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RECENT PROGRESS IN ENUMERATION OF HYPERMAPS

ABSTRACT. We enumerate the isomorphism classes of hypermaps of a given genus $g \leqslant 6$ and a given number of darts d. The hypermaps of a given genus g are distinguished up to orientation preserving isomorphisms. Our results depend on recent progress in counting rooted hypermaps, in particular by P. Zograf, M. Kazarian, A. Giorgetti and T. Walsh. These results can be interpreted as an enumeration of conjugacy classes of subgroups of the free Fuchsian group of rank two with a genus restriction.

§1. INTRODUCTION

An *oriented map* is a 2-cell decomposition of a closed orientable surface with a fixed global orientation. Oriented hypermaps are generalisations of oriented maps. While maps are 2-cell embeddings of graphs, hypermaps can be viewed as embeddings of hypergraphs in closed orientable surfaces. Walsh in [30] considered a model of a hypermap, where the underlying hypergraph is described via the corresponding bicoloured bipartite graph B. In his paper a hypermap is represented as a map with the underlying graph B.

In the context of algebraic geometry, hypermaps are called *dessins*, see [16]. Automorphisms of a hypermap are map isomorphism preserving the 2-colouring and orientation. The darts of a hypermap are identified with

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the edges of the corresponding bipartite bicoloured map. A hypermap is *rooted* if one of its darts, an edge in the bipartite model, is distinguished as a root. The automorphism group of a rooted map is by definition trivial.

The map enumeration problem has a long history, with origins dating back to 1963 when Tutte derived a closed formula for the number of rooted spherical maps [27]. Another significant point in the development of map enumeration was the formula counting unrooted spherical maps (up to isomorphism), derived by Liskovets about 20 years later [17]. For more recent results on map enumeration we refer the reader to [21, 22].

In what follows we consider the following two enumeration problems:

Problem 1. Given a genus $g \geqslant 0$ and an integer d, determine the number $h_g(d)$ of rooted hypermaps of genus g with d darts.

Problem 2. Given a genus $g \geqslant 0$ and an integer d, determine the number $U_g(d)$ of isomorphism classes of hypermaps (unrooted hypermaps) of genus g with d darts.

Walsh solved Problem 1 in [30] by determining the number $h_0(d)$ of spherical hypermaps with d darts. The solution of Problem 2 for the sphere can be obtained from a result by Bousquet–Mélou and Schaeffer counting planar 2-constellations [3].

The toroidal instance of Problem 1 was solved by Arquès in [2] by determining the numbers $h_1(d)$. The numbers of isomorphism classes of toroidal hypermaps, that is, the numbers $U_1(d)$, are determined in [23]. Recently Kazarian and Zograf [14] have determined the generating functions for $h_g(d)$ up to unknown coefficients of a polynomial of degree $5g-5$. In fact, their method gives an algorithm for determining the missing coefficients, which can be applied provided g is small. In a personal communication [32] Zograf gave explicit descriptions of the generating functions for the coefficients $h_2(d)$ and $h_3(d)$, the cases of genus 2 and 3. Independently, Giorgetti and Walsh [7] used a different approach and derived the generating functions for $h_g(d)$ in another equivalent form.

A method introduced in [22, 23], in combination with the new results on enumeration of rooted hypermaps, allows us to solve the problem of counting unrooted hypermaps (dessins) of small genera: here we present formulae for all genera up to six. More precisely, by [22, 23] the number of unrooted oriented hypermaps of a given genus g with d darts can be determined explicitly whenever the the numbers $h_\gamma(m)$ are known for each m dividing d and each $\gamma \leqslant g$ (see Theorem 5 for details). Since the numbers $h_\gamma(m)$ are known, we are able to determine the numbers $U_g(d)$ for $g \leqslant 6$.

The respective formulae are presented below in Theorems 8–12. Tables 2–7 containing the numbers $h_g(d)$ and $U_g(d)$ for $g \leqslant 6$ and $d \leqslant 36$ can be found at the end of the paper. The formulae giving $U_g(d)$ for $2 \leqslant g \leqslant 6$ are new.

The results we have derived can also be expressed in group theoretical language. Specifically, $h_g(n)$ gives the number of subgroups of index d and genus g in a free Fuchsian group of rank two, regarded as the universal triangle group $\Delta(\infty, \infty, \infty) = \langle x, y, z \mid xyz = 1\rangle$ acting on the hyperbolic plane $\mathbf{H}^2$, while $U_g(d)$ gives the number of conjugacy classes of such subgroups. Note that the number of subgroups of a given index in the free group of rank two was computed in a classical paper by M. Hall [12], while the conjugacy classes of subgroups of a given index in the free group were enumerated by Liskovets [36] (also see [15,20,26]). These results determine the numbers of rooted and of unrooted hypermaps counted regardless of genus. More details on the correspondence between subgroups of the free group of rank two and hypermaps will be given in the next section.

§2. Hypermaps on orbifolds

Hypermaps on surfaces. An *oriented combinatorial hypermap* is a triple $\mathcal{H} = (D; R, L)$, where D is a finite set of darts (also called brins, blades or bits) and R, L are permutations of D such that $\langle R, L\rangle$ is transitive on D. The orbits of R are called *hypervertices*, the orbits of L are called *hyperedges* and the orbits of RL are called *hyperfaces*. The *degree* of a hypervertex (hyperedge, hyperface) is the size of the respective orbit.

Let $|D| = d$. Denote by v, e and f the numbers of hypervertices, hyperedges and hyperfaces. Then the genus g of $\mathcal{H}$ is given by the Euler-Poincaré formula, as follows:

$$v + e + f - d = 2 - 2g.$$

Given hypermaps $\mathcal{H}_i = (D_i; R_i, L_i)$ for $i = 1, 2$, a mapping $\psi : D_1 \to D_2$ such that $R_2\psi = \psi R_1$ and $L_2\psi = \psi L_1$ is called a *morphism* (or a *covering*) $\mathcal{H}_1 \to \mathcal{H}_2$. Note that each morphism between hypermaps is by definition an epimorphism. If $\psi : \mathcal{H}_1 \to \mathcal{H}_2$ is a bijection, ψ is an *isomorphism*. The isomorphisms $\mathcal{H} \to \mathcal{H}$ form a group $\mathrm{Aut}(\mathcal{H})$ of *automorphisms* of $\mathcal{H}$. It is easily seen that $\mathrm{Aut}(\mathcal{H})$ acts semiregularly on D; equivalently, the stabiliser of a dart is trivial. A hypermap $\mathcal{H}$ is called *rooted* if one element x of D is chosen to be a root. Morphisms between rooted hypermaps take roots onto roots. It follows that a rooted hypermap admits no non-trivial automorphisms.

