Записки научных
семинаров ПОМИ
Том 445, 2016 г.

G. V. Kuz'mina

GEOMETRIC FUNCTION THEORY. JENKINS RESULTS. THE METHOD OF MODULES OF CURVE FAMILIES

Abstract. Results and applications of the method of modules in geometric function theory are presented. The method was originated by J. A. Jenkins,and further development proceeded in works of the Leningrad–St.Petersburg mathematical school. A retrospective description of the origin of the method is given, and the determining role of Jenkins in the development of the method of the extremal metric is pointed out.

Dedicated to the memory of James Allister Jenkins

The survey is organized in following way.

In the Introduction, a brief account of the history of Geometric Function Theory is given.

In Secs. 1 and 2 of this survey, basic definitions and facts of the theory of modules and the theory of quadratic differentials are described; we preserve the terminology from the Jenkins monograph [41]. These notions and facts are given for completeness of the presentation; they are used throughout in the following parts of this survey everywhere.

In Sec. 3, a short description of the General Coefficient Theorem of Jenkins and some applications of this theorem are given.

Section 4, is devoted to results of the method of modules of curve families, Sec. 5 is concerned with some applications of this method.

Section 6 deals with results of Jenkins related to the symmetrization method and various other questions.

At the end of this review, a list of Jenkins' articles and a list of cited works of other authors are presented.

In the sequel, the following notation is used: $\mathbb{C}$ is the complex plane, $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is the Riemann sphere, $U_R = \{z : |z| = R\}, U_1 := U$, $U(a, \epsilon) = \{z : |z - a| < \epsilon\}$, $C = \{z : |z| = 1\}$.

Key words and phrases: extremal metric, quadratic differential, trajectory, module of curve family, reduced module, extremal decomposition.

Let $M(D,a)$ be the reduced module of a simply connected domain D with respect to a point $a \in D : M(D,a) = 1/2\pi \log R(D,a)$, where $R(D,a)$ is the conformal radius of the domain D with respect to the point a if $a \neq \infty$, $M(D,\infty) = -1/2\pi \log R(D,\infty)$. By $M(D,a,b)$, the reduced module of the bigon D with respect to its vertices a and b is denoted.

INTRODUCTION: GEOMETRIC FUNCTION THEORY. THE ROLE OF J. A. JENKINS IN THE DEVELOPMENT OF THIS THEORY

The objects of study in geometric function theory (for short,the GFT) are classes of functions defined in given simply connected or multiply connected domains or on a Riemann surface. A distinctive characteristic of this theory is that it considers the functions in such classes mainly as mappings possessing some specific geometric properties. The essential role in the problems of the GFT belongs to univalent functions, these functions realize one-to-one mappings. Univalent mappings possess a number of important extremal properties in various general classes of conformal mappings.

Much attention in the GFT has been paid to the following objects. Let S denote the family of functions $f(z)$ regular and univalent in the disk $|z| < 1$ with the expansion in a neighborhood of the origin given by

$$f(z) = z + c_2 z^2 + c_3 z^3 + \dots . \qquad (0.1)$$

Let D be a domain on the z-sphere containing the point at infinity.Let $\Sigma(D)$ denote the family of functions $f(z)$ meromorphic and univalent in D with the Laurent expansion in a neighborhood of infinity given by

$$f(z) = z + \alpha_0 + \alpha_1 z^{-1} + \dots . \qquad (0.2)$$

In particular, if D is the simply connected domain $|z| > 1$, we denote the last class merely by Σ.

The beginning of the GFT has been made in 1907 in works of Koebe on functions in the class S. In the middle of the last century, the theory of univalent functions has already been a sufficiently advanced mathematical discipline in which powerful methods applicable to general families of univalent functions were developed. The first of the deeper methods to be applied in the theory of univalent functions was the parametric method due to Loewner [212]. Grötzsch was first who treated the theory of univalent functions in a unified manner by a single method, namely, by the method of the extremal metric. Several years later, Grunsky [181] treated a number of the same problem by the method of contour integration. Finally, Schiffer [226–228] developed a variational method for treating extremal problems for univalent functions. Schaffer and Spencer [223–225] gave another variant of the method of interior variations. Goluzin [164] applied his version of this method to various extremal problems in the theory of univalent functions. A characteristic of these methods and the results obtained by these methods are given in the Introduction to the monograph of Jenkins [41]; this monograph will repeatedly mentioned in the sequel.

Note that a Bieberbach typothesis influenced the initial development of these classical methods. Namely, in 1916 L.Bieberbach conjectured that in the class S for all $n \geqslant 2$ we have the inequality

$$|c_n| \leqslant n$$

and the equality $|c_n| = n$ occurs only for the Koebe functions $K_\epsilon(z) = z/(1-\epsilon z)^2, |\epsilon| = 1$. The functions $K_\epsilon(z)$ map the disk $|z| = 1$ onto the whole plane with a radial slit.

In the early 50s, a new method arose in the theory of univalent functions, namely, the method of symmetrization. An efficient approach to symmetrization for function theoretic problems was introduced by Pólya and Szegö [218]. This method was initially used in the works of Hayman [183].

That was the situation in the theory of univalent functions when the method of the extremal metric appeared. A more complete description is given in [41, Introduction]. Fundamental forwards steps for the creation of this method are due to Jenkins. The basis of the method of the extremal metric is the Grötzsch method of strips.

In the Jenkins monograph [41], the following estimate of Grötzsch's results is given.

"His approach, called by him the method of strips, represents a very essential improvement over the primitive length-area proofs, operating with the characteristic conformal invariants of doubly-connected domains and quadrangles. He readily obtained most of the then known results and in an outstanding series of papers [169–180] obtained many interesting new results, attacking with equal facility problems for simply-connected domains and for domains of finite connectivity. Notable also are his contributions to the theory of domains of infinite connectivity. ...It is difficult to understand the slowness with which proper recognition came to him. Even to-day, when one feels that his work must be universally known, we find his results being explicitly credited to others ... Perhaps the best measure of the brilliance of his accomplishment is the effort required for some mathematicians at the present time, working with the best tools now available, to rediscover his results, obtained twenty-five years ago and more."

Another direction in the development of the extremal metric method was presented by work of Ahlfors [124] which is also an improvement over the length-area proof.

