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ON THE DISTRIBUTION OF POINTS WITH ALGEBRAICALLY CONJUGATE COORDINATES IN A NEIGHBORHOOD OF SMOOTH CURVES

Abstract. Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a continuously differentiable function on a finite interval $J \subset \mathbb{R}$, and let $\boldsymbol{\alpha} = (\alpha_1, \alpha_2)$ be a point with algebraically conjugate coordinates such that the minimal polynomial P of α_1, α_2 is of degree $\leqslant n$ and height $\leqslant Q$. Denote by $M^n_\varphi(Q, \gamma, J)$ the set of points $\boldsymbol{\alpha}$ such that $|\varphi(\alpha_1) - \alpha_2| \leqslant c_1 Q^{-\gamma}$. We show that for $0 < \gamma < 1$ and any sufficiently large Q there exist positive values $c_2 < c_3$, where $c_i = c_i(n)$, $i = 1, 2$, that are independent of Q and such that $c_2 \cdot Q^{n+1-\gamma} < \# M^n_\varphi(Q, \gamma, J) < c_3 \cdot Q^{n+1-\gamma}$.

§1. Introduction

First of all, let us introduce some useful notation. Let n be a positive integer and $Q > 1$ be a sufficiently large real number. Consider a polynomial $P(t) = a_n t^n + \ldots + a_1 t + a_0 \in \mathbb{Z}[t]$. Denote by $H(P) = \max\limits_{0 \leqslant j \leqslant n} |a_j|$ the height of the polynomial P, and by $\deg P$ the degree of the polynomial P. We define the following class of integer polynomials with bounded height and degree:

$$\mathcal{P}_n(Q) := \{P \in \mathbb{Z}[t] : \deg P \leqslant n, H(P) \leqslant Q\}.$$

Denote by $\# S$ the cardinality of a finite set S and by $\mu_k S$ the Lebesgue measure of a measurable set $S \subset \mathbb{R}^k$, $k \in \mathbb{N}$. Furthermore, denote by $c_j > 0$ positive constants independent of Q. We are also going to use the Vinogradov symbol $A \ll B$, which means that there exists a constant $c > 0$ such that $A \leqslant c \cdot B$. We will also write $A \asymp B$ if $A \ll B$ and $B \ll A$. Now let us introduce the concept of an algebraic point. A point $\boldsymbol{\alpha} = (\alpha_1, \alpha_2)$ is called an *algebraic point* if α_1 and α_2 are roots of the same irreducible polynomial $P \in \mathbb{Z}[t]$. The polynomial P of the smallest degree $n \geqslant 2$ with relatively prime coefficients such that $P(\alpha_1) = P(\alpha_2) = 0$ is called the

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minimal polynomial of the algebraic point $\boldsymbol{\alpha}$. Denote by $\deg(\boldsymbol{\alpha}) = \deg P$ the degree of the algebraic point $\boldsymbol{\alpha}$ and by $H(\boldsymbol{\alpha}) = H(P)$ the height of the algebraic point $\boldsymbol{\alpha}$. Define the following set of algebraic points:

$$\mathbb{A}_n^2(Q) := \{\boldsymbol{\alpha} \in \mathbb{C}^2 : \deg \boldsymbol{\alpha} \leqslant n,\, H(\boldsymbol{\alpha}) \leqslant Q\}.$$

Further, denote by $\mathbb{A}_n^2(Q,D) := \mathbb{A}_n^2(Q) \cap D$ the set of algebraic points lying in some domain $D \subset \mathbb{R}^2$. Problems related to calculating the number of integer points in shapes and bodies in $\mathbb{R}^k$ can be naturally generalized to estimating the number of rational points in domains in Euclidean spaces. Let $f : J_0 \to \mathbb{R}$ be a continuously differentiable function defined on a finite open interval J_0 in $\mathbb{R}$. Define the following set:

$$N_f(Q,\gamma,J) := \Big\{ (p_1/q,\, p_2/q) \in \mathbb{Q}^2 : \\ 0 < q \leqslant Q, \quad p_1/q \in J, \quad |f(p_1/q) - p_2/q| < Q^{-\gamma} \Big\},$$

where $J \subset J_0$ and $0 \leqslant \gamma < 2$. In other words, the quantity $\#N_f(Q,\gamma,J)$ denotes the number of rational points with bounded denominators lying within a certain neighborhood of the curve parametrized by f. The problem is to estimate the value $\#N_f(Q,\gamma,J)$. In [7], Huxley proved that for functions $f \in C^2(J)$ such that $0 < c_4 := \inf\limits_{x \in J_0} |f''(x)| \leqslant c_5 := \sup\limits_{x \in J_0} |f''(x)| < \infty$ and an arbitrary constant $\varepsilon > 0$, the following upper bound holds:

$$\#N_f(Q,\gamma,J) \ll Q^{3-\gamma+\varepsilon}.$$

An estimate without ε in the exponent was obtained in 2006 in a paper by Vaughan and Velani [14]. One year later, Beresnevich, Dickinson, and Velani [1] proved a lower estimate of the same order:

$$\#N_f(Q,\gamma,J) \gg Q^{3-\gamma}.$$

This result was obtained using methods of metric theory introduced by Schmidt in [9]. In this paper, we consider a problem related to the distribution of algebraic points $\boldsymbol{\alpha} \in \mathbb{A}_n^2(Q)$ near smooth curves, which is a natural extension of the same problem formulated for rational points. Let $\varphi : J_0 \to \mathbb{R}$ be a continuously differentiable function defined on a finite open interval J_0 in $\mathbb{R}$ satisfying the conditions

$$\sup_{x \in J_0} |\varphi'(x)| := c_6 < \infty, \qquad \#\{x \in J_0 : \varphi(x) = x\} := c_7 < \infty. \tag{1.1}$$

Define the following set:

$$M_\varphi^n(Q,\gamma,J) := \left\{\boldsymbol{\alpha} \in \mathbb{A}_n^2(Q) : \alpha_1 \in J, \quad |\varphi(\alpha_1) - \alpha_2| < c_1 Q^{-\gamma}\right\},$$

where $c_1 = \left(\frac{1}{2} + c_6\right) \cdot c_8$ and $J \subset J_0$. This set contains algebraic points with bounded degree and height lying within some neighborhood of the curve parametrized by φ. Our goal is to estimate the value $\#M_\varphi^n(Q, \gamma, J)$. The first advancement in solving this problem for $0 < \gamma \leqslant \frac{1}{2}$ was made in 2014 in the paper [5]. We are going to state it in the following form: for any $Q > Q_0(n, J, \varphi)$ there exists a positive value $c_9 > 0$ such that $\#M_\varphi^n(Q, \gamma, J) > c_9 \cdot Q^{n+1-\gamma}$ for $0 < \gamma \leqslant \frac{1}{2}$. However, it should be noted that this result is not the best possible, since for the quantity $\#M_\varphi^n(Q, \gamma, J)$ an upper bound of order $Q^{n+1-\gamma}$ can be proved for $0 < \gamma < 1$. In this paper, we are going to fill this gap in the result of [5] by obtaining lower and upper bounds of the same order for $0 < \gamma < 1$. Our main result is as follows.

Theorem 1. *For any smooth function φ satisfying conditions* (1.1) *there exist positive values $c_2, c_3 > 0$ such that*

$$c_2 \cdot Q^{n+1-\gamma} < \#M_\varphi^n(Q, \gamma, J) < c_3 \cdot Q^{n+1-\gamma}$$

for $Q > Q_0(n, J, \varphi, \gamma)$, sufficiently large c_1, and $0 < \gamma < 1$.

The proof of Theorem 1 is based on the following idea. We consider the strip $L_\varphi^n(Q, \gamma, J) := \left\{\mathbf{x} \in \mathbb{R}^2 : x_1 \in J,\ |\varphi(x_1) - x_2| < c_1 Q^{-\gamma}\right\}$ and fill it using squares $\Pi = I_1 \times I_2$ with sides of length $\mu_1 I_1 = \mu_1 I_2 = c_8 Q^{-\gamma}$. In order to prove Theorem 1, we need to estimate the number of algebraic points lying in such a square Π. It should be mentioned that these estimates are highly relevant to several other problems in the metric theory of Diophantine approximation [6, 15]. Let us consider a more general case, namely, the case of a rectangle $\Pi = I_1 \times I_2$, where $\mu_1 I_i = c_8 Q^{-\gamma_i}$. We are now going to give an overview of results related to the distribution of algebraic points in rectangle Π. First of all, let us find the value of the parameter $\gamma_1 + \gamma_2$ such that a rectangle Π does not contain algebraic points $\boldsymbol{\alpha} \in \mathbb{A}_n^2(Q)$. The following Theorem 2 answers this question. The one-dimensional case of this problem was considered in [4].

Theorem 2. *For any fixed $p, q \in \mathbb{N}$ with $p < 2q$ there exists a rectangle Π_0 of size $\mu_2 \Pi_0 = c_{10}(p, q, n) \cdot Q^{-1}$, where*

$$c_{10}(p, q, n) = (2p(2q + 2p)^n(n + 1))^{-1} \cdot q^{n+1},$$

such that $\#\mathbb{A}_n^2(Q, \Pi_0) = 0$.

Proof. Consider the rectangle Π_0 with sides given by $I_{0,2} = \left(0; \frac{p}{q}\right)$ and $I_{0,1} = \left(\frac{p}{q}; \frac{p}{q} + c_{10} \cdot Q^{-1}\right)$. To prove Theorem 2, assume that there exists an algebraic point $\boldsymbol{\alpha} \in \mathbb{A}_n^2(Q, \Pi_0)$ with minimal polynomial P_1. Consider the resultant $R(P_1, P_2)$ of the polynomials P_1 and $P_2(t) = qt - p$. Since $\alpha_1 \neq \frac{p}{q}$ and $\alpha_2 \neq \frac{p}{q}$, we have $|R(P_1, P_2)| > 1$. On the other hand, from Feldman's lemma (Lemma 5) and the assumption $\boldsymbol{\alpha} \in \Pi_0$ we obtain that $|R(P_1, P_2)| < \frac{1}{2}$. This contradiction completes the proof. □

This simple result implies that if the size of a rectangle Π is sufficiently large, that is, $\mu_2 \Pi \gg Q^{-1}$, then we have $\#\mathbb{A}_n^2(Q, \Pi) \neq 0$, and we can consider lower bounds for this quantity. A bound of this type was obtained in [5]; it has the form

$$\#\mathbb{A}_n^2(Q, \Pi) > c_{11} \cdot Q^{n+1} \mu_2 \Pi. \tag{1.2}$$

In this paper, we obtain an upper bound for $\#\mathbb{A}_n^2(Q, \Pi)$. It is of the same order as estimate (1.2), which demonstrates that estimate (1.2) is asymptotically the best possible.

Theorem 3. *Let $\Pi = I_1 \times I_2$ be a rectangle with midpoint $\mathbf{d}$ and sides $\mu_1 I_i = c_8 Q^{-\gamma_i}$, $i = 1, 2$. Then for $0 < \gamma_1, \gamma_2 < 1$ and $Q > Q_0(n, \gamma, \mathbf{d})$, the estimate*

$$\#\mathbb{A}_n^2(Q, \Pi) < c_{12} \cdot Q^{n+1} \mu_2 \Pi$$

holds, where

$$c_{12} = 2^{3n+9} n^2 \rho_n(d_1) \rho_n(d_2) |d_1 - d_2|^{-1} \text{ and } \rho_n(x) = ((|x| + 1)^{n+1} - 1) \cdot |x|^{-1}.$$

It follows from Theorem 2 that for $1 < \gamma_1 + \gamma_2 < 2$ we cannot obtain estimate (1.2) for all rectangles Π. In particular, it is easy to show that certain neighborhoods of algebraic points of small height and small degree do not contain any other algebraic points $\boldsymbol{\alpha} \in \mathbb{A}_n^2(Q)$. This leads us to the definition of a set of small rectangles that are not affected by these "anomalous" points. Now let us introduce the concept of a (v_1, v_2)-*special* square.

Definition 1. *Let $\Pi = I_1 \times I_2$ be a square with midpoint $\mathbf{d}$, $d_1 \neq d_2$, and sides $\mu_1 I_1 = \mu_1 I_2 = c_8 Q^{-\gamma}$ such that $\frac{1}{2} < \gamma < 1$. We will say that the square Π satisfies the (l, v_1, v_2)-condition if $v_1 + v_2 = 1$ and there exist at most $\delta^3 \cdot 2^{l+3} Q^{1+2\lambda_{l+1}} \mu_2 \Pi$ polynomials $P \in \mathcal{P}_2(Q)$ of the form*

$P(t) = a_2t^2 + a_1t + a_0$ *satisfying the inequalities*

$$\begin{cases} |P(x_{0,i})| < h \cdot Q^{-v_i}, \quad i = 1, 2, \\ \delta Q^{\lambda_{l+1}} \leqslant |a_2| < \delta Q^{\lambda_l} \end{cases}$$

for some point $\mathbf{x}_0 \in \Pi$, *where* $\delta = 2^{-L-17}h^{-2} \cdot (d_1 - d_2)^2$, $L = \left[\frac{3-2\gamma}{1-\gamma}\right]$, *and*

$$\lambda_l = \begin{cases} 1 - \frac{(l-1)(1-\gamma)}{2}, \quad 1 \leqslant l \leqslant L+1, \\ \gamma - \frac{1}{2}, \quad l = L+2, \\ 0, \quad l \geqslant L+3. \end{cases} \tag{1.3}$$

Definition 2. *A square* $\Pi = I_1 \times I_2$ *with sides* $\mu_1 I_1 = \mu_1 I_2 = c_8 Q^{-\gamma}$ *such that* $\frac{1}{2} < \gamma < 1$ *is called a* (v_1, v_2)*-special square if it satisfies the* (l, v_1, v_2)*-condition for all* l *with* $1 \leqslant l \leqslant L+2$.

The following theorem can be proved for (v_1, v_2)-*special* squares.

Theorem 4. *For all* $\left(\frac{1}{2}, \frac{1}{2}\right)$*-special squares* $\Pi = I_1 \times I_2$ *with midpoints* $\mathbf{d}$, $d_1 \neq d_2$, *and sides* $\mu_1 I_1 = \mu_1 I_2 = c_8 Q^{-\gamma}$, *where* $\frac{1}{2} < \gamma < 1$ *and* $c_8 > c_0(n, \mathbf{d})$, *there exists a value* $c_{13} = c_{13}(n, \mathbf{d}, \gamma) > 0$ *such that*

$$\#\mathbb{A}_n^2(Q, \Pi) > c_{13} \cdot Q^{n+1} \mu_2 \Pi$$

for $Q > Q_0(n, \mathbf{d}, \gamma)$.

