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HIGHLY SYMMETRIC MAPS ON SURFACES WITH BOUNDARY

Abstract. The regular maps and the arc-transitive maps on surfaces with non-empty boundary are classified. It is shown that it is unrealistic to expect a similar classification of edge-transitive maps on such surfaces.

§1. Introduction

In topological graph theory, as in most other areas of mathematics, the search for the most symmetric objects is a major activity. The most symmetric maps on surfaces are the regular maps, those for which the automorphism group acts transitively on flags. In recent years, considerable efforts have been devoted to the problem of classifying the regular maps on a given compact surface, in both the orientable and non-orientable cases (see [2, 16, 17] for example). In 1985 Bryant and Singerman [1] laid the foundations of a theory of maps on surfaces with boundary, yet since then little attention seems to have been paid to the regular maps on these surfaces.

The main aim of this paper is to use algebraic map theory to classify such regular maps. Taken together, having a non-empty boundary and being regular are very restrictive conditions: by the first condition, at least one flag must have a vertex, edge or face meeting the boundary, so by the second condition every flag must have this property; from this, fairly easy group theory shows that the automorphism group must be cyclic or dihedral, and the classification follows by case-by-case analysis. These maps are all on the closed disc, so they are quotients of regular maps on the sphere; in the finite case (see Theorem 3.1) there are two mutually dual infinite families and six sporadic examples, and in the infinite case we obtain two examples, one of which is, in fact, a graph embedding rather than a map (see Section 3.5 for this technical distinction). Most of the maps we classify here also appear in the work of Li and Širáň [12] on regular

Key words and phrases: regular map, boundary, arc-transitive, edge-transitive, automorphism group.

maps whose automorphism group does not act faithfully on vertices, edges or faces.

This classification is extended in Section 7 to include regular hypermaps with non-empty boundary, objects which, through results of Köck and Singerman [10], are relevant to the study of algebraic curves defined over real algebraic number fields: all the examples arising are obtained in a simple way from the regular maps classified earlier.

In Theorem 5.1 we classify those maps with non-empty boundary which come close to being regular, in the sense that their automorphism group acts transitively on arcs but not on flags. Similar arguments show that most of these maps are on the closed disc, but there are also examples on the closed annulus, Möbius band and infinite strip.

Given these results, one might hope to obtain a similar classification by weakening the symmetry hypothesis a little further to include edge-transitive maps. However, group theoretic and combinatorial constructions are used in Section 6 to show that in this case the examples arising are so varied and so numerous that it is unrealistic to expect any useful classification.

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§2. Algebraic theory of maps

In this section we will briefly outline the algebraic theory of maps developed in more detail elsewhere (see [1,9], for example).

Each map $\mathcal{M}$ (possibly non-orientable or with non-empty boundary) determines a permutation representations of the group

$$\Gamma = \langle R_0, R_1, R_2 \mid R_i^2 = (R_0R_2)^2 = 1\rangle \cong V_4 * C_2,$$

on the set Φ of flags $\phi = (v, e, f)$ of $\mathcal{M}$, where v, e and f are a mutually incident vertex, edge and face. For each $\phi \in \Phi$ and each $i = 0, 1, 2$, there is at most one flag $\phi' \neq \phi$ with the same j-dimensional components as ϕ for each $j \neq i$ (possibly none if ϕ is a boundary flag). Define r_i to be the permutation of Φ transposing each ϕ with ϕ' if the latter exists, and fixing ϕ otherwise. (See Figs. 2 and 2 for the former and latter cases. In Fig. 2, as in all diagrams, the broken line represents part of the boundary of the map.). Since $r_i^2 = (r_0r_2)^2 = 1$ there is a permutation representation

$$\theta : \Gamma \to G := \langle r_0, r_1, r_2 \rangle \leqslant \mathrm{Sym}\,\Phi$$

of Γ on Φ, given by $R_i \mapsto r_i$.

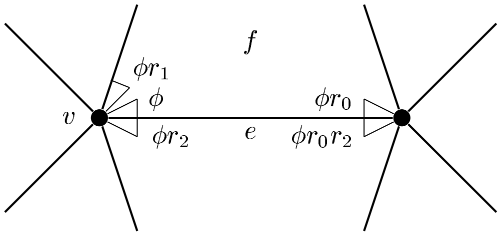

Fig. 1. Generators r_i of G acting on a flag $\phi = (v, e, f)$.

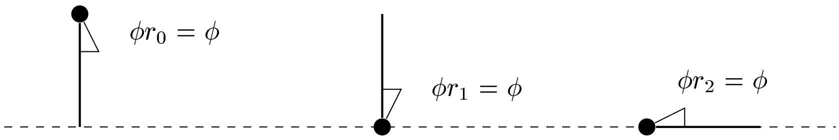

Fig. 2. Flags fixed by r_0, r_1 and r_2.

Conversely, any permutation representation of Γ on a set Φ determines a map $\mathcal{M}$ in which the vertices, edges and faces are identified with the orbits on Φ of the subgroups $\langle R_1, R_2\rangle \cong D_\infty$, $\langle R_0, R_2\rangle \cong V_4$ and $\langle R_0, R_1\rangle \cong D_\infty$, incident when they have non-empty intersection.

The map $\mathcal{M}$ is connected if and only if Γ acts transitively on Φ, as we will always assume. In this case the stabilisers in Γ of flags $\phi \in \Phi$ form a conjugacy class of subgroups $M \leqslant \Gamma$, called *map subgroups*. The map $\mathcal{M}$ is finite (has finitely many flags) if and only if M has finite index in Γ, and it has non-empty boundary if and only if some r_i has fixed points in Φ.

