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PRIMITIVE MONODROMY GROUPS OF RATIONAL FUNCTIONS WITH ONE MULTIPLE POLE

Abstract. We classify primitive monodromy groups of rational functions of the form P/Q, where Q is a polynomial with no multiple roots and $\deg P > \deg Q + 1$. There are 17 families of such functions which are not Belyi functions. Only one family from the list contains functions that have five critical values. All the remaining families consist of functions with at most four critical values and constitute one-dimensional strata in the Hurwitz space. We compute the action of the braid group on generators of their monodromy groups and draw the corresponding megamaps.

The result extends the classification of primitive edge rotation groups of weighted trees obtained by the author and Zvonkin and is also a generalization of the classification of primitive monodromy groups of polynomials obtained by P. Müller.

§1. Introduction

Let $f : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ be a rational function with a single pole of multiplicity greater than one. We consider such functions up to linear-fractional transformations $z \mapsto (az + b)/(cz + d)$ of the argument and assume that the multiple pole is placed at $z = \infty$. Then the function has the form $f(z) = P(z)/Q(z)$, where $P, Q \in \mathbb{C}[z]$, $\deg P > \deg Q + 1$ and Q has no multiple roots.

We are interested in primitive monodromy groups of such functions. The monodromy group of a generic rational function is most likely to be either alternating or symmetric. We say that a primitive monodromy group of degree n is *special* if it is different from $\mathbf{A}_n$ and $\mathbf{S}_n$.

Rational functions with only one multiple pole and three critical values are Belyi functions corresponding to weighted trees. The monodromy group of a Belyi function can be interpreted as the edge rotation group of the corresponding dessin. All special primitive monodromy groups of Belyi functions of weighted trees were classified in [11]. Theorem 1 below lists

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all possible special primitive monodromy groups of functions with only one multiple pole, which have at least four critical values.

A covering of the sphere S^2 of degree n, branched over k points, can be described by k permutations $a_1, a_2, \dots, a_k \in \mathbf{S}_n$ satisfying $a_1 a_2 \dots a_k = 1$. We say that two sets of generators $(a_1, a_2, \dots a_k)$ and $(a'_1, a'_2, \dots a'_k)$ are equivalent if there exists a permutation $\pi \in \mathbf{S}_n$ such that $a'_i = \pi^{-1} a_i \pi$ for all i. We are also interested in flexible equivalence of coverings (see [5]), which can be described using the Hurwitz braid group.

The spherical braid group (or Hurwitz braid group) on k strands is the group H_k generated by elements $\sigma_1, \dots \sigma_{k-1}$ satisfying the following relations:

$$\sigma_i \sigma_j = \sigma_j \sigma_i \quad \text{when} \quad |i-j| \geqslant 2,$$
$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1},$$
$$\sigma_1 \sigma_2 \dots \sigma_{k-2} \sigma_{k-1}^2 \sigma_{k-2} \dots \sigma_2 \sigma_1 = \mathrm{id}\,.$$

The Hurwitz group H_k acts on the sets of generators $(a_1, a_2, \dots a_k)$ as follows:

$$\sigma_i(a_i) = a_{i+1}, \quad \sigma_i(a_{i+1}) = a_{i+1}^{-1} a_i a_{i+1}, \quad \sigma_i(a_j) = a_j \text{ when } j \neq i, i+1.$$

A family of functions with four branch points (or *critical values*), which corresponds to an orbit of H_4 on the monodromy generators, constitutes a one-dimensional stratum in the Hurwitz space, with a Belyi function naturally defined on this stratum (see [8, 5]). The dessin d'enfant corresponding to this Belyi function is called a *megamap* and can be defined by the triple of permutations $\Sigma = \sigma_1^2$, $A = \sigma_2^2$, $\Phi = \sigma_2^{-1} \sigma_3^2 \sigma_2$ acting on the sets of generators. The elements $\Sigma, A, \Phi \in H_4$ satisfy $\Sigma A \Phi = \mathrm{id}$.

In this paper we also calculate the action of the Hurwitz group and draw megamaps for all families of functions with only one multiple pole and exactly four critical values.

§2. Main result

Theorem 1. *A complete list of special primitive monodromy groups of rational functions $f : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ with a single pole of multiplicity greater than one and with at least four critical values, their branch data* (*passports*) *and the numbers of non-equivalent sets of generators, is given in Table* 1.

Remarks. 1. In [11] we labeled special primitive monodromy groups of functions with three critical values by $n.m$, where n is the degree of the