In 1946, Ahlfors and Beurling [125] gave an important new formulation of the extremal metric method.

The important role in the development of the method of the extremal metric is due to Teichmüller [247-249]. We cite Jenkins' expression [41].

"On the one hand he made explicit the close relationship of this method with Differential Geometry. (However this idea is present implicity and in some places even implicitly in the work of Grötzsch, see especially [177 III, 179].) Even more important was his discovery, based on his study of Grötzsch's results on his own work on quasiconformal mapping, of the essential role played by quadratic differentials. In this connection he formulated a notable principle giving the manner in which quadratic differentials are associated with the solutions of extremal problems particularly in so far as the singularities of the quadratic differential correspond to given data of the extremal problem."

Recall that the Teichmüller principle consists in the following assertion. If in an extremal problem it is assumed that a certain point is fixed and there are no other restrictions, then the quadratic differential has a simple pole at that point. If an addition it is required that the function under consideration in the problem has fixed values for its first n derivatives at this point (in terms of the corresponding local parameter), then the quadratic differential has a pole of order $n+1$ at this point. More generally, the highest derivatives occurrence may not be required to be fixed but some condition on its region of variation may be desired.

However, Teichmüller did not prove any general result that realized this principle in concrete form.

One of the most general results of the method of the extremal metric and all the GFT is the General Coefficient Theorem of Jenkins (for short, the GCT;[41], Theorem 4.1). A more general form of the GCT was given in [50]. The GCT realizes the Teichḿller principle for a wide range of extremal problems.

Briefly about the GCT and its applications will be said in Sec. 3.

Almost simultaneously with the GCT, Jenkins [39 I] discovered the general principle that establishes an equivalence of a module problem for several curve classes and a problem on extremal decomposition of a Riemann surface into a family of domains associated with given curve classes. This principle was the basis for a new method of the GFT referred to as the method of modules of curve families (shortly, the module method or MM). This method was established by the St.Petersburg mathematical school.Results of the module method and its applications are the central theme of this survey.

At the present time, the method of the extremal metric is a general method in the theory of functions.

Along with the development of the extremal metric method, the classical methods of the GFT have also been perfected. For example, a general and rather heuristic form of the area method was worked out. The Lebedev monograph [210] is devoted to this method. The logical completion of the Loewner-Kufarev theory was shaped finally in the studies of Pommerenke [219, 220] and Gutlyanskii [182]. In the series of work by Goryanov, the semigroup aspect of the Loewner–Kufarev method was developed and applied [165–168].

The theory of quadratic differentials led to simplification of the proof and completeness of results of the variational method. The efficiency of the combination of the extremal metric method and the method of symmetrization was shown already in the first investigations by these methods.

To the present time, the method of symmetrization gained obtained unexpected applications and development. Also, with the help of the polarization method Dubinin obtained the solution of the Gonchar problem on condenser capacity, which that induced the interest to this method [133]. Working on the Gonchar problem concerning the harmonic measure, Dubinin created the method of dissymmetrization [132]. Contrary to the classical results, dissymmetrization of a symmetric condenser does not increase its capacity. Polarization and dissymmetrization are widely used in modern investigations.

One of the new symmetrization approaches is the piecewise separating symmetrization of condensers and domains, developed by Dubinin [134]. In a number of cases, the results obtained by this symmetrization can be derived by the method of modules.

A number of investigations due to Dubinin and his school are devoted to asymptotic properties of the capacity of generalized condensers under degeneration of its plates and some applications of this approach [136,137,143]. The indicated approach is parallel to the extremal metric approach to the concept of a reduced module.

For the questions mentioned above see the surveys articles [135, 141], the monograph [138].

On the background of development of new methods, a classical method showed itself unexpectedly. In 1984, L. de Branges [128, 129] proved the Bieberbach hypothesis with the help of the Loewner method [212], which completed almost 70s years history of the existence of this hypothesis. A sufficiently unusual history of de Branges' proof is presented in [161].

In the last decades, in the works of Dubinin and his pupils, a considerable advance was obtained in classical and modern problems for polynomials and entire functions. This progress was attained due to the application of univalent function theory and also potential theory and symmetrization. For this question, see the survey article [139].

1. MODULES AND EXTREMAL LENGTHS

1.1. In Secs.1 and 2, many definitions and facts from [41] are given without references to [41].

Since we shall discuss families of curves on a Riemann surface, we start with the definition of a conformally invariant metric.

Let $\mathfrak{R}$ be a Riemann surface. We say that *conformally invariant metric* $\rho(z)|dz|$ is defined on $\mathfrak{R}$ if every local uniformizing parameter z on $\mathfrak{R}$ gives rise to a real nonnegative measurable function $\rho(z)$ satisfying the following two conditions.

(1) If γ is a rectifiable curve in a planar parametric neighborhood for z, then the integral $\int\limits_{\gamma} \rho(z)|dz|$ exists as a Lebesgue-Stieltjes integral (the case where this integral is equal to $+\infty$ is not excluded).

(2) At every point of intersection of two neighborhood on $\mathfrak{R}$ that are related to local parameters z and z^*, for the corresponding functions $\rho(z)$

and $\rho^*(z^*)$ we have

$$\rho^*(z^*) = \rho(z)|dz/dz^*|.$$

A curve on a Riemann surface $\mathfrak{R}$ is said to be *locally rectifiable* if for every closed arc of this curve lying entirely in some neighborhood on $\mathfrak{R}$ in which a local parameter z is defined, the corresponding arc on the z-plane is rectifiable.

The notion of conformally invariant metric allows us to introduce the length of curves on $\mathfrak{R}$, and also the module and extremal length of a family of curves, which yield a general pattern of defining conformal invariants. We shall use the following L-definition of a module (see [41]).