The group G is called the *monodromy group* $\mathrm{Mon}\,\mathcal{M}$ of $\mathcal{M}$. The *automorphism group* $A = \mathrm{Aut}\,\mathcal{M}$ of $\mathcal{M}$ is the centraliser of G in $\mathrm{Sym}\,\Phi$. We have $A \cong N/M$ where $N := N_\Gamma(M)$ is the normaliser of M in Γ. The map

$\mathcal{M}$ is called *regular* if A is transitive on Φ, or equivalently G is a regular permutation group, that is, M is normal in Γ; in this case

$$A \cong G \cong \Gamma/M,$$

and one can identify Φ with G, so that A and G are the left and right regular representations of G on itself. We will then let a, b and c denote the automorphisms of $\mathcal{M}$ corresponding to r_0, r_1 and r_2, respectively changing the vertex, edge or face of a particular flag (or fixing it in the case of a boundary flag), so that A has a presentation of the form

$$A = \langle a, b, c \mid a^2 = b^2 = c^2 = (ac)^2 = 1, \ldots \rangle.$$

The (classical) dual $D(\mathcal{M})$ of $\mathcal{M}$, a map on the same surface formed by transposing the roles of vertices and faces, corresponds to the image of M under the automorphism δ of Γ which fixes R_1, and transposes R_0 and R_2. The Petrie dual $P(\mathcal{M})$ embeds the same graph as $\mathcal{M}$, but the faces are transposed with Petrie polygons, closed zig-zag paths which alternately turn first right and first left at the vertices of $\mathcal{M}$; this operation corresponds to the automorphism π of Γ which transposes R_0 and R_0R_2, and fixes R_1 and R_2. Both of these operations D and P preserve regularity and automorphism groups, but P may change the underlying surface, for example by changing orientability and by eliminating or introducing boundary components.

§3. Regular maps on surfaces with boundary

A map $\mathcal{M}$, corresponding to a map subgroup M of Γ, has non-empty boundary if and only if some r_i $(i = 0, 1, 2)$ has a fixed point on the set Φ of flags of $\mathcal{M}$, or equivalently, some conjugate of R_i lies in M. For a regular map, corresponding to a normal subgroup M of Γ, this is equivalent to M containing R_i, that is, $r_i = 1$. In this case, a, b or c is the identity automorphism, so that the group $A = \operatorname{Aut} \mathcal{M} \cong \Gamma/M$ is cyclic or dihedral, generated by at most two involutions. We will consider the different cases, concentrating first on the finite maps, those for which M has finite index in Γ.

3.1. The case $b = 1$. We first treat the simplest case, when $b = 1$, so that A is a quotient of the group

$$\langle R_0, R_2 \mid R_i^2 = (R_0R_2)^2 = 1 \rangle \cong V_4,$$

with a presentation of the form

$$A = \langle a, c \mid a^2 = c^2 = (ac)^2 = 1, \ldots \rangle.$$

If a, c and 1 are distinct then $A = \langle a, c\rangle \cong V_4$, and one can take $\mathcal{M}$ to be an embedding $\mathcal{B}$ of the complete graph K_2 as a diameter $\overline{\mathbb{D}} \cap \mathbb{R}$ of the closed disc $\overline{\mathbb{D}} = \{z \in \mathbb{C} \mid |z| \leqslant 1\}$. This map $\mathcal{B}$ is shown in Fig. 3.1, with the broken line representing the boundary $\partial\overline{\mathbb{D}} = S^1$ of $\overline{\mathbb{D}}$.

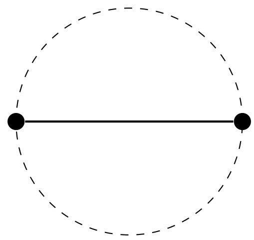

Fig. 3. The map $\mathcal{B}$, with $b = 1$, $A = \langle a, c\rangle \cong V_4$.

If $a = c \neq 1$ then $\mathcal{M}$ is the quotient $\mathcal{B}/\langle ac\rangle$ of $\mathcal{B}$ by a half-turn ac, that is, the embedding of a half-edge in $\overline{\mathbb{D}}$, with the vertex on the boundary and the rest of the half-edge along a radius to the centre (see Fig. 3.1). In this case $A \cong D_1$ (here we use this notation rather than C_2 for a group of order 2, since it is generated by a reflection rather than a half-turn).

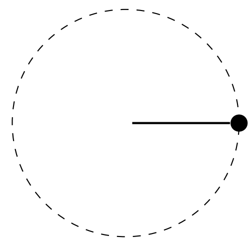

Fig. 4. The map $\mathcal{B}/\langle ac\rangle$, with $b = 1$, $A = \langle a\rangle = \langle c\rangle \cong D_1$.

If $a = 1$ but $c \neq 1$ then $\mathcal{M} = \mathcal{B}/\langle a\rangle$ is an embedding of a single half-edge as a diameter of $\overline{\mathbb{D}}$ (see Fig. 3.1), with $A = \langle c \mid c^2 = 1\rangle \cong D_1$.

If $c = 1$ but $a \neq 1$ then $\mathcal{M} = \mathcal{B}/\langle c\rangle$ is an embedding of K_2 on the boundary of $\overline{\mathbb{D}}$ (see Fig. 3.1), with $A = \langle a \mid a^2 = 1\rangle \cong D_1$.

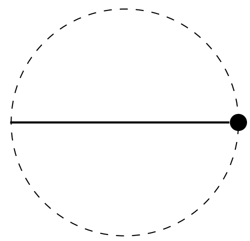

Fig. 5. The map $\mathcal{B}/\langle a\rangle$, with $a = b = 1$, $A = \langle c\rangle \cong D_1$.

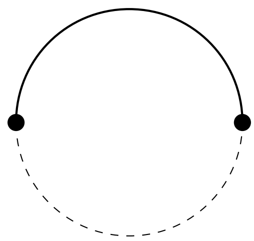

Fig. 6. The map $\mathcal{B}/\langle c\rangle$, with $b = c = 1$, $A = \langle a\rangle \cong D_1$.