	G	Passport	#	$g(M)$	$ER(M)$
4/6.1	$\mathbf{L}_2(5)$	$(1^2 2^2, 1^2 2^2, 1^2 2^2, 1^1 5^1)$	10	0	$\mathbf{A}_{10}$
4/6.2	$\mathbf{PGL}_2(5)$	$(1^2 2^2, 1^2 2^2, 2^3, 1^2 4^1)$	8	0	$\mathbf{A}_8$
4/6.3	$\mathbf{PGL}_2(5)$	$(1^2 2^2, 1^2 2^2, 1^2 4^1, 1^2 4^1)$	16	1	$8! \cdot 8!/2$
4/7.1	$\mathbf{L}_3(2)$	$(1^3 2^2, 1^3 2^2, 1^3 2^2, 7^1)$	2×7	0	$\mathbf{A}_7$
5/8.1	$\mathbf{ASL}_3(2)$	$(1^4 2^2, 1^4 2^2, 1^4 2^2, 1^4 2^2, 1^1 7^1)$	2×147	-	-
4/8.1	$\mathbf{ASL}_3(2)$	$(1^4 2^2, 1^4 2^2, 2^4, 1^1 7^1)$	2×7	0	$\mathbf{A}_7$
4/8.2	$\mathbf{ASL}_3(2)$	$(1^4 2^2, 1^4 2^2, 1^2 2^1 4^1, 1^1 7^1)$	2×14	0	$\mathbf{A}_{14}$
4/8.3	$\mathbf{ASL}_3(2)$	$(1^4 2^2, 1^4 2^2, 1^2 3^2, 1^1 7^1)$	2×21	1	$\mathbf{A}_{21}$
4/8.4	$\mathbf{PGL}_2(7)$	$(1^2 2^3, 1^2 2^3, 1^2 2^3, 1^2 6^1)$	18	1	$\mathbf{A}_{18}$
4/9.1	$\mathbf{AGL}_2(3)$	$(1^3 2^3, 1^3 2^3, 1^3 2^3, 1^1 8^1)$	2×16	0	$2^7 \cdot 4!$
4/10.1	$\mathbf{P\Gamma L}_2(9)$	$(1^4 2^3, 1^4 2^3, 2^5, 1^2 8^1)$	8	0	$4! \cdot 4!$
4/12.1	$\mathbf{M}_{11}$	$(1^4 2^4, 1^4 2^4, 1^4 2^4, 1^1 11^1)$	2×33	0	$\mathbf{A}_{33}$
4/12.2	$\mathbf{M}_{12}$	$(1^4 2^4, 1^4 2^4, 1^4 2^4, 1^1 11^1)$	2×22	0	$\mathbf{A}_{22}$
4/13.1	$\mathbf{L}_3(3)$	$(1^5 2^4, 1^5 2^4, 1^5 2^4, 13^1)$	4×13	0	$\mathbf{A}_{13}$
4/15.1	$\mathbf{L}_4(2)$	$(1^7 2^4, 1^7 2^4, 1^3 2^6, 15^1)$	2×5	0	$\mathbf{S}_5$
4/16.1	$\mathbf{AGL}_4(2)$	$(1^8 2^4, 1^4 2^6, 1^4 2^6, 1^1 15^1)$	2×15	0	$\mathbf{S}_{15}$
4/24.1	$\mathbf{M}_{24}$	$(1^8 2^8, 1^8 2^8, 1^8 2^8, 1^1 23^1)$	2×46	0	$\mathbf{A}_{46}$

Table 1. Special primitive monodromy groups of rational functions with a single pole of multiplicity greater than one and with at least four critical values. The meaning of the columns #, $g(M)$, and $ER(M)$ is explained below.

permutation group and m is a sequence number. To keep numbering in this paper consistent with [11] we label monodromy groups by $k/n.m$ where k is the number of critical values, n is the degree of the function and m is a sequence number. For example, 4/8.3 means "four critical values, group of degree 8, third covering".

2. For every critical value of a function f of degree n, the multiplicities of its preimages give us a partition of n. The collection of such partitions taken for all critical values is called the *passport*. A covering of the sphere S^2 of degree n branched over k points can be described by a tuple of k permutations $a_1, a_2, \ldots, a_k \in \mathbf{S}_n$ satisfying $a_1 a_2 \ldots a_k = 1$. The passport of the covering is a tuple of cycle structures of permutations a_i.

3. We call two set of generators $(a_1, a_2, \dots a_k)$ and $(a'_1, a'_2, \dots a'_k)$ *equivalent* if there is a permutation $\pi \in \mathbf{S}_n$ such that $a'_i = \pi^{-1} a_i \pi$ for all i. The column named # in Table 1 gives the numbers of non-equivalent sets of generators, that is, the sizes of the orbits of the action of the Hurwitz braid group H_k.

4. In most cases, for a partition of n in a passport, there is only one conjugacy class of elements with this cycle structure in the corresponding group G. In some cases, however, there are two or four such classes. For example, in the case 4/7.1, the group $\mathbf{L}_3(2)$ contains only one conjugacy class A of elements with cycle structure $1^3 2^2$ and exactly two classes B_1 and B_2 of elements with cycle structure 7^1. For both $i = 1, 2$ there are seven non-equivalent sets of generators (a_1, a_2, a_3, a_4) such that $a_1, a_2, a_3 \in A$ and $a_4 \in B_i$. In this case we write the total number of non-equivalent sets as 2×7.

5. For the functions with four critical values, the action of the Hurwitz group H_4 by the permutations Σ, A, Φ defines a dessin M which is called a *megamap*. The column $g(M)$ gives the genus and the column $ER(M)$ gives the edge rotation group of the megamap M. In cases 4/6.3, 4/9.1 and 4/10.1 the corresponding group $ER(M)$ is imprimitive, so we give only its order. All these megamaps are drawn in Figs. 1–15.

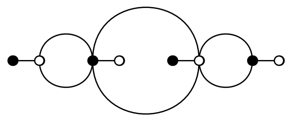

Fig. 1. Megamap 4/6.1: $g = 0$, $ER = \mathbf{A}_{10}$.

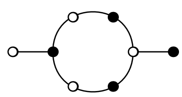

Fig. 2. Megamap 4/6.2: $g = 0$, $ER = \mathbf{A}_8$.

§3. Proof of the finiteness of the list

The proof of Theorem 1 proceeds as follows. First of all, we show that there is only a finite list of special primitive permutation groups which

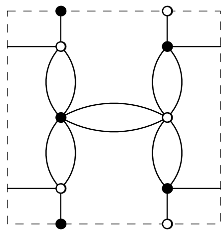

Fig. 3. Megamap 4/6.3: $g = 1$, $|ER| = 8! \cdot 8!/2 = 812\,851\,200$.

