{j\in\mathcal{M}} n_j = N$, and the partitions

$$\boldsymbol{\lambda} = (M^{n_M}, (M-1)^{n_{M-1}}, \dots, 1^{n_1}, 0^{n_0}),$$

where each number S appears n_S times (see Fig. 4).

The Bethe equations (31) for the phase model take the form

$$u_n^{-2(N+M+1)} = \frac{(-1)^{N-1}}{U^2} . \tag{56}$$

On solutions of the Bethe equations, the state vectors are the eigen-vectors of the correspondent transfer matrix (22) and of the Hamiltonian (11), respectively. It is convenient to express the eigenvalues $\Theta_N(v; \mathbf{u})$ of the

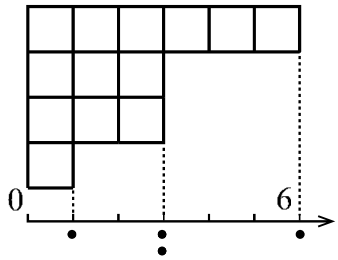

Fig. 4. A configuration of particles ($N = 4$) on a lattice ($M = 6$), the corresponding partition $\boldsymbol{\lambda} = (6^1, 5^0, 4^0, 3^2, 2^0, 1^1, 0^0) \equiv (6, 3, 3, 1)$, and its Young diagram.

transfer matrix in terms of the generating functions of complete symmetric functions (36):

$$\Theta_N^{\rm ph}(v; \mathbf{u}) = H(v^2; \mathbf{u}^{-2}) + v^{2(M+1)} H(v^{-2}; \mathbf{u}^2) \,. \tag{57}$$

The N-particle eigenenergies of the model are equal to

$$E_N^{\rm ph}(\mathbf{u}) = \frac{\partial}{\partial v^2} v^{M+1} \Theta_N^{\rm ph}(v; \mathbf{u})\Big|_{v=0} + \frac{\partial}{\partial v^{-2}} v^{-(M+1)} \Theta_N^{\rm ph}(v; \mathbf{u})\Big|_{v=\infty} \tag{58}$$

$$= h_1(\mathbf{u}^2) + h_1(\mathbf{u}^{-2}) = \sum_{k=1}^{N} \left(u_k^2 + u_k^{-2}\right) \,. \tag{59}$$

The scalar product of the state vectors (54) can be calculated by the Binet–Cauchy formula, which is a special case of the general formula (39):

$$\langle \Psi_N(\mathbf{v}) | \Psi_N(\mathbf{u}) \rangle = \sum_{\boldsymbol{\lambda} \subseteq \{M^N\}} S_{\boldsymbol{\lambda}}(\mathbf{v}^2) S_{\boldsymbol{\lambda}}(\mathbf{u}^{-2}) \tag{60}$$

$$= \frac{1}{\mathcal{V}_N(\mathbf{v}^2)\mathcal{V}_N(\mathbf{u}^{-2})} \det\Big(\frac{1 - (v_j/u_k)^{2(M+N)}}{1 - (v_j/u_k)^2}\Big)_{1 \leqslant j,k \leqslant N} \,. \tag{61}$$

The norms of the eigenvectors can be obtained from Eqs. (45) and (46) putting $\mathfrak{G}_s = M + 1$:

$$\mathcal{N}_{\rm ph}^2(\mathbf{u}) = \frac{U^{2(M+1)} (M + 1 + N)^{N-1} (M + 1)}{\mathcal{V}_N(\mathbf{u}^2)\mathcal{V}_N(\mathbf{u}^{-2})} \,. \tag{62}$$