If $a = b = c = 1$ then $\mathcal{M}$ is the trivial map $\mathcal{T} = \mathcal{B}/\langle a, c\rangle$ with one flag, an embedding of a half-edge along part of the boundary of $\overline{\mathbb{D}}$ (see Fig. 3.1), with $A \cong C_1$.

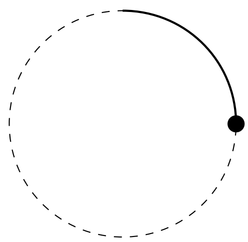

Fig. 7. The map $\mathcal{T}$, with $a = b = c = 1$, $A \cong C_1$.

3.2. The case $a = 1$. Now suppose that $a = 1$, so that A is a finite quotient of the infinite dihedral group

$$\langle R_1, R_2 \mid R_1^2 = R_2^2 = 1\rangle \cong D_\infty.$$

We have already dealt with the cases where $b = 1$, so we may assume that $b \neq 1$. If $c \neq 1$ then

$$A = \langle b, c \mid b^2 = c^2 = (bc)^n = 1\rangle \cong D_n$$

for some $n \in \mathbb{N}$ (the vertex-valency), and $\mathcal{M}$ is an embedding $\mathcal{A}_n$ of a semi-star map in $\overline{\mathbb{D}}$, with a single vertex of valency $n \geqslant 1$ at the centre, and n half-edges along radii to the nth roots of unity on the boundary. This includes the case $n = 1$, where $b = c$. The map $\mathcal{A}_4$ is shown in Fig. 8.

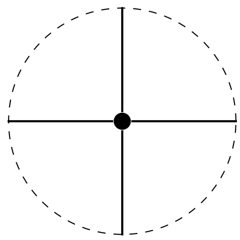

Fig. 8. The map $\mathcal{A}_n$, with $a = 1$, $A = \langle b, c\rangle \cong D_n$, where $n = 4$.

If $c = 1$ but $b \neq 1$ then $\mathcal{M}$ is an embedding $\mathcal{D}$ of a single vertex and two half-edges along part of the boundary of $\overline{\mathbb{D}}$ (see Fig. 9), with $A = \langle b \mid b^2 = 1\rangle \cong D_1$.

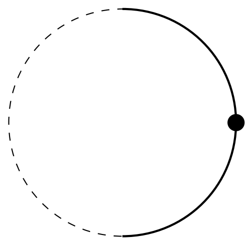

Fig. 9. The map $\mathcal{D}$, with $a = c = 1$, $A = \langle b\rangle \cong D_1$.

3.3. The case $c = 1$. When $c = 1$ we obtain the duals $D(\mathcal{M})$ of the maps $\mathcal{M}$ arising when $a = 1$. If $a, b \neq 1$ then

$$A = \langle a, b \mid a^2 = b^2 = (ab)^n = 1\rangle \cong D_n$$

for some $n \in \mathbb{N}$ (the face-valency), and $\mathcal{M}$ is an embedding $\mathcal{C}_n = D(\mathcal{A}_n)$ of a circuit of n vertices and n edges around the boundary of $\overline{\mathbb{D}}$. The map $\mathcal{C}_4$ is shown in Fig. 10.

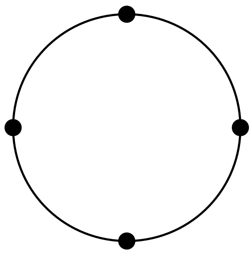

Fig. 10. The map $\mathcal{C}_n$, with $c = 1$, $A = \langle a, b\rangle \cong D_n$, where $n = 4$.

The cases where $a = 1$ or $b = 1$ have already been dealt with, so this completes the classification of the finite maps.

3.4. The list of finite regular maps. To summarise, we have proved the following theorem:

Theorem 3.1. *The finite regular maps with non-empty boundary are all on the closed disc $\overline{\mathbb{D}}$. They are as follows:*

- *an infinite family $\{\mathcal{A}_n \mid n \geqslant 1\}$, each embedding a semi-star of valency n, with automorphism group $\langle b, c\rangle \cong D_n$,*
- *an infinite family $\{\mathcal{C}_n \mid n \geqslant 1\}$, each embedding a circuit of n vertices and n edges around the boundary $\partial\overline{\mathbb{D}} = S^1$ of $\overline{\mathbb{D}}$, with automorphism group $\langle a, b\rangle \cong D_n$,*
- *an embedding $\mathcal{B}$ of K_2 as a diameter of $\overline{\mathbb{D}}$, with automorphism group $\langle a, c\rangle \cong V_4$,*
- *the quotients $\mathcal{B}/\langle a\rangle$, $\mathcal{B}/\langle c\rangle$ and $\mathcal{B}/\langle ac\rangle$ of $\mathcal{B}$ by subgroups of* Aut $\mathcal{B}$ *of order 2, each with automorphism group D_1,*
- *an embedding $\mathcal{D}$ of a vertex and two half-edges along the boundary of $\overline{\mathbb{D}}$, with automorphism group D_1,*

- *the trivial map $\mathcal{T}$, embedding a half-edge along the boundary of $\overline{\mathbb{D}}$, with automorphism group C_1.*

By considering the automorphism δ of Γ which transposes R_0 and R_2 and fixes R_1, we find that the classical duality D transposes these regular maps with boundary in pairs as follows, leaving the others invariant:

$$\mathcal{B}/\langle a\rangle \longleftrightarrow \mathcal{B}/\langle c\rangle, \quad \mathcal{A}_n \longleftrightarrow \mathcal{C}_n.$$

The Petrie duality P, corresponding to the automorphism π transposing R_0 and R_0R_2 and fixing R_1 and R_2, induces the following pairings:

$$\mathcal{B}/\langle a\rangle \longleftrightarrow \mathcal{B}/\langle ac\rangle, \quad \mathcal{A}_n \longleftrightarrow P(\mathcal{A}_n),$$

where $P(\mathcal{A}_n)$ is an embedding of an n-valent semi-star in the sphere, formed by extending its embedding $\mathcal{A}_n$ in $\overline{\mathbb{D}}$ to the Riemann sphere $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$.

