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CALCULATION OF BELYĬ FUNCTIONS FOR TREES WITH WEIGHTED EDGES

Abstract. The paper presents a technique for the automatic calculation of Belyĭ functions for trees with weighted edges.

§1. Introduction

Let $P(z)$ and $Q(z)$ be polynomials with complex coefficients; the ratio $B(z) = P(z)/Q(z)$ is an example of a *Belyĭ function* provided that

$$B'(z) = 0 \Rightarrow B(z) = 0 \vee B(z) = 1. \tag{1}$$

An introduction to the general theory of Belyĭ functions can be found in many sources (for example, in [6]); only a few facts required to understand this paper are reproduced here.

The preimage $B^{-1}([0,1])$ is a *plane graph*, possibly with multiple edges, called by A. Grothendieck a *dessin d'enfant*. The vertices of this graph are the preimages of the *critical values* 0 and 1; we shall assume that the vertices are colored white and black respectively. Values of z at which $B'(z)$ vanishes, and also multiple poles of $B(z)$, are called *critical points*.

The complement of $B^{-1}([0,1])$ consists of connected open regions called *faces*: there is one unbounded, *outer* face, and other bounded, *inner* faces, and each of the latter contains a zero of the polynomial $Q(z)$. Thus $B(z)$ defines a *planar map*.

According to the general theory, every connected plane graph is isomorphic to some graph obtained in this way from a suitable Belyĭ function; however actually finding this function turns out to be a difficult computational problem.

Paper [14] presents a fairly full survey of diverse methods used for such calculations (up to 2014; some more recent publications are [1, 4]), and it also classifies them. In particular, what the authors of [11] call *inductive complex analytic methods* are supposed to construct (numerical approximations to) a Belyĭ function for a given graph from Belyĭ functions for simpler graphs. Such simpler graphs can be obtained in several ways. In

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[2, 5, 3] a vertex in a tree is split into two vertices of smaller multiplicities. In [20], on the other hand, an edge of a tree is contracted, producing a vertex with larger multiplicity but reducing the number of vertices and edges. Having mentioned these two approaches, applicable only to trees without multiple edges, the authors of [14] write:

> **Question 3.6.** – *Can an inductive complex analytic method be employed to compute more complicated Belyĭ maps in practice?*
>
> In particular, the iterative method by Couveignes and Granboulan to find a good starting value seems to rely on intuition involving visual considerations; can these be made algorithmically precise?

It should be remarked that in [20] no human assistance is required: a correct starting approximation is constructed entirely by computer (see [10]).

In this paper we give an affirmative answer to the above question by presenting a fully automated technique for finding (arbitrary accurate approximations to) Belyĭ functions for maps in which every inner face is bounded by just two parallel edges; such maps can be described as *trees with weighted edges*, an edge of weight w representing w parallel edges. The interest in weighted trees is partially due to the relationship between their Belyĭ functions and the problem of finding polynomials $S(z)$ and $T(z)$ with prescribed multiplicities of zeroes for which the difference $S(z) - T(z)$ has the minimal possible degree (see, e.g., [12]).

The input to the proposed algorithm consists of a combinatorial description of a weighted tree; no additional information/assumptions, such as the size of the *orbit*, the *field of definition*, or the *monodromy group* of the corresponding Belyĭ function (coinciding with the *edge rotation group* of the tree) is required. The output consists of (approximate) positions of black and white vertices, and the accuracy can be made arbitrarily high. In the case when the field of definition has sufficiently small degree, this allows one, using the LLL algorithm [7] (or similar techniques), to find the field and an exact Belyĭ function with algebraic coefficients (see the examples in Appendices I and II and in [9, 10]).

§2. Informal description of the method

In the new method the simpler graphs are obtained by cutting a certain edge. In order to be able to do this, we consider a slightly different class of plane graphs. Namely, it is easy to verify that if $B(z)$ satisfies (1), then

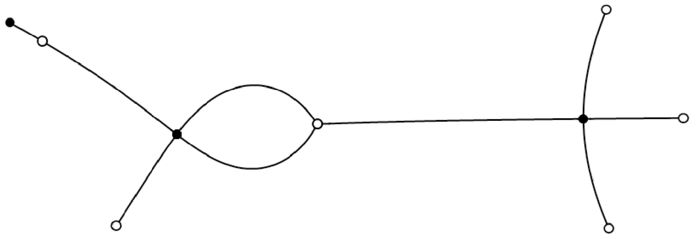

Fig. 1. Tree G with an edge of weight 2.

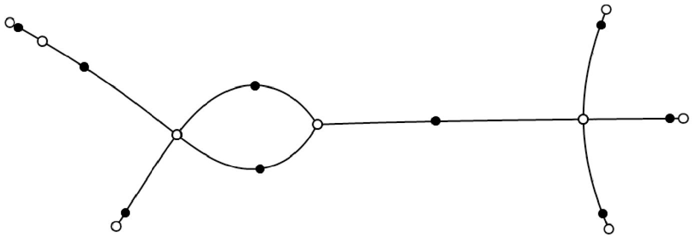

Fig. 2. Graph $\widetilde{G}$ resulting from G by subdividing edges.