Let Γ be a family of locally rectifiable curves on a

Riemann surface $\mathfrak{R}$, and let P be the class of conformally invariant metrics $\rho(z)|dz|$ defined on $\mathfrak{R}$ and such that $\rho(z)$ is square integrable in the z-plane for every local uniformizing parameter $z = x + iy$; we assume that the quantities

$$A_\rho(\mathfrak{R}) = \iint\limits_{\mathfrak{R}} \rho^2(z)dxdy, L_\rho(\Gamma) = \inf_{\gamma\in\Gamma} \int\limits_{\gamma} \rho(z)|dz|$$

are not equal to 0 or ∞ simultaneously. Let P_L be the subclass of P defined as follows: for $\rho \in P_L$ and $\gamma \in \Gamma$ we have

$$\int\limits_{\gamma} \rho(z)|dz| \geqslant 1.$$

If the set P_L is not void, then the quantity

$$M(\Gamma) = \inf_{\rho\in P_L} A_\rho(R)$$

is called *the module of the family* Γ. If P is not void and P_L is void, then we put $M(\Gamma) = \infty$. The reciprocal of $M(\Gamma)$ is the extremal length of Γ. If $M(\Gamma) \neq \infty$, then every metric in P_L is said to be admissible. If there exists a metric $\rho^*(z)|dz|$ in P_L such that

$$M(\Gamma) = A_{\rho^*}(\mathfrak{R}),$$

then it is called *an extremal metric of the module problem for the family* Γ.

Most fundamental of the basic properties of modules is the fact that they are conformally invariant. When an extremal metric exists, it is essentially unique (see [41], Theorems 2.1 and 2.2).

1.2. Mention simple examples of modules of curve families.

Theorem 1.1. *Let Q be a quadrangle with vertices denoted by $1, 2, 3, 4$ taken in the natural order on the boundary of Q. Let Γ be the class of locally rectifiable curves in Q joining the sides 12 and 34. The quadrangle can be mapped conformally onto a rectangle R with vertices A_1, A_2, A_3, A_4 so that $1, 2, 3, 4$ correspond respectively to these vertices. Let A_1A_2 have length a, A_2A_3 length b. Then Γ has module $m(\Gamma)$ equal to a/b.*

Theorem 2.2. *Let D be a doubly-connected domain lying in the w-plane for which neither boundary component is reduced to a point. Let Γ be the class of rectifiable Jordan curves lying in D and separating its boundary components, and let Γ' be the class of locally rectifiable curves lying in D and joining its boundary components. The domain D can be mapped conformally onto the circular ring in the z-plane defined by*

$$r_1 < |z| < r_2 \quad (0 < r_1 < r_2).$$

Then Γ has module $m(\Gamma)$ equal to $\frac{1}{2\pi} \log \frac{r_2}{r_1}$, and Γ' has module $m(\Gamma')$ equal to $2\pi / \log \frac{r_2}{r_1}$.

In the terms of modules of doubly-connected domains, well-known lemmas of Grötzsch are formulated with elegance ([41],Theorems 2.6 and 2.7).

Lemma 1.1. *(The first lemma of Grötzsch.) Let $D_i, i = 1, \ldots, n$,be nonoverlapping quadrangles lying in the circular ring $r_1 < |z| < r_2$ $(0 < r_1 < r_2)$, each with a pair of opposite sides on the two bounding circles of that ring. Let D_i have the module M_i for the class of curves joining this pair of opposite sides. Then*

$$\sum_{i=1}^{n} M_i \leqslant 2\pi / \log(r_2/r_1).$$

Equality occurs if and only if the quadrangles D_i are obtained from the ring by radial decomposition so that the sum of areas of the quadrangles is equal to the area of the ring.

The second lemma of Grötzsch establishes an appropriate extremal property for decomposition of a circular ring into nonoverlapping concentric rings.

1.3. The definition of the module of a family of curves can be extended in various ways. One such generalization is related to the notion of the reduced module of a simply connected domain with respect to an interior point.

Let D be a simply connected domain of hyperbolic type, and z_0 a point of D. For $\epsilon > 0$ sufficiently small, the set $D(\epsilon) = D \setminus \overline{U(z_0, \epsilon)}$ is a doubly connected domain. Let $M(D(\epsilon))$ be the module of this domain for the class of curves that separate the boundary component of $D(\epsilon)$. The reduced module of D with respect to z_0 is defined as follows:

$$M(D, z_0) = \lim_{\epsilon \to 0} \{M(D(\epsilon)) + \frac{1}{2\pi} \log \epsilon\}.$$

Let $R(D, z_0)$ be the conformal radius of the domain D with respect to z_0. Then

$$M(D, z_0) = \frac{1}{2\pi} \log R(D, z_0),$$

if $z_0 \neq \infty$, $M(D, \infty) = -\frac{1}{2\pi} \log R(D, \infty)$.

Now we give the definition of the reduced module of a bigon with nonzero integer angles at its vertices, suggested by Emel'yanov (see [148, 198, 156, 238]).

Let D be a simply connected domain of hyperbolic type with two distinguished boundary elements $\tilde{a}_1$ and $\tilde{a}_2$ with supports at different or coinciding points a_1 and a_2 (for definiteness, let $a_1, a_2 \in \mathbb{C}$). We assume that D satisfies the following condition $(*)$: if $\zeta = g(z)$ is the conformal homeomorphism of D onto the strip $-h/2 < \operatorname{Im} \zeta < h/2$ that satisfies $\operatorname{Re} g(\tilde{a}_1) = -\infty, \operatorname{Re} g(\tilde{a}_2) = +\infty$, and ϵ_1 and ϵ_2 are sufficiently small positive numbers, then in the connected component $\Delta_k(\epsilon_k)$ of $D \cap U(a_k, \epsilon_k)$ having $\tilde{a}_k$ as a boundary element we have the relation

$$g(z) = (-1)^{k-1}\{A_k \log(z - a_k) + \tilde{g}_k(z)\}, k = 1, 2,$$

where $A_k > 0$, and $\tilde{g}_k(z)$ is a regular function. It is clear that $\phi_k = h/A_k$ is the interior angle of D at the boundary element $\tilde{a}_k$.