§2. Auxiliary statements

For a polynomial P with roots $\alpha_1, \ldots, \alpha_n$, let

$$S(\alpha_i) := \left\{ x \in \mathbb{R} : |x - \alpha_i| = \min_{1 \leqslant j \leqslant n} |x - \alpha_j| \right\}.$$

Furthermore, from now on, we assume that the roots of the polynomial P are sorted by the distance from $\alpha_i = \alpha_{i,1}$:

$$|\alpha_{i,1} - \alpha_{i,2}| \leqslant |\alpha_{i,1} - \alpha_{i,3}| \leqslant \ldots \leqslant |\alpha_{i,1} - \alpha_{i,n}|.$$

Lemma 1. *Let* $x \in S(\alpha_i)$. *Then*

$$|x - \alpha_i| \leqslant n|P(x)| \cdot |P'(x)|^{-1}, \quad |x - \alpha_i| \leqslant 2^{n-1}|P(x)| \cdot |P'(\alpha_i)|^{-1},$$

$$|x - \alpha_i| \leqslant \min_{1 \leqslant j \leqslant n} \left(2^{n-j}|P(x)| \cdot |P'(\alpha_i)|^{-1} \cdot |\alpha_{i,1} - \alpha_{i,2}| \ldots |\alpha_{i,1} - \alpha_{i,j}| \right)^{1/j}.$$

The first inequality follows from the inequality

$$|P'(x)|\cdot|P(x)|^{-1}\leqslant\sum_{j=1}^{n}|x-\alpha_{i,j}|^{-1}\leqslant n|x-\alpha_{i,1}|^{-1}.$$

For a proof of the second and the third inequalities, see [8, 3].

Lemma 2 (see [2]). *Let $I\subset\mathbb{R}$ be an interval, and let $A\subset I$ be a measurable set, $\mu_1 A\geqslant\frac{1}{2}\mu_1 I$. If for all $x\in A$ the inequality $|P(x)|<c_{14}\cdot Q^{-w}$ holds for some $w>0$, then*

$$|P(x)|<6^n(n+1)^{n+1}\cdot c_{14}\cdot Q^{-w}$$

for all points $x\in I$, where $n=\deg P$.

Lemma 3 (see [16]). *Let δ, η_1, η_2 be real positive numbers, and let $P_1,P_2\in\mathbb{Z}[t]$ be irreducible polynomials of degrees at most n such that* $\max(H(P_1),H(P_2))<K$. *Let $J_i\subset\mathbb{R}$, $i=1,2$, be intervals of sizes $\mu_1 J_i=K^{-\eta_i}$. If for some $\tau_1,\tau_2>0$ and for all $\mathbf{x}\in J_1\times J_2$ the inequalities* $\max(|P_1(x_i)|,|P_2(x_i)|)<K^{-\tau_i}$ *hold, then*

$$\tau_1+\tau_2+2+2\max(\tau_1+1-\eta_1,0)+2\max(\tau_2+1-\eta_2,0)<2n+\delta \quad (2.1)$$

for $K>K_0(\delta)$.

Lemma 4 (see [8]). *Let $P\in\mathbb{Z}[t]$ be a reducible polynomial, $P=P_1\cdot P_2$,* $\deg P=n\geqslant 2$. *Then*

$$H(P_1)H(P_2)\asymp H(P).$$

Lemma 5 (see [10]). *For any subset of roots $\alpha_{i_1},\dots,\alpha_{i_s}$, $1\leqslant s\leqslant n$, of a polynomial $P(t)=a_nt^n+\ldots+a_1t+a_0$, we have*

$$\prod_{j=1}^{s}|\alpha_{i_j}|\leqslant(n+1)2^n\cdot H(P)\cdot|a_n|^{-1}.$$

Lemma 6. *Let $G=G(\mathbf{d},\mathbf{K})$, where $|d_1-d_2|>\varepsilon_1>0$, be a set of points* $\mathbf{b}=(b_1,b_0)\in\mathbb{Z}^2$ *such that*

$$|b_1d_i+b_0|\leqslant K_i,\quad i=1,2. \quad (2.2)$$

Then

$$\#G\leqslant\left(4\varepsilon_1^{-1}K_1+1\right)\cdot(4K_2+1).$$

Proof. Without loss of generality, we assume that $K_1 \geqslant K_2$. Consider the system of equations

$$b_1 d_i + b_0 = l_i, \quad i = 1, 2, \tag{2.3}$$

in two variables. It is clear that for $|l_i| \leqslant K_i$ any solution of the system (2.3) satisfies (2.2). Thus, our problem is reduced to estimating the number of integer solutions of the system (2.3) with different values $|l_i| \leqslant K_i$, $i = 1, 2$. Let us consider the difference of the equations (2.3): $b_1(d_1 - d_2) = l_1 - l_2$. Then for $|l_i| \leqslant K_1$ we obtain

$$|b_1| \leqslant (|l_1| + |l_2|) \cdot |d_1 - d_2|^{-1} \leqslant 2\varepsilon_1^{-1} K_1.$$

This inequality implies that all possible values of b_1 lie in the interval $J_1 = \left(-2\varepsilon_1^{-1} K_1, 2\varepsilon_1^{-1} K_1\right)$. Let us fix the value of $b_1 \in J_1$ and consider the system (2.3) for two different combinations $(b_1, b_{0,0})$ and $(b_1, b_{0,j})$. In this case, the system (2.3) can be transformed as follows:

$$|b_{0,0} - b_{0,j}| = |l_{1,0} - l_{1,j}| \leqslant 2K_i, \quad i = 1, 2.$$

These inequalities imply that for a fixed b_1, all possible values of b_0 lie in the interval $J_0(b_1) = (b_{0,0} - 2K_2,\, b_{0,0} + 2K_2)$. Remembering that $b_1, b_0 \in \mathbb{Z}$, we have

$$\#G \leqslant (\mu_1 J_1 + 1) \cdot (\mu_1 J_0 + 1) = \left(4\varepsilon_1^{-1} K_1 + 1\right) \cdot (4K_2 + 1). \qquad \square$$

§3. Proof of Theorem 3

Assume that $\#\mathbb{A}_n^2(Q, \Pi) \geqslant c_{12} \cdot Q^{n+1} \mu_2 \Pi$. Taking an algebraic point $\boldsymbol{\alpha} \in \mathbb{A}_n^2(Q, \Pi)$ with minimal polynomial P, let us construct an estimate for the polynomial P at points d_1, d_2. Since $\alpha_i \in I_i$, we have

$$|P^{(k)}(\alpha_i)| \leqslant \sum_{j=k}^{n} \tfrac{j!}{(j-k)!} \cdot |a_j| \cdot |\alpha_i|^{j-k} < \tfrac{n!}{(n-k)!} \cdot \rho_n(d_i) \cdot Q,$$

for $1 \leqslant k \leqslant n$ and $Q > Q_0$. From these estimates and the Taylor expansion of P in the intervals I_i, $i = 1, 2$, we obtain the following inequality:

$$\begin{aligned} |P(d_i)| &\leqslant \sum_{k=1}^{n} \left| \tfrac{1}{k!} P^{(k)}(\alpha_i)(d_i - \alpha_i)^k \right| \\ &< \sum_{k=1}^{n} 2^{-k} \tbinom{k}{n} \rho_n(d_i) \cdot Q\mu_1 I_i \leqslant 2^n \rho_n(d_i) \cdot Q\mu_1 I_i. \end{aligned} \tag{3.1}$$

Let us fix the vector $\mathbf{A}_1 = (a_n, \ldots, a_2)$, where $a_n, \ldots, a_2$ are the coefficients of the polynomial $P \in \mathcal{P}_n(Q)$. Denote by $\mathcal{P}_n(Q, \mathbf{A}_1) \subset \mathcal{P}_n(Q)$ the subclass of polynomials P that have the same vector of coefficients $\mathbf{A}_1$ and satisfy (3.1). The number of subclasses $\mathcal{P}_n(Q,\mathbf{A}_1)$ is equal to the number of vectors $\mathbf{A}_1$, which can be estimated as follows for $Q > Q_0$:

$$\#\{\mathbf{A}_1\} = (2Q+1)^{n-1} < 2^n \cdot Q^{n-1}. \tag{3.2}$$

It should also be noted that every point of the set $\mathbb{A}_n^2(Q,\Pi)$ corresponds to a polynomial $P \in \mathcal{P}_n(Q)$ that satisfies (3.1). On the other hand, every polynomial $P \in \mathcal{P}_n(Q)$ satisfying (3.1) corresponds to at most n^2 points of the set $\mathbb{A}_n^2(Q,\Pi)$. This allows us to write

$$c_{11} \cdot Q^{n+1}\mu_2\Pi < \#\mathbb{A}_n^2(Q,\Pi) \leqslant n^2 \sum_{\mathbf{A}_1} \#\mathcal{P}_n(Q,\mathbf{A}_1).$$

Thus, by estimate (3.3) and Dirichlet's principle applied to vectors $\mathbf{A}_1$ and polynomials P satisfying (3.1), there exists a vector $\mathbf{A}_{1,0}$ such that

$$\#\mathcal{P}_n(Q,\mathbf{A}_{1,0}) \geqslant c_{12} \cdot 2^{-n}n^{-2}Q^2\mu_2\Pi. \tag{3.3}$$

Let us find an upper bound for the value $\#\mathcal{P}_n(Q,\mathbf{A}_{1,0})$. To do this, we fix some polynomial $P_0 \in \mathcal{P}_n(Q,\mathbf{A}_{1,0})$ and consider the difference between the polynomials P_0 and $P_j \in \mathcal{P}_n(Q,\mathbf{A}_{1,0})$ at points d_i, $i = 1, 2$. From estimate (3.1) it follows that

$$|P_0(d_i) - P_j(d_i)| = |(a_{0,1} - a_{j,1})d_i + (a_{0,0} - a_{j,0})| \leqslant 2^{n+1}\rho_n(d_i) \cdot Q\mu_1 I_i.$$

Thus the number of different polynomials $P_j \in \mathcal{P}_n(Q,\mathbf{A}_{1,0})$ does not exceed the number of integer solutions of the following system:

$$|b_1 d_i + b_0| \leqslant 2^{n+1}\rho_n(d_i) \cdot Q\mu_1 I_i, \quad i = 1, 2.$$

Now let us use Lemma 6 for $K_i = 2^{n+1}\rho_n(d_i) \cdot Q\mu_1 I_i$. Since $\mu_1 I_i = c_8 Q^{-\gamma_i}$ and $\gamma_i < 1$, we have $K_i \geqslant 2^{n+1}\rho_n(d_i)c_8 \cdot Q^{1-\gamma_i} > \max\{\varepsilon_1, 1\}$ for $Q > Q_0$. This implies that

$$j \leqslant 2^{2n+8}|d_1 - d_2|^{-1}\rho_n(d_1)\rho_n(d_2) \cdot Q^2\mu_2\Pi.$$

It follows that $\#\mathcal{P}_n(Q,\mathbf{A}_{1,0}) \leqslant 2^{2n+8}|d_1 - d_2|^{-1}\rho_n(d_1)\rho_n(d_2) \cdot Q^2\mu_2\Pi$, which contradicts inequality (3.3) for

$$c_{12} = 2^{3n+9}n^2\rho_n(d_1)\rho_n(d_2)|d_1 - d_2|^{-1}.$$

This leads to the estimate

$$\#\mathbb{A}_n^2(Q,\Pi) < c_{12} \cdot Q^{n+1}\mu_2\Pi.$$

§4. Proof of Theorem 4

4.1. The main lemma.

Lemma 7. *Let* $\Pi = I_1 \times I_2$ *be a square with midpoint* $\mathbf{d}$, $d_1 \neq d_2$, *and sides* $\mu_1 I_1 = \mu_1 I_2 = c_8 Q^{-\gamma}$, *where* $\frac{1}{2} < \gamma < 1$ *and* $c_8 > c_0(n, \mathbf{d})$. *Given positive values* v_1, v_2 *such that* $v_1 + v_2 = n - 1$, *let* $L = L_n(Q, \delta_n, \mathbf{v}, \Pi)$ *be the set of points* $\mathbf{x} \in \Pi$ *such that there exists a polynomial* $P \in \mathcal{P}_n(Q)$ *satisfying the following system of inequalities:*

$$\begin{cases} |P(x_i)| < h_n \cdot Q^{-v_i}, \\ \min\limits_i \{|P'(x_i)|\} < \delta_n \cdot Q, \quad i = 1, 2, \end{cases} \tag{4.1}$$

where $h_n = \sqrt{\frac{3}{2}(|d_1| + |d_2|) \cdot \max(1, 3|d_1|, 3|d_2|)^{n^2}}$. *If* Π *is a* $\left(\frac{v_1}{n-1}, \frac{v_2}{n-1}\right)$*-special square, then*

$$\mu_2 L < \tfrac{1}{4} \cdot \mu_2 \Pi$$

for $\delta_n < \delta_0(n, \mathbf{d})$ *and* $Q > Q_0(n, \mathbf{v}, \mathbf{d}, \gamma)$.