The generating function of random walks on $\mathrm{Simp}_{(N)}(\mathbb{Z}^{M+1})$ takes a form similar to that of the generalized phase model, see (53):

$$F_{\mathrm{ph}}^{(N)}(\mathbf{l},\mathbf{j}|t) = \sum_{\{\mathbf{u}\}} \frac{e^{t\sum\limits_{k=1}^{N}(u_k^2+u_k^{-2})}}{\mathcal{N}_{\mathrm{ph}}^2(\mathbf{u})} S_{\boldsymbol{\lambda}_L}(\mathbf{u}^2) S_{\boldsymbol{\lambda}_R}(\mathbf{u}^{-2}) \,, \tag{63}$$

where the summation is over all independent solutions of Eqs. (56). The partition $\boldsymbol{\lambda}_R$ is $\big(M^{j_M},(M-1)^{j_{M-1}},\dots,1^{j_1},0^{j_0}\big)$, where each number S appears j_S times and $j_0+j_1+\ldots+j_D=N$. The partition $\boldsymbol{\lambda}_L$ is $\big(M^{l_M},(M-1)^{l_{M-1}},\dots,1^{l_1},0^{l_0}\big)$, and $l_0+l_1+\ldots+l_D=N$.

§7. Continuous-time quantum walks

A continuous-time quantum walk is a unitary process that describes the quantum-mechanical analog of the classical random walk process. The dynamics of continuous-time quantum walks is determined by the Hamiltonian evolution of a particle on a lattice. The Hamiltonian of the phase model describes strongly correlated bosons on a chain and can be chosen as a generator of continuous-time quantum walks. After the substitution $\tau \to -it$, the generating function of classical random walks (63) has the meaning of the transition amplitude $\mathcal{F}^{(N)}(\mathbf{l},\mathbf{j}|t)$ of a *continuous-time quantum walker* from the state $|l_0,l_1,\dots,l_M\rangle$ at time $t=0$ to the state $|j_0,j_1,\dots,j_M\rangle$ at a specific time t:

$$F_{\mathrm{ph}}^{(N)}(\mathbf{l},\mathbf{j}|-it) \equiv \mathcal{F}^{(N)}(\mathbf{l},\mathbf{j}|t)\,.$$

For the transition amplitude, the condition $\sum\limits_{\mathbf{j}} |\mathcal{F}^{(N)}(\mathbf{l},\mathbf{j}|t)|^2 = 1$ holds.

In the one-dimensional case $M=1$, the detailed analysis [38,39] of the Bethe equations (56) allows one to rewrite the answer for the transition amplitude on a segment $[0,N]$ in the form

$$\begin{aligned}\mathcal{F}^{(N)}(l,j|t) &\equiv \mathcal{F}^{(N)}\big((l,N-l),(N-j,j)|t\big)\\ &= \frac{2}{N+1}\sum_{k=1}^{N} e^{-itE_k} \sin\left[\frac{\pi k(j+1)}{N+1}\right] \sin\left[\frac{\pi k(l+1)}{N+1}\right]\,,\end{aligned} \tag{64}$$

with the Bloch spectrum

$$E_k = -2\cos\left(\frac{\pi k}{N+1}\right)\,. \tag{65}$$

In Fig. 5 we show the probability $|\mathcal{F}^{(N)}(l,j|t)|^2$ for the initial state $j = l = N/2$ and $F_t(j;l|0)|_{t=0} = \delta_{jl}$.

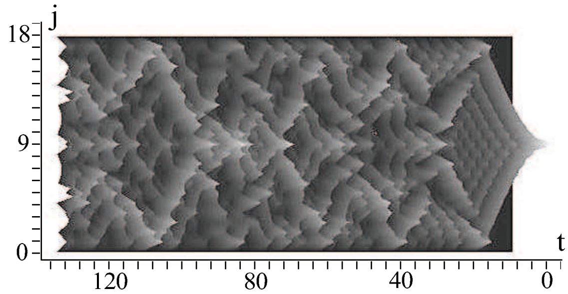

Fig. 5. The probability $|\mathcal{F}^{(N)}(l,j|t)|^2$ for $N = 18$ as a surface diagram in the $t-j$ plane. The height is indicated with a gray scale such that black means zero height.