By *a surface* we mean a connected, orientable surface without boundary. A *topological map* is a 2-cell decomposition of a surface. Usually, maps on surfaces are described as 2-cell embeddings of graphs. Oriented combinatorial maps are hypermaps $(D; R, L)$ such that L is a fixed-point-free involution. Walsh observed that oriented hypermaps can be viewed as particular maps. Namely, he demonstrated a one-to-one correspondence [30, Lemma 1] between hypermaps and (oriented) 2-coloured bipartite maps. This means that one of the two global orientations of the underlying surface is fixed, and, moreover, we assume that the colouring of vertices, say by black and white colours, is preserved by morphisms between maps. The correspondence is given as follows. Let $\mathcal{M}$ be a 2-coloured bipartite map on an orientable surface S with a fixed global orientation. We let D be the set of edges of $\mathcal{M}$. The orientation of S induces at each black vertex v of $\mathcal{M}$ a cyclic permutation R_v of the edges incident with v. In this way a permutation $R = \prod R_v$ of D is defined. Similarly, the orientation of S determines a cyclic permutation L_u at each white vertex u. Set $L = \prod L_u$. We then have a unique hypermap $(D; R, L)$ corresponding to $\mathcal{M}$. Conversely, given a hypermap $(D; R, L)$ we first define a bipartite 2-colored graph X whose edges are elements of D, black vertices are orbits of R and white vertices are orbits of L. An edge $x \in D$ is incident with a (black or white) vertex u if $x \in u$. The permutations R and L induce local rotations of arcs outgoing from black and white vertices, respectively. It is well known (see Gross and Tucker [8, Section 3.2]) that the system of rotations determines a 2-cell embedding of X into an orientable surface. The surface S is defined by taking the cycles of the product RL as the boundary walks of faces of the underlying map: S can be explicitly obtained by gluing a 2-cell to each such boundary walk. By its construction S is endowed with an orientation consistent with the way R and L permute the darts at vertices. It is worth mentioning that the idea of describing maps by means of rotations dates back to the 19th century, and can be traced in works of Hamilton and Heffter.

In a similar way, an oriented 2-coloured bipartite map is called *rooted* if one of the edges is selected to be a root. Morphisms between rooted 2-coloured bipartite maps take a root onto a root.

There is yet another way to describe hypermaps. Let $\mathcal{H} = (D; R, L)$ be a hypermap. Clearly, the permutation group $\langle R, L\rangle$ is an epimorphic image of the free product $\Delta = C * C \cong \langle \rho\rangle * \langle \lambda\rangle$ of two infinite cyclic groups. The group Δ acts on D via the epimorphism taking $\rho \mapsto R$ and $\lambda \mapsto L$. Thus by using some standard results in permutation group theory each hypermap can be described by a subgroup $F \leqslant \Delta$ [6,11,28,29]. The subgroup F, called a *hypermap subgroup*, can be identified with the stabiliser of a dart in the action of Δ on D. Since the action of Δ on D is transitive, the number of darts $|D| = d$ coincides with index $[\Delta : F]$ of F in Δ. Given $F \leqslant \Delta$ the corresponding hypermap can be constructed as an *algebraic hypermap* $\mathcal{H}(\Delta/F) = (D; R, L)$, where $D = \{xF \mid x \in \Delta\}$ is the set of left cosets of F in Δ, and the action of R, L on D is defined by $R(xF) = (\rho x)F$, $L(xF) = (\lambda x)F$. Note that the group Δ is sometimes called a universal group. More precisely, Δ is identified with the triangle group $T(\infty,\infty,\infty) = \langle x, y, z \mid xyz = 1\rangle$ acting on the hyperbolic plane $\mathbf{H}^2$ by orientation-preserving isometries (see G. Jones, D. Singerman [11]). In this case $\mathbf{H}^2/\Delta$ is a thrice punctured sphere and $\mathbf{H}^2/F$ is a punctured orientable surface whose genus g coincides with the genus of the corresponding hypermap. In what follows we will refer to g as the *genus of the subgroup* F.

We summarise the above discussion in the following propositions.

Proposition 1. *The following objects are in one-to-one correspondence:*

(1) *rooted* 2*-coloured bipartite maps of genus* g *with* d *edges,*
(2) *rooted hypermaps* $(D; R, L)$ *of genus* g *with* $|D| = d$*,*
(3) *subgroups of the group* $\Delta = T(\infty,\infty,\infty)$ *of index* d *and genus* g*.*

Part (1) $\Leftrightarrow$ (2) follows from Walsh [30]. Part (2) $\Leftrightarrow$ (3) is in ([5,11]).

It is well known that isomorphic hypermaps have conjugate hypermap subgroups. Hence isomorphism classes of hypermaps correspond to conjugacy classes of subgroups.

Proposition 2. *The following objects are in one-to-one correspondence:*

(1) *isomorphism classes of* 2*-coloured bipartite maps of genus* g *with* d *edges,*
(2) *isomorphism classes of hypermaps* $(D; R, L)$ *of genus* g *with* $|D| = d$*,*
(3) *conjugacy classes of subgroups of index* d *and genus* g *in the group* $\Delta = T(\infty,\infty,\infty)$*.*

Remark. Following Belyi [34] and Grothendieck [9] we know that a 2-coloured bipartite map, viewed as a topological realisation of a hypermap, can be endowed with the structure of a Riemann surface. In this context 2-coloured bipartite maps are called dessins.

Regular coverings. Let $\psi : \mathcal{H}_1 \to \mathcal{H}_2$ be a morphism of hypermaps. The covering transformation group consists of the automorphisms α of $\mathcal{H}_1$

satisfying the condition $\psi = \psi \circ \alpha$. A morphism $\psi : \mathcal{H}_1 \to \mathcal{H}_2$ will be called *regular* if the covering transformation group acts transitively on the fibre $\psi^{-1}(x)$ over a dart x of $\mathcal{H}_2$. All regular morphisms defined on a hypermap $\mathcal{H} = (D; R, L)$ can be constructed by taking a semi-regular subgroup $G \leqslant \mathrm{Aut}(\mathcal{H})$ and letting $\bar{D}$ be the set of orbits of G, with $\bar{R}[x] = [Rx]$ and $\bar{L}[x] = [Lx]$. Then the natural projection $x \mapsto [x]$ defines a regular covering $\mathcal{H} \to \bar{\mathcal{H}}$, where $\bar{\mathcal{H}} = (\bar{D}, \bar{R}, \bar{L})$. When we replace combinatorial hypermaps with their associated bipartite maps, a morphism between two hypermaps extends to a branched covering between the underlying surfaces, possibly with branch points at the vertices and faces. Thus morphisms between hypermaps are also called *coverings*.