3.5. The infinite case. In the infinite dihedral group D_∞, the only normal subgroup of infinite index is the identity subgroup, so if we try to extend the preceding classification of regular maps with non-empty boundary to the infinite case, the only automorphism groups which arise are the groups

$$A = \langle b, c \mid b^2 = c^2 = 1\rangle \cong D_\infty$$

where $a = 1$, and

$$A = \langle a, b \mid a^2 = b^2 = 1\rangle \cong D_\infty$$

where $c = 1$. In the first case the corresponding map is an embedding $\mathcal{A}_\infty$ of a semi-star with countably infinite valency, such that $\mathcal{A}_n \cong \mathcal{A}_\infty/\langle (bc)^n\rangle$ for each $n \in \mathbb{N}$. In the second case we have its dual, an embedding $\mathcal{C}_\infty$ of an infinite path, such that $\mathcal{C}_n \cong \mathcal{C}_\infty/\langle (ab)^n\rangle$ for each n. In both cases we can take the underlying surface to be the closed disc, as in all the finite cases.

In the case of $\mathcal{A}_\infty$ it is simplest initially to take this disc to be the closed hemisphere $\overline{\mathbb{H}} = \{z \in \mathbb{C} \mid \operatorname{Im} z \geqslant 0\} \cup \{\infty\}$ of the Riemann sphere $\widehat{\mathbb{C}} = \mathbb{P}^1(\mathbb{C}) = \mathbb{C} \cup \{\infty\}$; there is a single vertex at ∞, with half-edges along the lines $\operatorname{Re} z = n$ to the boundary points $n \in \mathbb{Z}$. Applying the Möbius transformation $f : z \mapsto (z - i)/(z + i)$ (a rotation of the sphere $\widehat{\mathbb{C}}$ of order 3) sends $\overline{\mathbb{H}}$ to the closed unit disc $\overline{\mathbb{D}}$; the vertex ∞ is sent to 1, and the half-edges $\operatorname{Re} z = n$ $(n \in \mathbb{Z})$ are sent to arcs of circles meeting the boundary S^1 perpendicularly at 1 and at $f(n) = (n - i)/(n + i)$ (or

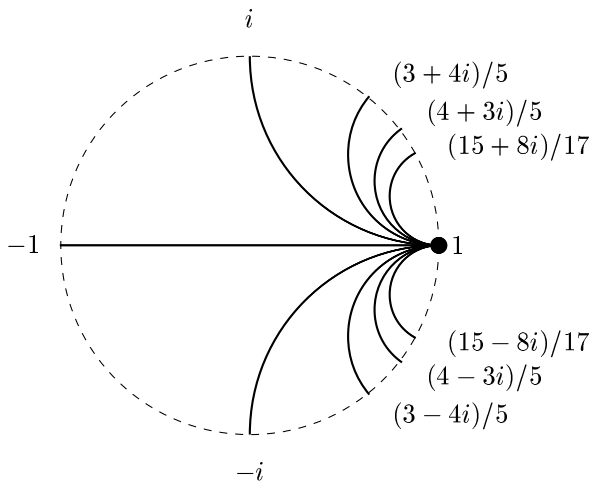

Fig. 11. The map $\mathcal{A}_\infty$, with $a = 1$, $G = \langle b, c\rangle \cong D_\infty$.

to a diameter from 1 to -1 when $n = 0$). This map is shown in Fig. 11, where only those half-edges with $|n| \leqslant 4$ are indicated. As $n \to \pm\infty$ the half-edges accumulate at the boundary point 1.

In the case of $\mathcal{C}_\infty$, one can use the same Möbius transformation f to send a doubly infinite path graph along $\mathbb{R} \subset \overline{\mathbb{H}}$, with vertices at the integers, to an isomorphic graph along the subset $S^1 \setminus \{1\}$ of the boundary of $\overline{\mathbb{D}}$, with vertices at the points $(n-i)/(n+i)$ for $n \in \mathbb{Z}$. This is shown in Fig. 12, where only those vertices with $|n| \leqslant 4$ are indicated. As $n \to \pm\infty$ the vertices accumulate at the boundary point 1.

This embedding is isomorphic to the dual $D(\mathcal{A}_\infty)$ of $\mathcal{A}_\infty$; in fact, if we translate the half-edges in $\overline{\mathbb{H}}$ by 1/2, so that they are given by $\operatorname{Re} z = n + \frac{1}{2}$ for $n \in \mathbb{Z}$, and then apply the Möbius transformation f, the image *is* the dual of $\mathcal{A}_\infty$, with vertices at the points $f(n+\frac{1}{2}) = (n+\frac{1}{2}-i)/(n+\frac{1}{2}+i)$ on S^1, again accumulating at 1. As in the finite case, the Petrie duality P leaves $\mathcal{C}_\infty$ invariant, while transposing $\mathcal{A}_\infty$ with an embedding of the same infinite semi-star graph in the sphere.