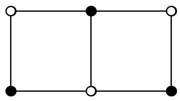

Fig. 4. Megamaps 4/7.1 and 4/8.1: $g = 0$, $ER = \mathbf{A}_7$.

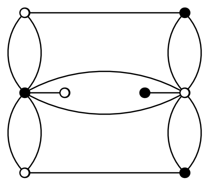

Fig. 5. Megamap 4/8.2: $g = 0$, $ER = \mathbf{A}_{14}$.

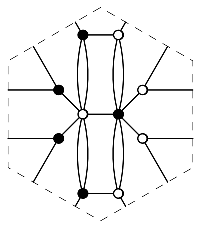

Fig. 6. Megamap 4/8.3: $g = 1$, $ER = \mathbf{A}_{21}$.

contain a permutation with cycle structure $1^t(n-t)^1$ and which may appear as monodromy groups of coverings of genus 0. After that, we find all

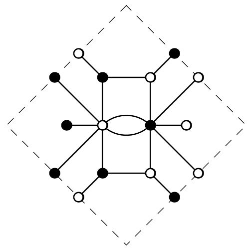

Fig. 7. Megamap 4/8.4: $g = 1$, $ER = \mathbf{A}_{18}$.

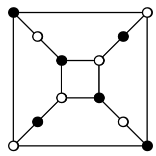

Fig. 8. Megamap 4/9.1: $g = 0$, $|ER| = 2^7 \cdot 4! = 3072$.

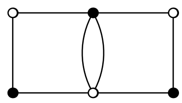

Fig. 9. Megamap 4/10.1: $g = 0$, $|ER| = 4! \cdot 4! = 576$.

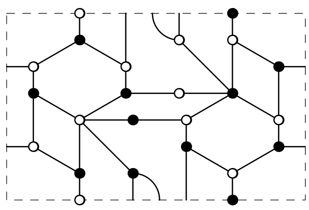

Fig. 10. Megamap 4/12.1: $g = 1$, $ER = \mathbf{A}_{33}$.

possible non-equivalent sets of generators for these groups using the GAP system (see [1]).

To determine primitive monodromy groups of functions with a single multiple pole we use the following theorem.

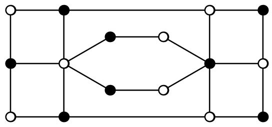

Fig. 11. Megamap 4/12.2: $g = 0$, $ER = \mathbf{A}_{22}$.

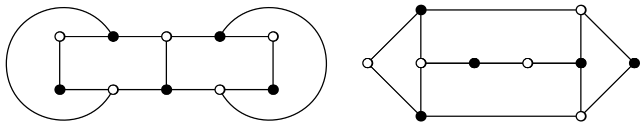

Fig. 12. Megamaps 4/13.1: $g = 0$, $ER = \mathbf{A}_{13}$.

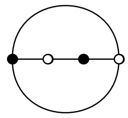

Fig. 13. Megamap 4/15.1: $g = 0$, $ER = \mathbf{S}_5$.

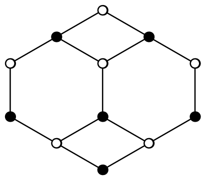

Fig. 14. Megamap 4/16.1: $g = 0$, $ER = \mathbf{S}_{15}$.

Theorem 2 (G. Jones, [3]). *Let G be a primitive permutation group of degree n not equal to $\mathbf{S}_n$ or $\mathbf{A}_n$. Suppose that G contains a permutation with cycle structure $1^t(n-t)^1$. Then $t \leqslant 2$, and one of the following holds:*

0. *$t = 0$ and either*
   (a) *$\mathbf{C}_p \subseteq G \subseteq \mathbf{AGL}_1(p)$, with $n = p$ prime, or*
   (b) *$\mathbf{PGL}_d(q) \subseteq G \subseteq \mathbf{P\Gamma L}_d(q)$, with $n = (q^d - 1)/(q-1)$ and $d \geqslant 2$ for some prime power $q = p^e$, or*
   (c) *$G = \mathbf{L}_2(11)$, $\mathbf{M}_{11}$ or $\mathbf{M}_{23}$, with $n = 11$, 11 or 23 respectively.*
1. *$t = 1$ and either*

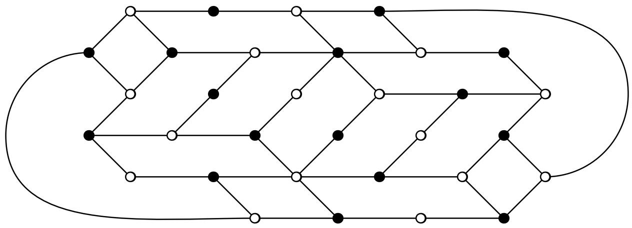

Fig. 15. Megamap 4/24.1: $g = 0$, $ER = \mathbf{A}_{46}$.

(a) $\mathbf{AGL}_d(q) \subseteq G \subseteq \mathbf{A\Gamma L}_d(q)$, *with* $n = q^d$ *and* $d \geqslant 1$ *for some prime power* $q = p^e$, *or*
(b) $G = \mathbf{L}_2(p)$ *or* $\mathbf{PGL}_2(p)$, $n = p + 1$ *for some prime* $p \geqslant 5$, *or*
(c) $G = \mathbf{M}_{11}$, $\mathbf{M}_{12}$ *or* $\mathbf{M}_{24}$, *with* $n = 12$, 12 *or* 24 *respectively.*

2. $t = 2$ *and* $\mathbf{PGL}_2(q) \subseteq G \subseteq \mathbf{P\Gamma L}_2(q)$, *with* $n = q + 1$ *for some prime power* $q = p^e$.