$\widetilde{B}(z) = 4B(z) - 4B^2(z)$ is also a Belyĭ function. The plane graph $\widetilde{G}$ corresponding to $\widetilde{B}(z)$ can be obtained from the plane graph G corresponding to $B(z)$ by subdividing each edge of the latter graph into two edges by inserting into it a new vertex (see an example in Figs 1–2). Moreover, white and black vertices of G become white vertices of the graph $\widetilde{G}$, *keeping their coordinates*. Thus, having constructed a Belyĭ function for $\widetilde{G}$, we can then easily find the required function $B(z)$.

We can select any edge of G having some weight w and cut each of the w black vertices of $\widetilde{G}$ originating from the selected edge of G into two black vertices of degree 1. The graph $\widetilde{G}$ splits into two subgraphs $\widetilde{G}_{\mathrm{L}}$ and $\widetilde{G}_{\mathrm{R}}$ (see Fig. 3; there, and also in Figs. 1–2, the graphs are shown in their "true" geometrical forms arising from the corresponding Belyĭ functions, though in fact such a splitting is performed on a purely graph-theoretical level). For them we iteratively find the corresponding Belyĭ functions $B_{\mathrm{L}}(z)$ and $B_{\mathrm{R}}(z)$ and then on their basis we construct a Belyĭ function for $\widetilde{G}$.

To be able to do this we use a precomputed catalog of Belyĭ functions

$$\mathbf{B}_{m,w,n}(z) = (z - \mathbf{z}_{\mathrm{L}})^m (z - \mathbf{z}_{\mathrm{R}})^n / \mathbf{Q}^2_{m,w,n}(z) \qquad (2)$$

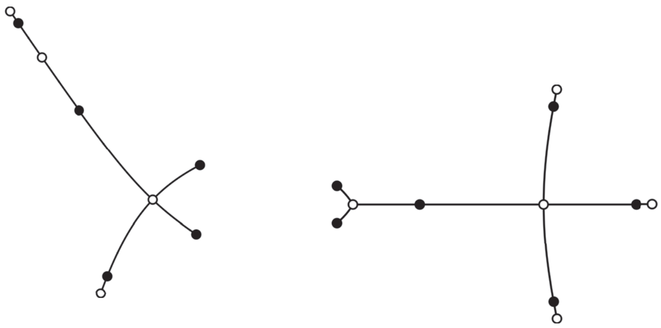

Fig. 3. The result of cutting the graph $\widetilde{G}$ at vertices inserted into the edge of weight 2 of the tree G.

for *canonical graphs* $C_{m,w,n}$. Each of these graphs has two white vertices of degrees m and n respectively, w black vertices of degree 2 each connected to both the white vertices, and also $m - 2w + n$ black vertices of degree 1 all lying on the boundary of the outer face (Fig. 4 depicts the graph $C_{8,2,10}$).

Let $B_{\mathrm{L}}(z) = F_{\mathrm{L}} P_{\mathrm{L}}(z)/Q_{\mathrm{L}}(z)$ and $B_{\mathrm{R}}(z) = F_{\mathrm{R}} P_{\mathrm{R}}(z)/Q_{\mathrm{L}}(z)$ where F_{L} and F_{R} are some numeric factors and $P_{\mathrm{L}}(z)$, $Q_{\mathrm{L}}(z)$, $P_{\mathrm{R}}(z)$, and $Q_{\mathrm{L}}(z)$ are polynomials with leading coefficients 1 and of degrees p_{L}, q_{L}, p_{R}, q_{R} respectively. We shall use $\mathbf{B}_{m,w,n}(z)$ with $m = p_{\mathrm{L}} - q_{\mathrm{L}}$, $n = p_{\mathrm{R}} - q_{\mathrm{R}}$.

Combining B_{L}, B_{R}, and $\mathbf{B}_{m,w,n}(z)$ is based on considering the preimages $B_{\mathrm{L}}^{-1}([0,\infty])$, $B_{\mathrm{R}}^{-1}([0,\infty])$ and $\mathbf{B}_{m,w,n}^{-1}([0,\infty])$. Visually, *inside* a sufficiently small disks centered at a white vertex of degree m the preimage

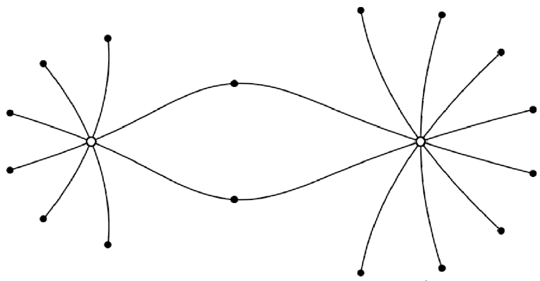

Fig. 4. Graph $C_{8,2,10}$ as the preimage $\mathbf{B}_{8,2,10}^{-1}([0,1])$.