Suppose that D satisfies condition $(*)$. Let Γ be the class of rectifiable curves in D that join the sides of D. We denote by $S_k(\epsilon_k)$ the arc of the circle $|z - a_k| = \epsilon_k$ contained in the boundary of the domain $\Delta_k(\epsilon_k)$. Let $D(\epsilon_1, \epsilon_2)$ be a quadrangle in D with opposite sides $S_k(\epsilon_k), k = 1, 2$. Let $\Gamma(\epsilon_1, \epsilon_2)$ denote the class of locally rectifiable curves in $D(\epsilon_1, \epsilon_2)$ that separate the sides $S_1(\epsilon_1)$ and $S_2(\epsilon_2)$, and let be the module of $D(\epsilon_1, \epsilon_2)$

for the class $\Gamma(2)(\epsilon_1, \epsilon_2)$. The limit

$$M(D) := \lim_{\epsilon_1, \epsilon_2 \to 0} \{M(D(\epsilon_1, \epsilon_2)) + \sum_{k=1}^{2} \phi_k^{-1} \log \epsilon_k\}$$

is called *the reduced module of the bigon D for the class* Γ and is denoted by $M(D, a_1, a_2)$.

Another extension of the notion of the module of curve families is that of the reduced module of a triangle suggested by Solynin [230,238]. Necessary conditions of the existence of this reduced module were investigated also by Emel'yanov [153].

1.4. Important for applications to the theory of univalent functions is the extension obtained by considering simultaneously a number of curve families.The module defined in this manner is a function rather than a number. Jenkins [39 I, II; 41] has proved the existence of an extremal metric in situations of considerable generality. Namely, Jenkins has established the general principle which states the relationship between quadratic differentials and a class of modules for multiple curves families. This principle was a basis of the module method. This result of Jenkins is cited in Sec. 4.2.

2. QUADRATIC DIFFERENTIALS

The notion of quadratic differential is one of the most important notions in more recent geometric function theory. Quadratic differentials without this notion and an explicit general analytic definition have already been presented in Grötzsh's earlier papers, as well as in Schiffer's fundamental lemma from 1938 and its applications to extremal problems in conformal mapping. Teichmüller made quadratic differentials an independent notion and formulated his general principle (see the Introduction). In investigations by Jenkins, the great attention is paid to the theory of quadratic differentials.

Below we give some facts from quadratic differentials theory, following the presentation in [41]. Many results on quadratic differentials are collected in a later monography of Strebel [242].

2.1. Let $\mathfrak{R}$ be a Riemann surface.*A quadratic differential on* $\mathfrak{R}$ is an entity which assigns, to every local uniformizing parameter z on $\mathfrak{R}$, a function $Q(z)$ meromorphic in the neighborhood for the parameter z and satisfying the following condition. If z^* is another local uniformizing parameter on

$\mathfrak{R}$ and $Q^*(z^*)$ is the corresponding function associated with $z*$, and if the neighborhoods for z and z^* overlap, then on the intersection of these neighborhoods we have

$$Q^*(z^*) = Q(z)(dz/dz^*)^2.$$

A point $P \in \mathfrak{R}$ is called *a zero or a pole of order μ of the differential* $Q(z)dz^2$ if for every local uniformizing parameter z,P is represented by a point having this property with respect to $Q(z)$. The zeros and poles of $Q(z)dz^2$ are called critical points. The set of zeros and simple poles of $Q(z)dz^2$ will be denoted by C, and the set of poles of order $\mu \geqslant 2$ will be denoted by H.

A maximal regular curve on $\mathfrak{R}$ on which $Q(z)dz^2 > 0$ (respectively, $Q(z)dz^2 < 0$) is called *a trajectory* (respectively, *an orthogonal trajectory*) of $Q(z)dz^2$.

The trajectories and orthogonal trajectories are intrinsically associated with a given quadratic differential, i.e., they do not depend on the specific choice of a local uniformizing parameter.

2.2. For the first time, the local structure of trajectories has been described by Teichmuller [247] without proof. The first detailed presentation under the additional (inessential) condition of hyperellipticity of the quadratic differential in question has been given by Schaeffer and Spencer [225]. Another proof was suggested by Jenkins [21], who considered the case of a quadratic differential on a Riemann surface. In the case of the Riemann sphere, the local and global structure of trajectories of a quadratic differential is described by Jensen in a chapter of the Pommerenke book [220]. Jensen's treatment uses the conformal mappings to reduce the quadratic differential to a form as simple as possible. In this connection, see also the Strebel monograph [242].

The structure of trajectories near the points of $\mathfrak{R} \setminus H$ is described by the following two theorems [41].

Theorem 2.1. *For any point $P \in \mathfrak{R} \setminus (C \cup H)$ there exists a neighborhood N of P on $\mathfrak{R}$ and a homeomorphism of N onto the disk $|w| < 1$ that takes the maximal open arc of every trajectory in N to a segment on which* $\operatorname{Im} w$ *is constant.*

Thus, each point of $R \setminus (C \cup H)$ belongs to a unique trajectory of the differential $Q(z)dz^2$, which is either an open arc or a closed Jordan curve on $\mathfrak{R}$.

Theorem 2.2. *For each point $P \in C$ of order $\mu(\mu > 0$ if P is a zero and $\mu = -1$ if P is a simple pole), there exists a neighborhood N of P on $\mathfrak{R}$ and a homeomorphism of N onto the disk $|w| < 1$ that takes the maximal open arc of every trajectory on N to an open arc on which $\operatorname{Im} w^{(\mu+2)/2}$ is constant. There are $\mu + 2$ trajectories with limiting endpoints at P; their limiting tangential directions at P are spaced at equal angles of opening $2\pi/(\mu + 1)$.*

The behavior of the trajectories near the points belonging to H turns out to be much more complicated. We give a reduced version of Theorems 3.3 and 3.4 in [41].

Theorem 2.3. *Suppose that $P \in H$ is a pole of order $\mu \geqslant 2$, and let z be a local parameter such that P corresponds to $z = 0$. Let $\epsilon > 0$ be sufficiently small.*

I. *Let $\mu = 2$, and(for some choice of a branch of the square root) let*

$$Q(z)^{1/2} = (a + bi)z^{-1}(1 + b_1 z + \dots), \quad a, b \in R, a + bi \neq 0,$$

in the vicinity of $z = 0$. Asymptotically, the image of every trajectory meeting the disk $|z| < \epsilon$ behaves as a logarithmic spiral for $a \neq 0, b \neq 0$ and as a rectilinear ray for $a \neq 0, b = 0$. If $a = 0$, then the image of every trajectory meeting the circle $|z| = \epsilon$ is a closed Jordan curve lying in the circular annulus $\epsilon - 0(\epsilon) < |z| < \epsilon + 0(\epsilon)$.