Proof. Since $d_1 \neq d_2$, we may assume that for $Q > Q_0$ the inequality

$$|x_1 - x_2| > \varepsilon_1 = \tfrac{|d_1 - d_2|}{2} \tag{4.2}$$

is satisfied for every point $\mathbf{x} \in \Pi$. Let us introduce some additional notation. For a polynomial P, let $\mathcal{A}(P)$ denote the set of roots of P. Denote by L_1 and L_2 the sets of points $\mathbf{x} \in \Pi$ such that there exists an irreducible polynomial $P \in \mathcal{P}_n(Q)$ satisfying (4.1) and the condition $|P'(x_1)| < \delta_n Q$ or $|P'(x_2)| < \delta_n Q$, respectively, and let L_3 denote the set of points $\mathbf{x} \in \Pi$ such that (4.1) is satisfied for some reducible polynomial $P \in \mathcal{P}_n(Q)$. Clearly, we have $L = L_1 \cup L_2 \cup L_3$. The case of irreducible polynomials will be the most difficult one and requires the largest part of the proof. Let us start by considering this case, deriving estimates for the measures $\mu_1 L_1$ and $\mu_1 L_2$. Without loss of generality, let us assume that $|P'(x_1)| < \delta_n Q$, i.e., consider the set L_1. In this case, the main idea is to split an interval T_i, which contains all possible values of P' at points $\mathbf{x} \in \Pi$, into subintervals $T_{i,1}$, $T_{i,2}$, $T_{i,3}$ and to estimate the measure of the set of solutions of the system (4.1) for $|P'(x_i)| \in T_{i,k}$, $k = 1, 2, 3$. This splitting is performed as follows: for $i = 1, 2$,

$$\begin{aligned}
T_{i,1} &= \Big[0; \quad 2c_{15} \cdot Q^{\frac{1}{2}-\frac{v_i}{2}}\Big),\\
T_{i,2} &= \Big[2c_{15} \cdot Q^{\frac{1}{2}-\frac{v_i}{2}}; \quad Q^{\frac{1}{2}-\frac{(n-2)v_i}{2(n-1)}\cdot\theta(n)}\Big),\\
T_{1,3} &= \Big[Q^{\frac{1}{2}-\frac{(n-2)v_1}{2(n-1)}\cdot\theta(n)}; \quad \delta_n \cdot Q\Big),\\
T_{2,3} &= \Big[Q^{\frac{1}{2}-\frac{(n-2)v_2}{2(n-1)}\cdot\theta(n)}; \quad \rho_{n+1}(d_2) \cdot Q\Big),
\end{aligned}$$

where $\theta(n) = 0$ if $n \leqslant 3$ and $\theta(n) = 1$ if $n > 3$. Without loss of generality, let us assume that $|d_1| < |d_2|$. We would like to verify that if a polynomial $P \in \mathcal{P}_n(Q)$ satisfies the condition

$$|P'(x_i)| \geqslant 2c_{15} \cdot Q^{\frac{1}{2}-\frac{v_i}{2}}, \tag{4.3}$$

then the values $|P'(\alpha_i)|$ can be estimated as follows:

$$\tfrac{1}{2}|P'(x_i)| \leqslant |P'(\alpha_i)| \leqslant 2|P'(x_i)|, \quad i = 1, 2, \tag{4.4}$$

where $x_i \in S(\alpha_i)$ and $c_{15} = 2^{n-1}n(n-1) \cdot \max\{h_n, 1\} \cdot \max\{1, \rho_{n-1}(d_2)\}$. Let us write the Taylor expansion of P':

$$P'(x_i) = P'(\alpha_i) + P''(\alpha_i)(x_i - \alpha_i) + \ldots + \tfrac{1}{(n-1)!} \cdot P^{(n)}(\alpha_i)(x_i - \alpha_i)^{n-1}. \tag{4.5}$$

Using Lemma 1 and estimates (4.1), (4.3), we obtain

$$|x_i - \alpha_i| \leqslant nh_n c_{15}^{-1} \cdot Q^{-\frac{v_i+1}{2}} < Q^{-\frac{v_i+1}{2}}, \quad |\alpha_i| \leqslant |x_i| + \tfrac{1}{2} < |d_2| + 1$$

for $Q > Q_0$. Let us estimate every term in (4.5) in the following way:

$$\left|\tfrac{1}{(k-1)!} \cdot P^{(k)}(\alpha_i)(x_i - \alpha_i)^{k-1}\right| < \tbinom{k-1}{n-1} \cdot n(n-1)\rho_{n-1}(d_2) \cdot Q^{\frac{1}{2}-\frac{v_i}{2}},$$

for $Q > Q_0$ and $2 \leqslant k \leqslant n$. Thus, we can write

$$\begin{aligned}
\left|\sum_{k=2}^{n} \tfrac{1}{(k-1)!} \cdot P^{(k)}(\alpha_i)(x_i - \alpha_i)^{k-1}\right| &< 2^{n-1}n(n-1)\rho_{n-1}(d_2) \cdot Q^{\frac{1}{2}-\frac{v_i}{2}}\\
&< \tfrac{1}{2}|P'(x_i)|.
\end{aligned}$$

Substituting this inequality into (4.5) yields estimates (4.4). This means that for $|P'(x_i)| \in T_{i,3}$ and $|P'(x_i)| \in T_{i,2}$ we have $|P'(\alpha_i)| \in \overline{T}_{i,3}$ and

$|P'(\alpha_i)| \in \overline{T}_{i,2}$, respectively, where

$$\overline{T}_{1,3} = \left[\tfrac{1}{2} Q^{\frac{1}{2} - \frac{(n-2)v_1}{2(n-1)} \cdot \theta(n)}; \quad 2\delta_n \cdot Q\right),$$
$$\overline{T}_{2,3} = \left[\tfrac{1}{2} Q^{\frac{1}{2} - \frac{(n-2)v_2}{2(n-1)} \cdot \theta(n)}; \quad 2\rho_{n+1}(d_2) \cdot Q\right),$$
$$\overline{T}_{i,2} = \left[c_{15} \cdot Q^{\frac{1}{2} - \frac{v_i}{2}}; \quad 2 \cdot Q^{\frac{1}{2} - \frac{(n-2)v_i}{2(n-1)} \cdot \theta(n)}\right), \quad i = 1, 2.$$

Let us consider the case $|P'(\alpha_i)| \in \overline{T}_{i,3}$, $i = 1, 2$. We are going to use induction on the degree of polynomials P.

The base of the induction: polynomials of the second degree. Let us consider the system (4.1) for $n = 2$. For given $u_{2,1}, u_{2,2} > 0$ satisfying the condition $u_{2,1} + u_{2,2} = 1$, let $L' = L_2(Q, \delta_2, \mathbf{u}_2, \Pi)$ be the set of points $\mathbf{x} \in \Pi$ such that there exists a polynomial $P \in \mathcal{P}_2(Q)$ satisfying the system of inequalities

$$\begin{cases} |P(x_i)| < h_2 \cdot Q^{-u_{2,i}}, \\ \min\limits_i \{|P'(x_i)|\} < \delta_2 \cdot Q, \quad i = 1, 2. \end{cases} \tag{4.6}$$

Let us prove that for all $(u_{2,1}, u_{2,2})$-*special* squares Π satisfying the conditions of Lemma 7, the estimate

$$\mu_2 L' < \tfrac{1}{4} \cdot \mu_2 \Pi$$

holds for $\delta_2 < \delta_0(\mathbf{d}, \gamma)$ and $Q > Q_0(\mathbf{u}_2, \gamma, \mathbf{d})$. Let $P(t) = a_2 t^2 + a_1 t + a_0$. First, note that the definition of a $(u_{2,1}, u_{2,2})$-*special* square implies that for $Q > Q_0$ there exists at most

$$\delta 2^{l+3} c_5^2 Q^{1-2\gamma} < \delta 2^{l+3} c_5^2 Q^{-\varepsilon} < 1$$

polynomials $P \in \mathcal{P}_2(Q)$ satisfying $|a_2| < \delta Q^{\gamma - \frac{1}{2}}$ and (4.6). Therefore, from now on we are going to assume that $|a_2| \geqslant \delta Q^{\gamma - \frac{1}{2}}$. By the third inequality of Lemma 1, for every polynomial P satisfying the system (4.6) at a point $\mathbf{x} \in \Pi$, we have the following estimates:

$$|x_i - \alpha_i| < \left(|P(x_i)||a_2|^{-1}\right)^{\frac{1}{2}} < \delta^{1/2} h_2^{1/2} \cdot Q^{-\frac{2\gamma + 2u_{2,i} - 1}{4}} < \tfrac{\varepsilon_1}{8}, \tag{4.7}$$

where $Q > Q_0$ and $x_i \in S(\alpha_i)$, $i = 1, 2$. Thus, from (4.7) and (4.2) we obtain that the distance between the roots α_1 and α_2 of the polynomial P satisfies

$$|\alpha_1 - \alpha_2| > |x_1 - x_2| - |x_1 - \alpha_1| - |x_2 - \alpha_2| > \tfrac{3}{4} \cdot \varepsilon_1.$$

This leads to the following lower bound for $|P'(\alpha_i)|$:

$$|P'(\alpha_i)| = |a_2| \cdot |\alpha_1 - \alpha_2| > \tfrac{3}{4} \cdot \varepsilon_1 \cdot |a_2|. \tag{4.8}$$

An upper bound for $|P'(\alpha_i)|$ can be obtained from the Taylor expansion of the polynomial P':

$$|P'(\alpha_i)| \leqslant |P'(x_i)| + |P''(x_i)| \cdot |x_i - \alpha_i| \leqslant |P'(x_i)| + \tfrac{\varepsilon_1}{4} \cdot |a_2|.$$

Hence, by (4.8) and (4.6) we have

$$|a_2| < 4\varepsilon_1^{-1} \cdot \min_i \{|P'(x_i)|\} < 4\delta_2\varepsilon_1^{-1} \cdot Q. \tag{4.9}$$

Now let us turn to the estimation of $\mu_2 L'$. From Lemma 1 and the estimates (4.8) it follows that $L' \subset \bigcup_{P \in \mathcal{P}_2(Q)} \sigma_P$, where

$$\sigma_P = \{\mathbf{x} \in \Pi : \quad |x_i - \alpha_i| < 2h_2\varepsilon_1^{-1}Q^{-u_{2,i}}|a_2|^{-1}, \quad i = 1, 2\}. \tag{4.10}$$

Simple calculations show that for $c_8 > 2^4 h_2 \varepsilon_1^{-1}\delta^{-1}$ and $|a_2| > \delta Q^{\gamma - \frac{1}{2}}$ we have

$$\mu_2\sigma_P \leqslant 2^4 h_2^2 \varepsilon_1^{-2} Q^{-1} |a_2|^{-2} \leqslant \tfrac{2^8 h_2^2}{\varepsilon_1^2 \delta^2} \cdot Q^{-2\gamma} < \tfrac{1}{4} \cdot \mu_2 \Pi.$$

Let $\mathcal{P}_2(Q, l) \subset \mathcal{P}_2(Q)$ be a subclass of polynomials defined as follows:

$$\mathcal{P}_2(Q, l) = \{P \in \mathcal{P}_2(Q) : \delta Q^{\lambda_{l+1}} \leqslant |a_2| < \delta Q^{\lambda_l}\},$$

where λ_l is defined by (1.3) and $\delta = 2^{-L-17} h_2^{-2} \cdot (d_1 - d_2)^2$, $L = \left[\frac{3-2\gamma}{1-\gamma}\right]$. Thus, by (4.9) it follows that for $|a_2| > \delta Q^{\gamma - \frac{1}{2}}$ and $\delta_2 = \frac{4\delta}{\varepsilon_1}$ we have

$$\mu_2 L' \leqslant \mu_2 \bigcup_{P \in \mathcal{P}_2(Q)} \sigma_P \leqslant \sum_{l=1}^{L+1} \sum_{P \in \mathcal{P}_2(Q,l)} \mu_2 \sigma_P.$$

From the definition of a $(u_{2,1}, u_{2,2})$-*special* square it follows that the number of polynomials $P \in \mathcal{P}_2(Q, l)$ satisfying (4.6) does not exceed

$$\delta^3 \cdot 2^{l+3} Q^{1+2\lambda_{l+1}} \mu_2 \Pi. \tag{4.11}$$

Hence, by estimates (4.10) and (4.11) we have

$$\mu_2 L_2 \leqslant 2^8 \varepsilon_1^{-2} h_2^2 \delta Q^{-1} \mu_2 \Pi \cdot \sum_{l=1}^{L+1} 2^{l+3} Q^{1+2\lambda_{l+1} - 2\lambda_{l+1}} \leqslant \tfrac{1}{4} \cdot \mu_2 \Pi.$$

The induction step: reduction of the degree of the polynomial. Let us return to the proof of Lemma 7. For $|P'(\alpha_i)| \in \overline{T}_{i,3}$, $i = 1, 2$, we consider the following system of inequalities:

$$\begin{cases} |P(x_i)| < h_n \cdot Q^{-v_i}, \quad i = 1, 2, \\ \frac{1}{2} Q^{\frac{1}{2} - \frac{(n-2)v_1}{2(n-1)} \cdot \theta(n)} \leqslant |P'(\alpha_1)| < 2\delta_n \cdot Q, \\ \frac{1}{2} Q^{\frac{1}{2} - \frac{(n-2)v_2}{2(n-1)} \cdot \theta(n)} \leqslant |P'(\alpha_2)| < 2\rho_{n+1}(d_2) \cdot Q. \end{cases} \tag{4.12}$$

Denote by $L_{3,3}$ the set of points $\mathbf{x} \in \Pi$ such that there exists a polynomial $P \in \mathcal{P}_n(Q)$ satisfying the system (4.12). By Lemma 1, it follows that $L_{3,3} \subset \bigcup\limits_{P \in \mathcal{P}_n(Q)} \bigcup\limits_{\boldsymbol{\alpha} \in \mathcal{A}^2(P)} \sigma_P(\boldsymbol{\alpha})$, where

$$\sigma_P(\boldsymbol{\alpha}) := \left\{ \mathbf{x} \in \Pi : \quad |x_i - \alpha_i| < 2^{n-1} h_n Q^{-v_i} |P'(\alpha_i)|^{-1}, \quad i = 1, 2 \right\}. \tag{4.13}$$

This means that the following estimate for $\mu_2 L_{3,3}$ holds:

$$\mu_2 L_{3,3} \leqslant \mu_2 \bigcup_{P \in \mathcal{P}_n(Q)} \bigcup_{\boldsymbol{\alpha} \in \mathcal{A}^2(P)} \sigma_P(\boldsymbol{\alpha}) \leqslant \sum_{P \in \mathcal{P}_n(Q)} \sum_{\boldsymbol{\alpha} \in \mathcal{A}^2(P)} \mu_2 \sigma_P(\boldsymbol{\alpha}).$$

Together with the sets $\sigma_P(\boldsymbol{\alpha})$, consider the following expanded sets:

$$\begin{aligned} \sigma'_P(\boldsymbol{\alpha}) &= \sigma'_{P,1}(\alpha_1) \times \sigma'_{P,2}(\alpha_2) \\ &= \left\{ \mathbf{x} \in \Pi : \quad |x_i - \alpha_i| < c_{16} Q^{-u_{i,n-1}} |P'(\alpha_i)|^{-1} \right\}, \end{aligned} \tag{4.14}$$

where $u_{i,n-1} = \frac{(n-2)v_i}{n-1}$, $i = 1, 2$. It is easy to see that the measure of the expanded set $\sigma'_P(\boldsymbol{\alpha})$ is smaller than the measure of the square Π for $Q > Q_0$. Using (4.13) and (4.14), we find that the measures of the sets $\sigma_P(\boldsymbol{\alpha})$ and $\sigma'_P(\boldsymbol{\alpha})$ are connected as follows:

$$\mu_2 \sigma_P(\boldsymbol{\alpha}) \leqslant 2^{2n-2} h_n^2 c_{16}^{-2} \cdot Q^{-1} \mu_2 \sigma'_P(\boldsymbol{\alpha}). \tag{4.15}$$

For a fixed a, let $\mathcal{P}_n(Q, a) \subset \mathcal{P}_n(Q)$ denote the subclass of polynomials with the leading coefficient a:

$$\mathcal{P}_n(Q, a) = \{ P \in \mathcal{P}_n(Q) : P(t) = a t^n + \ldots + a_0 \}.$$

Since $-Q \leqslant a \leqslant Q$, the number of subclasses $\mathcal{P}_n(Q, a)$ is equal to

$$\# \{a\} = 2Q + 1. \tag{4.16}$$

We are going to use Sprindžuk's method of essential and nonessential domains [8]. Consider a family of sets $\sigma'_P(\boldsymbol{\alpha})$, $P \in \mathcal{P}_n(Q, a)$. A set $\sigma'_{P_1}(\boldsymbol{\alpha}_1)$ is called *essential* if for every set $\sigma'_{P_2}(\boldsymbol{\alpha}_2)$, $P_2 \neq P_1$, the inequality

$$\mu_2 \left(\sigma'_{P_1}(\boldsymbol{\alpha}_1) \cap \sigma'_{P_2}(\boldsymbol{\alpha}_2) \right) < \tfrac{1}{2} \cdot \mu_2 \sigma'_{P_1}(\boldsymbol{\alpha}_1)$$

is satisfied. Otherwise, the set $\sigma'_{P_1}(\boldsymbol{\alpha}_1)$ is called *nonessential*.