Maps and hypermaps on orbifolds. Given a regular covering $\psi : \mathcal{H} \to \mathcal{K}$, let x be a hypervertex, hyperface or hyperedge of $\mathcal{K}$. Let $\mathcal{H}$ be of genus g, let $\mathcal{K}$ be of genus γ and let $G \leqslant \mathrm{Aut}(\mathcal{H})$ be the covering transformation group. Denote by S_g the underlying surface associated with $\mathcal{H}$. The ratio of degrees $b(x) = deg(\tilde{x})/deg(x)$, where $\tilde{x} \in \psi^{-1}(x)$ is a lifting of x along ψ, will be called the *branch index* of x. By transitivity of the action of the group of covering transformations, the branch index is a well-defined positive integer independent of the choice of the lift $\tilde{x}$. Hence $x \mapsto b(x)$, $x \in V(\mathcal{K}) \cup E(\mathcal{K}) \cup F(\mathcal{K})$, is well defined on the union of the sets of hypervertices, hyperedges and hyperfaces. Writing all the values $b(x) > 1$ in non-decreasing order we get an integer sequence $m_1, m_2, \ldots, m_r$. In this way a *quotient orbifold* S_g/G with signature $[\gamma; m_1, m_2, \ldots, m_r]$ is defined. For our purposes we define a topological 2-dimensional orbifold $O = O[\gamma; m_1, \ldots, m_r]$ to be a closed orientable surface of genus γ with a distinguished (finite) set of points $\mathcal{B}$, called branch points, and an integer function assigning to each $x \in \mathcal{B}$ an integer $b(x) \geqslant 2$. A 2-coloured bipartite map of genus γ is a map on the orbifold O provided the following two conditions are satisfied:

(1) no branch point $x \in \mathcal{B}$ lies on an edge,
(2) each face contains at most one branch point $x \in \mathcal{B}$.

The signature of an orbifold associated with a regular covering of hypermaps coincides with the signature of an orbifold determined by the corresponding regular covering of Walsh's 2-coloured bipartite maps. Note also that a regular covering $\psi : \mathcal{H} \to \mathcal{K}$ extends (uniquely) to a regular covering $S_g \to S_g/G$, where g is the genus of $\mathcal{H}$ and G is the group of covering transformations. The concept of a map on an orbifold naturally generalises the concept of a map on a (closed) surface, because ordinary maps are just maps on orbifolds with an empty set of branch points.

Let O be an orbifold with signature $[\gamma; m_1, m_2, \ldots, m_r]$. The *orbifold fundamental group* $\pi_1(O)$ is a Fuchsian group

$$\begin{aligned}\pi_1(M,\sigma) &= F[\gamma; m_1, m_2, \ldots, m_r]\\ &= \Big\langle a_1, b_1, a_2, b_2, \ldots, a_\gamma, b_\gamma, e_1, \ldots, e_r \mid \prod_{i=1}^{\gamma}[a_i, b_i]\prod_{j=1}^{r} e_j = 1,\\ &\qquad\qquad e_1^{m_1} = \cdots = e_r^{m_r} = 1\Big\rangle. \qquad (2.1)\end{aligned}$$

Let $\mathcal{H} \to \mathcal{H}/G = \mathcal{K}$ be a regular covering between hypermaps with a covering transformation group G, and suppose that $\mathcal{H}$ is finite. Let the signature of the orbifold corresponding to $\mathcal{K} = \mathcal{H}/G$ be $[\gamma; m_1, m_2, \ldots, m_r]$. Then the Euler characteristic of the underlying surface of $\mathcal{H}$ is given by the Riemann–Hurwitz equation:

$$\chi = |G|\Big(2 - 2\gamma - \sum_{i=1}^{r}\Big(1 - \frac{1}{m_i}\Big)\Big). \qquad (2.2)$$

§3. General counting formula

A group epimorphism is called *order-preserving* if it preserves the orders of elements of finite order. Given a closed orientable surface S_g of genus g and a cyclic orbifold $O = S_g/Z_\ell$ we denote by $\mathrm{Epi_o}(\pi_1(O), Z_\ell)$ the number of order-preserving epimorphisms $\pi_1(O) \to Z_\ell$. The following theorem gives a general counting formula for the numbers of unrooted hypermaps of given genus. Based on an approach from [20], the following general counting formula is derived in [23].

Theorem 3. *Let S_g be a closed orientable surface of genus g. Let $h_O(d)$ be the number of rooted hypermaps with d darts on a cyclic orbifold $O = S_g/Z_\ell$.*

Then the number of unrooted hypermaps of genus g having n darts is

$$U_g(n) = \frac{1}{n}\sum_{\substack{\ell\mid n\\ \ell d = n}}\ \sum_{O\in \mathrm{Orb}(S/Z_\ell)} h_O(d).\mathrm{Epi_o}(\pi_1(O), Z_\ell),$$

where the second sum runs through all cyclic orbifolds S_g/Z_ℓ.

The numbers of rooted hypermaps on cyclic orbifolds can be expressed in terms of numbers of rooted hypermaps on surfaces. Let $\mathcal{H}$ be a rooted hypermap on an orbifold O such that $\mathcal{H} = \tilde{\mathcal{H}}/Z_\ell = (D; R, L)$ is a quotient of a finite map $\tilde{\mathcal{H}}$ on a surface S_g. Thus $O = S_g/G$, where $G \cong Z_\ell$ is a discrete cyclic group of orientation-preserving symmetries of S_g of order ℓ. It follows that each branch index of the branched covering $S_g \to O$ is a divisor of ℓ. We can write $O = O[\gamma; 2^{q_2}, \ldots, \ell^{q_\ell}]$, where $q_j \geqslant 0$, and j^{q_j} indicates that there are q_j branch points of index j for each $j = 2, \ldots, \ell$. The genera γ and g are related by the Riemann–Hurwitz equation

$$2 - 2g = \ell\Big(2 - 2\gamma - \sum_{j=2}^{\ell} q_j\big(1 - \frac{1}{j}\big)\Big).$$

We use the convention $h_\gamma(d) = h_{[\gamma;\,\varnothing]}(d)$ denoting the number of rooted hypermaps with d darts on a closed surface of genus g. Clearly, the exponential notation $O = O[\gamma; 2^{q_2}, \ldots, \ell^{q_\ell}]$ can be used for any oriented orbifold (not necessarily cyclic) provided the indices of branch points are bounded by ℓ.

Given integers $x_1, x_2, \ldots, x_q$ and $y \geqslant x_1 + x_2 + \cdots + x_q$ we denote by

$$\binom{y}{x_1, x_2, \ldots, x_q} = \frac{y!}{x_1!x_2!\ldots x_q!(y - \sum\limits_{j=1}^{q} x_j)!},$$

the multinomial coefficient.

Proposition 4. [23] *The number of rooted hypermaps with d darts on an orbifold*

$$O = O[\gamma; 2^{q_2}, \ldots, \ell^{q_\ell}]$$

is

$$h_O(d) = \binom{d+2-2\gamma}{q_2, q_3, \ldots, q_\ell} h_\gamma(d). \tag{3.1}$$

Combining Proposition 4 and Theorem 3 one gets the following theorem, see [23].

Theorem 5. *The number of unrooted hypermaps with* n *darts on a closed surface* S_g *of genus* g *is given by*

$$U_g(n) = \frac{1}{n} \sum_{\substack{\ell | n \\ \ell d = n}} \sum_{\substack{O \in \mathrm{Orb}(S/Z_\ell) \\ O = O[\gamma; 2^{q_2}, 3^{q_3}, \ldots, \ell^{q_\ell}]}} \mathrm{Epi_o}(\pi_1(O), Z_\ell) \binom{d+2-2\gamma}{q_2, q_3, \ldots, q_\ell} h_\gamma(d), \tag{3.2}$$

where the second sum runs through all cyclic orbifolds S_g/Z_ℓ.