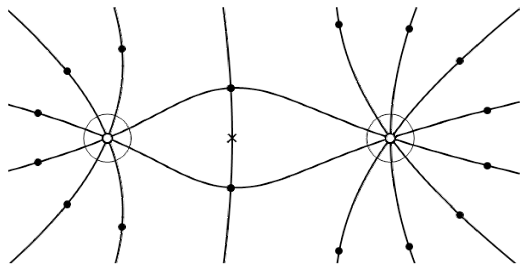

Fig. 5. Preimage $\mathbf{B}_{8,2,10}^{-1}([0,\infty])$, the small cross marks the position of the pole.

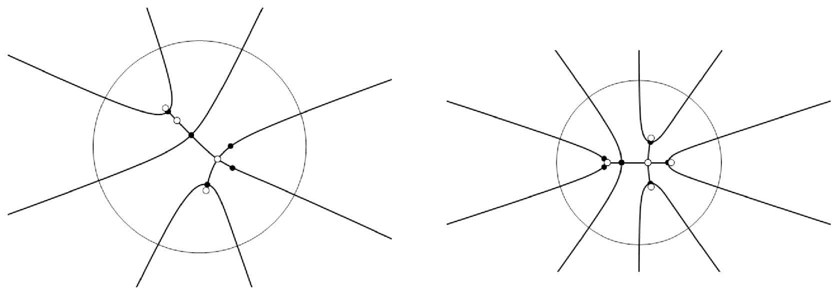

Fig. 6. Preimages $B_{\mathrm{L}}^{-1}([0,\infty])$ and $B_{\mathrm{R}}^{-1}([0,\infty])$.

$\mathbf{B}_{m,w,n}^{-1}([0,\infty])$ looks like m almost straight lines forming angles of size $2\pi/m$ (see an example in Fig. 5). On the other hand, *outside* a sufficiently large disk centered at the origin the preimage $B_{\mathrm{L}}^{-1}([0,\infty])$ looks like m almost straight lines with the same angles of size $2\pi/m$ between them (see Fig. 6). Geometrically, we cut the large disk from the preimage $B_{\mathrm{L}}^{-1}([0,\infty])$ and rotate it (if required), then properly scale and substitute the scaled disk for the corresponding small disk in the preimage $\mathbf{B}_{m,w,n}^{-1}([0,\infty])$. Naturally, a similar operation is performed for $B_{\mathrm{R}}^{-1}([0,\infty])$ and the other small disk in the preimage $\mathbf{B}_{m,w,n}^{-1}([0,\infty])$.

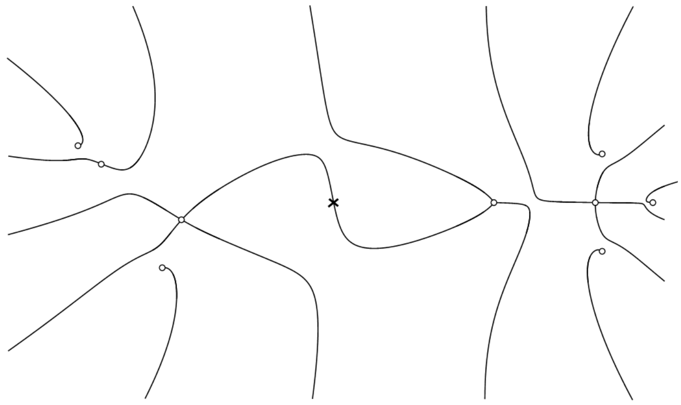

Fig. 7. Preimage $\breve{A}^{-1}([0,\infty])$.

Symbolically, we consider the function

$$\breve{A}(z) = \frac{M_{\mathrm{L}}^{q_{\mathrm{L}}-p_{\mathrm{L}}} M_{\mathrm{R}}^{q_{\mathrm{R}}-p_{\mathrm{R}}} P_{\mathrm{L}}(M_{\mathrm{L}} \times (z - \mathbf{z}_{\mathrm{L}})) P_{\mathrm{R}}(M_{\mathrm{R}} \times (z - \mathbf{z}_{\mathrm{R}}))}{Q_{\mathrm{L}}(M_{\mathrm{L}} \times (z - \mathbf{z}_{\mathrm{L}})) Q_{\mathrm{R}}(M_{\mathrm{R}} \times (z - \mathbf{z}_{\mathrm{R}})) \mathbf{Q}^2_{m,w,n}(z)} \tag{3}$$