The case of essential sets. From the definition of essential sets we immediately have that

$$\sum_{P\in\mathcal{P}_n(Q,a)} \sum_{\substack{\boldsymbol{\alpha}\in\mathcal{A}^2(P):\\ \sigma'_P(\boldsymbol{\alpha}) \text{ is essential}}} \mu_2\sigma'_P(\boldsymbol{\alpha}) \leqslant 2\mu_2\Pi. \tag{4.17}$$

Then inequalities (4.15), (4.16), and (4.17) for $c_{16} = 2^{n+5}h_n$ allow us to write

$$\begin{aligned} &\sum_a \sum_{P\in\mathcal{P}_n(Q,a)} \sum_{\substack{\boldsymbol{\alpha}\in\mathcal{A}^2(P):\\ \sigma'_P(\boldsymbol{\alpha}) \text{ is ess.}}} \mu_2\sigma_P(\boldsymbol{\alpha}) \\ &\qquad \leqslant 2^{-10}\cdot \sum_{P\in\mathcal{P}_n(Q,a)} \sum_{\substack{\boldsymbol{\alpha}\in\mathcal{A}^2(P):\\ \sigma'_P(\boldsymbol{\alpha}) \text{ is ess.}}} \mu_2\sigma'_P(\boldsymbol{\alpha}) < \tfrac{1}{288}\cdot\mu_2\Pi. \end{aligned} \tag{4.18}$$

The case of nonessential sets. If a set $\sigma'_{P_1}(\boldsymbol{\alpha}_1)$ is *nonessential*, then the family contains another set $\sigma'_{P_2}(\boldsymbol{\alpha}_2)$ such that

$$\mu_2\left(\sigma'_{P_1}(\boldsymbol{\alpha}_1)\cap\sigma'_{P_2}(\boldsymbol{\alpha}_2)\right) > \tfrac{1}{2}\mu_2\sigma'_{P_1}(\boldsymbol{\alpha}_1).$$

Consider the difference $R = P_2 - P_1$, which is a polynomial of degree $\deg R \leqslant n-1$ and height $H(R) \leqslant 2Q$. Let us estimate the polynomials R and R' at points $\mathbf{x} \in \left(\sigma'_{P_1}(\boldsymbol{\alpha}_1)\cap\sigma'_{P_2}(\boldsymbol{\alpha}_2)\right)$. From the Taylor expansions of the polynomials P_j in the intervals $\sigma'_{P_1,i}(\alpha_{1,i})\cap\sigma'_{P_2,i}(\alpha_{2,i})$, $i,j=1,2$, estimates (4.12), (4.14), and the equality $u_{i,n-1} = \frac{(n-2)v_i}{n-1}$ we have

$$\begin{aligned} |P_j(x_i)| &\leqslant \sum_{k=1}^{n}\left|\tfrac{1}{k!}P_j^{(k)}(\alpha_{j,i})(x_i-\alpha_{j,i})^k\right| \\ &\leqslant \sum_{k=1}^{n}\tbinom{k}{n}\cdot\rho_n c_{16}^k\cdot Q^{-u_{i,n-1}} \leqslant \rho_n(d_2)(1+c_{16})^n\cdot Q^{-u_{i,n-1}} \end{aligned}$$

for $Q > Q_0$. Now we can write

$$|R(x_i)| < |P_1(x_i)| + |P_2(x_i)| < 2\rho_n(d_2)(1+c_{16})^n\cdot Q^{-u_{i,n-1}}. \tag{4.19}$$

Similarly, the Taylor expansions of the polynomials P'_j, $j=1,2$, in the intervals $\sigma'_{P_1,i}(\alpha_{1,i})\cap\sigma'_{P_2,i}(\alpha_{2,i})$, estimates (4.12), (4.14), and the equality $u_{i,n-1} = \frac{(n-2)v_i}{n-1}$ allow us to write

$$|P_j(x_i)| < n^2\rho_n(d_2)(1+c_{16})^{n-1}\cdot|P'_j(\alpha_{j,i})|.$$

From these estimates and inequalities (4.12), it easily follows that

$$\min_i \{|R'(x_i)|\} < 4n^2\rho_n(d_2)(1+c_{16})^{n-1}\delta_n \cdot Q. \tag{4.20}$$

Inequalities (4.19) and (4.20) hold for every point $\mathbf{x} \in \sigma'_{P_1}(\boldsymbol{\alpha}_1) \cap \sigma'_{P_2}(\boldsymbol{\alpha}_2)$. Since $\mu_1\left(\sigma'_{P_1,i}(\alpha_{1,i}) \cap \sigma'_{P_2,i}(\alpha_{2,i})\right) > \frac{1}{2}\mu_1\sigma'_{P_1,i}(\alpha_{1,i})$ for $i = 1, 2$, from Lemma 2 it follows that for every point $\mathbf{x} \in \sigma'_{P_1}(\boldsymbol{\alpha}_1)$ the inequalities

$$|R(x_i)| < c_{17} \cdot Q^{-u_{i,n-1}}, \quad \min_i \{|R'(x_i)|\} < c_{18}\delta_n \cdot Q \tag{4.21}$$

hold, where $c_{17} = 6^n(n+1)^{n+1} \cdot 2\rho_n(d_2)(1+c_{16})^n$ and

$$c_{18} = 6^n(n+1)^{n+1} \cdot 2n^2\rho_n(d_2)(1+c_{16})^{n-1}.$$

Denote by L' the set of points $\mathbf{x} \in \Pi$ such that there exists a polynomial $R \in \mathcal{P}_{n-1}(Q_1)$ satisfying the following system of inequalities:

$$\begin{cases} |R(x_i)| < c_{19}h_{n-1} \cdot Q_1^{-u_{i,n-1}}, \quad u_{i,n-1} > 0, \\ \min\limits_i\{|R'(x_i)|\} < \delta_{n-1} \cdot Q_1, \\ u_{1,n-1} + u_{2,n-1} = n-2, \quad i = 1, 2, \end{cases}$$

where $Q_1 = 2Q$, $c_{19} = \max\limits_i \left\{2^{u_{i,n-1}}\right\} c_{17}h_{n-1}^{-1}$, and $\delta_{n-1} = 2c_{18} \cdot \delta_n$. It should be mentioned that if a polynomial $R(t) = a_1t - a_0$ is linear, then by Lemma 1 we obtain

$$\left|x_i - \tfrac{a_0}{a_1}\right| \ll Q_1^{-u_{i,n-1}} < \tfrac{\varepsilon}{4}, \quad i = 1, 2,$$

for $Q_1 > Q_0$. Hence, we immediately have $|x_1 - x_2| < \varepsilon$, which contradicts condition 2 for the polynomial Π. Estimates (4.21) imply that the inclusion

$$\bigcup_{P \in \mathcal{P}_n(Q,a)} \bigcup_{\substack{\boldsymbol{\alpha} \in \mathcal{A}^2(P): \\ \sigma'_P(\boldsymbol{\alpha}) \text{ is noness.}}} \sigma'_P(\boldsymbol{\alpha}) \subset L'$$

is satisfied for all a. Thus, by the induction assumption, we obtain that

$$\sum_a \sum_{P \in \mathcal{P}_n(Q,a)} \sum_{\substack{\boldsymbol{\alpha} \in \mathcal{A}^2(P): \\ \sigma'_P(\boldsymbol{\alpha}) \text{ is noness.}}} \mu_2\sigma_P(\boldsymbol{\alpha}) \leqslant \mu_2 L' \leqslant \tfrac{1}{288} \cdot \mu_2\Pi, \tag{4.22}$$

for a sufficiently small constant δ_n and $Q > Q_0$. Then estimates (4.18) and (4.22) allow us to write

$$\mu_2 L_{3,3} \leqslant \sum_a \sum_{P\in\mathcal{P}_n(Q,a)} \sum_{\substack{\boldsymbol{\alpha}\in\mathcal{A}^2(P):\\ \sigma'_P(\boldsymbol{\alpha}) \text{ is ess.}}} \mu_2\sigma_P(\boldsymbol{\alpha}) + \sum_a \sum_{P\in\mathcal{P}_n(Q,a)} \sum_{\substack{\boldsymbol{\alpha}\in\mathcal{A}^2(P):\\ \sigma'_P(\boldsymbol{\alpha}) \text{ is noness.}}} \mu_2\sigma_P(\boldsymbol{\alpha}) \leqslant \tfrac{1}{144}\cdot\mu_2\Pi.$$

The case of the subintervals $T_{1,n}$ and $T_{2,n}$. For $|P'(\alpha_i)| \in \overline{T}_{i,2}$, $i = 1, 2$, we have the following system of inequalities:

$$\begin{cases} |P(x_i)| < h_n\cdot Q^{-v_i}, \\ c_{15}\cdot Q^{\frac12-\frac{v_i}{2}} \leqslant |P'(\alpha_i)| < 2Q^{\frac12-\frac{(n-2)v_i}{2(n-1)}\cdot\theta(n)}, \quad i=1,2. \end{cases} \tag{4.23}$$

Denote by $L_{2,2}$ the set of points $\mathbf{x}\in\Pi$ such that there exists a polynomial $P\in\mathcal{P}_n(Q)$ satisfying (4.23). By Lemma 1, the set $L_{2,2}$ is contained in the union $\bigcup\limits_{P\in\mathcal{P}_n(Q)} \bigcup\limits_{\boldsymbol{\alpha}\in\mathcal{A}^2(P)} \sigma_P(\boldsymbol{\alpha})$, where

$$\sigma_P(\boldsymbol{\alpha}) = \left\{\mathbf{x}\in\Pi : |x_i-\alpha_i| < 2^{n-1}h_n c_{15}^{-1} Q^{-\frac{v_i+1}{2}}, \quad i=1,2\right\}. \tag{4.24}$$

In this case, we cannot use induction, since the degree of the polynomial cannot be reduced. Let us estimate the measure of the set $L_{2,2}$ by a different method. Without loss of generality, we may assume that $v_1 \leqslant v_2$. Let us cover the square Π by a system of disjoint rectangles $\Pi_k = J_{k,1}\times J_{k,2}$, where $\mu_1 J_{k,i} = Q^{-\frac{v_i+1}{2}+\varepsilon_{2,i}}$, $i = 1, 2$. The number of rectangles Π_k can be estimated as follows:

$$\begin{aligned} k &\leqslant 4\max\left\{\frac{\mu_1 I_1}{\mu_1 J_{k,1}}, 1\right\}\cdot\max\left\{\frac{\mu_1 I_2}{\mu_1 J_{k,2}}, 1\right\} \\ &= \begin{cases} 4Q^{\frac{n+1}{2}-\varepsilon_{2,1}-\varepsilon_{2,2}}\mu_2\Pi, & \gamma < \frac{v_i+1}{2}, \\ 4Q^{\frac{v_2+1}{2}-\varepsilon_{2,2}}\mu_1 I_2, & \gamma \geqslant \frac{v_1+1}{2}. \end{cases} \end{aligned} \tag{4.25}$$

We will say that a polynomial P belongs to Π_k if there is a point $\mathbf{x}\in\Pi_k$ such that inequalities (4.23) are satisfied. Let us prove that a rectangle Π_k cannot contain two irreducible polynomials $P\in\mathcal{P}_n(Q)$. Assume the converse: the system of inequalities (4.23) holds for some irreducible polynomials P_j at some point $\mathbf{x}_j\in\Pi_k$, $j = 1, 2$. This means that for $Q > Q_0$

and all points $\mathbf{x} \in \Pi_k$, the estimates

$$|x_i - \alpha_{j,i}| \leqslant |x_i - x_{j,i}| + |x_{j,i} - \alpha_{j,i}| \leqslant 2 \cdot Q^{-\frac{v_i+1}{2}+\varepsilon_{2,i}} < Q^{-\frac{v_i+1}{2}+2\varepsilon_{2,i}} \quad (4.26)$$

are satisfied, where $x_{j,i} \in S(\alpha_{j,i})$. Let us estimate the absolute values $|P_j(x_i)|$, $i, j = 1, 2$, where $\mathbf{x} \in \Pi_k$. From the Taylor expansions of P_j in the interval $J_{k,i}$ and estimates (4.23), (4.26) we obtain that

$$|P_j(x_i)| \leqslant \rho_n(d_2) 3^n \cdot Q^{-v_i + \frac{v_i}{2(n-1)} + (n-1)\varepsilon_{2,i}} < Q^{-v_i + \frac{v_i}{2(n-1)} + n\varepsilon_{2,i}},$$

for $Q > Q_0$ and $\varepsilon_{2,i} < \frac{v_i}{2(n-1)}$. Applying Lemma 3 for $\eta_i = \frac{v_i+1}{2} - 2\varepsilon_{2,i}$, $\tau_i = v_i - \frac{v_i}{2(n-1)} - n \cdot \varepsilon_{2,i}$, $i = 1, 2$, and $\varepsilon_{2,i} = \frac{v_i}{8(n-1)}$ leads to the inequality

$$\tau_1 + \tau_2 + 2 + 2(\tau_1 + 1 - \eta_1) + 2(\tau_2 + 1 - \eta_2) > 2n + \tfrac{1}{4}.$$

This contradiction shows that there is at most one irreducible polynomial $P \in \mathcal{P}_n(Q)$ that belongs to the rectangle Π_k. Hence, by inequalities (4.26) and (4.25) for $Q > Q_0$, we can estimate the measure of the set $L_{2,2}$ as follows:

$$\mu_2 L_{2,2} \leqslant \sum_{\Pi_k} \mu_2 \sigma_P(\boldsymbol{\alpha}) \ll Q^{-\varepsilon_{2,2}} \mu_2 \Pi < \tfrac{1}{144} \cdot \mu_2 \Pi.$$

The case of a small derivative. Let us discuss the case where $|P'(x_i)| \in T_{i,1}$, $i = 1, 2$. In this case, we can show that if for some polynomial P and a point $\mathbf{x} \in \Pi$ inequalities (4.1) are satisfied for $|P'(x_i)| \in T_{i,1}$, then by Lemma 1 we have

$$\left| P''(\alpha_i)(x_i - \alpha_i) + \ldots + \tfrac{1}{(n-1)!} \cdot P^{(n)}(\alpha_i)(x_i - \alpha_i)^{n-1} \right| < c_{15} Q^{\frac{1}{2} - \frac{v_1}{2}}.$$

Using the Taylor expansion of the polynomial P' and this estimate, we obtain

$$|P'(\alpha_i)| < 3c_{15} \cdot Q^{\frac{1}{2} - \frac{v_i}{2}},$$

which contradicts our assumption. Denote by $L_{1,1}$ the set of points $\mathbf{x} \in \Pi$ such that there exists a polynomial $P \in \mathcal{P}_n(Q)$ satisfying

$$\begin{cases} |P(x_i)| < h_n \cdot Q^{-v_i}, \\ |P'(\alpha_i)| < 4c_{15} \cdot Q^{\frac{1}{2} - \frac{v_i}{2}}, \quad i = 1, 2. \end{cases} \quad (4.27)$$

The polynomials $P \in \mathcal{P}_n(Q)$ will be classified according to the distribution of their roots and the size of the leading coefficient. This classification was introduced by Sprinžuk in [8]. Let $\varepsilon_3 > 0$ be a sufficiently small constant.