II. *Let $\mu \geqslant 3$. Then the image of every trajectory having a limiting endpoint at $z = 0$ tends to this point along $(\mu - 2)$ directions equally spaced at angles of $2\pi/(\mu - 2)$. The image of every trajectory meeting the disk $|z| < \epsilon$ tends to $z = 0$ in at least one sense. If the image of a certain trajectory lies entirely in the disk $|z| < \epsilon$, then it tends to $z = 0$ in two adjacent limiting directions.*

2.3. When we consider the global structure of trajectories of the differential $Q(z)dz^2$, an important part is played by *the set* Φ defined as the union of all trajectories of $Q(z)dz^2$ that have a limiting end point in the set C. The elements of Φ are called *critical trajectories* of the differential $Q(z)dz^2$. Let $\overline{\Phi}$ denote the closure of Φ.

The first general result on the global structure of trajectories was obtained by Jenkins and Spencer [4], where it was shown that in the case of a hyperelliptic quadratic differential, the structure of trajectories is described in terms of domains of four types (the definition of these basic types of domains is given below) together with a finite number of domains in which some of the trajectories belonging to the family Φ are everywhere dense. Later, Jenkins applied the same arguments to positive quadratic differential on a finite Riemann surface. A *quadratic differential on a finite Riemann surface* $\mathfrak{R}$ *is positive* if, in terms of a boundary uniformizing paramerer z, the function $Q(z)$ is regular and positive on the segment of the real axis corresponding to the boundary points of $\mathfrak{R}$ with the exception of the zeros of $Q(z)$ (these zeros are necessarily of even order).

Any positive quadratic differential is automatically regular in the boundary of $\mathfrak{R}$. To make the formulations shorter, we agree that every quadratic differential on a closed Riemann surface (in particular, on the z-sphere) is positive.

The following lemma of an algebraic nature [41, Lemma3.2] establishes a property of quadratic differentials, which is important for applications.

Lemma 2.1. *Consider a positive quadratic differential on a finite Riemann surface* $\mathfrak{R}$ *of genus* g *with* n *boundary components; let* p *be the total order of the poles of this differential and* q *the total order of its zeros, where each zero on the boundary (necessarily of even order) is counted with half of its multiplicity. Then* $p-q=4-4g-2n$.

It follows that in the case $\mathfrak{R}=\overline{\mathbb{C}}$ we have $p-q=4$.

In the definitions of basic types of domains, below $\mathfrak{R}$ is a finite Riemann surface and $Q(z)dz^2$ is a quadratic differential on $\mathfrak{R}$.*An* F*-set* K *with respect to this differential* is a subset of R such that each trajectory of $Q(z)dz^2$ that meets K lies entirely in K. The inner closure of a set K is defined as the interior of the closure of K and is denoted by $\hat{K}$. The inner closure of an F-set is also an F-set.

A ring, circular, strip, end, or density domain for the differential $Q(z)dz^2$ is a maximal connected open F-set possessing the following properties.

(1)*A ring domain* D contains no points of the set $C\cup H$ and is swept out by trajectories of $Q(z)dz^2$, each being a closed Jordan curve. For a suitable choice of a pure imaginary constant c, the function $w = \exp\{c \int Q(z)^{1/2} dz\}$ conformally maps D onto the circular annulus $r_1 < |w| < r_2$.

(2) *A circular domain* C contains a unique double pole A of $Q(z)dz^2$, and $C \setminus A$ is swept out by trajectories of $Q(z)dz^2$, each being a closed Jordan curve separating A from the boundary of C. For a suitable choice of a pure imaginary constant c, the function $w = \exp\{c \int Q(z)^{1/2} dz\}$ extended by zero to the point A conformally maps C onto the disk $|w| < R$ and takes A to $w = 0$.

(3)*A strip domain* S contains no points of the set $C \cup H$ and is swept out by trajectories of $Q(z)dz^2$, each having a limiting endpoint in one direction at a point $A \in H$ and a limiting endpoint in the other direction at some point $B \in H$ (possibly coinciding with A). The function $\zeta = \int Q(z)^{1/2} dz$ conformally maps the domain S onto the strip $a < \operatorname{Im} w < b$.

The local structure of the trajectories of the differential $Q(z)dz^2$ implies that A and B must be poles of $Q(z)dz^2$ of order $\geqslant 2$.

(4) *An end domain* E contains no points of the set $C \cup H$ and is swept out by trajectories of $Q(z)dz^2$ each having a limiting endpoint at one and same point $A \in H$ in each of the two possible directions. The function $\zeta = \int Q(z)^{1/2} dz$ conformally maps the domain E onto the upper or the lower half-plane of the ζ-plane (depending on the choice of a branch of the square root).

The point A must be a pole of $Q(z)dz^2$ of order $\geqslant 3$.

A density domain F contains no points of the set H, and $F \setminus C$ is swept out by trajectories of $Q(z)dz^2$,each being everywhere dense in F.

2.4. The global structure of trajectories is described by the Basic Structure Theorem (for short,the BST) in [41]. Here we give a short version of this theorem.

Theorem 2.4. *Let* $\mathfrak{R}$ *be a finite Riemann surface and* $Q(z)dz^2$ *be a positive quadratic differential on* $\mathfrak{R}$. *Assume that this configuration is not conformally equivalent to any of the following possible cases:* (1) $\mathfrak{R}$ *is the* z*-sphere,* $Q(z)dz^2 = dz^2$; (2) $\mathfrak{R}$ *is the* z*-sphere,* $Q(z)dz^2 = Ke^{i\alpha}dz^2/z^2$, $K > 0$, α *is real;* (3) $\mathfrak{R}$ *is a torus,* $Q(z)dz^2$ *is regular on* $\mathfrak{R}$. *Then* $\mathfrak{R} \setminus \overline{\Phi}$ *consists of a finite number of ring, circular, strip, and end domains.*

Each pole of $Q(z)dz^2$ *of order* $\mu = 2$ *has a neighborhood contained in a circular domain, or a neighborhood covered by the inner closures of finitely many strip domains, and each pole of order* $\mu \geqslant 3$ *has a neighborhood* *covered by the inner closures of $\mu - 2$ end domains and finitely many (possibly, none) strip domains.*

The inner closure $\hat{\Phi}$ of the set Φ need not be empty. If $\hat{\Phi} \neq \emptyset$, then $\mathfrak{R}$ contains domains in which every trajectory is everywhere dense.