The numbers $\mathrm{Epi_o}(\pi_1(O), Z_\ell)$ were computed by the authors in [22] in terms of some standard arithmetical functions. The following section surveys results on $\mathrm{Epi_o}(\pi_1(O), Z_\ell)$.

§4. Number of epimorphisms from a Fuchsian group onto a cyclic group

As one can see from Theorems 3 and 5, to derive an explicit formula for the number of unrooted hypermaps with a given genus and a given number of darts, one needs to deal with the number $\mathrm{Epi_o}(\pi_1(O), \mathsf{Z}_\ell)$ of order-preserving epimorphisms from $\pi_1(O)$ onto a cyclic group Z_ℓ. These numbers are calculated using some number-theoretical machinery in [22]. In what follows we recall some relevant results used in later computations. An arithmetic function, called by Liskovets the *orbicyclic arithmetic function* [18], is a multivariate integer function defined in [22] by

$$E(m_1, m_2, \ldots, m_r) = \frac{1}{m} \sum_{k=1}^{m} \Phi(k,\, m_1) \cdot \Phi(k,\, m_2) \ldots \Phi(k,\, m_r),$$

where $\Phi(k,\, m)$ is the Von Sterneck function defined by

$$\Phi(x, n) = \frac{\phi(n)}{\phi(\frac{n}{(x,n)})} \, \mu\left(\frac{n}{(x,n)}\right),$$

(x, n) is the greatest common divisor of x and n, and ϕ and μ are, respectively, the Euler and Möbius functions. It was shown by O. Hölder that $\Phi(x, n)$ coincides with the Ramanujan sum

$$\sum_{\substack{1 \leqslant k \leqslant n \\ (k,\, n)=1}} \exp\Bigl(\frac{2\, ikx}{n}\Bigr),$$

see Apostol [1, p. 164] and [24]. For more information about the Ramanujan sum the reader is referred to [19].

Recall that the Jordan multiplicative function $\phi_k(n)$ of order k can be defined as follows:

$$\phi_k(n) = \sum_{d|n} \mu\left(\frac{n}{d}\right) d^k.$$

The following proposition, generalising a statement of Harvey [10], is proved in [22].

Proposition 6. *Let $\Gamma = F[g; m_1, \ldots, m_r]$ be a Fuchsian group of signature $[g; m_1, \ldots, m_r]$. Denote by $m = l.c.m.\,(m_1, \ldots, m_r)$ the least common multiple of $m_1, \ldots, m_r$ and let m divide ℓ. Then the number of order-preserving epimorphisms from the group Γ onto a cyclic group Z_ℓ is given by the formula*

$$\mathrm{Epi_o}(\Gamma, Z_\ell) = m^{2g}\phi_{2g}(\ell/m)E(m_1, m_2, \ldots, m_r).$$

In particular, if $\Gamma = F[g; \varnothing]$ is a surface group of genus g we have

$$\mathrm{Epi_o}(\Gamma, Z_\ell) = \phi_{2g}(\ell).$$

For practical use it is sometimes more convenient to use the multiplicative form of the function $E(m_1, m_2, \ldots, m_r)$ derived in [18] as follows.

First let us assume that all periods m_j are powers p^{a_j} of the same prime p. Since E is a symmetric multivariate function, we may assume that the exponents form a non-increasing sequence:

$$a_1 = a_2 = \cdots = a_s = a > a_{s+1} \geqslant a_{s+2} \geqslant \cdots \geqslant a_r.$$

Set $v = \sum_{j=2}^{r} (a_j - 1)$, so in particular $v = 0$ if $r = 1$. In [18] Liskovets proved that

$$E(p^{a_1}, p^{a_2}, \ldots, p^{a_r}) = (p-1)^{r-s+1} p^v \frac{(p-1)^{s-1} + (-1)^s}{p}.$$

Now let us consider general case. Set $m = l.c.m.(m_1, m_2, \ldots, m_r)$. For any prime p dividing m define $E_p(m_1, m_2, \ldots, m_r) = E(p^{a_1}, p^{a_2}, \ldots, p^{a_r})$, where p^{a_j} is the highest power of p which divides m_j, for $j = 1, 2, \ldots, r$. Then by [18, p. 160]

$$E(m_1, m_2, \ldots, m_r) = \prod_{\substack{p|m \\ p \text{ prime}}} E_p(m_1, m_2, \ldots, m_r).$$

Hence one can easily determine the numbers $\mathrm{Epi_o}(\pi_1(O), Z_\ell)$ for surfaces of genera at most 3, compare with [4,35]. The numbers $\mathrm{Epi_o}(\pi_1(O), Z_\ell)$ have been determined up to genus 101 by Karabáš [13]. An orbifold $O =$

$O[\gamma; m_1, \ldots, m_r]$ will be called *g-admissible* if it can be represented as a quotient orbifold $O = S_g/\mathbb{Z}_\ell$, where S_g is an orientable surface of genus g surface and Z_ℓ is a cyclic group of automorphisms of S_g.

Proposition 7 ([13]). *The admissible cyclic orbifolds O of genus at most 3 and the corresponding numbers of order preserving epimorphisms $\pi_1(O) \to Z_\ell$ are summarised in Table 1.*

Table 1.

genus	ℓ	Orbifold O	$\mathrm{Epi}_o(\pi_1(O), Z_\ell)$	genus	ℓ	Orbifold O	$\mathrm{Epi}_o(\pi_1(O), Z_\ell)$
1	ℓ	$[1; \varnothing]$	$\phi_2(\ell)$	3	2	$[2; \varnothing]$	15
1	2	$[0; 2^4]$	1	3	2	$[1; 2^4]$	4
1	3	$[0; 3^3]$	2	3	2	$[1; 2^8]$	1
1	4	$[0; 4^2, 2]$	2	3	3	$[1; 3^2]$	18
1	6	$[0; 6, 3, 2]$	2	3	3	$[0; 3^5]$	10
2	1	$[2; \varnothing]$	1	3	4	$[1; 2^2]$	12
2	2	$[1; 2^2]$	4	3	4	$[0; 2^3, 4^2]$	2
2	2	$[0; 2^6]$	1	3	4	$[0; 4^4]$	8
2	3	$[0; 3^4]$	6	3	6	$[0; 2, 3^2, 6]$	2
2	4	$[0; 2^2, 4^2]$	2	3	6	$[0; 2^2, 6^2]$	2
2	5	$[0; 5^3]$	12	3	7	$[0; 2^2, 7^3]$	30
2	6	$[0; 2^2, 3^2]$	2	3	8	$[0; 4, 8^2]$	8
2	6	$[0; 3, 6^2]$	2	3	9	$[0; 3, 9^2]$	12
2	8	$[0; 2, 8^2]$	4	3	12	$[0; 2, 12^2]$	4
2	10	$[0; 2, 5, 10]$	4	3	12	$[0; 3, 4, 12]$	4
3	1	$[3; \varnothing]$	1	3	14	$[0; 2, 7, 14]$	4

§5. Enumeration of rooted hypermaps of given genus

In this section we survey known results concerning the numbers $h_g(d)$. Set $h_{d,g} = h_g(d)$ and let

$$F_g(x) = \sum_{d=1}^{\infty} h_{d,g} x^d,$$

be the corresponding generating function. Setting

$$x = \frac{t}{(1+2t)^2},$$

Kazarian and Zograf [14] have determined $F_g(x)$ as a rational function of t. In general, for $g \geqslant 1$ they proved that

$$F_g(x) = \widetilde{F}_g(t) = \frac{t^{2g+1}P_g(t)}{(1+t)^{4g-3}(1-2t)^{5g-3}},$$

where $P_g(t)$ is a polynomial of degree $5g-5$.