For every polynomial $P \in \mathcal{P}_n(Q)$ of degree m with $3 \leqslant m \leqslant n$, we define numbers $\rho_{1,j}$ and $\rho_{2,j}$, $2 \leqslant j \leqslant m$, as solutions of the following equations:

$$|\alpha_{1,1} - \alpha_{1,j}| = Q^{-\rho_{1,j}}, \qquad |\alpha_{2,1} - \alpha_{2,j}| = Q^{-\rho_{2,j}}.$$

Let us also define vectors $\mathbf{k}_i = (k_{i,2}, \dots, k_{i,m}) \in \mathbb{Z}^{m-1}$ as solutions of the inequalities

$$(k_{i,j} - 1) \cdot \varepsilon_3 \leqslant \rho_{i,j} < k_{i,j} \cdot \varepsilon_3, \quad i = 1, 2, \quad j = \overline{2, m}.$$

Clearly, we have $m(m-1)$ pairs of vectors $\mathbf{k}_1, \mathbf{k}_2$ that correspond to a polynomial $P \in \mathcal{P}_n(Q)$ of degree m with $2 \leqslant m \leqslant n$ depending on the choice of the roots $\alpha_{1,1}$ and $\alpha_{1,2}$. Let us define subclasses of polynomials $\mathcal{P}_m(Q, \mathbf{k}_1, \mathbf{k}_2, u) \subset \mathcal{P}_n(Q)$ as follows. A polynomial P of degree m with $2 \leqslant m \leqslant n$ belongs to the subclass $\mathcal{P}_m(Q, \mathbf{k}_1, \mathbf{k}_2, u)$ if (1) the pair of vectors $(\mathbf{k}_1, \mathbf{k}_2)$ correspond to the polynomial P for some pair of roots α_1, α_2; (2) the leading coefficient of P is bounded as follows: $Q^u \leqslant |a_m| < Q^{u+\varepsilon_3}$, where $u \in \mathbb{Z} \cdot \varepsilon_3$. Let us estimate the number of different subclasses $\mathcal{P}_m(Q, \mathbf{k}_1, \mathbf{k}_2, u)$. Since $1 \leqslant |a_m| \leqslant Q$, the following estimate holds: $0 \leqslant u \leqslant 1 - \varepsilon_3$. On the other hand, we can write $Q \gg |\alpha_{j_1} - \alpha_{j_2}| \gg H(P)^{-m+1} \gg Q^{-m+1}$, where $\alpha_{j_1}, \alpha_{j_2}$ are roots of the polynomial P, which leads to the estimate $-\frac{1}{\varepsilon_3} + 1 \leqslant k_{i,j} \leqslant \frac{m-1}{\varepsilon_3}$. Thus, an integer vector $\mathbf{k}_i = (k_{i,2}, \dots, k_{i,n})$ can assume at most $\left(m\varepsilon_3^{-1} - 1\right)^{m-1}$ values. Now, the number of subclasses $\mathcal{P}_m(Q, \mathbf{k}_1, \mathbf{k}_2, u)$ can be estimated as follows:

$$\# \{m, \mathbf{k}_1, \mathbf{k}_2, u\} \leqslant n c_{20}^2 \cdot (\varepsilon_3^{-1} + 1), \tag{4.28}$$

where $c_{20} = \sum\limits_{i=2}^{n} \left(i\varepsilon_3^{-1} - 1\right)^{i-1}$. Define values $p_{i,j}$, $i = 1, 2$, as follows:

$$\begin{cases} p_{i,j} = (k_{i,j+1} + \ldots + k_{i,m}) \cdot \varepsilon_3, & 1 \leqslant j \leqslant m-1, \\ p_{i,j} = 0, & j = m. \end{cases} \tag{4.29}$$

Consider polynomials P belonging to the same subclass $\mathcal{P}_m(Q, \mathbf{k}_1, \mathbf{k}_2, u)$. For these polynomials, we can write the following estimates for their derivatives at a root α_i:

$$\begin{gathered} Q^{u-p_{i,1}} \leqslant |P'(\alpha_i)| = |a_m| \cdot |\alpha_{i,1} - \alpha_{i,2}| \ldots |\alpha_{i,1} - \alpha_{i,m}| \leqslant Q^{u-p_{i,1}+m\varepsilon_3}, \\ |P^{(j)}(\alpha_i)| \leqslant \tfrac{m!}{(m-j)!} \cdot Q^{u-p_{i,j}+m\varepsilon_3}. \end{gathered} \tag{4.30}$$

Since we are concerned only with polynomials satisfying the system (4.27), we may assume that the following inequalities hold for at least one value of l:

$$Q^{u-p_{1,i}} \leqslant |P'(\alpha_i)| < 4c_{15}Q^{\frac{1}{2}-\frac{v_i}{2}}, \quad i=1,2.$$

This condition implies that

$$p_{1,1} > u + \tfrac{v_1-1}{2}, \qquad p_{2,1} > u + \tfrac{v_2-1}{2}. \tag{4.31}$$

Now let us estimate the measure of the set $L_{1,1}$. From Lemma 1 it follows that $L_{1,1} \subset \bigcup\limits_{m,\mathbf{k}_1,\mathbf{k}_2,u} \bigcup\limits_{P\in\mathcal{P}_m(Q,\mathbf{k}_1,\mathbf{k}_2,u)} \bigcup\limits_{\boldsymbol{\alpha}\in\mathcal{A}^2(P)} \sigma_P(\boldsymbol{\alpha})$, where

$$\sigma_P(\boldsymbol{\alpha}) := \Big\{\mathbf{x}\in\Pi : |x_i-\alpha_i| \leqslant \min_{2\leqslant j\leqslant m}\Big(\tfrac{2^{m-j}h_nQ^{-v_i}}{|P'(\alpha_{i,1})|}\cdot|\alpha_{i,1}-\alpha_{i,2}|\ldots|\alpha_{i,1}-\alpha_{i,j}|\Big)^{1/j}, \quad i=1,2\Big\}.$$

This, together with the earlier notation (4.29) and estimates (4.30), yields

$$\sigma_P(\boldsymbol{\alpha}) := \Big\{\mathbf{x}\in\Pi : |x_i-\alpha_i| \leqslant \tfrac{1}{2}\cdot\min_{2\leqslant j\leqslant m}\Big((2^mh_n)^{1/j}\cdot Q^{\frac{-u-v_i+p_{i,j}}{j}}\Big),\ i=1,2\Big\}$$

for $P\in\mathcal{P}_m(Q,\mathbf{k}_1,\mathbf{k}_2,u)$. The numbers $j=m_1$ and $j=m_2$ in the formula above provide the best estimates for the roots α_1 and α_2, respectively, if the following inequalities are satisfied:

$$(2^mh_n)^{1/m_i}\cdot Q^{\frac{-u-v_i+p_{i,m_i}}{m_i}} \leqslant (2^mh_n)^{1/k}\cdot Q^{\frac{-u-v_i+p_{i,k}}{k}}, \quad 1\leqslant k\leqslant m, \quad i=1,2. \tag{4.32}$$

Then

$$\sigma_P(\boldsymbol{\alpha}) := \Big\{\mathbf{x}\in\Pi : |x_i-\alpha_i| < \tfrac{1}{2}\cdot(2^mh_n)^{1/m_i}\cdot Q^{\frac{-u-v_i+p_{i,m_i}}{m_i}}, \quad i=1,2\Big\}. \tag{4.33}$$

Cover the square Π by a system of disjoint rectangles $\Pi_{m_1,m_2} = J_{m_1}\times J_{m_2}$, where $\mu_1 J_{m_i} = Q^{-\frac{u+v_i-p_{i,m_i}}{m_i}+\varepsilon_4}$. The number of rectangles Π_{m_1,m_2} can be estimated as follows:

$$\#\Pi_{m_1,m_2} \leqslant 4Q^{\frac{u+v_1-p_{1,m_1}}{m_1}+\frac{u+v_2-p_{2,m_2}}{m_2}-2\varepsilon_4}\cdot\mu_2\Pi. \tag{4.34}$$

Let us show that a rectangle Π_{m_1,m_2} cannot contain two irreducible polynomials belonging to the same subclass $\mathcal{P}_m(Q,\mathbf{k}_1,\mathbf{k}_2,u)$. Assume the converse: let inequalities (4.27) hold for some irreducible polynomial

$P_j \in \mathcal{P}_m(Q, \mathbf{k}_1, \mathbf{k}_2, u)$ and some point $\mathbf{x}_j \in \Pi_{m_1,m_2}$, $j = 1, 2$. Then for all points $\mathbf{x} \in \Pi_{m_1,m_2}$, we obtain

$$|x_i - \alpha_{j,i}| \leqslant |x_i - x_{j,i}| + |x_{j,i} - \alpha_{j,i}| \leqslant 2 \cdot Q^{-\frac{u+v_i-p_{i,m_i}}{m_i}+\varepsilon_4} < Q^{-\frac{u+v_i-p_{i,m_i}}{m_i}+2\varepsilon_4}, \tag{4.35}$$

where $x_{j,i} \in S(\alpha_{j,i})$ and $Q > Q_0$. Let us estimate $|P_j(x_i)|$, $i, j = 1, 2$, where $\mathbf{x} \in \Pi_{m_1,m_2}$. From the Taylor expansions of the polynomials P_j in the intervals J_{m_i} and inequalities (4.30), (4.35), (4.32) for $Q > Q_0$ we obtain that

$$|P_j(x_i)| \leqslant \rho_m(d_2) \cdot 3^m \cdot Q^{-v_1+m\varepsilon_4+m\varepsilon_3} < Q^{-v_1+(m+1)\varepsilon_4+m\varepsilon_3}.$$

Apply Lemma 3 for $\eta_i = \frac{u+v_i-p_{i,m_i}}{m_i} - 2\varepsilon_4$ and $\tau_i = v_i - (m+1)\varepsilon_4 - m\varepsilon_3$, $i = 1, 2$. Then for $\varepsilon_3 = \frac{1}{12m}$ and $\varepsilon_4 = \frac{1}{4(3m+1)}$ we have

$$\tau_1 + \tau_2 + 2 = (n-1) + 2 - 2m\varepsilon_3 - 2m\varepsilon_4 = n + 1 - \tfrac{1}{6} - 2(m+1)\varepsilon_4,$$

$$2(\tau_i + 1 - \eta_i) = 2v_i + 2 - \tfrac{1}{6} - \tfrac{2(u+v_i-p_{i,m_i})}{m_i} - 2m \cdot \varepsilon_4.$$

Let us estimate the expression $2(\tau_i + 1 - \eta_i)$ using inequalities (4.31):

$$2(\tau_i + 1 - \eta_i) \geqslant \begin{cases} v_i + 2 - u + \frac{2p_{i,m_i}}{m} - \frac{1}{6} - 2m\varepsilon_4, & m_i \geqslant 2, \\ v_i + 1 - \frac{1}{6} - 2m\varepsilon_4, & m_i = 1, \end{cases}$$
$$\geqslant v_i + 1 - \tfrac{1}{6} - 2m \cdot \varepsilon_4.$$

Substituting these expressions into (2.1) yields

$$\tau_1 + \tau_2 + 2 + 2(\tau_1 + 1 - \eta_1) + 2(\tau_2 + 1 - \eta_2) > 2m + \tfrac{1}{2},$$

which is a contradiction. This means that there is at most one irreducible polynomial $P \in \mathcal{P}_m(Q, \mathbf{k}_1, \mathbf{k}_2, u)$ belonging to the rectangle Π_{m_1,m_2}. Now, by inequalities (4.28) and (4.33) for $Q > Q_0$, the measure of the set $L_{1,1}$ can be estimated as follows:

$$\mu_2 L_{1,1} \leqslant \sum_{m,\mathbf{k}_1,\mathbf{k}_2,u} \sum_{\Pi_{m_1,m_2}} \mu_2 \sigma_P \ll Q^{-2\varepsilon_4} \cdot \mu_2 \Pi < \tfrac{1}{72} \cdot \mu_2 \Pi.$$

Mixed cases. All mixed cases have the same structure and can be proved using Lemma 3 and the ideas described above, see [17]. Thus, we have $L_1 \subset \bigcup_{1 \leqslant i,j \leqslant 3} L_{i,j}$, which leads to the following estimate:

$$\mu_2 L_1 \leqslant \sum_{1 \leqslant i,j \leqslant 3} \mu_2 L_{i,j} \leqslant 9 \cdot \tfrac{1}{144} \cdot \mu_2 \Pi = \tfrac{1}{16} \cdot \mu_2 \Pi.$$

Similarly, $\mu_2 L_2 \leqslant \frac{1}{16} \cdot \mu_2 \Pi$. These estimates conclude the proof of Lemma 7 in the case of irreducible polynomials.