3.6. A technical distinction. We have refrained from calling $\mathcal{C}_\infty$ a map, since it fails condition M5 for a map with boundary, stated in [1]. This requires that any face (connected component of the complement of the

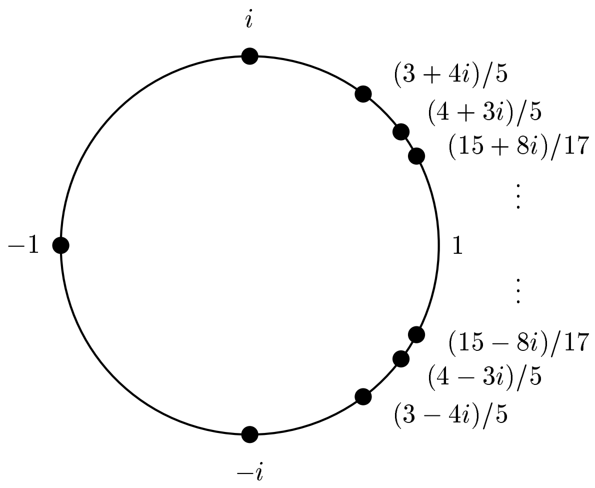

Fig. 12. The embedding $\mathcal{C}_\infty$, with $c=1$, $G=\langle a,b\rangle\cong D_\infty$.

graph) which meets the boundary should be homeomorphic to a half-disc, the quotient of an open disc by a reflection, so that it meets the boundary along an open interval. Although $\mathcal{A}_\infty$ satisfies this condition, $\mathcal{C}_\infty$ does not, since its unique face consists of the open disc $\mathbb{D}$ together with the boundary point 1.

This condition M5 is also required in the case of finite maps, for instance to avoid examples such as the embedding of K_2 in an annulus, shown in Fig. 13, and similarly the embedding of K_2 in a Möbius band formed in the usual way by cutting across the annulus, twisting it by a half-turn, and rejoining. In each case, although the unique face is simply connected, it meets the boundary in two open intervals, rather than one.

The labelling of the flags in Figures 13 and 14 shows that algebraically there is no distinction between these two embeddings and the map $\mathcal{B}$ in Fig. 3.1, which also embeds K_2: in all three cases, the monodromy group G is a Klein four-group, where r_0 and r_2 are the permutations $(12)(34)$ and $(14)(23)$ of the flags, and r_1 is the identity. However, these three embeddings are topologically very different, so in order to preserve a bijection between isomorphism classes of algebraic and topological maps, it is necessary impose the condition M5 for maps with boundary.

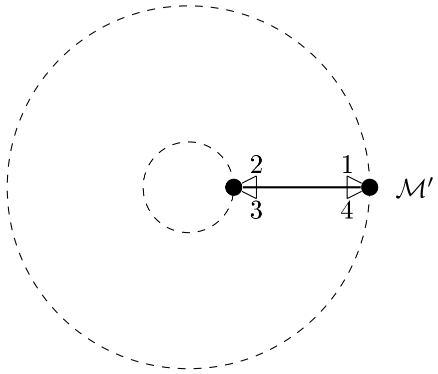

Fig. 13. An embedding of K_2 in a closed annulus.

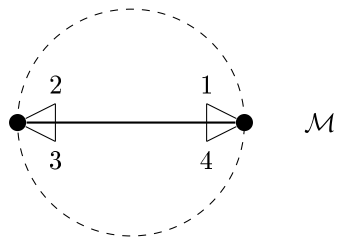

Fig. 14. The map $\mathcal{B}$.

§4. Canonical doubles

Let Γ^+ denote the orientation-preserving subgroup of index 2 in Γ, consisting of the elements represented by words of even length in the generators R_i. A map $\mathcal{M}$ is orientable and without boundary if and only if the corresponding conjugacy class of map subgroups $M \leqslant \Gamma$ are contained in Γ^+. Otherwise, when $\mathcal{M}$ is non-orientable or with non-empty boundary, the *canonical double* of $\mathcal{M}$ is defined to be the map $\mathcal{M}^{\mathrm{d}}$ corresponding to the conjugacy class of subgroups $M^+ = M \cap \Gamma^+$ of Γ. This is an orientable map without boundary, and the index 2 inclusion $M^+ < M$ induces a double covering $\mathcal{M}^{\mathrm{d}} \to \mathcal{M}$ branched only over the boundary of $\mathcal{M}$. Equivalently, $\mathcal{M}^{\mathrm{d}}$ has an orientation-reversing automorphism of order 2 with quotient map $\mathcal{M}$, and with its fixed-points corresponding to the boundary points of $\mathcal{M}$.

If $\mathcal{M}$ is a regular map then M is normal in Γ, and hence so is M^+, so $\mathcal{M}^{\mathrm{d}}$ is regular. If $\mathcal{M}$ is a map on the closed disc $\overline{\mathbb{D}}$, then $\mathcal{M}^{\mathrm{d}}$ is a map on the Riemann sphere $\widehat{\mathbb{C}}$. It follows that the regular maps with non-empty boundary which we have classified are all quotients, by reflections, of regular maps on the sphere. In each case, one sheet of the covering is $\mathcal{M}$, and the other is the map obtained by inverting $\mathcal{M}$ in the unit circle S^1.

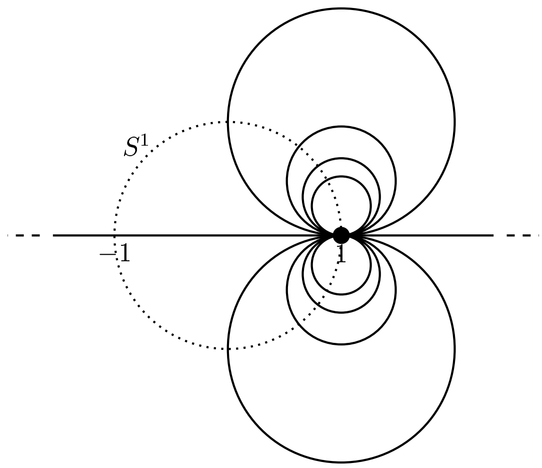

Fig. 15. The double of $\mathcal{A}_\infty$, with S^1 shown (dotted).