The classification in Theorem 2 contains several infinite series of groups. Our goal is to show that the additional condition of being a monodromy group of a covering of genus $g = 0$ leaves us with only a finite number of groups.

The case $t = 0$ of genus 0 corresponds to polynomials; the complete classification of the corresponding groups was given in [7]. All the possible special primitive monodromy groups of polynomials with more than two finite critical values correspond to the cases 4/7.1, 4/13.1 and 4/15.1 in Table 1.

All the sets of generators of genus 0 for affine groups (the case 1(a) of Theorem 2) are listed in [6]. Sets of generators with four or more elements such that one of them has the cycle structure $1^1(n-1)^1$ give us the cases 5/8.1, 4/8.1, 4/8.2, 4/8.3, 4/9.1 and 4/16.1 in Table 1.

The case 1(c) of Theorem 2 contains three groups: $\mathbf{M}_{11}$, $\mathbf{M}_{12}$ and $\mathbf{M}_{24}$. In the remaining cases 1(b) and 2 the group G is a subgroup of $\mathbf{P\Gamma L}_2(q)$ acting on points of the projective line $\mathbb{P}^1(\mathbb{F}_q)$. We need the following lemma; its proof can be found in [7] or [9].

Lemma 3. *Let x be a permutation. Denote by $c(x)$ the number of independent cycles of x, and by $c_1(x)$ the number of fixed points of x.*

(i) *Let G be a permutation group and suppose that for every non-identity permutation $g \in G$ we have $c_1(g) \leqslant C$. Then for every* *non-identity permutation $x \in G$ of order d we have*

$$c(x) \leqslant \frac{n-C}{d} + C.$$

(ii) *Let $G = \mathbf{P\Gamma L}_2(q)$ act on the points of the projective line $\mathbb{P}^1(\mathbb{F}_q)$, where $q = p^m$ for a prime p. Then for any non-identity element $x \in G$ we have*

$$c_1(x) \leqslant \begin{cases} 2 & \textit{if } m = 1, \\ q^{1/l} + 1 & \textit{if } m > 1, \textit{ and } l > 1 \textit{ is the minimal divisor of } m. \end{cases}$$

Let $a_1, a_2, \dots a_k \in G$ satisfying $a_1 a_2 \dots a_k = 1$ define a covering $f : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ of degree n. Then by the Riemann–Hurwitz formula

$$(k-2)n + 2 = \sum_{i=1}^{k} c(a_i).$$

Let f be a rational functions of degree $n = q + 1$ with a single multiple pole; suppose that the monodromy group of f is $G \subseteq \mathbf{P\Gamma L}_2(q)$ and that the ramification over ∞ is given by the permutation a_k. Then $c_1(a_k) \leqslant 3$, and by Lemma 3 we have

$$(k-2)(q+1) + 2 = \sum_{i=1}^{k} c(a_i) \leqslant 3 + \sum_{i=1}^{k-1} \left(\frac{q+1-(\sqrt{q}+1)}{\operatorname{ord}(a_i)} + (\sqrt{q}+1) \right)$$

$$\leqslant 3 + \sum_{i=1}^{k-1} \left(\frac{q+1-(\sqrt{q}+1)}{2} + (\sqrt{q}+1) \right) = 3 + (k-1)\left(\frac{q+\sqrt{q}}{2} + 1 \right).$$

Hence

$$(k-1)(q+1) \leqslant q + 2 + (k-1)\left(\frac{q+\sqrt{q}}{2} + 1 \right)$$

$$(k-1)\frac{q-\sqrt{q}}{2} \leqslant q + 2.$$

For $k \geqslant 4$ it follows that $q - 3\sqrt{q} - 4 \leqslant 0$, i.e., $q \leqslant 16$.

§4. Generators and Hurwitz group action

In the previous section, we have shown that there is a finite list of groups that appear as special primitive monodromy groups of functions with a single multiple pole and at least four critical values. In particular, all such groups have degree $\leqslant 24$.

We used a program written for the GAP computer system to check all the primitive permutation groups of degree up to 24. For every group G we found all the sets of generators of a monodromy of genus 0 and computed the braid group orbits. This section presents the results of calculations. Since the complete lists of the orbit elements are too long, we give only one representative for each orbit. To restore the whole orbit it is sufficient to apply Σ, A and Φ.

The corresponding megamaps were shown above. All of them are invariant under the color exchange which, in some cases, requires choosing a particular permutation as the permutation corresponding to the faces of the megamap. For example, in the case 4/6.2 the faces are described by the permutation A, while in case 4/6.3 the faces are described by the permutation Σ.

However, in most cases, by choosing any of the permutations Σ, A or Φ as the permutation corresponding to the faces of the megamap we obtain isomorphic dessins. Such dessins are called self-dual.

$\mathbf{L}_2(5)$ of order 60 PrimitiveGroup(6,1)

4/6.1. $(1^2 2^2, 1^2 2^2, 1^2 2^2, 1^1 5^1)$. Orbit size: 10.

(1) $a_1 = (1,6)(4,5)$ $a_2 = (1,6)(2,3)$ $a_3 = (3,5)(4,6)$

The Hurwitz group H_4 acts by permutations

$$\begin{aligned} \Sigma &= (2,4,8,3,5)(7,10,9) \\ A &= (1,2,5)(3,6,4,9,10) \\ \Phi &= (1,3,7,9,2)(4,6,8) \end{aligned}$$

The corresponding (self-dual) dessin of genus 0 is shown in Fig. 1; its edge rotation group is $\mathbf{A}_{10}$.