The case of reducible polynomials. In this section, we are going to estimate the measure of the set L_3. Clearly, the results of Lemma 3 do not apply directly to this case. Let a polynomial P of degree n be a product of several (not necessarily different) irreducible polynomials $P_1, P_2, \ldots, P_s$, $s \geqslant 2$, where $\deg P_i = n_i \geqslant 1$ and $n_1 + \ldots + n_s = n$. Then, by Lemma 4, we have

$$H(P_1) \cdot H(P_2) \cdot \ldots \cdot H(P_s) \leqslant c_{19} H(P) \leqslant c_{19} Q.$$

On the other hand, by the definition of height, we have $H(P_i) \geqslant 1$, and thus $H(P_i) \leqslant c_{19} Q$, $i = 1, \ldots, s$. Denote by $L_3(k, \varepsilon_5)$ the set of points $\mathbf{x} \in \Pi$ such that there exists a polynomial $P \in \mathcal{P}_k(Q_1)$ satisfying the inequality

$$|R(x_1) R(x_2)| < h_n^2 Q_1^{-k+\varepsilon_5}. \tag{4.36}$$

If a polynomial P satisfies inequalities (4.1) at a point $\mathbf{x} \in \Pi$, we can write

$$|P(x_1)P(x_2)| = |P_1(x_1)P_1(x_2)| \cdot \ldots \cdot |P_s(x_1)P_s(x_2)| \leqslant h_n^2 Q^{-n+1}.$$

Since $n = n_1 + \ldots + n_s$ and $s \geqslant 2$, it is easy to see that at least one of the inequalities

$$\begin{aligned} &|P_i(x_1)P_i(x_2)| \leqslant h_n^2 Q^{-n_i+\gamma}, \quad n_i \geqslant 2, \\ &|P_i(x_1)P_i(x_2)| \leqslant h_n^2 Q^{-\gamma}, \quad n_i = 1, \quad i = 1, \ldots, s, \end{aligned} \tag{4.37}$$

is satisfied at the point $\mathbf{x}$. Hence, $\mathbf{x} \in L_3(n_j, \gamma)$ for $n_j \geqslant 2$ or $\mathbf{x} \in L_3(1, 1-\gamma)$, and we have

$$L_3 \subset \left(\bigcup_{k=2}^{n-1} L_3(k, \gamma) \right) \cup L_3(1, 1-\gamma).$$

Let us estimate the measure of the set $L_3(k, \gamma)$, $2 \leqslant k \leqslant n-1$. Denote by $L_3^1(k, t)$ the set of points $(x_1, x_2) \in \Pi$ such that there exists a polynomial $P \in \mathcal{P}_k(Q_1)$ satisfying the inequalities

$$\begin{cases} |P(x_1)| < h_n^2 Q_1^t, \quad |P(x_2)| < h_n^2 Q_1^{-k+1-t}, \\ \min\limits_i \{|P'(\alpha_i)|\} < \delta_k Q_1, \quad x_i \in S(\alpha_i), \quad i = 1, 2; \end{cases} \tag{4.38}$$

and by $L_3^2(k,t)$, the set of points $(x_1,x_2)\in\Pi$ such that there exists a polynomial $P\in\mathcal{P}_k(Q_1)$ satisfying the inequalities

$$\begin{cases}|P(x_1)|<h_n^2Q_1^t, & |P(x_2)|<h_n^2Q_1^{-k+\frac{1+\gamma}{2}-t},\\ |P'(\alpha_i)|>\delta_kQ_1, & x_i\in S(\alpha_i),\quad i=1,2.\end{cases}\tag{4.39}$$

By the definition of the set $L_3(k,\gamma)$, it is easy to see that

$$L_3(k,\gamma)\subset\left(\bigcup_{i=0}^{N_1}L_3^1(k,1-i(1-\gamma))\right)\cup\left(\bigcup_{i=0}^{N_2}L_3^2(k,1-i(1-3\gamma)/2)\right),$$

where $N_1=\left[\frac{2+k-\gamma}{1-\gamma}\right]$ and $N_2=\left[\frac{4+2k-2\gamma}{1-3\gamma}\right]$. The system (4.38) is a system of the form (4.1). Furthermore, since the polynomials $P\in\mathcal{P}_k(Q_1)$ are irreducible and $k<n$, we can apply the above arguments for a sufficiently small constant δ_k and $Q_1>Q_0$ to obtain the following estimate:

$$\mu_2L_3^1(k,t)<\tfrac{1}{24n(N_1+1)}\cdot\mu_2\Pi.\tag{4.40}$$

Now let us estimate the measure of the set $L_3^2(k,t)$. From Lemma 1 it follows that $L_3^2(k,t)$ is contained in a union $\bigcup\limits_{P\in\mathcal{P}_k(Q)}\bigcup\limits_{\boldsymbol{\alpha}\in\mathcal{A}^2(P)}\sigma_P(\boldsymbol{\alpha},t)$, where

$$\sigma_P(\boldsymbol{\alpha},t):=\left\{\mathbf{x}\in\Pi:\begin{array}{l}|x_1-\alpha_1|\leqslant 2^{k-1}h_n^2\cdot Q^t\cdot|P'(\alpha_1)|^{-1},\\ |x_2-\alpha_2|\leqslant 2^{k-1}h_n^2\cdot Q^{-k+\frac{1+\gamma}{2}-t}\cdot|P'(\alpha_2)|^{-1}\end{array}\right\}.$$

Let us estimate the value of the polynomial P at the central point $\mathbf{d}$ of the square Π. The Taylor expansion of the polynomial P can be written as follows:

$$P(d_i)=P'(\alpha_i)(d_i-\alpha_i)+\tfrac{1}{2}P''(\alpha_i)(d_i-\alpha_i)^2+\ldots+\tfrac{1}{k!}\cdot P^{(k)}(\alpha_i)(d_i-\alpha_i)^k.\tag{4.41}$$

If the polynomial P satisfies (4.39), it follows that

$$\begin{aligned}|d_1-\alpha_1|&\leqslant|d_1-x_{0,1}|+|x_{0,1}-\alpha_1|\leqslant\mu_1I_1+2^{k-1}h_n^2\delta_k^{-1}\cdot Q_1^{t-1},\\ |d_2-\alpha_2|&\leqslant|d_2-x_{0,2}|+|x_{0,2}-\alpha_2|\leqslant\mu_1I_2+2^{k-1}h_n^2\delta_k^{-1}\cdot Q_1^{-k+\frac{1+\gamma}{2}-t-1}.\end{aligned}\tag{4.42}$$

Without loss of generality, let us assume that $t\geqslant -k+\frac{1+\gamma}{2}-t$. Then we can rewrite estimates (4.42) as follows:

$$|d_1-\alpha_1|\leqslant\begin{cases}c_{21}\cdot\mu_1I_1, & t<1-\gamma,\\ c_{21}\cdot Q_1^{t-1}, & 1-\gamma\leqslant t\leqslant 1,\end{cases}\qquad|d_2-\alpha_2|\leqslant\mu_1I_2,$$

where $c_{21} = 2^{k-1} h_n^2 \delta_k^{-1} + c_8$. Using these inequalities and expression (4.41) allows us to write

$$|P(d_1)| < \begin{cases} c_{22} \cdot Q_1 \cdot \mu_1 I_1, & t < 1-\gamma, \\ c_{22} \cdot Q_1^t, & 1-\gamma \leqslant t < 1, \end{cases} \qquad |P(d_2)| < c_{22} \cdot Q_1 \cdot \mu_1 I_2. \tag{4.43}$$

Fix a vector $\mathbf{A}_1 = (a_k, \dots, a_2)$, where $a_k, \dots, a_2$ will denote the coefficients of the polynomial $P \in \mathcal{P}_k(Q_1)$. Consider the subclass $\mathcal{P}_k(\mathbf{A}_1)$ of polynomials P that satisfy (4.39) and have the same vector of coefficients $\mathbf{A}_1$. For $Q_1 > Q_0$, the number of such classes can be estimated as follows:

$$\#\{\mathbf{A}_1\} = (2Q_1 + 1)^{k-1} < 2^k Q_1^{k-1}. \tag{4.44}$$

Let us estimate the value $\#\mathcal{P}_k(\mathbf{A}_1)$. Take a polynomial $P_0 \in \mathcal{P}_k(\mathbf{A}_1)$ and consider the difference between the polynomials P_0 and $P_j \in \mathcal{P}_k(\mathbf{A}_1)$ at points d_i, $i = 1, 2$. By (4.43), we have

$$\begin{aligned} |P_0(d_1) - P_j(d_1)| &= |(a_{0,1} - a_{j,1})d_1 + (a_{0,0} - a_{j,0})| \\ &\leqslant \begin{cases} 2c_{22} \cdot Q_1 \mu_1 I_1, & t < 1-\gamma, \\ 2c_{22} \cdot Q_1^t, & 1-\gamma \leqslant t \leqslant 1, \end{cases} \end{aligned}$$

$$|P_0(d_2) - P_j(d_2)| = |(a_{0,1} - a_{j,1})d_2 + (a_{0,0} - a_{j,0})| \leqslant 2c_{22} \cdot Q_1 \mu_1 I_2.$$

This implies that the number of different polynomials $P_j \in \mathcal{P}_k(\mathbf{A}_1)$ does not exceed the number of integer solutions to the system

$$|b_1 d_i + b_0| \leqslant K_i, \quad i = 1, 2,$$

where $K_2 = 2c_{22} \cdot Q_1 \mu_1 I_2$ and $K_1 = 2c_{22} \cdot Q_1 \mu_1 I_1$ if $t < 1-\gamma$ and $K_1 = 2c_{22} \cdot Q_1^t$ if $1-\gamma \leqslant t \leqslant 1$. It is easy to see that $K_i \geqslant 2c_{22} \cdot Q_1^{1-\gamma} > Q_1^{\varepsilon}$ for $Q_1 > Q_0$. Thus, by Lemma 6, we have

$$\#\mathcal{P}_k(\mathbf{A}_1) \leqslant \begin{cases} 2^7 \varepsilon_1^{-1} \cdot Q_1^2 \cdot \mu_2 \Pi, & t < 1-\gamma, \\ 2^7 \varepsilon_1^{-1} \cdot Q_1^{t+1} \cdot \mu_1 I_2, & 1-\gamma \leqslant t \leqslant 1. \end{cases}$$

This estimate and inequality (4.44) mean that the number N of polynomials $P \in \mathcal{P}_k(Q_1)$ satisfying the system (4.39) can be estimated as follows:

$$N \leqslant \begin{cases} 2^{k+7} \varepsilon_1^{-1} \cdot Q_1^{k+1} \cdot \mu_2 \Pi, & t < 1-\gamma, \\ 2^{k+7} \varepsilon_1^{-1} \cdot Q_1^{k+t} \cdot \mu_1 I_2, & 1-\gamma \leqslant t \leqslant 1. \end{cases} \tag{4.45}$$

On the other hand, the measure of the set $\sigma_P(\boldsymbol{\alpha}, t)$ satisfies the inequality

$$\mu_2\sigma_P(\boldsymbol{\alpha}, t) \leqslant \begin{cases} 2^{2k}h_n^4\delta_k^{-2}\cdot Q_1^{-k-2+\frac{1+\gamma}{2}}, & t < 1-\gamma, \\ 2^{2k}h_n^4\delta_k^{-2}\cdot Q_1^{-k-1-t+\frac{1+\gamma}{2}}\cdot \mu_1 I_1, & 1-\gamma \leqslant t \leqslant 1. \end{cases} \quad (4.46)$$

Then, by estimates (4.45) and (4.46), for $Q_1 > Q_0$ we can write

$$\mu_2 L_3^2(k,t) \leqslant 2^{3k+7}\delta_k^{-2}h_n^4\varepsilon_1^{-1}Q_1^{-\frac{1-\gamma}{2}}\mu_2\Pi < \tfrac{1}{24n(N_2+1)}\cdot\mu_2\Pi. \quad (4.47)$$

Inequalities (4.40) and (4.47) lead to the following estimate for the measure of the set $L_3(k)$, $2 \leqslant l \leqslant n-1$:

$$\begin{aligned}\mu_2 L_3(k,\gamma) &\leqslant \sum_{i=0}^{N_1}\mu_2 L_3^1(k, 1-i(1-\gamma)) + \sum_{i=0}^{N_2}\mu_2 L_3^2(k, 1-i(1-3\gamma)/2) \\ &\leqslant \tfrac{1}{12n}\cdot\mu_2\Pi.\end{aligned}$$

Now let us estimate the measure of the set $L_3(1, 1-\gamma)$. For every point $\mathbf{x} \in L_3(1, 1-\gamma)$ there exists a rational point $\frac{a_0}{a_1}$ such that

$$\left|x_1 - \tfrac{a_0}{a_1}\right|\cdot\left|x_2 - \tfrac{a_0}{a_1}\right| < h_n^2 Q_1^{-\gamma}|a_1|^{-2}.$$

Since $|x_1 - x_2| > \varepsilon_1$, one of the values $\left|x_i - \frac{a_0}{a_1}\right|$, $i = 1, 2$, is greater than $\frac{\varepsilon_1}{2}$. Thus we consider the sets

$$\sigma_i\left(a_0/a_1\right) := \left\{\mathbf{x}\in\Pi : \left|x_i - \tfrac{a_0}{a_1}\right| \leqslant 2h_n^2\varepsilon_1^{-1}Q_1^{-\gamma}|a_1|^{-2}\right\}, \quad i = 1, 2. \quad (4.48)$$

Simple calculations show that for $c_8 > 4h_n^2\varepsilon_1^{-1}$ we have

$$\mu_2\sigma_i\left(a_0/a_1\right) \leqslant 4h_n^2\varepsilon_1^{-1}c_8 Q_1^{-2\gamma} \leqslant \mu_2\Pi.$$

Let us define the following sets:

$$\sigma_i = \bigcup_{1\leqslant a_0, a_1 \leqslant Q_1}\sigma_i\left(a_0/a_1\right), \quad i = 1, 2.$$

It is easy to see that $L_3(1, 1-\gamma) \subset \sigma_1 \cup \sigma_2$ and we need to estimate the measure of the sets σ_1 and σ_2. For a fixed value a_1, let us consider the set $N(a_1) := \{a_0 \in \mathbb{Z} : \sigma_i\left(a_0/a_1\right) \neq \varnothing\}$. The cardinality of this set can be estimated in the following way:

$$\#N(a_1) \leqslant \begin{cases} 3\mu_1 I_i\cdot|a_1|^{-1}, & (\mu_1 I_i)^{-1} \leqslant |a_1| \leqslant Q_1, \\ 2, & 1 \leqslant |a_1| \leqslant (\mu_1 I_i)^{-1}. \end{cases}$$

These inequalities together with (4.48) imply that

$$\mu_2\sigma_i \leqslant \sum_{1\leqslant|a_1|\leqslant Q_1} N(a_1)\cdot\mu_2\sigma_i\,(a_0/a_1)$$
$$\leqslant 8h_n^2c_8\varepsilon_1^{-1}Q_1^{-2\gamma}\sum_{1\leqslant|a_1|\leqslant(\mu_1 I_i)^{-1}}|a_1|^{-2}$$
$$+\,12h_n^2\varepsilon_1^{-1}Q_1^{-\gamma}\mu_2\Pi\sum_{(\mu_1 I_i)^{-1}\leqslant|a_1|\leqslant Q_1}|a_1|^{-1}\leqslant 2\pi^2c_8h_n^2\varepsilon_1^{-1}Q_1^{-2\gamma}$$
$$+\,12h_n^2\varepsilon_1^{-1}Q_1^{-\gamma}\ln Q\mu_2\Pi\leqslant\tfrac{1}{24n}\cdot\mu_2\Pi$$

for $Q_1 > Q_0$ and $c_8 > 96n\pi^2h_n^2\varepsilon_1^{-1}$. Then

$$\mu_2L_3(1,1-\gamma)\leqslant\tfrac{1}{12n}\cdot\mu_2\Pi,$$

and, finally,

$$\mu_2L_3\leqslant\sum_{k=2}^{n-1}\mu_2L_3(k,\gamma)+\mu_2L_3(1,1-\gamma)\leqslant\tfrac{n-1}{12n}\cdot\mu_2\Pi\leqslant\tfrac{1}{12}\cdot\mu_2\Pi.$$