The question of whether every trajectory had a point set closure which was either an arc or a Jordan curve, i.e., whether conversely there could be recurrent trajectories was considered by Schaeffer and Spencer [225]. They showed in particular that there could be no recurrent trajectory in the case of a differential with one or two poles and obtained the same result for a particular type of the meromorphic quadratic differential with three poles. They expected and were trying to prove that this was the general situation.

Jenkins proved that the only general circumstances in which one can affirm the absence of recurrent trajectories for positive quadratic differentials on finite Riemann surface are in the case of schlichtartig domains and when the total number of poles and boundary components is at most three. In the case $\mathfrak{R} = \overline{\mathbb{C}}$, the Three Pole Theorem is as follows [41].

Theorem 2.5. *Let $Q(z)dz^2$ be a quadratic differential on $\overline{\mathbb{C}}$ having at most three distinct poles. Then the set $\hat{\Phi}$ is empty.*

Note that for the quadratic differentials

$$Q(z)dz^2 = e^{i\alpha}[(z^2-1)(z-a)]^{-1}dz^2, \quad \alpha \in \mathfrak{R},$$

with four distinct poles $\pm 1, a, \infty$, the set $\hat{\Phi}$ is empty only for countably many values of α. In each of these cases,$\overline{\Phi}$ consists of two analytic arcs connecting some pair of points among $\{-1, 1, a, \infty\}$, and the domain $\overline{\mathbb{C}} \setminus \overline{\Phi}$ realizes the maximum of the conformal module in the corresponding family of doubly connected domains on $\overline{\mathbb{C}}$.

Theorem 2.4 has turned out to be sufficient for many applications and for the proof of the GCT,but it leaves open the question of the structure of trajectories in domains containing an everywhere dense trajectory. An answer to this question is given by the Extended Form of the Basic Structure Theorem obtained in [51]. Let $\Lambda \in \Phi$ be the union of all trajectories of $Q(z)dz^2$ one of whose limiting endpoints is a point of C, and the other one is a point of $C \cup H$.

Theorem 2.6. *Let the conditions of Theorem* 2.4 *be fulfilled. Then* $R \setminus \overline{\Lambda}$ *consists of a finite number of ring, circular, strip, end, and density domains.*

2.5. The facts concerning the structure of trajectories of quadratic differentials are widely used in the GFT. In many investigations, poles of the associated quadratic differential are free parameters. The facts on the structure of the trajectories in some cases allow one to establish a symmetry in the arrangement of these poles, which leads to the solution of the problem considered. One of these facts is the following lemma of Pirl [217].

Lemma 2.2. *Let* $Q(z)dz^2$ *be a meromorphic quadratic differential on* $\overline{\mathbb{C}}$. *Let* γ *be a critical trajectory of this differential,* a *and* b *be the limiting endpoints of* $\gamma, a \neq b$. *Assume that the segment* $[a, b]$ *has no common points with* γ *and that on the domain bounded by the curve* γ *and the segment* $[a, b]$ *critical points of* $Q(z)dz^2$ *are not present. Then on the interval* (a, b) *at least one point of tangency with a trajectory of* $Q(z)dz^2$ *is present.*

A recent example of the usage of this lemma is the work [157] devoted to the Vuorinen problem.

3. THE GENERAL COEFFICIENT THEOREM AND ITS APPLICATIONS

3.1. As was already noted, one of the most general results of the method of the extremal metric is the General Coefficient Theorem of Jenkins ([41], Theorem 4.1). This theorem (for short,the GCT) is the central topic of the monograph [41], a more general form of the GCT was given in [50]. The GCT realizes the Teichmuller principle for a wide range of extremal problems.

Within the limits of the present survey, we restrict ourselves to a general characterization of that theorem. In the GCT one considers a positive quadratic differential $Q(z)dz^2$ on a finite Riemann surface $\mathfrak{R}$, a family $\boldsymbol{\Delta}$ of domains Δ_j on $\mathfrak{R}$ admissible with respect to this differential, and an admissible family $\mathbf{f}$ of functions f_j associated with $\boldsymbol{\Delta}$. It is assumed that $Q(z)dz^2$ has poles $P_1, \dots, P_n$ of order at least 2.

By *an admissible family* $\boldsymbol{\Delta}$ *of domains* $\Delta_j, j = 1, \dots, k$, *on* $\mathfrak{R}$ *with respect to* $Q(z)dz^2$, we mean the complement on $\mathfrak{R}$ of the union of a finite set of trajectories of $Q(z)dz^2$each of which is either closed or has a limiting end point in each sense at a point of C, possible end points of these trajectories and a finite number of arcs in $\mathfrak{R}\setminus H$ on closures of trajectories.

According to this definition, every point of H is interior to a domain Δ_j.

An admissible family $\mathbf{f}$ *of functions* f_j, $j = 1, \ldots, k$, associated with Δ is a family, with the following properties: (1) the functions f_j conformally map the domains Δ_j onto nonoverlapping domains on $\mathfrak{R}$; (2) if A is a pole of the differential $Q(z)dz^2$ in Δ_j, then $f_j(A) = A$; (3) if A is a pole of $Q(z)dz^2$ in Δ_j of order at least 2 and A is mapped by the local parameter z to the point at infinity, then the coefficients of the expansions of the functions $Q(z)$ and $f_j(z)$ in terms of the same parameter are subject to certain normalization conditions; (4) finally, the family $\mathbf{f}$ satisfies some conditions of a topological nature.

The GCT provides an inequality for a certain functional; the latter involves coefficients of the expansions of $Q(z)$ and $f_l(z)$ near the poles P_j, $j = 1, \ldots, n$, and a statement on the equality cases in this inequality.