$\mathbf{PGL}_2(5)$ of order 120 PrimitiveGroup(6,2)

4/6.2. $(1^2 2^2, 1^2 2^2, 2^3, 1^2 4^1)$. Orbit size: 8.

(1) $a_1 = (1,3)(2,5)(4,6)$ $a_2 = (2,5)(3,6)$ $a_3 = (1,6)(4,5)$

The Hurwitz group H_4 acts by permutations

$$\begin{aligned} \Sigma &= (1,2)(3,6)(4,7,8) \\ A &= (1,3,7)(2,4,8,6,5) \\ \Phi &= (1,4)(2,5,3)(6,7) \end{aligned}$$

The corresponding dessin of genus 0 is shown in Fig. 2; its edge rotation group is $\mathbf{A}_8$.

$\mathbf{PGL}_2(5)$ of order 120 PrimitiveGroup(6,2)

4/6.3. $(1^2 2^2, 1^2 2^2, 1^2 4^1, 1^2 4^1)$. Orbit size: 16.

(1) $a_1 = (2,4)(5,6)$ $a_2 = (1,4)(3,6)$ $a_3 = (1,2,3,4)$

The Hurwitz group H_4 acts by permutations

$$\begin{aligned} \Sigma &= (1,2,5)(3,8,7,6,14)(4,10,13) \\ A &= (1,3,9,10)(2,6)(4,11,7,15,14,12)(5,13,16,8) \\ \Phi &= (1,4,12,6)(2,7,11,13)(3,5)(8,16,10,9,14,15) \end{aligned}$$

The corresponding dessin of genus 1 is shown in Fig. 3; its edge rotation group is imprimitive of order $8! \cdot 8!/2 = 812\,851\,200$.

$\mathbf{L}_3(2)$ of order 168 PrimitiveGroup(7,5)

4/7.1. $(1^3 2^2, 1^3 2^2, 1^3 2^2, 7^1)$. Orbit size: 2×7.

The group $\mathbf{L}_3(2)$ contains two conjugacy classes of elements of order 7, which give two sets of generators with this passport (see Remark 4 on page 15). Each of these sets consists of seven elements; the Hurwitz group acts transitively on each. Here are the representatives for each set:

(1_a) $a_1 = (3,5)(6,7)$ $a_2 = (1,7)(2,5)$ $a_3 = (1,5)(3,4)$

(1_b) $a_1 = (1,6)(2,3)$ $a_2 = (1,7)(2,5)$ $a_3 = (2,6)(4,5)$

The Hurwitz group H_4 acts on both orbits by the same permutations

$$\begin{aligned} \Sigma &= (1,2,5)(3,4)(6,7) \\ A &= (1,3)(2,6)(4,5,7) \\ \Phi &= (1,4,6)(2,7)(3,5) \end{aligned}$$

The corresponding (self-dual) dessin of genus 0 is shown in Fig. 4; its edge rotation group is $\mathbf{A}_7$.

$\mathbf{ASL}_3(2)$ of order 1344 PrimitiveGroup(8,3)

5/8.1. $(1^4 2^2, 1^4 2^2, 1^4 2^2, 1^4 2^2, 1^1 7^1)$. Orbit size: 2×147.

This is the only case in our list when the functions have more than four (namely, five) critical values. The group $\mathbf{ASL}_3(2)$ contains two conjugacy classes of elements of order 7, and the braid group action has two orbits of size 147.

(1_a) $a_1 = (5,6)(7,8)$ $a_2 = (2,3)(5,8)$ $a_3 = (1,8)(4,5)$ $a_4 = (1,8)(3,6)$

(1_b) $a_1 = (5,6)(7,8)$ $a_2 = (2,3)(6,7)$ $a_3 = (1,5)(4,8)$ $a_4 = (1,5)(3,7)$

4/8.1. $(1^4 2^2, 1^4 2^2, 2^4, 1^1 7^1)$. Orbit size: 2×7.

The group $\mathbf{ASL}_3(2)$ contains two conjugacy classes of elements of order 7, which give two orbits with this passport:

(1_a) $a_1 = (3,4)(7,8)$ $a_2 = (1,7)(4,6)$ $a_3 = (1,7)(2,6)(3,5)(4,8)$

(1_b) $a_1 = (1,7)(4,6)$ $a_2 = (3,4)(7,8)$ $a_3 = (1,8)(2,3)(4,5)(6,7)$

The Hurwitz group H_4 acts on both orbits by the same permutations

$$\begin{aligned} \Sigma &= (1,2,5)(3,4)(6,7) \\ A &= (1,3)(2,4,7)(5,6) \\ \Phi &= (1,4)(2,6)(3,5,7) \end{aligned}$$

The corresponding (self-dual) dessin of genus 0 is the same as the megamap in case 4/7.1 (Fig. 4; its edge rotation group is $\mathbf{A}_7$).

4/8.2. $(1^4 2^2, 1^4 2^2, 1^2 2^1 4^1, 1^1 7^1)$. Orbit size: 2×14.

(1_a) $a_1 = (5,6)(7,8)$ $a_2 = (1,8)(3,6)$ $a_3 = (1,4,5,8)(2,6)$

(1_b) $a_1 = (1,8)(3,6)$ $a_2 = (5,6)(7,8)$ $a_3 = (1,6,4,7)(2,3)$

The Hurwitz group H_4 acts on both orbits by the same permutations

$$\begin{aligned} \Sigma &= (1,2,5)(4,8)(9,12,13)(10,14) \\ A &= (1,3,8)(2,6,4,9,13,14,7)(5,10,11) \\ \Phi &= (1,4,6)(2,7,10)(3,5,11,14,12,9,8) \end{aligned}$$

The corresponding (self-dual) dessin of genus 0 is shown in Fig. 5; its edge rotation group is $\mathbf{A}_{14}$.