In the finite case, the regular maps on the sphere are well-known. However, none of the maps corresponding to the five platonic solids arises in this context. The double of $\mathcal{A}_n$ is the hosohedron (or beach-ball), denoted by $\{2, n\}$ in the notation of [4], with two n-valent vertices, n edges and n digonal faces, while $\mathcal{C}_n^{\mathrm{d}}$ is the dihedron $\{n, 2\}$, with two n-gonal faces separated by a circuit of n vertices and n edges. More generally, in any case such as $\mathcal{C}_n$, $\mathcal{C}_\infty$, $\mathcal{B}/\langle c\rangle$, $\mathcal{D}$ or $\mathcal{T}$, where the embedded graph is contained in the boundary of the disc, the double is an embedding of the same graph in the sphere. The double of $\mathcal{B}$ is the hosohedron (also a dihedron) $\{2, 2\}$, though the corresponding reflection differs from those yielding $\mathcal{A}_2$ or $\mathcal{C}_2$. The double of $\mathcal{B}/\langle a\rangle$ is the dihedron $\{1, 2\}$, the spherical embedding of a single vertex and a loop, while that of $\mathcal{B}/\langle ac\rangle$ is an embedding of a semi-star of valency 2. Finally $\mathcal{A}^{\mathrm{d}}_{\infty}$ is a pencil of circles in the Riemann sphere, all mutually tangent at the vertex 1; nine of these circles, including the real projective line $\mathbb{P}^1(\mathbb{R}) = \mathbb{R}\cup\{\infty\}$, are shown in Fig. 15, together with S^1 (a dotted line).

§5. Arc-transitive maps

A map $\mathcal{M}$ is arc-transitive if and only if $N\langle R_2\rangle = \Gamma$, where N is the normaliser $N_\Gamma(M)$ of M in Γ. This condition implies that $|\Gamma : N| \leqslant 2$, so either $N = \Gamma$ and $\mathcal{M}$ is regular, or N is a normal subgroup of index 2 in Γ, not containing R_2. We have dealt with regular maps, so in this section we will assume that $\mathcal{M}$ is arc-transitive but not regular, so that N has index 2. There are seven such subgroups N, of which four do not contain R_2, namely the normal closures in Γ of the following sets Σ:

(1) $\Sigma = \{R_0R_1, R_1R_2\}$.
(2) $\Sigma = \{R_0, R_1R_2\}$;
(3) $\Sigma = \{R_0, R_1\}$;
(4) $\Sigma = \{R_1, R_0R_2\}$;

The map $\mathcal{M}$ has non-empty boundary if and only if M contains a conjugate of some R_i for $i = 0, 1$ or 2. This conjugate is then in N, so $R_i \in N$ since N is normal in Γ. This eliminates case (1), since in this case $N = \Gamma^+$, which does not contain R_i for any i.

The simplest of the remaining cases is case (2), where the Reidemeister-Schreier process [13, §II.4] shows that

$$N = \langle R_0, S := R_1R_2 \mid R_0^2 = 1\rangle \cong C_2 * C_\infty.$$

Now M must be normal in N, and must contain a reflection. Since R_2 acts by conjugation on N by commuting with R_0 and inverting S, the reflections in N are the conjugates in N of R_0, so $R_0 \in M$; thus N/M is cyclic, and M is the normal closure in N of $\{R_0\}$ or of $\{R_0, S^n\}$ for some $n = |N : M| \geqslant 1$. The action of R_2 by conjugation on R_0 and S shows that it normalises each such subgroup M, which is therefore normal in Γ. This contradicts our assumption that $\mathcal{M}$ is not regular, so case (2) does not arise.

In case (3) the Reidemeister-Schreier process shows that

$$N = \langle R_0, R_1, R_3 := R_1^{R_2} \mid R_i^2 = 1\rangle \cong C_2 * C_2 * C_2,$$

where R_2 commutes with R_0 and transposes R_1 and R_3. (See Fig. 3.1 for the map $\mathcal{N} = \mathcal{B}/\langle a\rangle$ corresponding to N.) Again, M is normal in N and contains a reflection. The reflections in N are the conjugates in N of R_0, R_1 and R_3, so M must contain at least one of these three generators, and hence N/M is a cyclic or dihedral group. If $R_0 \in M$ then the only possibility for M not to be normal in Γ is for it to contain just one of R_1 and R_3; this gives two subgroups M of index 2 in N, conjugate in Γ and corresponding to an arc-transitive but non-regular map $\mathcal{M} = \mathcal{E}$ on $\overline{\mathbb{D}}$, with a vertex on the boundary and two half-edges extending across the interior to boundary points (see Fig. 16).

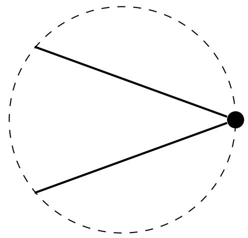

Fig. 16. The map $\mathcal{E}$.

We may therefore assume that $R_0 \notin M$, so $R_i \in M$ for $i = 1$ or 3. If both $R_1, R_3 \in M$ then M is normal in Γ, against our assumption. If $R_1 \in M$ but $R_3 \notin M$ then provided $|N : M|$ is finite $\mathcal{M}$ is a map $\mathcal{F}_n$ which embeds an n-gon in $\overline{\mathbb{D}}$, where $N/M \cong D_n$ for some $n \geqslant 1$, with vertices on the boundary and edges in the interior; the same applies, with a conjugate subgroup M, if $R_3 \in M$ but $R_1 \notin M$. (See Fig. 17 for $\mathcal{F}_4$.) This includes the case $n = 1$, where the embedded graph consists of a single boundary vertex attached to a loop in the interior of the disc.

If $|N : M|$ is infinite the situation is similar to that for the embedding $\mathcal{C}_\infty$ shown in Fig. 12: we have an embedding $\mathcal{F}_\infty$ of an infinite path in $\overline{\mathbb{D}}$, with the same vertices as $\mathcal{C}_\infty$, accumulating at 1, but now connected by edges in the interior of the disc, rather than its boundary (see Fig. 18).