where M_{L} and M_{R} are sufficiently large numerical factors. Fig. 7 shows an example of such a preimage $\breve{A}^{-1}([0,\infty])$ (for small values of M_{L} and M_{R} – otherwise the details would be invisible due to the scaling). The two white vertices of $C_{m,w,n}$ "split" into the shifted white vertices of $\widetilde{G}_{\mathrm{L}}$ and $\widetilde{G}_{\mathrm{R}}$ which kept their multiplicities; the pole of $\mathbf{B}^{-1}_{m,w,n}(z)$ also kept its multiplicity and position. As for the black vertices of $\widetilde{G}_{\mathrm{L}}$, $\widetilde{G}_{\mathrm{R}}$, and $C_{m,w,n}$ (those that corresponded to the critical points of second order), they were destroyed. However, selecting larger values for M_{L} and M_{R} we can make preimage $\breve{A}^{-1}([0,\infty])$ globally looking arbitraly like $\mathbf{B}^{-1}_{m,w,n}([0,\infty])$ and locally, in the vicinities of points $\mathbf{z}_{\mathrm{R}}$ and $\mathbf{z}_{\mathrm{R}}$, looking arbitraly like $B_{\mathrm{L}}^{-1}([0,\infty])$ and $B_{\mathrm{R}}^{-1}([0,\infty])$ respectively. This allows us to restore the destroyed critical points of second order by applying Newton iterations; let $A(z)$ denote the result of such "adjusting" of function $\breve{A}(z)$.

Topologically, the preimage $A^{-1}([0,\infty])$ has the same structure as the preimage $B^{-1}([0,\infty])$ where $B(z)$ is the desired Belyĭ function for $\widetilde{G}$. However, there is a great distinction between $A(z)$ and $B(z)$, namely, the critical values of $A(z)$ at the restored critical points differ from 1. Due to the scaling and extra factors in (3) these values are close to

$$\frac{M_{\mathrm{L}}^{q_{\mathrm{L}}-p_{\mathrm{L}}} M_{\mathrm{R}}^{q_{\mathrm{R}}-p_{\mathrm{R}}} P_{\mathrm{R}}(M_{\mathrm{R}} \times (\mathbf{z}_{\mathrm{L}} - \mathbf{z}_{\mathrm{R}}))}{F_{\mathrm{L}} Q_{\mathrm{R}}(M_{\mathrm{R}} \times (\mathbf{z}_{\mathrm{L}} - \mathbf{z}_{\mathrm{R}}))\mathbf{Q}^2_{m,w,n}(\mathbf{z}_{\mathrm{L}})} \tag{4}$$

(at critical points near $\mathbf{z}_{\mathrm{L}}$) or to

$$\frac{M_{\mathrm{R}}^{q_{\mathrm{R}}-p_{\mathrm{R}}} M_{\mathrm{L}}^{q_{\mathrm{L}}-p_{\mathrm{L}}} P_{\mathrm{L}}(M_{\mathrm{L}} \times (\mathbf{z}_{\mathrm{R}} - \mathbf{z}_{\mathrm{L}}))}{F_{\mathrm{R}} Q_{\mathrm{L}}(M_{\mathrm{L}} \times (\mathbf{z}_{\mathrm{R}} - \mathbf{z}_{\mathrm{L}}))\mathbf{Q}^2_{m,w,n}(\mathbf{z}_{\mathrm{R}})} \tag{5}$$

(at critical points near $\mathbf{z}_{\mathrm{R}}$). Selecting correlated values of M_{L} and M_{R} we can make both (4) and (5) equal to a certain (small positive) number c_0. Let $A_{c_0}(z)$ be the corresponding "adjusted" function. It satisfies the following weaker counterpart of (1):

$$A'_c(z) = 0 \Rightarrow A_c(z) = 0 \vee A_c(z) = 1 \vee A_c(z) = c \tag{6}$$

for $c = c_0$.

The function $A_{c_0}(z)$ can be taken as an initial point for constructing $B(z)$. Namely, treating c as a parameter, we start incrementing its value and finding the corresponding function $A_c(z)$ satisfying (6). The desired function $B(z)$ will be just $A_1(z)$.

§3. Some technical details

The edges of the desired weighted tree are numbered in an arbitrary way, and the input to the algorithm consists of two lists:

- edges listed around the outer face (each edge is encountered twice);
- a list of the weights of the edges.

Any edge can be used for splitting the graph $\widetilde{G}$, but it is reasonable to have the two resulting parts with approximately the same number of edges.

The functions $A_c(z)$ are represented by the following information:

- zeroes $z_1(c), \dots, z_f(c)$;
- multiplicities of these zeroes $d_1, \dots, d_f$;
- critical points $u_1(c), \dots, u_g(c)$ such that $A_c(u_k(c)) = 1$, $k = 1, \dots, g$;
- critical points $v_1(c), \dots, v_h(c)$ such that $A_c(v_k(c)) = c$, $k = 1, \dots, h$;
- poles $y_1(c), \dots, y_{w-1}(c)$;

- a constant factor F.