This proves Lemma 7 in the case of reducible polynomials. Combining the obtained estimates for the different cases yields the final estimate

$$\mu_2L\leqslant\mu_2L_1+\mu_2L_2+\mu_2L_3\leqslant\tfrac{1}{4}\cdot\mu_2\Pi. \qquad \square$$

Remark. Note that in the case of reducible polynomials we do not use the inequality $\min_i\{|P'(x_i)|\} < \delta_nQ$. This means that the set L_3 is the set of points $\mathbf{x}\in\Pi$ such that there exists a reducible polynomial $P\in\mathcal{P}_n(Q)$ satisfying the inequalities

$$|P(x_i)| < h_nQ^{-v_i},\quad i=1,2.$$

4.2. The final part of the proof. Let us use Lemma 7 to conclude the proof. Consider the set $B_1 = \Pi\setminus L_n(Q,\delta_n,\mathbf{v},\Pi)$ for $n\geqslant 2$, $v_1=v_2=\frac{n-1}{2}$, $Q > Q_0$, and a sufficiently small constant δ_n. From Lemma 7 it follows that

$$\mu_2B_1\geqslant\tfrac{3}{4}\cdot\mu_2\Pi. \tag{4.49}$$

Now we will prove that for every point $\mathbf{x}\in\Pi$ there exists a polynomial $P\in\mathcal{P}_n(Q)$ such that

$$|P(x_i)|\leqslant h_n\cdot Q^{-\frac{n-1}{2}},\quad i=1,2.$$

By Minkowski's linear forms theorem [9], for every point $\mathbf{x} \in \Pi$ there exists a nonzero polynomial $P(t) = a_n t^n + \ldots + a_1 t + a_0 \in \mathbb{Z}[t]$ satisfying

$$|P(x_i)| \leqslant h_n \cdot Q^{-\frac{n-1}{2}}, \quad |a_j| \leqslant \max(1, 3|d_1|, 3|d_2|)^{-n-1} \cdot Q, \quad i = 1,2, \quad 2 \leqslant j \leqslant n.$$

One can easily verify that $|a_1| < Q$ and $|a_0| < Q$; hence $P \in \mathcal{P}_n(Q)$. Then, by the remark after Lemma 7, we can say that for every point $\mathbf{x}_1 \in B_1$ there exists an irreducible polynomial $P_1 \in \mathcal{P}_n(Q)$ such that

$$\begin{cases} |P_1(x_{1,i})| < h_n \cdot Q^{-\frac{n-1}{2}}, \\ |P_1'(x_{1,i})| > \delta_n \cdot Q, \quad i = 1,2. \end{cases}$$

Consider the roots α_1, α_2 of the polynomial P_1 such that $x_{1,i} \in S(\alpha_i)$. By Lemma 1, we have

$$|x_{1,i} - \alpha_i| \leqslant n h_n \delta_n^{-1} Q^{-\frac{n+1}{2}}, \quad i = 1,2. \tag{4.50}$$

Let us prove that $\alpha_1, \alpha_2 \in \mathbb{R}$. Assume the converse: let $\alpha_i \in \mathbb{C}$, then the number $\overline{\alpha_i}$ complex conjugate to α_i is also a root of the polynomial P_1, and $x_{1,i} \in S(\overline{\alpha_i})$. Hence, from estimates (4.50) and Lemma 5 we have

$$|P'(\alpha_i)| \leqslant |a_n||\overline{\alpha_i} - \alpha_i| \leqslant c_{24} \cdot Q^{-\frac{n-1}{2}}.$$

On the other hand, the Taylor expansion of the polynomial P_1 in the interval $S(\alpha_i)$ implies that

$$|P'(\alpha_i)| \geqslant \tfrac{1}{2}\delta_n \cdot Q.$$

These two inequalities contradict each other. Let us choose a maximal system of algebraic points $\Gamma = \{\boldsymbol{\gamma}_1, \ldots, \boldsymbol{\gamma}_t\} \subset \mathbb{A}_n^2(Q)$ satisfying the condition that the rectangles $\sigma(\boldsymbol{\gamma}_k) = \{|x_i - \gamma_{k,i}| < n\delta_n^{-1} Q^{-\frac{n+1}{2}},\ i = 1,2\}$, $1 \leqslant k \leqslant t$, do not intersect. Furthermore, let us introduce the expanded rectangles

$$\sigma'(\boldsymbol{\gamma}_k) = \left\{ |x_i - \gamma_{k,i}| < 2n h_n \delta_n^{-1} Q^{-\frac{n+1}{2}}, \quad i = 1,2 \right\}, \quad k = \overline{1,t}, \tag{4.51}$$

and show that

$$B_2 \subset \bigcup_{k=1}^{t} \sigma'(\boldsymbol{\gamma}_k). \tag{4.52}$$

To prove this fact, we are going to show that for any point $\mathbf{x}_1 \in B_1$ there exists a point $\boldsymbol{\gamma}_k \in \Gamma$ such that $\mathbf{x}_1 \in \sigma'(\boldsymbol{\gamma}_k)$. Since $\mathbf{x}_1 \in B_1$, there is a point $\boldsymbol{\alpha}$ satisfying inequalities (4.50). Thus, either $\boldsymbol{\alpha} \in \Gamma$ and $\mathbf{x}_1 \in \sigma'(\boldsymbol{\alpha})$, or there exists a point $\boldsymbol{\gamma}_k \in \Gamma$ satisfying

$$|\alpha_i - \gamma_{k,i}| \leqslant n h_n \delta_n^{-1} Q^{-\frac{n+1}{2}}, \quad i = 1, 2,$$

which implies that $\mathbf{x}_1 \in \sigma'(\boldsymbol{\gamma}_k)$. Hence, from (4.49), (4.51), and (4.52) we have

$$\tfrac{3}{4} \cdot \mu_2 \Pi \leqslant \mu_2 B_1 \leqslant \sum_{k=1}^{t} \mu_2 \sigma_1(\boldsymbol{\gamma}_k) \leqslant t \cdot 2^6 n^2 h_n^2 \delta_n^{-2} Q^{-n-1},$$

which yields the estimate

$$\#\mathbb{A}_n^2(Q, \Pi) \geqslant t \geqslant c_{13} \cdot Q^{n+1} \mu_2 \Pi.$$

§5. Proof of Theorem 1

Now we can prove Theorem 1, which is the main result of the paper. Consider the set

$$L_\varphi(Q, \gamma, J) := \left\{ \mathbf{x} \in \mathbb{R}^2 : x_1 \in J, \quad |\varphi(x_1) - x_2| < c_1 Q^{-\gamma} \right\}.$$

Clearly, $M_\varphi^n(Q, \gamma, J) = L_\varphi(Q, \gamma, J) \cap \mathbb{A}_n^2(Q)$, and our problem is reduced to estimating the number of algebraic points in the set $\mathbb{A}_n^2(Q)$ lying within the strip $L_\varphi(Q, \gamma, J)$.

5.1. The lower bound. The lower bound for $0 < \gamma \leqslant \frac{1}{2}$ was obtained in [5], which allows us to consider only the case where $\frac{1}{2} < \gamma < 1$. Note that the distance between algebraically conjugate numbers is bounded from below, meaning that a certain neighborhood of the line $\varphi_1(x) = x$ must be excluded from consideration. Let us consider the set

$$D_0 := \left\{ x \in J : |\varphi(x) - x| < \tfrac{\varepsilon_1}{2} \right\},$$

where $\varepsilon_1 > 0$ is a small positive constant. Since the number of points $x \in J$ such that $\varphi(x) = x$ is finite, for a sufficiently small constant ε_1 we have $\mu_1 D_0 < \frac{1}{4} \mu_1 J$. Instead of the interval J, let us consider the set $J \setminus D_0 = \bigcup_k J_k$, $k \leqslant c_5 + 1$. The measure of this set is larger than $\frac{3}{4} \mu_1 J$. For every interval $J_k = [b_{k,1}, b_{k,2}]$, let us consider the strip $L_\varphi(Q, \gamma, J_k)$ and estimate the cardinality of the set $L_\varphi(Q, \gamma, J_k) \cap \mathbb{A}_n^2(Q)$. Let us divide the strip $L_\varphi(Q, \gamma, J_k)$ into subsets E_j as follows:

$$E_j := \left\{ \mathbf{x} \in \mathbb{R}^2 : x_1 \in J_{k,j}, \quad |\varphi(x_1) - x_2| < c_1 Q^{-\gamma} \right\},$$

where $J_{k,j} = [y_{j-1}, y_j]$, $y_0 = b_{k,1}$, and $y_{j+1} = y_j + c_8 Q^{-\gamma}$. The number t_k of subsets E_j can be estimated in the following way:

$$t_k \geqslant \mu_1 J_k \cdot (\mu_1 J_{k,j})^{-1} - 1 \geqslant \tfrac{1}{2} \cdot c_8^{-1} Q^{\gamma} \mu_1 J_k. \tag{5.1}$$

For $\overline{\varphi}_j = \frac{1}{2}\left(\max\limits_{x \in J_{k,j}} \varphi(x) + \min\limits_{x \in J_{k,j}} \varphi(x)\right)$, consider the squares defined as

$$\Pi_j := \left\{\mathbf{x} \in \mathbb{R}^2 : x_1 \in J_{k,j}, \quad \left|\overline{\varphi}_j - x_2\right| < \tfrac{1}{2} c_8 Q^{-\gamma}\right\}.$$

Since the function φ is continuously differentiable on the interval J, and $\max\limits_{x \in J} |\varphi'(x)| < c_6$, we obtain by the mean value theorem that

$$\left|\max_{x \in J_{k,j}} \varphi(x) - \min_{x \in J_{k,j}} \varphi(x)\right| < c_6 \cdot c_8 Q^{-\gamma},$$

which implies that the square Π_j is contained in the subset E_j. Thus, every set E_j defines the respective square $\Pi_j = I_{j,1} \times I_{j,2}$ of size $\mu_2 \Pi_j = c_8^2 Q^{-2\gamma}$. Let us estimate the number of $\left(\frac{1}{2}, \frac{1}{2}\right)$-*special* squares Π_j. To obtain this estimate, let us derive an upper bound on the number of squares Π_j satisfying the $\left(l, \frac{1}{2}, \frac{1}{2}\right)$-*condition* for $1 \leqslant l \leqslant L + 2$. For polynomials $P \in \mathcal{P}_2(Q)$ of the form $P(t) = a_2 t^2 + a_1 t + a_0$ satisfying the conditions

$$\delta Q^{\lambda_{l+1}} \leqslant |a_2| < \delta Q^{\lambda_l}, \quad |P(x_i)| < h \cdot Q^{-\frac{1}{2}}, \quad i = 1, 2, \tag{5.2}$$

denote by $\mathcal{P}_2(Q, l, D)$ the subclass of polynomials $P \in \mathcal{P}_2(Q)$ satisfying inequalities (5.2) at some point $\mathbf{x} \in D \subset \mathbb{R}^2$. By definition, if a square Π_j satisfies the $\left(l, \frac{1}{2}, \frac{1}{2}\right)$-*condition*, then the following inequality holds:

$$\#\mathcal{P}_2(Q, l, \Pi_j) \leqslant \delta^3 \cdot 2^{l+3} Q^{1+2\lambda_{l+1}} \mu_2 \Pi_j.$$

Consider the expanded sets $E_s = \bigcup\limits_{i=j_s}^{j_s+T(l)} E_i$ composed of $T(l)$ subsets E_j, where

$$T(l) = c_{24} Q^{\gamma - \lambda_l}, \quad c_{24} = \tfrac{1}{8} \cdot \delta^{-1} c_8^{-1} \left(|d_1| + |d_2| + \varepsilon_1\right)^{-1} \cdot \min\left\{c_6, \varepsilon_1^{-1}\right\}, \tag{5.3}$$

and $j_1 = 1$, $j_{s+1} = j_s + T(l) + 1$. By inequality (5.1), the number of expanded sets can be estimated as follows:

$$s \leqslant t_k \cdot T(l)^{-1} \leqslant c_8 T(l)^{-1} Q^{\gamma} \mu_1 J_k.$$

Now let us show that at least $\left(1 - 2^{-l-3}\right) \cdot T(l)$ squares $\Pi_j \subset E_s$ satisfy the $\left(l, \frac{1}{2}, \frac{1}{2}\right)$-*condition*. By the definition of the set E_s, for every point $\mathbf{x} \in E_s$

we obtain

$$x_1 \in I_1, \qquad \mu_1 I_1 = c_8 \cdot c_{24} Q^{-\lambda_l}. \tag{5.4}$$

On the other hand, since φ is continuously differentiable on the interval J and $\max_{x\in J} |\varphi'(x)| < c_6$, we have $E_s \subset \Pi$, where $\Pi = I_1 \times I_2$ and $\mu_1 I_2 = c_6 \mu_1 I_1$. Thus $\#\mathcal{P}_2(Q,l,E_s) \leqslant \#\mathcal{P}_2(Q,l,\Pi)$, and we only need to estimate the quantity $\#\mathcal{P}_2(Q,l,\Pi)$. By the third inequality of Lemma 1, for every polynomial $P \in \mathcal{P}_2(Q,l,\Pi)$ satisfying the system (5.2) at a point $\mathbf{x}_0 \in \Pi$, the inequalities

$$|x_{0,i} - \alpha_i| < \left(|P(x_{0,i})| \cdot |a_2|^{-1}\right)^{-\frac{1}{2}} < h^{\frac{1}{2}} \cdot Q^{-\frac{1}{4}} < \tfrac{\varepsilon_1}{8} \tag{5.5}$$

are satisfied for $Q > Q_0$ and $x_{0,i} \in S(\alpha_i)$. From (5.5) and the condition $|x_1 - x_2| > \varepsilon_1$, we obtain the following lower bound for $|P'(\alpha_i)|$:

$$|P'(\alpha_i)| = |a_2| \cdot |\alpha_1 - \alpha_2| > \tfrac{3}{4} \cdot \varepsilon_1 \cdot |a_2|. \tag{5.6}$$

Moreover, from inequalities (5.5) we have

$$|P'(x_{0,i})| \leqslant |a_2| \cdot (|\alpha_1 - x_{0,i}| + |\alpha_2 - x_{0,i}|) \leqslant \left(|d_1| + |d_2| + \tfrac{1}{2}\varepsilon_1\right) \cdot |a_2|, \tag{5.7}$$

where $\mathbf{d}$ is the midpoint of the rectangle Π. Let us estimate the polynomials $P \in \mathcal{P}_2(Q,l,\Pi)$ at a point $\mathbf{d} \in \Pi$. From the Taylor expansion of the polynomial P in the interval I_i and inequalities (5.2), (5.8) we have

$$|P(d_i)| < (|d_1| + |d_2| + \varepsilon_1) \cdot |a_2| \cdot \mu_1 I_i. \tag{5.8}$$