4/8.3. $(1^4 2^2, 1^4 2^2, 1^2 3^2, 1^1 7^1)$. Orbit size: 2×21.

(1_a) $a_1 = (3,5,7)(4,6,8)$ $a_2 = (2,4)(6,8)$ $a_3 = (1,4)(5,8)$

(1_b) $a_1 = (3,7,5)(4,8,6)$ $a_2 = (1,2)(3,4)$ $a_3 = (1,8)(3,6)$

The Hurwitz group H_4 acts on both orbits by the same permutations

$$\begin{aligned} \Sigma &= (1,2,5)(3,8,15)(4,9,10,14,13,20,16)(6,12,19,7)(11,17,21,18) \\ A &= (1,3,9)(2,6)(4,8,12)(5,10,17)(7,13)(15,21)(18,20) \\ \Phi &= (1,4,6)(2,7,14,10)(3,5,11,18,13,19,12)(8,16,20,21)(9,15,17) \end{aligned}$$

The corresponding dessin of genus 1 is shown in Fig. 6; its edge rotation group is $\mathbf{A}_{21}$.

$\mathbf{PGL}_2(7)$ of order 336 PrimitiveGroup(8,5)

4/8.4. $(1^2 2^3, 1^2 2^3, 1^2 2^3, 1^2 6^1)$. Orbit size: 18.

(1) $a_1 = (2,8)(4,6)(5,7)$ $a_2 = (1,5)(3,7)(4,8)$ $a_3 = (1,5)(2,4)(6,7)$

The Hurwitz group H_4 acts by permutations

$$\begin{aligned}\Sigma &= (1,2)(3,7,12)(4,8,15,10,14,13,17)(5,9,16)(11,18)\\ A &= (1,3,8)(2,5,10,15,12,18,6)(4,9)(7,13)(11,14,16)\\ \Phi &= (1,4,5)(2,6,11,9,17,13,3)(7,14,18)(8,12)(10,16)\end{aligned}$$

The corresponding (self-dual) dessin of genus 1 is shown in Fig. 7; its edge rotation group is $\mathbf{A}_{18}$.

$\mathbf{AGL}_2(3)$ of order 432 PrimitiveGroup(9,7)

4/9.1. $(1^3 2^3, 1^3 2^3, 1^3 2^3, 1^1 8^1)$. Orbit size: 2×16.

The group $\mathbf{AGL}_2(3)$ contains two conjugacy classes of elements of order 8, which give two orbits with this passport:

(1_a) $a_1 = (4,7)(5,8)(6,9)$ $a_2 = (1,5)(3,8)(6,7)$ $a_3 = (1,5)(2,6)(3,4)$

(1_b) $a_1 = (4,7)(5,8)(6,9)$ $a_2 = (1,8)(3,5)(4,9)$ $a_3 = (1,8)(2,4)(5,7)$

The Hurwitz group H_4 acts on both orbits by permutations

$$\begin{aligned}\Sigma &= (1,2)(3,7,11)(4,8,13)(5,9,14)(6,10,12)(15,16)\\ A &= (1,3,8)(2,5,10)(4,9)(6,7)(11,15,13)(12,14,16)\\ \Phi &= (1,4,5)(2,6,3)(7,12,15)(8,11)(9,13,16)(10,14)\end{aligned}$$

The corresponding (self-dual) dessin of genus 0 is shown in Fig. 8; its edge rotation group is imprimitive of order $2^7 \cdot 4! = 3072$.

$\mathbf{P\Gamma L}_2(9)$ of order 1440 PrimitiveGroup(10,7)

4/10.1. $(1^4 2^3, 1^4 2^3, 2^5, 1^2 8^1)$. Orbit size: 8.

(1) $a_1 = (1,3)(2,10)(5,8)$ $a_2 = (1,7)(2,8)(3,9)$
$a_3 = (1,9)(2,10)(3,4)(5,6)(7,8)$

The Hurwitz group H_4 acts by permutations

$$\begin{aligned}\Sigma &= (1,2)(3,5,8)(6,7)\\ A &= (1,3,6,4)(2,5)(7,8)\\ \Phi &= (1,4,7,5)(2,3)(6,8)\end{aligned}$$

The corresponding dessin of genus 0 is shown in Fig. 9; its edge rotation group is imprimitive of order $4! \cdot 4!$.

$\mathbf{M}_{11}$ of order 7920 acting on 12 points PrimitiveGroup(12,1)

4/12.1. $(1^4 2^4, 1^4 2^4, 1^4 2^4, 1^1 11^1)$. Orbit size: 2×33.

The group $\mathbf{M}_{11}$ contains two conjugacy classes of elements of order 11, which give two orbits with this passport:

$$\begin{array}{llll} (1_a) & a_1 = (2,7)(5,9)(6,12)(10,11) & (1_b) & a_1 = (2,7)(5,9)(6,12)(10,11) \\ & a_2 = (3,6)(4,10)(5,8)(9,11) & & a_2 = (3,12)(4,11)(5,10)(8,9) \\ & a_3 = (1,9)(3,11)(4,7)(6,12) & & a_3 = (1,10)(2,11)(3,6)(5,12) \end{array}$$

On the first orbit the Hurwitz group H_4 acts by permutations

$$\begin{aligned} \Sigma &= (1,2,5)(3,8,16)(4,9,18)(6,13)(7,14,21,27,22) \\ &\quad (10,17,11)(12,15,19)(20,25,32)(23,30,33)(24,28,26)(29,31) \\ A &= (1,3,9)(2,6,14)(4,10)(5,7,15)(8,12,11)(13,19,25) \\ &\quad (16,17,24,31,30)(18,23,28)(20,26,21)(22,29,32)(27,33) \\ \Phi &= (1,4,11,19,6)(2,7)(3,5,12)(8,17)(9,16,23)(10,18,24) \\ &\quad (13,20,14)(15,22,25)(21,28,33)(26,32,31)(27,30,29) \end{aligned}$$

The corresponding (self-dual) dessin of genus 1 is shown in Fig. 10; its edge rotation group is $\mathbf{A}_{33}$.