In other words, the following equations should hold:

$$A_c(z) = \frac{F\prod_{k=1}^{f}(z - z_k(c))^{d_k}}{(\prod_{k=1}^{w-1}(z - y_k(c)))^2}, \tag{7}$$

$$A_c(u_k(c)) = 1, \quad k = 1, \ldots, g, \tag{8}$$

$$A_c'(u_k(c)) = 0, \quad k = 1, \ldots, g, \tag{9}$$

$$A_c(v_k(c)) = c, \quad k = 1, \ldots, h, \tag{10}$$

$$A_c'(v_k(c)) = 0, \quad k = 1, \ldots, h. \tag{11}$$

We treat (7) as a definition of $A_c(z)$ and (8)–(11) as a system of $2g + 2h$ equations among $f + g + h + w - 1$ numbers, the z's, u's, v's, and y's. According to Euler's theorem on plane graphs $f+g+h+w-1 = 2g+2h+1$, so we are free to incorporate one more equation; we will demand that the centre of gravity of the white vertices should lie at the origin:

$$d_1w_1(c) + \cdots + d_fw_f(c) = 0. \tag{12}$$

Treating c as an independent variable, we can differentiate (8)–(12) by it:

$$\frac{\mathrm{d}}{\mathrm{d}c}A_c(u_k(c)) = 0, \qquad k = 1, \ldots, g, \tag{13}$$

$$\frac{\mathrm{d}}{\mathrm{d}c}A_c'(u_k(c)) = 0, \qquad k = 1, \ldots, g, \tag{14}$$

$$\frac{\mathrm{d}}{\mathrm{d}c}A_c(v_k(c)) = 0, \qquad k = 1, \ldots, h, \tag{15}$$

$$\frac{\mathrm{d}}{\mathrm{d}c}A_c'(v_k(c)) = 0, \qquad k = 1, \ldots, h, \tag{16}$$

$$d_1\frac{\mathrm{d}}{\mathrm{d}c}w_1(c) + \cdots + d_f\frac{\mathrm{d}}{\mathrm{d}c}w_f(c) = 0. \tag{17}$$

We need to solve system (13)–(17) of differential equations for c in $[c_0, 1]$; initial values $z_1(c_0), \ldots, z_f(c_0)$, $y_1(c_0), \ldots, y_{w-1}(c_0)$ are taken from the function A_{c_0}, and initial values $u_1(c_0), \ldots, u_g(c_0)$, $v_1(c_0), \ldots, v_h(c_0)$ can be found from equations (8)–(10).

In principle, system (13)–(17) could be solved by many programs for the boundary problem of ordinary differential equations. However, the presence of algebraic equations (8)–(12) gives a supplementary tool f or checking and improving the accuracy of the solution.

Suppose that we have found solutions of (13)–(17) for an increasing sequence of values of c: $c_1, \dots, c_l$. Then we can construct polynomials $Z_1(c), \dots, Z_f(c)$, $U_1(c), \dots, U_g(c)$, $V_1(c), \dots, V_h(c)$, $Y_1(c), \dots, Y_{w-1}(c)$ either having degrees $k-1$ and interpolating values of $z_1(c), \dots, z_f(c)$, $u_1(c), \dots, u_g(c), v_1(c), \dots, v_h(c), y_1(c), \dots, y_{w-1}(c)$ found for $c = c_1, \dots, c_l$, or having smaller degrees and best fitting these values. After that we can extrapolate our solution to $c_k + \Delta$. The choice of Δ is subject to two conditions:

(i) neither the zeroes, nor the critical points, nor the poles can move too far;

(ii) the extrapolated values should satisfy (8)–(12) sufficiently well.

More formally, for (i) the following conditions should be met for a certain *security parameter* α:

$$|Z_k(c_l+\Delta) - Z_k(c_l)| \leq \alpha \min_{j \neq k} \{|Z_k(c_l) - Z_j(c_l)|\}, \tag{18}$$

$$|U_k(c_l+\Delta) - U_k(c_l)| \leq \alpha \min_{j \neq k} \{|U_k(c_l) - U_j(c_l)|\}, \tag{19}$$

$$|V_k(c_l+\Delta) - V_k(c_l)| \leq \alpha \min_{j \neq k} \{|V_k(c_l) - V_j(c_l)|\}, \tag{20}$$

$$|Y_k(c_l+\Delta) - Y_k(c_l)| \leq \alpha \min_{j \neq k} \{|Y_k(c_l) - Y_j(c_l)|\}. \tag{21}$$

Similarly, for (ii) the distances to the nearest pole, critical point or zero should be compared (in general, for a different security parameter β) with the discrepancies of the corresponding solutions of equations $A_{c_k+\Delta}(z) = 1$, $A_{c_k+\Delta}(z) = c$, and $A'_{c_k+\Delta}(z) = 0$. The smaller are the values of α and β, the less are the chances of losing the desired structure of the preimage $A^{-1}_{c+\Delta}([0,\infty])$ by jumping to another tree; however, smaller values of α and β also imply a smaller value of Δ and hence a larger number of steps for reaching $c = 1$.

Having found (as large as possible) an admissible value of Δ we can, by Newton's method, improve the extrapolated solution of system (8)–(11) for $c = c_l + \Delta$ to any desired precision; this is important for the accuracy of the forthcoming extrapolations.