Let us consider the continuous-time quantum walks on a segment $[0,N]$ in the bias potential Δ defined by the Hamiltonian, see [40]:

$$\widehat{H}_\Delta = \left(\phi_0^\dagger\phi_1 + \phi_0\phi_1^\dagger\right) + 2\Delta N_0\,, \tag{66}$$

where N_0 is the number operator acting in the zero node. The transition amplitude satisfies the equation

$$-i\frac{d}{dt}\mathcal{F}_\Delta^{(N)}(l,j|t) = \mathcal{F}_\Delta^{(N)}(l,j+1|t) + \mathcal{F}^{(N)}(l,j-1|t) - 2\Delta j\mathcal{F}_\Delta^{(N)}(l,j|t)\,, \tag{67}$$

for a fixed l. The boundary conditions are

$$\mathcal{F}_\Delta^{(N)}(l,j|t) = 0 \quad \text{for} \quad j,l = -1, N+1,$$

and the initial condition is $\mathcal{F}_\Delta^{(N)}(l,j|0) = \delta_{jl}$. Naturally, the transition amplitude (64) is the solution of this equation for the zero bias potential $\Delta = 0$.

For nonzero Δ, Eq. (67) can also be solved exactly. The solution is expressed via the Lommel polynomials $B(\nu,\lambda;m)$:

$$\mathcal{F}_\Delta^{(N)}(l,j|t) = \sum_{k=1}^{N+1} e^{-itE_k}\frac{B(\nu_k,\Delta^{-1};j)B(\nu_k,\Delta^{-1};l)}{\sum\limits_{n=1}^{N} B^2(\nu_k,\Delta^{-1};n)}. \tag{68}$$

The Lommel polynomials are polynomials of degree $m-1$ in Δ and in ν and can be expressed in terms of Bessel functions of the first and second kind:

$$B(\nu,\Delta^{-1};m) = \frac{\pi\lambda}{2}\left[J_\nu(\Delta^{-1})Y_{\nu+m}(\Delta^{-1}) - J_{\nu+m}(\Delta^{-1})Y_\nu(\Delta^{-1})\right].$$

These polynomials satisfy the recurrence relation

$$B(\nu,\Delta^{-1};m+1) + B(\nu,\Delta^{-1};m-1) = 2\Delta(m+\nu)B(\nu,\Delta^{-1};m).$$

The boundary conditions given below (67) mean that $B(\nu,\Delta^{-1};N+2)=0$, and $N+1$ roots of this equation are ν_k. The spectrum E_k is now

$$E_k = -\Delta(\nu_k+1). \tag{69}$$

For small $N\Delta$, this spectrum is similar to the Bloch spectrum (65). There is a crossover around $N\Delta \approx 1$ into a linear spectrum $E_k \approx \Delta k$ for $N\Delta \gg 1$. This latter kind of spectrum is also called the Wannier–Stark ladder.

For $N\Delta \ll 1$, we thus find that the probability $|\mathcal{F}_\Delta^{(N)}(l,j|t)|^2$ behaves very similarly to that for $\Delta = 0$ shown in Fig. 6. Around $N\Delta \approx 1$, there is a transition to a completely different behavior. For increasing $N\Delta$, the probability $|\mathcal{F}_\Delta^{(N)}(l,j|t)|^2$ becomes localized around the center line. We show $|\mathcal{F}_\Delta^{(N)}(l,j|t)|^2$ for $N\Delta = 3$ in Fig. 6, where these features of the solution are clearly visible. For increasing $N\Delta$, the number of localized

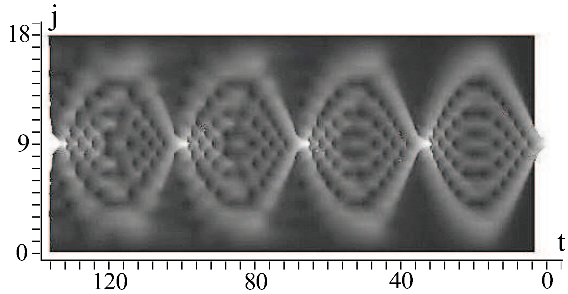

Fig. 6. The probability $|\mathcal{F}_\Delta^{(N)}(l,j|t)|^2$ for $N=18$, $j=9$, and $\Delta^{-1}=6$.

states increases, and the probability becomes more and more localized around the center line (i.e., the initial value). These properties have been experimentally verified for quantum walks in optical lattices [22].

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