On the second orbit the Hurwitz group H_4 acts by permutations

$$\Sigma' = \Sigma^{-1}, \quad A' = A^{-1}, \quad \Phi' = A\Sigma$$

and the corresponding dessin is mirror-symmetric to the dessin shown in Fig. 10.

$\mathbf{M}_{12}$ of order 95 040 PrimitiveGroup(12,2)

4/12.2. $(1^4 2^4, 1^4 2^4, 1^4 2^4, 1^1 11^1)$. Orbit size: 2×22.

The group $\mathbf{M}_{12}$ contains two conjugacy classes of elements of order 11, which give two orbits with this passport:

$$\begin{array}{llll} (1_a) & a_1 = (1,11)(3,9)(5,6)(7,12) & (1_b) & a_1 = (1,8)(5,10)(6,9)(7,11) \\ & a_2 = (1,8)(5,10)(6,9)(7,11) & & a_2 = (1,11)(3,9)(5,6)(7,12) \\ & a_3 = (1,4)(2,11)(3,9)(6,7) & & a_3 = (1,3)(2,12)(4,8)(5,9) \end{array}$$

The Hurwitz group H_4 acts on each orbit by permutations

$$\Sigma = (1,2,5,12,10)(3,8)(4,9,16)(6,7)(11,18,22)(13,14)(15,17)(19,21,20)$$

$$A = (1,3,9)(2,6)(4,11,19,14,7)(5,13)(8,10,17)(12,20,15)(16,18)(21,22)$$

$$\Phi = (1,4,6)(2,7,13)(3,10)(5,14,20)(8,15,21,18,9)(11,16)(12,17)(19,22)$$

The corresponding (self-dual) dessin of genus 0 is shown in Fig. 11; its edge rotation group is $\mathbf{A}_{22}$.

$\mathbf{L}_3(3)$ of order 5616 PrimitiveGroup(13,7)

4/13.1. $(1^5 2^4, 1^5 2^4, 1^5 2^4, 13^1)$. Orbit size: 4×13.

The group $\mathbf{L}_3(3)$ contains four conjugacy classes of elements of order 13, which give four orbits with this passport:

$$\begin{array}{llll}
(1_a) & a_1 = (1,6)(2,11)(4,13)(9,12) & (1_b) & a_1 = (3,13)(5,12)(6,7)(8,11) \\
 & a_2 = (2,3)(4,9)(7,12)(10,13) & & a_2 = (1,10)(6,12)(7,13)(8,9) \\
 & a_3 = (3,9)(5,7)(6,11)(8,12) & & a_3 = (1,9)(2,12)(4,5)(6,11) \\
(1_c) & a_1 = (1,2)(3,8)(6,11)(9,12) & (1_d) & a_1 = (1,2)(5,13)(6,12)(9,11) \\
 & a_2 = (2,4)(3,9)(5,8)(7,12) & & a_2 = (1,5)(3,10)(4,11)(12,13) \\
 & a_3 = (2,9)(3,10)(4,13)(11,12) & & a_3 = (3,12)(5,8)(7,9)(11,13)
\end{array}$$

The Hurwitz group H_4 acts on the orbits (a) and (b) by permutations

$$\begin{aligned}
\Sigma &= (1,2,5)(3,4)(6,8,11)(7,9,12)(10,13) \\
A &= (1,3)(2,6,9)(4,5,8)(7,10,11)(12,13) \\
\Phi &= (1,4,6)(2,7,8)(3,5)(9,11,13)(10,12),
\end{aligned}$$

and on the orbits (c) and (d) by permutations

$$\begin{aligned}
\Sigma &= (1,2,4)(3,6,10)(5,9,8)(7,12)(11,13) \\
A &= (1,3,7)(2,5)(4,8,13)(6,11)(9,12,10) \\
\Phi &= (1,12,5)(2,8)(3,4,11)(6,13,9)(7,10)
\end{aligned}$$

The corresponding (self-dual) dessins of genus 0 are shown in Fig. 12; their edge rotation group is $\mathbf{A}_{13}$.

A different action of the braid group is considered in [4]; it also gives two different dessins in this case.

$\mathbf{L}_4(2)$ of order 20 160 PrimitiveGroup(15,4)

4/15.1. $(1^7 2^4, 1^7 2^4, 1^3 2^6, 15^1)$. Orbit size: 2×5.

The group $\mathbf{L}_4(2)$ contains two conjugacy classes of elements of order 15, which give two orbits with this passport:

$$
\begin{aligned}
(1_a) \quad a_1 &= (1,13)(2,14)(4,8)(7,11) \\
a_2 &= (1,10)(3,8)(4,15)(6,13) \\
a_3 &= (1,7)(3,5)(8,12)(9,11)(10,14)(13,15) \\
(1_b) \quad a_1 &= (1,13)(2,14)(4,8)(7,11) \\
a_2 &= (1,8)(2,11)(5,12)(6,15) \\
a_3 &= (1,15)(2,10)(3,5)(4,12)(7,9)(11,13)
\end{aligned}
$$

The Hurwitz group H_4 acts on each orbit by permutations

$$
\begin{aligned}
\Sigma &= (1,2)(3,4) \\
A &= (1,3)(2,4,5) \\
\Phi &= (1,4)(2,5,3)
\end{aligned}
$$

The corresponding dessin of genus 0 is shown in Fig. 13; its edge rotation group is $\mathbf{S}_5$.