Independently, Giorgetti and Walsh [7] have investigated the same generating function

$$H_g(x) = \sum_{d=1}^{\infty} h_{d,g}x^d, \ H_g(x) = F_g(x)$$

in a different way. They put $x = \mu(1-2\mu)$, with $\mu = 0$ when $x = 0$, and considered a rational expression of $H_g(x)$ in terms of μ. The main idea is to express $H_g(x)$ for $g > 1$ as

$$H_g(x) = \widetilde{H}_g(\mu) = 4\mu^3(\mu(1-2\mu))^{2g-2}(1-4\mu)^{3-5g}(1-\mu)^{3-4g}D_g(\mu),$$

where $D_g(\mu)$ is a polynomial of degree $5g-6$.

The formulae for $H_g(x)$, where $g = 0, 1, 2, 3, 4, 5$ and 6, were derived explicitly by Giorgetti and Walsh in [7]. More precisely, one gets

(o) $H_0(x) = \frac{(1-3\mu)\mu}{(1-2\mu)^2}$,

(i) $H_1(x) = \frac{\mu^3}{(1-4\mu)^2(1-\mu)}$,

(ii) $H_2(x) = \frac{4\mu^3(\mu(1-2\mu))^2D_2(\mu)}{(1-4\mu)^7(1-\mu)^5}$,

(iii) $H_3(x) = \frac{4\mu^3(\mu(1-2\mu))^4D_3(\mu)}{(1-4\mu)^{12}(1-\mu)^9}$,

(iv) $H_4(x) = \frac{4\mu^3(\mu(1-2\mu))^6D_4(\mu)}{(1-4\mu)^{17}(1-\mu)^{13}}$,

(v) $H_5(x) = \frac{4\mu^3(\mu(1-2\mu))^8D_5(\mu)}{(1-4\mu)^{22}(1-\mu)^{17}}$,

(vi) $H_6(x) = \frac{4\mu^3(\mu(1-2\mu))^{10}D_6(\mu)}{(1-4\mu)^{27}(1-\mu)^{21}}$,

where

$$\begin{aligned} D_2(\mu) &= 2 - 15\mu + 48\mu^2 - 77\mu^3 + 51\mu^4, \\ D_3(\mu) &= 45 - 552\mu + 3360\mu^2 - 13168\mu^3 + 35172\mu^4 - 61872\mu^5 \\ &\quad + 61676\mu^6 - 13164\mu^7 - 36888\mu^8 + 28496\mu^9, \\ D_4(\mu) &= 2016 - 30456\mu + 239697\mu^2 - 1320920\mu^3 + 5541192\mu^4 \\ &\quad - 17597520\mu^5 + 39814032\mu^6 - 53553072\mu^7 + 1281984\mu^8 \\ &\quad + 170357328\mu^9 - 389268768\mu^{10} + 442844592\mu^{11} \end{aligned}$$

$$
\begin{aligned}
&- 243313744\mu^{12} + 15509760\mu^{13} + 32375616\mu^{14},\\
D_5(\mu) &= 151200 - 2490480\mu + 21738240\mu^2 - 141393220\mu^3\\
&+ 761835465\mu^4 - 3336459144\mu^5 + 11016156244\mu^6\\
&- 23295865824\mu^7 + 7568059872\mu^8 + 165542511744\mu^9\\
&- 761565230016\mu^{10} + 2000782619136\mu^{11} - 3552865706240\mu^{12}\\
&+ 4243997599488\mu^{13} - 2962590413376\mu^{14} + 338393916800\mu^{15}\\
&+ 1403096348736\mu^{16} - 1163002515456\mu^{17} + 239043447552\mu^{18}\\
&+ 61742404608\mu^{19},\\
D_6(\mu) &= 17107200 - 284717376\mu + 2485496880\mu^2 - 17314508592\mu^3\\
&+ 112079088144\mu^4 - 626336383104\mu^5 + 2630924485729\mu^6\\
&- 6580517850696\mu^7 - 4043551301232\mu^8 + 138473163256176\mu^9\\
&- 813298324826016\mu^{10} + 3098312828500416\mu^{11}\\
&- 8736443315384448\mu^{12} + 18704646148809216\mu^{13}\\
&- 29719458122609664\mu^{14} + 31734000656779264\mu^{15}\\
&- 13439214645718272\mu^{16} - 22997164994372352\mu^{17}\\
&+ 54283457920223232\mu^{18} - 55010184951564288\mu^{19}\\
&+ 28025505345377280\mu^{20} - 2073822560019456\mu^{21}\\
&- 4933663711730688\mu^{22} + 1584534210564096\mu^{23}\\
&+ 178054771302400\mu^{24}.
\end{aligned}
$$

Since $x = \frac{t}{(1+2t)^2}$ in the work of Kazarian and Zograf [14, 33], and $x = \mu(1-2\mu)$ in that of Giorgetti and Walsh [7], we get the following useful relation

$$\frac{t}{(1+2t)^2} = \mu(1-2\mu).$$

Taking into account the initial data we obtain $\mu = \frac{t}{1+2t}$. This gives the following correspondence between the Kazarian–Zograf and Giorgetti-Walsh generating functions:

$$\widetilde{F}_g(t) = \widetilde{H}_g\left(\frac{t}{1+2t}\right).$$

We will use the Giorgetti–Walsh formulae to enumerate the rooted hypermaps of each genus $g \leq 6$. The results of our calculations coincide with those obtained by Kazarian and Zograf, we checked it up to genus three.

As already mentioned, the explicit formulae for the coefficients $h_{0,d}$ were obtained by Walsh in [30], and for $h_{1,d}$ by Arquès in [2].

§6. Counting unrooted hypermaps of genus at most three

In this section we apply the above results to calculate the numbers of unrooted hypermaps with a given number of darts on the surfaces of genus two and three. For the sake of completeness we also summarise the known resuts for the sphere and for the torus.