A catalog of Belyĭ functions (2) for canonical graphs was precomputed in the following way. Paper [13] explicitly describes, among other things, Belyĭ functions for trees having $m-w$ and $n-w$ edges of weight 1 adjacent to the ends of an edge of weight w. Such a Belyĭ function can be constructed by the method described above from $\mathbf{B}_{m,w,n}(z)$. But it is possible to reverse the order of things by considering the boundary problem for (13)–(17) with the initial values of the unknown functions for $c = 1$.

Acknowledgments

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Appendix I. Examples of calculated Belyĭ functions

A large number of weighted trees are considered in [19]. For many of them the authors indicate *fields of definition* of their Belyĭ functions, and corresponding *monodromy groups*. For other trees these data are missing in [19] but they were partly found later by other authors. In particular, according to [15] H. Monien did it for the two trees from orbit 24.1 (here and below we use the numbering of orbits introduced in [19]) and J. Voight did it for the 10 trees from orbit 24.1.

The techniques described in the present paper were used to fill in some other gaps in [19]. In particular, the authors of that paper write:

> We believe that the orbit 12.8 is defined over $\mathbb{Q}(\sqrt{-11})$ and the orbit 12.3 is defined over a cubic extension of $\mathbb{Q}(\sqrt{-11})$. ... We believe that this orbit [12.9] is defined over a quadratic extension of $\mathbb{Q}(\sqrt{-11})$.

Calculations performed by the author confirmed these conjectures.

Fig. 8 exhibits "true" geometric form of 3 (of a total of 6) trees from orbit 12.3 (the other three are their mirror images). These trees have 10 edges of weight 1 and one edge of weight 2. The cubic extension predicted in [19] can be defined by the polynomial $z^3 + (1 + \sqrt{-11})z^2 + 8$, and the coefficients of the numerator and the denominator of corresponding Belyĭ functions are explicitly given in [9]. According to [19], the monodromy group of these trees, which acts on 12 edges of the trees, is isomorphic to M_{11}. Thus, in this case M_{11} acts not by its natural action on 11 points but by its primitive action on 12 points (for example, on 12 cosets of its subgroup $\mathrm{PSL}(2, 11)$).

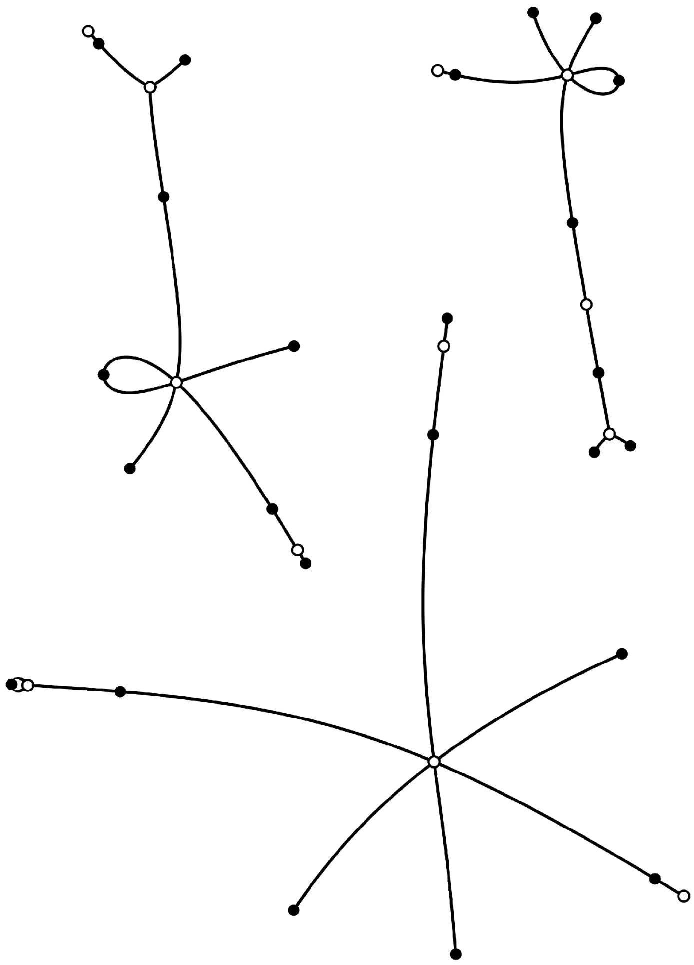

Fig. 8. Weighted trees from orbit 12.3 of [19].

Fig. 9 exhibits the "true" geometric form of one of the two trees from the orbit 12.8 (the other tree is just the mirror image). The coefficients (belonging, as predicted in [19], to $\mathbb{Q}(\sqrt{-11})$) of the numerator and the denominator of a coressponding Belyĭ function can be found [9]. According to [19], the monodromy group of this function is the Mathieu group M_{12}.