Finally, in case (4) the Reidemeister-Schreier process shows that

$$N = \langle R_1, R_3 := R_1^{R_2}, R_4 := R_0R_2 \mid R_i^2 = 1\rangle \cong C_2 * C_2 * C_2,$$

where R_2 transposes R_1 and R_3, and commutes with R_4. (See Fig. 3.1 for the corresponding map $\mathcal{N} = \mathcal{B}/\langle ac\rangle$.) This subgroup N is obtained from

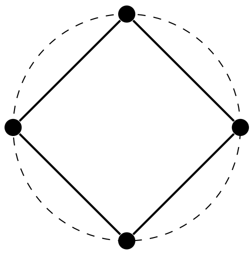

Fig. 17. The map $\mathcal{F}_n$, where $n = 4$.

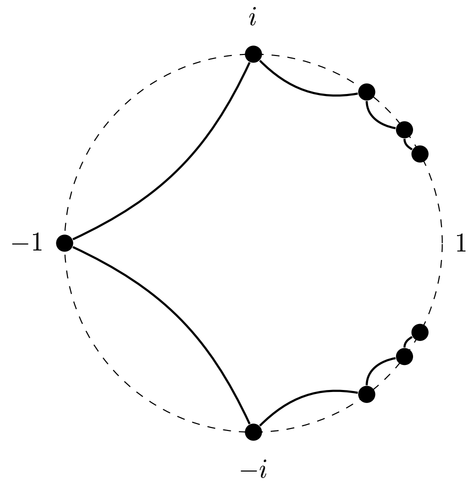

Fig. 18. The embedding $\mathcal{F}_\infty$.

that in case (3) by applying the automorphism π of Γ corresponding to the Petrie operation P, so the arc-transitive maps arising are the Petrie duals of those in case (3), embedding the same graphs. The reflections in N are now the conjugates in N of R_1 and R_3, so M contains one but not both of these two generators. If M contains either R_1 or R_3, and also R_4, we obtain the map $\mathcal{G}$ in Fig. 19, the Petrie dual $P(\mathcal{E})$ of the map $\mathcal{E}$ in Fig. 16.

If M contains R_1 or R_3, but not R_4, then in the finite case we obtain a map $\mathcal{H}_n = P(\mathcal{F}_n)$ which embeds a circuit of n vertices and edges where

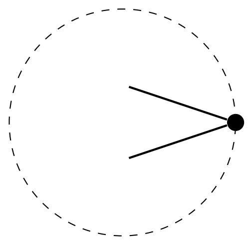

Fig. 19. The map $\mathcal{G}$.

$N/M \cong D_n$ for some $n \geqslant 1$; its vertices are on the boundary of an annulus or Möbius band as n is even or odd. (See Fig. 20, where the left and right sides of the rectangular strip are identified orientably or non-orientably in these two cases.) In the infinite case the pattern is the same, but with the strip extending infinitely far in both directions; the resulting map $\mathcal{H}_\infty = P(\mathcal{F}_\infty)$ embeds an infinite path with vertices on alternate boundary components, and edges in the interior. The automorphism group is isomorphic to D_∞, realised as the frieze group p2mg, with the cyclic subgroup of index 2 generated by a glide reflection, and the involutions either reflections or half-turns.

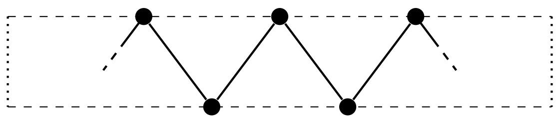

Fig. 20. The map $\mathcal{H}_n$.

To summarise, we have proved:

Theorem 5.1. *The arc-transitive finite maps with non-empty boundary are:*

- *the regular maps listed in Theorem* 3.1*;*
- *the Petrie dual pair of maps $\mathcal{E}$ and $\mathcal{G} = P(\mathcal{E})$, which embed a boundary vertex and two semi-edges in $\overline{\mathbb{D}}$, with automorphism group D_1;*
- *an infinite family $\{\mathcal{F}_n \mid n \geqslant 1\}$, each embedding a circuit of n boundary vertices and n interior edges in $\overline{\mathbb{D}}$, with automorphism group D_n,*

- *an infinite family* $\{\mathcal{H}_n = P(\mathcal{F}_n) \mid n \geqslant 1\}$*, each embedding a circuit of* n *boundary vertices and* n *interior edges in a closed annulus or Möbius band as* n *is even or odd, with automorphism group* D_n*.*

In the infinite case we have the embedding $\mathcal{F}_\infty$ and the map $\mathcal{H}_\infty$, which embed an infinite path with boundary vertices and interior edges in the closed disc or infinite strip, with automorphism groups isomorphic to D_∞.

§6. Edge-transitive maps

One might hope to obtain similar classifications of maps with boundary which satisfy slightly weaker conditions than regularity or arc-transitivity, such as edge-transitivity. The following examples show that this is unrealistic.

A map $\mathcal{M}$, corresponding to a conjugacy class in Γ of map subgroups M, is edge-transitive if and only if $NE = \Gamma$, where $N = N_\Gamma(M)$ as before, and $E := \langle R_0, R_2\rangle \cong V_4$. There are 14 conjugacy classes of subgroups $H \leqslant \Gamma$ satisfying $HE = \Gamma$, corresponding to the 14 types of edge-transitive maps classified by Graver and Watkins in [6] (see also [18] for finite realisations of these types). As just one example, let H be the normal closure in Γ of R_1, a normal subgroup of index 4 playing the role of $N_\Gamma(M)$ for the edge- but not vertex- or face-transitive maps, denoted by type 3 in [6]. This group has a presentation

$$H = \langle S_0 := R_1,\, S_1 := R_1^{R_0},\, S_2 := R_1^{R_2},\, S_3 := R_1^{R_0R_2} \mid S_i^2 = 1\rangle,$$

so $H \cong C_2 * C_2 * C_2 * C_2$ and the normal subgroups M of H are the kernels of epimorphisms $\theta : H \to B$ where B is any group generated by at most four involutions.