6.1. The sphere. For each $\ell > 1$ there is only one possible action of the cyclic group Z_ℓ on the sphere S. The corresponding orbifold O has signature $[0; \ell, \ell]$, and by Proposition 7 we have $\mathrm{Epi_o}(\pi_1(O), Z_\ell) = \phi(\ell)$. By Theorem 5 we obtain

$$U_0(d) = \tfrac{1}{d}\big(h_0(d) + \sum_{\substack{\ell \mid d,\ \ell > 1 \\ \ell m = d}} \phi(\ell)\tbinom{m+2}{2} h_0(m)\big), \tag{6.1}$$

where the numbers $h_0(m)$ of spherical rooted hypermaps with m darts were determined by Walsh [30] as follows:

$$h_0(m) = \frac{3 \cdot 2^{m-1}}{(m+1)(m+2)}\binom{2m}{m}. \tag{6.2}$$

Inserting (6.2) into (6.1) we get the following formula, see [23], counting the spherical unrooted hypermaps with d darts:

$$U_0(d) = \frac{1}{d}\Big(\frac{3 \cdot 2^{d-1}}{(d+1)(d+2)}\binom{2d}{d} + \sum_{\substack{\ell \mid d,\ \ell > 1 \\ \ell m = d}} 3 \cdot 2^{m-2}\binom{2m}{m}\phi(\ell)\Big).$$

The numbers of rooted and unrooted spherical hypermaps with up to 30 darts are given in Table 1 at the end of this paper.

Note that the numbers $U_0(d)$ were also determined in an equivalent form by Bosquet–Melou and Schaeffer [3], in terms of unrooted planar 2-constellations formed by d polygons.

6.2. The torus. In this section we derive an explicit formula for counting unrooted maps on the torus. The list of 1-admissible orbifolds and the corresponding numbers $Epi_o(\pi_1(O), Z_\ell)$ were derived in Proposition 7. Rooted toroidal maps were enumerated by Arquès in [2]. He proved that

$$h_1(d) = \frac{1}{3}\sum_{k=0}^{d-3} 2^k(4^{d-2-k}-1)\binom{d+k}{k}. \tag{6.3}$$

Inserting (6.3) into Theorem 5 we obtain the following formula, derived in [23], giving the number $U_1(d)$ of oriented unrooted toroidal hypermaps with d darts:

$$\frac{1}{d}\Bigg(\binom{\frac{d+4}{2}}{4}h_0\left(\frac{d}{2}\right) + 2\binom{\frac{d+6}{3}}{3}h_0\left(\frac{d}{3}\right) + 6\binom{\frac{d+8}{4}}{3}h_0\left(\frac{d}{4}\right) + 12\binom{\frac{d+12}{6}}{3}h_0\left(\frac{d}{6}\right) + \sum_{\substack{\ell\,|\,d,\\ \ell m=d}} \phi_2(\ell)h_1(m)\Bigg),$$

where ϕ_2 is the Jordan multiplicative function of the second order, and $h_0(m)$ and $h_1(m)$ are respectively determined by (6.2) and (6.3).

6.3. The surfaces of genus 2 and 3. By using the general counting formula (3.2) and the lists of the numbers $\mathrm{Epi_o}(\pi_1(O), Z_\ell) = \phi(\ell)$ (see Proposition 7), where O ranges through all 2- and 3-admisible orbifolds, we get the following two theorems.

Theorem 8. *The number of oriented unrooted hypermaps with d darts on a surface of genus two is given by the formula*

$$\frac{1}{d}\big(h_2(d) + 4h_{[1;2^2]}(d/2) + h_{[0;2^6]}(d/2) + 6h_{[0;3^4]}(d/3) + 2h_{[0;2^2,4^2]}(d/4) + 12h_{[0;5^3]}(d/5)2h_{[0;2^2,3^2]}(d/6) + 2h_{[0;3,6^2]}(d/6) + 4h_{[0;2,8^2]}(d/8) + 4h_{[0;2,5,10]}(d/10)\big),$$

where $h_O(m)$ is defined in (3.1) and $h_g(m)$ is the number of rooted hypermaps of genus g with m darts.

Theorem 9. *The number of oriented unrooted hypermaps with d darts on a surface of genus three is given by the formula*

$$\frac{1}{d}\big(h_3(d) + 15h_2(d/2) + 4h_{[1;2^4]}(d/2) + h_{[0;2^8]}(d/2) + 18h_{[1;3^2]}(d/3) + 10h_{[0;3^5]}(d/3) + 12h_{[1;2^2]}(d/4) + 2h_{[0;2^3,4^2]}(d/4) + 8h_{[0;4^4]}(d/4) + 2h_{[0;2,3^2,6]}(d/6) + 2h_{[0;2^2,6^2]}(d/6) + 30h_{[0;7^3]}(d/7) + 8h_{[0;4,8^2]}(d/8) + 12h_{[0;3,9^2]}(d/9) + 4h_{[0;2,12^2]}(d/12) + 4h_{[0;3,4,12]}(d/12) + 6h_{[0;2,7,14]}(d/14)\big),$$

where $h_O(m)$ *is defined in* (3.1) *and* $h_g(m)$ *is the number of rooted hypermaps of genus* g *with* m *darts.*

6.4. The surfaces of genus 4, 5 **and** 6. The general counting formula (3.2) and the list of the numbers $\mathrm{Epi_o}(\pi_1(O), Z_\ell)$ given in [21, Lemma 4.5], where O ranges through all 4-admisible orbifolds, give the following counting formula.

Theorem 10. *The number of oriented unrooted hypermaps with d darts on a surface of genus four is given by the formula*

$$\frac{1}{d}\Big(h_4(d) + 16h_{[2;2^2]}(d/2) + 4h_{[1;2^6]}(d/2) + h_{[0;2^{10}]}(d/2) + 80h_2(d/3) + 18h_{[1;3^3]}(d/3) + 22h_{[0;3^6]}(d/3) + 32h_{[1;4^2]}(d/4) + 8h_{[0;2,4^4]}(d/4) + 2h_{[0;2^4,4^2]}(d/4) + 52h_{[0;5^4]}(d/5) + 32h_{[1;2^2]}(d/6) + 2h_{[0;2,6^3]}(d/6) + 2h_{[0;2^2,3^3]}(d/6) + 6h_{[0;3^2,6^2]}(d/6) + 2h_{[0;2^3,3,6]}(d/6) + 4h_{[0;2^2,8^2]}(d/8) + 18h_{[0;9^3]}(d/9) + 12h_{[0;5,10^2]}(d/10) + 4h_{[0;2^2,5^2]}(d/10) + 4h_{[0;3,12^2]}(d/12) + 4h_{[0;4,6,12]}(d/12) + 8h_{[0;3,5,15]}(d/15) + 8h_{[0;2,16^2]}(d/16) + 6h_{[0;2,9,18]}(d/18)\Big),$$

where $h_O(m)$ *is defined in* (3.1) *and* $h_g(m)$ *is the number of rooted hypermaps of genus* g*.with* m *darts.*

Similar arguments using the lists of 5-admissible and 6-admissible orbifolds O and the corresponding numbers $\mathrm{Epi_o}(\pi_1(O), Z_\ell)$ [13] give the following results.