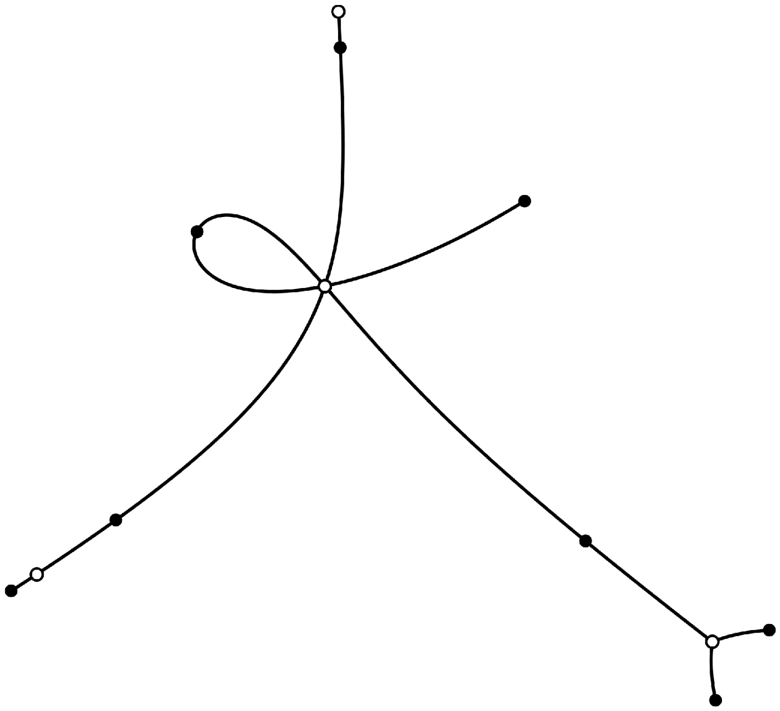

Fig. 9. Weighted tree from orbit 12.8 of [19].

Fig. 10 exhibits the "true" geometric form of two (out of 4) trees from the orbit 12.9 (the other two are just their mirror images). The four corresponding Belyĭ functions can be defined as follows (see also [9]):

$$B(z) = N(z)/D(z), \tag{22}$$

where

$$N(z) = \sum_{k=0}^{12} P_k(\alpha) z^k, \qquad D(z) = z + 737, \tag{23}$$

$$\begin{aligned} P_0(\alpha) = {} & -1757429320513160002684949258191137 \\ & +8224680925652794493000295154114112\alpha \\ & -472843059717435088551180208584324\alpha^2 \end{aligned}$$

$$
\begin{aligned}
&\qquad\qquad +2102628674271161395154494882 90240\alpha^3,\\
P_1(\alpha) &= -25514030092120654387882863271 7736\\
&\qquad\qquad +302579692841461836412742626764096\alpha\\
&\qquad\qquad -21712651518795608710598573445624\alpha^2\\
&\qquad\qquad +7478122967938442476339415883120\alpha^3,\\
P_2(\alpha) &= -27209388079446547124727590193 18\\
&\qquad\qquad +2246594943216494692191618673920\alpha\\
&\qquad\qquad -181004968056890191444460294724\alpha^2\\
&\qquad\qquad +54475523354906984166717565872\alpha^3,\\
P_3(\alpha) &= -79603032247164867829774985 28\\
&\qquad\qquad +3156321302003169859715782912\alpha\\
&\qquad\qquad -359787190968054909915940128\alpha^2\\
&\qquad\qquad +7174725802417664895610976 0\alpha^3,\\
P_4(\alpha) &= 2643763408614374367837285\\
&\qquad\qquad -1463690344750320681068659 2\alpha\\
&\qquad\qquad +771290967010943957558904\alpha^2\\
&\qquad\qquad -370493809653334324224960\alpha^3,\\
P_5(\alpha) &= 3218086171834420578120 0 - 375512603120897910370 56\alpha\\
&\qquad\qquad +259015803656393148868 8\alpha^2 - 917691302646153943776\alpha^3,\\
P_6(\alpha) &= 3733064478492711516 - 443174089168478976\alpha\\
&\qquad\qquad +17801784756196056\alpha^2 + 2732572643029152\alpha^3,\\
P_7(\alpha) &= -46059956689934880 + 25405769479875840\alpha\\
&\qquad\qquad -2393404361881632\alpha^2 + 594658818300096\alpha^3,\\
P_8(\alpha) &= -2519416580967 - 18799500924864\alpha\\
&\qquad\qquad +906679294092\alpha^2 - 493342261632\alpha^3,\\
P_9(\alpha) &= -7089311592 + 12635149120\alpha - 908020344\alpha^2 + 324743408\alpha^3,\\
P_{10}(\alpha) &= -34487046 + 33014784\alpha - 2556180\alpha^2 + 821040\alpha^3,\\
P_{11}(\alpha) &= 0,\\
P_{12}(\alpha) &= 3,
\end{aligned}
$$

and for α one should take each of the four numbers $1 \pm 2\sqrt{-2} \pm \sqrt{-11}$. According to [19], in each case the monodromy group of the function $B(z)$ is the Mathieu group M_{12}.