$\mathbf{AGL}_4(2) = 2^4.\mathbf{L}_4(2)$ of order 322 560 PrimitiveGroup(16,11)

4/16.1. $(1^8 2^4, 1^4 2^6, 1^4 2^6, 1^1 15^1)$. Orbit size: 2×15.

The group $\mathbf{AGL}_4(2)$ contains two conjugacy classes of elements of order 15, which give two orbits with this passport:

$$
\begin{aligned}
(1_a) \quad a_1 &= (2,13)(4,15)(5,10)(7,12) \\
a_2 &= (1,5)(3,14)(4,9)(7,10)(8,13)(12,16) \\
a_3 &= (1,5)(2,14)(3,11)(6,10)(7,15)(9,13) \\
(1_b) \quad a_1 &= (2,13)(4,15)(5,10)(7,12) \\
a_2 &= (1,10)(2,15)(4,6)(7,16)(8,9)(11,13) \\
a_3 &= (1,10)(2,9)(3,11)(4,12)(5,6)(13,14)
\end{aligned}
$$

The Hurwitz group H_4 acts on each orbit by permutations

$$
\begin{aligned}
\Sigma &= (1,2,5)(3,8,4)(6,10,14)(7,11)(9,13)(12,15) \\
A &= (1,3)(2,6,11)(4,9,10)(5,7,12,13,8)(14,15) \\
\Phi &= (1,4,6)(2,7)(3,5)(8,9)(10,13,15)(11,14,12)
\end{aligned}
$$

The corresponding dessin of genus 0 is shown in Fig. 14; its edge rotation group is $\mathbf{S}_{15}$.

$\mathbf{M}_{24}$ of order 244 823 040 PrimitiveGroup(24,1)

4/24.1. $(1^8 2^8, 1^8 2^8, 1^8 2^8, 1^1 23^1)$. Orbit size: 2×46.

The group $\mathbf{M}_{24}$ contains two conjugacy classes of elements of order 23, which give two orbits with this passport:

(1_a) $a_1 = (1,11)(2,7)(3,9)(5,17)(8,15)(13,18)(14,22)(19,24)$

$a_2 = (1,21)(3,23)(6,13)(7,11)(9,14)(15,19)(16,24)(17,18)$

$a_3 = (1,12)(4,6)(7,13)(8,10)(11,20)(14,22)(15,23)(17,19)$

(1_b) $a_1 = (1,11)(2,7)(3,9)(5,17)(8,15)(13,18)(14,22)(19,24)$

$a_2 = (1,2)(3,22)(5,13)(6,18)(8,24)(9,23)(11,21)(16,19)$

$a_3 = (1,6)(2,20)(3,14)(4,18)(8,13)(9,24)(10,15)(12,21)$

The Hurwitz group H_4 acts on the first orbit by permutations

$$\begin{aligned}
\Sigma &= (1,2,5,12,10)(3,8,18)(4,9)(6,15,26)(7,16)(11,21,27)\\
&\quad (13,24,14)(17,28,38)(19,20)(22,33)(23,34,44)(25,37,43)\\
&\quad (29,39,46)(30,41,32)(31,35,45)(36,40,42)\\
A &= (1,3,9)(2,6,16)(4,11,15)(5,13)(7,17,29,40,24)(8,10,20)\\
&\quad (12,22,34)(14,25,33)(18,30,21)(19,23,35)(26,28)\\
&\quad (27,32,38)(31,42,46)(36,37)(39,41)(43,45,44)\\
\Phi &= (1,4,6)(2,7,13)(3,10)(5,14,22)(8,19,31,39,30)(9,18,11)\\
&\quad (12,23,20)(15,27,28)(16,26,17)(21,32)(24,36,25)(29,38,41)\\
&\quad (33,43,34)(35,44)(37,42,45)(40,46)
\end{aligned}$$

The corresponding (self-dual) dessin of genus 0 is shown in Fig. 15; its edge rotation group is $\mathbf{A}_{46}$.

On the second orbit the Hurwitz group H_4 acts by permutations

$$\Sigma' = \Sigma^{-1}, \quad A' = A^{-1}, \quad \Phi' = A\Sigma$$

and the corresponding dessin is mirror-symmetric to the dessin shown in Fig. 15.

§5. Concluding remarks

Examining the list of megamaps presented in this paper we can make the following observations.

1. All these megamaps are invariant under the vertex color exchange. Moreover, some of them are self-dual.

2. In all our cases the action of the braid group on the set of generators $\{(g_1, g_2, \ldots, g_k) \mid g_i \in C_i\}$ is transitive for every tuple of conjugacy classes $(C_1, C_2, \ldots, C_k)$, so we have one megamap for every tuple.

3. Only in three cases (4/6.3, 4/9.1 and 4/10.1) is the action of the braid group imprimitive.

4. Only in three cases (4/10.1, 4/15.1 and 4/16.1) does the action of the braid group contain odd permutations.

Do these observations reflect some general phenomena? Is it possible to find criteria which could predict these properties of megamaps without explicit calculation? Megamaps are interesting objects of study which are not yet well understood.

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