Fix a number a and consider the subclass of polynomials P with the same leading coefficient:

$$\mathcal{P}_2(Q,l,\Pi,a) := \{P \in \mathcal{P}_2(Q,l,\Pi) : a_2 = a\}.$$

It is clear that the inequality $\#\mathcal{P}_2(Q,l,\Pi,a) > 0$ holds only if conditions (5.2) are satisfied. Hence, the number of classes under consideration can be estimated as follows:

$$\#\{a\} \leqslant \delta Q^{\lambda_l}. \tag{5.9}$$

Now let us estimate the number of polynomials in subclass $\mathcal{P}_2(Q,l,\Pi,a)$. Choose a polynomial $P_0 \in \mathcal{P}_2(Q,l,\Pi,a)$ and consider the differences between the polynomials P_0 and $P_j \in \mathcal{P}_2(Q,l,\Pi,a)$ at the point $\mathbf{d}$. From estimates (5.8) it follows that

$$|P_0(d_i) - P_j(x_{0,i})| = |(a_{0,1} - a_{j,1})d_i + (a_{0,0} - a_{j,0})| \leqslant 2c_{25} \cdot |a| \cdot \mu_1 I_i,$$

where $c_{25} = |d_1| + |d_2| + \varepsilon_1$. Thus, the number of different polynomials $P_j \in \mathcal{P}_2(Q, l, \Pi, a)$ does not exceed the number of integer solutions of the following system:

$$|b_1 d_i + b_0| \leqslant 2c_{25} \cdot |a| \cdot \mu_1 I_i, \quad i = 1, 2.$$

Let us apply Lemma 6 with $K_i = 2c_{25} \cdot |a| \cdot \mu_1 I_i$. From estimates (5.2) and (5.4), we can easily verify that $4\varepsilon_1^{-1} K_1 < 1$ and $4K_2 < 1$, which leads to the inequality

$$\#\mathcal{P}_2(Q, l, \Pi, a) \leqslant 1. \tag{5.10}$$

Hence, from inequality (5.9) we obtain the estimate

$$\#\mathcal{P}_2(Q, l, \Pi) = \sum_a \#\mathcal{P}_2(Q, l, \Pi, a) \leqslant \delta Q^{\lambda_l}. \tag{5.11}$$

Let us consider the case where $1 \leqslant l \leqslant L + 1$. Assume that the inequality

$$\#\mathcal{P}_2(Q, l, \Pi_j) > \delta^3 \cdot 2^{l+3} Q^{1+2\lambda_{l+1}} \mu_2 \Pi_j \tag{5.12}$$

holds for $2^{-l-3} \cdot T(l)$ squares Π_j. By Lemma 1, for a polynomial $P \in \mathcal{P}_2(Q)$ the set of points $\mathbf{x}$ satisfying (5.2) is contained in the following set:

$$\sigma_P := \left\{ |x_i - \alpha_i| \leqslant hQ^{-\frac{1}{2}} \cdot |P'(\alpha_i)|^{-1}, \quad x_i \in S(\alpha_i), \quad i = 1, 2 \right\}.$$

From (5.2) and (5.6) it is easy to see that the measure of the set σ_P is at most half the size of Π_j for $1 \leqslant l \leqslant L + 1$ and $c_8 > h\delta^{-1}\varepsilon_1^{-1}$. Therefore, no polynomial $P \in \mathcal{P}_2(Q)$ satisfies inequalities (5.2) at three points that lie inside three different squares Π_j. Since $\Pi_j \subset E_j \subset E \subset \Pi$, we have $\bigcup_j \Pi_j \subset \Pi$. Then, by our assumption and the inequality $\#\mathcal{P}_2(Q, l, \Pi_j) \geqslant 0$, we get

$$\#\mathcal{P}_2(Q, l, \Pi) \geqslant \sum_{i=j_s}^{j_s+T(l)} \#\mathcal{P}_2(Q, l, \Pi_i) \geqslant \tfrac{1}{2^{l+3}} \cdot T(l) \cdot \#\mathcal{P}_2(Q, l, \Pi_j).$$

From inequalities (5.3) and (5.12) for $1 \leqslant l \leqslant L$, we obtain:

$$\#\mathcal{P}_2(Q, l, \Pi) \geqslant c_{24}\delta^3 \cdot c_8^2 \cdot Q^{1-\gamma+2\lambda_{l+1}-\lambda_l} > \delta Q^{\lambda_l},$$

for $c_8 > 8\delta^{-1} c_{25} \cdot \left(\min\left\{c_6, \varepsilon_1^{-1}\right\}\right)^{-1}$. This inequality contradicts estimate (5.11). For $l = L + 1$, we can use inequalities (5.3) and (5.12) to obtain

$$\begin{aligned}\#\mathcal{P}_2(Q, l, \Pi) &\geqslant c_{24}\delta^3 \cdot c_8^2 \cdot Q^{\gamma-\lambda_{L+1}} > \delta Q^{\gamma-1+\frac{1-\gamma}{2}\cdot\left[\frac{3-2\gamma}{1-\gamma}\right]} \\ &\geqslant \delta Q^{\gamma-1+\frac{3-2\gamma}{2}-\frac{1-\gamma}{2}} > \delta Q^{\frac{\gamma}{2}},\end{aligned}$$

for $c_8 > 8\delta^{-1}c_{25} \cdot \left(\min\left\{c_6, \varepsilon_1^{-1}\right\}\right)^{-1}$. On the other hand, estimates (5.11) imply that

$$\#\mathcal{P}_2(Q,l,\Pi) \leqslant \delta Q^{\lambda_{L+1}} = \delta Q^{1-\frac{1-\gamma}{2}\cdot\left[\frac{3-2\gamma}{1-\gamma}\right]} \leqslant \delta Q^{1-\frac{3-2\gamma}{2}} = \delta Q^{\gamma-\frac{1}{2}} < \delta Q^{\frac{\gamma}{2}}$$

for $\gamma < 1$, which contradicts the previous inequality. This argument proves that the number of squares $\Pi_j \subset E_s$ satisfying the $\left(l, \frac{1}{2}, \frac{1}{2}\right)$-*condition* for $1 \leqslant l \leqslant L+1$ is larger than $\left(1 - 2^{-l-3}\right) \cdot T(l)$. The case $l = L+2$ needs to be treated differently. From Lemma 1 and inequalities (5.6) it follows that the set of points $\mathbf{x}$ satisfying inequalities (5.2) for some polynomial P is contained in the set

$$\sigma_P := \left\{|x_i - \alpha_i| \leqslant h\varepsilon_1^{-1} \cdot Q^{-\frac{1}{2}} \cdot |a_2|^{-1}, \quad i = 1,2\right\}$$

and the measure of the set σ_P is larger than the size of the square Π_j. This means that a single polynomial can belong to a large number of different sets $\mathcal{P}_2(Q,l,\Pi_j)$. Let us estimate this number for a fixed polynomial $P \in \mathcal{P}_2(Q,l,\Pi)$. Since the side of the square σ_P is larger than the width of the strip $L_\varphi(Q,\gamma,J_k)$, we have

$$\#\{\Pi_j : P \in \mathcal{P}_2(Q,l,\Pi_j)\} \leqslant 2h\varepsilon_1^{-1}c_8^{-1} \cdot Q^{\gamma-\frac{1}{2}} \cdot |a_2|^{-1}.$$

Now, from inequalities (5.11) and estimates (5.10) we can obtain that

$$\begin{aligned}\# \bigcup_{P\in\mathcal{P}_2(Q,l,\Pi)} \{\Pi_j : P \in \mathcal{P}_2(Q,l,\Pi_j)\} &\leqslant 2h\varepsilon_1^{-1}c_8^{-1} \cdot Q^{\gamma-\frac{1}{2}} \sum_{1\leqslant|a_2|<\delta Q^{\gamma-\frac{1}{2}}} |a_2|^{-1} \\ &\leqslant 2^4\varepsilon_1^{-1}hc_8^{-1}\left(\gamma - \tfrac{1}{2}\right)Q^{\gamma-\frac{1}{2}}\ln Q < \tfrac{1}{2^{l+3}} \cdot T(l),\end{aligned}$$

for $\gamma < 1$ and $Q > Q_0$. This implies that the inequality $\#\mathcal{P}_2(Q,l,\Pi) > 0$ can only be satisfied for $2^{-l-3} \cdot T(l)$ squares $\Pi_j \subset E_s$, and, therefore, the number of squares $\Pi_j \subset E_s$ satisfying the $\left(l, \frac{1}{2}, \frac{1}{2}\right)$-*condition* for $l = L+2$ is larger than $\left(1 - 2^{-l-3}\right) \cdot T(l)$. Now it follows from inequality (5.1) that the number of squares $\Pi_j \in L_\varphi(Q,\gamma,J_k)$ satisfying the $\left(l, \frac{1}{2}, \frac{1}{2}\right)$-*condition* for $1 \leqslant l \leqslant L+2$ is larger than $\left(1 - \frac{1}{2^{l+3}}\right) \cdot t_k$. Thus, we have

$$\sum_{\substack{P_j,l:P_j \text{ satisfy} \\ (l,1/2,1/2)\text{-condition}}} 1 \geqslant \sum_{l=1}^{L+2}\left(1 - \tfrac{1}{2^{l+3}}\right) \cdot t_k = \left(L + 2 - \tfrac{1}{4} + \tfrac{1}{2^{L+3}}\right) \cdot t_k > \left(L + \tfrac{7}{4}\right) \cdot t_k.$$

Assume that the number of squares $\Pi_j \subset L_\varphi(Q,\gamma,J_k)$ that satisfy the $\left(l, \frac{1}{2}, \frac{1}{2}\right)$-*condition* for all l with $1 \leqslant l \leqslant L+2$ is smaller than $\frac{3}{4} \cdot t_k$. Then we have

$$\sum_{\substack{P_j,l:P_j \text{ satisfy} \\ (l,1/2,1/2)\text{-condition}}} 1 \leqslant \tfrac{3}{4} \cdot t_k \cdot (L+2) + \tfrac{1}{4} \cdot t_k \cdot (L+1) = \left(L + \tfrac{7}{4}\right) \cdot t_k,$$

which contradicts the previous estimate. Thus, there exist at least $\frac{3}{4} \cdot t_k$ $\left(\frac{1}{2}, \frac{1}{2}\right)$-*special* squares $\Pi_j \subset L_\varphi(Q,\gamma,J_k)$. These squares satisfy the conditions of Theorem 4, allowing us to write the following estimate:

$$\#\mathbb{A}_n^2(Q,\Pi_j) \geqslant c_{13} Q^{n+1} \mu_2 \Pi_j = c_{13} c_8^2 \cdot Q^{n+1-2\gamma}.$$

Inequality (5.1) and the upper bound on the number of $\left(\frac{1}{2}, \frac{1}{2}\right)$-*special* squares imply that

$$\begin{aligned} \# \left(L_\varphi(Q,\gamma,J_k) \cap \mathbb{A}_n^2(Q)\right) &\geqslant \tfrac{3}{4} c_{13} c_8^2 \cdot t_k \cdot Q^{n+1-2\gamma} \\ &\geqslant \tfrac{3}{8} c_{13} c_8 \cdot Q^{n+1-\gamma} \mu_1 J_k. \end{aligned}$$

These inequalities, in turn, lead us to the following lower bound on the cardinality $\#M_\varphi(Q,J,\gamma)$:

$$\begin{aligned} \#M_\varphi(Q,J,\gamma) &\geqslant \tfrac{3}{8} c_{13} c_8 \cdot Q^{n+1-\gamma} \sum_k \mu_1 J_k \\ &\geqslant \tfrac{9}{32} c_{13} c_8 \cdot \mu_1 J \cdot Q^{n+1-\gamma} = c_2 \cdot Q^{n+1-\gamma}. \end{aligned}$$

5.2. The upper bound. As in the previous section, let us divide the set $L_\varphi(Q,\gamma,J)$, $J = [b_1, b_2]$, into subsets E_j, $1 \leqslant j \leqslant t$:

$$E_j := \left\{\mathbf{x} \in \mathbb{R}^2 : x_1 \in J_j, \quad |\varphi(x_1) - x_2| < \left(\tfrac{1}{2} + c_6\right) \cdot c_8 Q^{-\gamma}\right\},$$

where

$$J_j = [y_{j-1}, y_j], \quad y_0 = b_1, \quad y_{j+1} = y_j + \left(\tfrac{1}{2} + \tfrac{3}{2} c_6\right) \cdot c_8 Q^{-\gamma},$$

and the number of subsets E_j satisfies the inequality

$$t \leqslant \mu_1 J \cdot (\mu_1 J_j)^{-1} \leqslant \left(\tfrac{1}{2} + \tfrac{3}{2} c_6\right)^{-1} \cdot c_8^{-1} Q^\gamma \mu_1 J. \tag{5.13}$$

Once again, for $\overline{\varphi}_j = \frac{1}{2}\Big(\max\limits_{x \in J_j} \varphi(x) + \min\limits_{x \in J_j} \varphi(x)\Big)$ let us consider the squares

$$\Pi_j := \left\{\mathbf{x} \in \mathbb{R}^2 : x_1 \in J_j, \quad \left|\overline{\varphi}_j - x_2\right| < \left(\tfrac{1}{2} + \tfrac{3}{2} c_6\right) \cdot c_8 Q^{-\gamma}\right\}.$$

Since the function φ is continuously differentiable on the interval J, and $\max\limits_{x \in J} |\varphi'(x)| < c_6$, it is easy to see that each subset E_j is contained in the respective square Π_j: $E_j \subset \Pi_j$, $1 \leqslant j \leqslant t$. Note that the squares Π_j satisfy the conditions of Theorem 3. Therefore, we have

$$\#\mathbb{A}_n^2(Q,\Pi_j) \leqslant c_{12}Q^{n+1}\mu_2\Pi_j = c_{12}c_8^2\left(\tfrac{1}{2}+\tfrac{3}{2}c_6\right)^2 \cdot Q^{n+1-2\gamma}.$$

These inequalities, together with estimate (5.13), lead to the following upper bound for $\#M_\varphi(Q,I,\gamma)$:

$$\#M_\varphi(Q,J,\gamma) \leqslant \sum_{j=1}^{t} \#\mathbb{A}_n^2(Q,\Pi_j) \leqslant c_{12}c_8\left(\tfrac{1}{2}+\tfrac{3}{2}c_3\right)\cdot \mu_1 J \cdot Q^{n+1-\gamma}$$
$$= c_3 \cdot Q^{n+1-\gamma}.$$

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