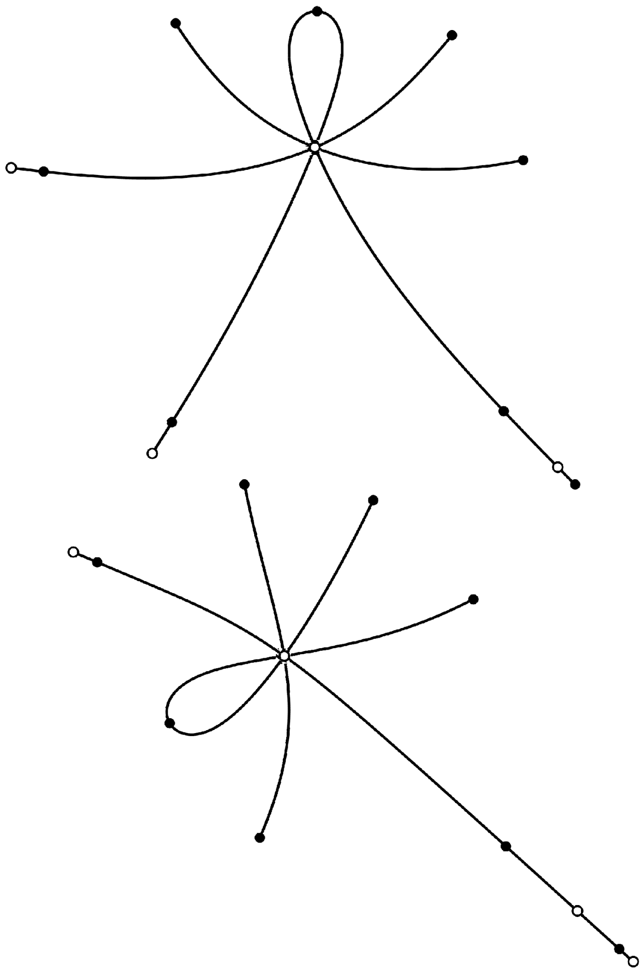

Fig. 10. Weighted trees from orbit 12.9 of [19].

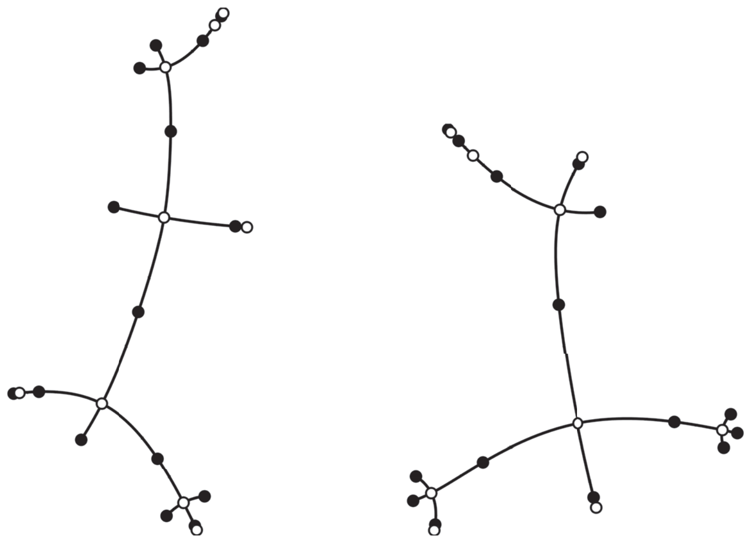

Fig. 11. Trees with Mathieu group M_{23} as their edge rotation groups.

Appendix II. An example of a calculated Shabat polynomial

At the end of 1990s the author wrote a program (outlined in [20] and available (in an improved form) in [10]) for calculating *generalized Chebyshev polynomials* (also known as *Shabat polynomials*) which are just Belyĭ functions for ordinary trees (i.e., those without multiple edges). This program allowed the author to find generalized Chebyshev polynomials for several trees for which it was known (in particular, from [18]) that their fields of definition have small degrees. A polynomial for trees having the Mathieu group M_{11} as their edge rotation group was published in [20]; a polynomial with edge rotation group M_{23} was presented in [8]; however, this passed unnoticed, and recently generalized Chebyshev polynomials for M_{11} and M_{23} were considered anew (see [4], [4, footnote at p. 360], and [12]).

Fig. 11 depicts trees with edge rotation group M_{23}; the coefficients of the corresponding polynomials are available at [8, 10]. Originally, the field of definition was found as a quadric extension of $\mathbb{Q}(\sqrt{-23})$ defined by a polynomial written explicitly, but having very large coefficients; later a simpler definition of the same field, namely $\mathbb{Q}\left(\sqrt{23/2-(5/2)\sqrt{-23}}\right)$, was found by M. A. Vsemirnov (see [6, Example 2.4.10]).

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