In the proof of the GCT, the key point is the invocation of the Basic Structure Theorem and of the extremal properties of the Q-metric $|d\zeta| = |Q(z)|^{1/2}|dz|$. In accordance with the BST, which is conformally invariant on $R \setminus H$, some special neighborhoods $U(P_j, L)$ of the points $P_j \in H$ are introduced(L is a real parameter, and the neighborhood $U(P_j, L)$ contracts to the point P_J as $L \to \infty$). Let $\Delta_j(L)$ be the domains obtained from Δ_j by deleting these neighborhoods,

$$\Delta_i(L) = \Delta_i \setminus \bigcup_{j=1}^{n} \overline{U}(P_j, L), \quad i = 1, \ldots, k.$$

For the areas in the Q-metric of the images of the domains $\Delta_i(L)$ under the mappings realized by the functions in $\mathbf{f}$, some lower and upper estimates are established in terms of the areas of $\Delta_i(L)$ in the same metric. Combination of the estimates obtained leads to the inequality of the GCT. Equality in this inequality occurs only if the mappings realized by the functions in $\mathbf{f}$ are isometric in the Q-metric and every trajectory of the differential $Q(z)dz^2$ is mapped again to a trajectory by the corresponding function in $\mathbf{f}$. Furthermore, no open set on $\mathfrak{R}$ can be exterior relative to $\cup_{i=1}^{k} f_i(\Delta_i)$.

The General Coefficient Theorem has passed through a number of consequent extensions and generalizations [23,41,50,60,62].The Extended Form of the GCT is presented in [50], where the normalization conditions of the GCT for admissible functions f_l are weakened, and result obtained is applied to a broader range of problems. The proof in [50] required of additional considerations related to the change of the uniformizing parameter.

By the GCT we shall always mean the Extended Form of the GCT.

The success in applying the GCT depends on the right choice of a differential $Q(z)dz^2$, an admissible family Δ of domains, and an admissible family $\mathbf{f}$ of functions.

The proof of the GCT in the case $\mathfrak{R} = \overline{\mathbb{C}}$ was reproduced by Jensen [220].

As the monograph of Jenkins [41] shows, the force of the GCT (already in its initial form)is such that it includes as corollaries practically all known results about of univalent functions. These results are presented with significant simplification and uniformity of proofs. The GCT has led to solution of new, by statement, extremal problems.

3.2. Dwell completely briefly on some applications of the GCT. By means of GCT, Jenkins established [48] significantly more complete results than those obtained previously in the class S_R of functions $f \in S$ with real coefficients $c_2, c_3, \dots$ in the expansion(1.1). In particular, he found a geometrically explicit condition determining the Koebe set, say, $K(S_R)$, for the class S_R (see Sec. 5.3). The region of values of $f(z_0)$ in the class S_R, where z_0 is an arbitrary point of U, is determined in [48] by an analogous condition.

In [49 I] Jenkins worked out in detail a low order version of the GCT and established a number of new results for the classes S and Σ. In [49 II] he obtained a number of sharp estimates for the coefficients in the classes S and M, where M is the class of functions $f(z)$, meromorphic and univalent in $|z| < 1$ with the expansion $f(z) = c_1 z + c_2 z^2 z^2 + \dots$ in a neighborhood of the origin.

These papers aroused great interest in the problem of estimating the coefficients for the functions in the class S and Σ for which certain coefficients satisfy prescribed conditions (for example, are real numbers).In this connection, we mentioned the works of Y.Kubota in which, with the help of the GCT, sharp estimates are found for $\operatorname{Re} \alpha_4$ in the class of functions $f(z) \in \Sigma$ with real coefficient α_1 [188], and for $\operatorname{Re} \alpha_5$ in the class of functions in Σ with real coefficients α_1 and α_2 in the expansion (1.1) [187]. The estimate obtained for $\operatorname{Re} \alpha_4$ is the first disproof of the conjecture that $|\alpha_n| \leqslant 2/(n+1)$ in the class Σ for even $n \geqslant 2$. Indicated results are not strengthened in the present time. Phelp [216] determined the range of (c_2, c_3, c_4) In the class S_R.

The GCT gave rise to uniqueness results in the theory of extremal problems connected with the coefficient problem of univalent functions [66].Using the uniqueness results, Babenko [126] and independently Pfluger [216] established the property of convexity of the corresponding sections of the nth body V_n in the class S, i.e., the region of values of the system $(c_2, \ldots, c_n)$of coefficients in this class. This property of the body V_n in the small had been established earlier by Duren and Schiffer [145].

As Jenkins indicates,the selection of special Riemann surfaces and quadratic differentials in the GCT gives rise to whole new classes of problems for univalent functions. In [41], Jenkins introduces the class $\Sigma(r)$ of functions from Σ, which map $|z| > 1$ onto a domain whose complement contains a domain with inner conformal radius with respect to the origin at least r, $0 < r < 1$. Concerning the results of Jenkins and other authors for the class $\Sigma(r)$, see Sec. 5.5.

For some applications of the GCT and related results, see the survey article of Jenkins [122]; some of these results are cited in Sec. 5.1.

4. METHOD OF MODULES OF CURVE FAMILIES. EXTREMAL DECOMPOSITION PROBLEMS

4.1 Even in the early works by Jenkins [20,22 I,II, 32] the efficiency of the notion of the module of a family of several curve classes in combination with results of the symmetrization method has been with the example of the solution of difficult extremal problems for univalent function theory. These results are mentioned in Sec. 5.1.

4.2 In [39 I, II] Jenkins established a general principle, which states the relationship between the quadratic differentials and an important class of modules for multiple curve families.It played a defining role for development of the method of modules of curve families. In this method, problems on the extremal decomposition are considered. These problems are related with finding the maximum of a functional defined on the family $\mathcal{D}$ of systems D_i associated with a family $\mathcal{H}$ of homotopy classes of curves H_i; this functional is a linear combination

$$\sum_i \alpha_i^2 M_i(D_i)$$

of functions of the domains D_i (modules or reduced modules of D_i associated with the classes H_i), the α_i^2 being real parameters.

Citing a theorem from [39, I], we preserve the Jenkins' formulations almost word for word. We need some definitions.