For any such subgroup M the normaliser $N = N_\Gamma(M)$ satisfies $N \geqslant H$, so $NE = \Gamma$ and the corresponding map $\mathcal{M}$ is edge-transitive. It has non-empty boundary if and only if M contains reflections; since the reflections in H are the conjugates in H of its generators S_i, this condition is equivalent to $S_i \in \ker\theta$ for some $i = 0, \ldots, 3$. The required groups B are therefore those generated by at most three involutions. This is a very wide class, including, for example, every non-abelian finite simple group except $U_3(3)$ (see [14]); most of these simple groups have many such generating triples which are inequivalent under automorphisms and hence correspond to different kernels M (for instance, Hall [8] showed that there are 19 for A_5). The resulting profusion of normal subgroups M of finite index in

H makes it impossible to envisage a reasonable classification of the corresponding finite edge-transitive maps with non-empty boundary. If the finiteness condition is relaxed the situation is even worse: Bernhard Neumann [15] showed that there are uncountably many isomorphism classes of 2-generator groups; since $C_2 * C_2 * C_2$ contains a free group of rank 2 as a subgroup of index 2, it follows that the same applies to groups generated by three involutions, so there are uncountably many isomorphism classes of edge-transitive maps with non-empty boundary.

Here is a method for constructing explicit examples of such maps, all of type 3. Take any map $\mathcal{M}$ with empty boundary, and colour its vertices black. Within the interior of each m-gonal face of $\mathcal{M}$, draw another m-gon, with white vertices midway between successive vertices of $\mathcal{M}$, and then join alternate black and white vertices around the face in zig-zag fashion to give a $2m$-gon. Now delete the edges of $\mathcal{M}$, and remove the interiors of the white m-gons, giving a boundary component within each face of $\mathcal{M}$. The edges of the $2m$-gons are the edges of a new bipartite map $\mathcal{M}^*$, with black and white vertices in the interior and boundary respectively. Alternate faces are either triangles meeting the boundary, or quadrilaterals in the interior (see Fig. 21, where $m = 4$). This process is reversible, so $\operatorname{Aut}\mathcal{M}^* = \operatorname{Aut}\mathcal{M}$. Since some vertices and faces meet the boundary, while others do not, $\mathcal{M}^*$ is neither vertex- nor face-transitive. The edges of $\mathcal{M}^*$ correspond bijectively to the flags of $\mathcal{M}$, so if $\mathcal{M}$ is regular then $\mathcal{M}^*$ is edge-transitive, of Graver-Watkins type 3. Since $\mathcal{M}$ can be any regular map without boundary, $\operatorname{Aut}\mathcal{M}^*$ can be isomorphic to any non-dihedral quotient of Γ.

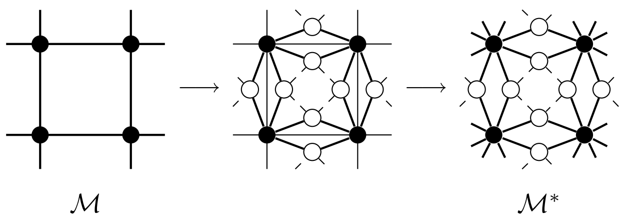

Fig. 21. Construction of edge-transitive maps.

§7. Regular hypermaps

In recent years the theory of maps has been extended to include hypermaps. Originally introduced by Cori [3] as a method for concisely representing 2-dimensional images, their study subsequently received a boost through their use (under the name *dessins d'enfants*) in Grothendieck's program [7] for relating the Galois theory of algebraic number fields and the Teichmüller theory of Riemann surfaces (see [5, 11] for accessible introductions). In this theory, compact oriented hypermaps represent projective algebraic curves defined over algebraic number fields, and those with non-empty boundary represent curves over real algebraic number fields (see the work of Köck and Singerman in [10] for more precise details).

Just as a map embeds a graph in a surface, a hypermap embeds a hypergraph, which is essentially a graph in which edges are allowed to be incident with any number of vertices. Algebraically, this corresponds to replacing the group $\Gamma \cong V_4 * C_2$ with the group

$$\Delta = \langle R_0, R_1, R_2 \mid R_i^2 = 1 \text{ for } i = 0, 1, 2\rangle \cong C_2 * C_2 * C_2,$$

thus omitting the defining relation $(R_0R_2)^2 = 1$ of Γ which restricts the valencies of the edges of a map to 1 or 2. In particular, every map can be regarded as a hypermap. Conversely, the most economical way of representing a hypermap is as its Walsh map [19], a bipartite map on the same surface, with its black and white vertices representing the vertices and edges of the hypermap, and edges between black and white vertices indicating incidence.

Apart from lacking a restriction on the valencies of edges, the algebraic theory of hypermaps is very similar to that for maps: thus regular hypermaps correspond to normal subgroups M of Δ, their automorphism groups are isomorphic to Δ/M, and those with non-empty boundary correspond to normal subgroups containing some R_i. As in the case of maps, it follows that regular hypermaps with non-empty boundary must have cyclic or dihedral automorphism groups, with a, b or $c = 1$. Indeed, when $a = 1$ or $c = 1$ the extra relation $(ac)^2 = 1$ for maps is redundant, so in these cases the hypermaps arising are exactly the same as the maps classified earlier, and one simply needs to reinterpret them as hypermaps. When $b = 1$ the omission of the relation $(ac)^2 = 1$ allows arbitrary dihedral groups, as in the cases where $a = 1$ or $c = 1$, but the hypermaps arising are just the vertex-edge duals of those for $a = 1$, the same hypermaps except that vertices and edges are transposed. (This corresponds to applying the automorphism of Δ which fixes R_2 and transposes R_0 and R_1, or equivalently to transposing the vertex-colours in the Walsh map.) Thus the regular hypermaps with non-empty boundary are those obtained in this way from the regular maps classified in Section 3. In particular, they can all be drawn on the closed disc.

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