Theorem 11. *The number of oriented unrooted hypermaps with d darts on a surface of genus five is given by the formula*

$$\frac{1}{d}\Big(h_5(d) + 63h_3(d/2) + 16h_{[2;2^4]}(d/2) + 4h_{[1;2^8]}(d/2) + h_{[0;2^{12}]}(d/2) + 54h_{[1;3^4]}(d/3) + 42h_{[0;3^7]}(d/3) + 240h_2(d/4) + 12h_{[1;2^4]}(d/4) + 32h_{[1;2,4^2]}(d/4) + 2h_{[0;2^5,4^2]}(d/4) + 8h_{[0;2^2,4^4]}(d/4) + 100h_{[1;5^2]}(d/5) + 54h_{[1;3^2]}(d/6) + 6h_{[0;2,3^3,6]}(d/6) + 2h_{[0;2^2,3,6^2]}(d/6) + 2h_{[0;2^4,3^2]}(d/6) + 6h_{[0;6^4]}(d/6) + 48h_{[1;2^2]}(d/8) + 8h_{[0;2,4,8^2]}(d/8) + 4h_{[0;2^2,10^2]}(d/10) + 90h_{[0;11^3]}(d/11) + 4h_{[0;6,12^2]}(d/12) + 8h_{[0;3,15^2]}(d/15) + 8h_{[0;2,20^2]}(d/20) + 10h_{[0;2,11,22]}(d/22)\Big),$$

where $h_O(m)$ is defined in (3.1) and $h_g(m)$ is the number of rooted hypermaps of genus g with m darts.

Theorem 12. *The number of oriented unrooted hypermaps with d darts on a surface of genus six is given by the formula*

$$\frac{1}{d}\Big(h_6(d) + 64h_{[3;2^2]}(d/2) + 16h_{[2;2^6]}(d/2) + 4h_{[1;2^10]}(d/2) + h_{[0;2^{14}]}(d/2) + 162h_{[2;3^2]}(d/3) + 90h_{[1;3^5]}(d/3) + 86h_{[0;3^8]}(d/3) + 32h_{[1;2^2,4^2]}(d/4) + 32h_{[0;4^6]}(d/4) + 8h_{[0;2^3,4^4]}(d/4) + 2h_{[0;2^6,4^2]}(d/4) + 624h_2(d/5) + 204h_{[0;5^5]}(d/5) + 72h_{[1;6^2]}(d/6) + 6h_{[0;2,3,6^3]}(d/6) + 10h_{[0;3^3,6^2]}(d/6) + 2h_{[0;2^4,6^2]}(d/6) + 2h_{[0;2^3,3^2]}(d/6) + 6h_{[0;2^2,3^4]}(d/6) + 186h_{[0;7^4]}((d)/7) + 16h_{[0;4^2,8^2]}(d/8) + 4h_{[0;2^3,8^2]}(d/8) + 24h_{[0;3^2,9^2]}(d/9) + 96h_{[1;2^2]}(d/10) + 12h_{[0;2,5^2,10]}(d/10) + 4h_{[0;3^2,4^2]}(d/12) + 4h_{[0;2,3,4,12]}(d/12) + 4h_{[0;2^2,12^2]}(d/12) + 132h_{[0;13^3]}(d/13) + 6h_{[0;2^2,7^2]}(d/14) + 30h_{[0;7,14^2]}(d/14) + 24h_{[0;5,15]}(d/15) + 16h_{[0;4,16^2]}(d/16) + 12h_{[0;3,18^2]}(d/18) + 8h_{[0;4,5,20]}(d/20) + 12h_{[0;3,7,21]}(d/21) + 8h_{[0;2,24^2]}(d/24) + 12h_{[0;2,13,26]}(d/26)\Big),$$

where $h_O(m)$ is defined in (3.1) and $h_g(m)$ is the number of rooted hypermaps of genus g with m darts.

Table 2. Numbers of rooted and unrooted hypermaps on the sphere with at most 36 darts.

Darts	No. of rooted hypermaps	No. of unrooted hypermaps
01	1	1
02	3	3
03	12	6
04	56	20
05	288	60
06	1584	291
07	9152	1310
08	54912	6975
09	339456	37746
10	2149888	215602
11	13891584	1262874
12	91287552	7611156
13	608583680	46814132
14	4107939840	293447817
15	28030648320	1868710728
16	193100021760	12068905911
17	1341536993280	78913940784
18	9390758952960	521709872895
19	66182491668480	3483289035186
20	469294031831040	23464708686960
21	3346270487838720	159346213738020
22	23981605162844160	1090073011199451
23	172667557172477952	7507285094455566
24	1248519259554840576	52021636161126702
25	9063324995286990848	362532999811480604
26	66032796394233790464	2539722940697502966
27	482722511571640123392	17878611539691757938
28	3539965084858694238208	126427324476844560112
29	26035872237025235042304	897788697828456380772
30	192014557748061108436992	6400485258395785352796
31	1419744002743239710867456	45798193636878700350566
32	10522808490920482562899968	328837765342188339724215
33	78169434503980727610114048	2368770742544870599309164
34	581928012418523194430849024	17115529777022135213432360
35	4340868416959794639538225152	124024811913136989701130840
36	32442279747804780990233051136	901174437439071256974607848

Table 3. Numbers of rooted and unrooted hypermaps on the torus with at most 36 darts.

Darts	No. of rooted hypermaps	No. of unrooted hypermaps
03	1	1
04	15	6
05	165	33
06	1611	285
07	14805	2115
08	131307	16533
09	1138261	126501
10	9713835	972441
11	81968469	7451679
12	685888171	57167260
13	5702382933	438644841
14	47168678571	3369276867
15	388580070741	25905339483
16	3190523226795	199408447446
17	26124382262613	1536728368389
18	213415462218411	11856420991413
19	1740019150443861	91579955286519
20	14162920013474475	708146055343668
21	115112250539595093	5481535740059577
22	934419385591442091	42473608898628639
23	7576722323539318101	329422709719100787
24	61375749135369153195	2557322884534185500
25	496747833856061953365	19869913354242478293
26	4017349254284543961771	154513432889706455145
27	32467023775647069984085	1202482362061007078175
28	262225359776626483309227	9365191420865873023026
29	2116714406654571321840981	72990151953605907649689
30	17077642118698511054318251	569254737292213025378571
31	137718253327424350825305429	4442524300884656478235659
32	1110121628423796225561242283	34691300888262396351206916
33	8945004369725873610785379669	271060738476541624829912533
34	72050204862659963828300327595	2119123672431330647024502021
35	580158674937809688551201527125	16575962141080276815748625439
36	4670100332161384829372940855979	129725009226706415775520829736

Table 4. Numbers of rooted and unrooted hypermaps of genus two with at most 36 darts.

Darts	No. of rooted hypermaps	No. of unrooted hypermaps
05	8	4
06	252	48
07	4956	708
08	77992	9807
09	1074564	119436
10	13545216	1355400
11	160174960	14561360
12	1805010948	150429819
13	19588944336	1506841872
14	206254571236	14732613116
15	2118399516180	141226638540
16	21310566266640	1331912032173
17	210636265153004	12390368538412
18	2050696768165560	113927616087252
19	19704531058696008	1037080582036632
20	187168609978022860	9358430685657218
21	1759888050471704664	83804192879934456
22	16398685297890141180	745394788170961932
23	151570887948878270348	6590038606472968276
24	1390769475046930549944	57948728145925503486
25	12677318864153808488340	507092754566152344372
26	114866214763196961698608	4417931337231617942004
27	1035084863819168419185504	38336476437747003381792
28	9280607880474962296968276	331450281447322431858738
29	82826281765786596848797216	2856078681578848167199904
30	736047153365477687155010772	24534905112199593482491548
31	6515177130047600059726385988	210167004195083872894399548
32	57458287482787782037108848928	1795571483837278068662501714
33	505004316878986204007116435068	15303161117545036486870040316
34	4424381841184163772620281181544	130128877681888658319586285764
35	38646910849128504231568725072824	1104197452832242978044820979944
36	336637773892407350719198338194844	9351049274789106814328551437162

The first 12 members of each sequence were computed by Walsh, see the sequences A214817 and A214819 in Sloan's Encyclopedia of Integer Sequences [25].

Table 5. Numbers of rooted and unrooted hypermaps of genus three with at most 36 darts.

Darts	No. of rooted hypermaps	No. of unrooted hypermaps
07	180	30
08	9132	1155
09	268980	29910
10	6010220	601364
11	112868844	10260804
12	1877530740	156469887
13	28540603884	2195431068
14	404562365316	28897471080
15	5422718644920	361514582340
16	69428442576136	4339280187364
17	855504181649448	50323775391144
18	10204459810035768	566914469842923
19	118364711625485256	6229721664499224
20	1340006035830921720	67000302262906866
21	14850353930248138104	707159710965012834
22	161502853638370415864	7341038807584085816
23	1727146533728893094604	75093327553430134548
24	18194375590933862966292	758098983024722532057
25	189080264025911947923500	7563210561036477916940
26	1940922056061010034996724	74650848310828035397344
27	19701557064420962393581236	729687298682257951832052
28	197942906403556061566996716	7069389514421460285584196
29	1970114245638125530899290580	67934973987521570031010020
30	19439135973954567991969413660	647971199131913428836824787
31	190275115717451197782353154992	6137906958627457992979134032
32	1848675554724680793176038604496	57771111085147274672337156264
33	17837763912982477086251258735424	540538300393408396560945218358
34	171011492126067571593754049882912	5029749768413762495690182157138
35	1629668927538321804193022741828400	46561969358237765834086364060880
36	15442910125865479229529011667731664	428969725718485640429202027454929

Compare the numbers in Table 5 with the sequences A214818 and A214820 in the Encyclopedia of Integer Sequences [25]

Table 6. Numbers of rooted and unrooted hypermaps of genus four with at most 36 darts.

Darts	No. of rooted hypermaps	No. of unrooted hypermaps
09	8064	900
10	579744	58032
11	23235300	2112300
12	684173164	57017238
13	16497874380	1269067260
14	344901105444	24635879496
15	6471056247920	431403755052
16	111480953909328	6967561712925
17	1792031518697232	105413618746896
18	27197316623478960	1510962076238986
19	393207192141924744	20695115375890776
20	5453210050430783640	272660503240047690
21	72949244341257096792	3473773540061130158
22	945523594111460363208	42978345198144175632
23	11918067649004916470640	518176854304561585680
24	146538779626167833263888	6105782484587260861256
25	1762112462707129510538640	70484498508285180442512
26	20768368282870029687839376	798783395497239872773008
27	240368024958405223433064588	8902519442903897358900492
28	2736299821653534456272141028	97724993630512562418847782
29	30681668858759894127714525252	1057988581336548073369466388
30	339282919442101898443749216780	11309430648070428892839507568
31	3704145932011843576043417342880	119488578451994954065916688480
32	39964865942865385063297950889824	1248902060714547710624818909977
33	426488837595688137917785681779808	12923904169566307209653526582082
34	4505169994980190400756661701929056	132504999852358593288691501546752
35	47140026269255416100164550166394896	1346857893407297602861844292181680
36	488894077989077432470427208027444912	13580391055252151499726431398068094

Table 7. Numbers of rooted and unrooted hypermaps of genus five with at most 36 darts.

Darts	No. of rooted hypermaps	No. of unrooted hypermaps
11	604800	54990
12	57170880	4764654
13	2936606400	225892800
14	108502598960	7750214770
15	3225186125460	215012412162
16	81861294718764	5116332159396
17	1840409325096500	108259372064500
18	37558997857897164	2086611028856442
19	708015469597497732	37263972084078828
20	12488421105878928700	624421056158964400
21	208161512148250424484	9912452959442487798
22	3304395638081490531324	150199801748445171324
23	50267199680265668419244	2185530420881116018228
24	736516493829967530909204	30688187243229908347917
25	10437808798822929984593100	417512351952917199390324
26	143579847174876616432522932	5522301814423518341906738
27	1922778363105897685775636508	71214013448366581415749056
28	25131774144239809681153633380	897563362294358670413127978
29	321313741483354251493720181436	11079784189081181085990351084
30	4026024070504885445987516470740	134200802350164006593186711922
31	49520952483083613251458914166776	1597450080099471395208352069896
32	598831540994207081864686094849544	18713485656068987355673572454932
33	7128322474534319390866713604688760	216009771955585436086944711426036
34	83625433414413168743681427153055368	2459571571012152235464897610954668
35	967844566522922616486129256587080600	27652701900654931899603693046233460
36	11060906643174942949121784485538024168	307247406754859529111624332164273023

The above tables were computed using MATHEMATICA, Ver. 8. The input numbers of rooted maps come from [2].

Table 8. Numbers of rooted and unrooted hypermaps of genus six with at most 36 darts.

Darts	No. of rooted hypermaps	No. of unrooted hypermaps
13	68428800	5263764
14	8099018496	578503836
15	511859777472	34123986582
16	22925949056640	1432872113513
17	815521082030784	47971828354752
18	24494440792190400	1360802282552400
19	645212095792089220	33958531357478380
20	15292175926873102956	764608796937519942
21	332150183310464271324	15816675395738446494
22	6702637985834037183508	304665363009300666760
23	126995200843857803023176	5521530471472078392312
24	2278149500006567629947864	94922895834007468383231
25	38954050134978747926573016	1558162005399149917112472
26	638403304977613386193366152	24553973268378095302926108
27	10074031934071102231202906148	373112293854485268614197848
28	153658174505132363683454644044	5487791946612027408828957093
29	2272899190645387594635333126300	78375834160185779125356314700
30	32696626257089371291804270484436	1089887541902981774320298641500
31	458548507259290795212121173292320	14791887330944864361681328170720
32	6282789494351752733963682019756896	196337171698492317310913800795248
33	84259058847630667707075246329668128	2553304813564565688093367462644732
34	1107884001406279366551657449722624608	32584823570772923227586875738243520
35	14302816838086914619821081571229279928	408651909659626131994888044899128296
36	181537182685287061468632215656362324072	5042699519035751717459257549806279480

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