Let $\mathfrak{R}$ be a finite Riemann surface, and let there be given a set $A = \{\alpha_k\}_{k=1}^n$ of distinct points. On $\mathfrak{R}' = \mathfrak{R} \setminus A$ we consider a free family $\mathcal{H} = \{H_k\}_{k=1}^{n_1+n_2}$ of homotopic classes of locally rectifiable curves of the following two types. *The first type* consists of classes H_i, $i = 1, \dots, n_1$, of closed Jordan curves not homotopic to zero on $\mathfrak{R}'$. If $\mathfrak{R}$ actually has boundary components, then *the second type* consists of classes $H_i, i = n_1+1, \dots, n_1+n_2$, of arcs on $\mathfrak{R}'$ connecting some boundary element of $\mathfrak{R}$. Let $\{\alpha_k\}_{k=1}^{n_1+n_2}$ be a system of positive numbers.

First consider the module problem $P(\alpha_1, \dots, \alpha_{n_1+n_2})$ consisting of finding the module $\mathcal{M}(\alpha_1, \dots, \alpha_{n_1+n_2})$ defined as $\inf \iint\limits_R \rho^2 dA$ in the class of metrics satisfied the condition

$$\int\limits_{\gamma_k} \rho|dz| \geqslant \alpha_k$$

for every rectifiable curve $\gamma_k \in H_k, k = 1, \dots, n_1 + n_2$.

Now consider a problem on extremal decomposition in an admissible family of domains associated with the family $\mathcal{H}$. This family is defined in following way.

We call a doubly-connected domain D lying on $\mathfrak{R}'$ associated with the homotopy class H of the first type if the class of simple closed curves lying in D and separating its boundary components is contained in H. In this case, we refer to the module of D for this class of curves as likewise associated with H. We call a quadrangle D lying on $\mathfrak{R}'$ associated with the homotopy class H of the second type if a pair of opposite sides of D lies respectively on the boundary components of $\mathfrak{R}$ joined by arcs in H and if the class of arcs lying in D and joining these sides is contained in H.In this case, we refer to the module of D for this class of curves as likewise associated with H.

By an admissible family $\mathcal{D}$ of domains associated with a free family of homotopy classes $H_i, i = 1, \dots, n_1 + n_2$, we mean a finite number of domains each associated with a class H_i (a doubly-connected domain or to whether quadrangle according as H_i is of first or second type)and no more than one associated with any such class.Let $M_k(D_k)$ be the module of D_k associated with the class $H_k, k = 1, \ldots, n_1 + n_2$.

We ask for the maximum of $\sum\limits_{k=1}^{n_1+n_2} \alpha_k^2 M_k(D_k)$ in the family $\mathcal{D}$.

The following theorem shows that these two values are the same, and a unique extremal configuration corresponds to a positive quadratic differential on $\mathfrak{R}$.

Theorem 4.1. *Let the previous conditions be fulfilled. Then for the module problem $P(\alpha_1, \ldots, \alpha_{n_1+n_2})$, there exists an extremal metric $\rho^*(w)|dw|$. This metric has the form $|Q(w)|^{1/2}|dw|$ where $Q(w)dw^2$ is a quadratic differential on $\mathfrak{R}$ regular apart from possible simple poles at the distinguished points.*

If $\mathfrak{R}$ is not a closed surface of genus 1 or a doubly-connected domain (in either case without distinguished points), then the trajectories of $Q(w)dw^2$ which have limiting end points at its finite critical points together with those which pass through distinguished points divide $\mathfrak{R}$ into an admissible family D^* of domains $D_i^*, i = 1, \ldots, n_1 + n_2$, associated with the given free family of homotopy classes H^i. If M_i^* is the associated module for the domain D_i^*, then

$$\mathcal{M}(a_1, \ldots, a_k) = \sum_{i=1}^{n_1+n_2} \alpha_i^2 M_i(D_i^*).$$

For an admissible family D of domains $D_i, i = 1, \ldots, n_1 + n_2$, associated with a given free family of homotopy classes H_i, if M_i is the associated module for the domain D_i, then

$$\sum_{i=1}^{n_1+n_2} \alpha_i^2 M_i(D_i) \leqslant \mathcal{M}(a_1, \ldots, \alpha_{n_1+n_2}). \tag{4.1}$$

Subject to the previous exclusions, equality in (4.1) may occur only for the family D^*.

The proof of Theorem 4.1 in [39 I] was obtained with the help of Schiffer's variational method, the proof in [39 II] is based on the method of the extremal metric only.

Similarly to the GCT, Theorem 4.1 establishes the determining role of quadratic differentials in the conformal mapping problems. Later, investigations of many authors were devoted to various questions of the theory of quadratic differentials including its role in problems on extremal decomposition and their connections with topology and differential geometry. Renelt [222] considered the problem on the greatest lower bound of the sum

$$\sum_i \alpha_i^2 M_i^{-1}(D_i)$$

(we use the former notation). In this connection, see the Jenkins' work [119]. Tamrazov [245] obtained a supplement to the GCT in the case where the associated quadratic differential $Q(z)dz^2$ does not have poles of order greater than 1.

Let us give some examples of another character. Many results on quadratic differentials are collected in the Strebel monograph [242]. A holomorphic quadratic differential on a compact Riemann surface such that all of its trajectories explaining the critical ones are closed is called *the Jenkins–Strebel differential* by some authors. In 1974, Strebel conjectured that on a compact Riemann surface such differentials are dense in the space of all holomorphic quadratic differentials. This was proved by Douady and Hubbard [131]. The properties of the Jenkins–Strebel differentials have been studied in many papers. We does not dwell on these papers.

Even in [39 I], it has been mentioned that the result of this paper can be extended to the case of a family $\mathcal{H}$ of homotopy classes H_i of curves on a Riemann surface R of three types: the family $\mathcal{H}$ contains, along with the classes H_i considered above,the classes H_l of closed curves homotopic to point contours at distinguished points $b_l \in R$. Properties of quadratic differentials with closed trajectories and second order poles were considered by Strebel. Mention one of his results. Let Q be a quadratic differential on a compact Riemann surface with closed trajectories which has double poles P_j. The critical trajectories will cut out certain simply connected domains D_j containing P_j. Let r_j be the conformal radius of D_j with respect to P_j in term of a given local parameter at P_j. Strebel proved the existence of a unique differential Q for which the ratios of the r_j have prescribed values (see [242]).

A simple proof of the general result which is the extension of Theorem 4.1 indicated above in the case of a planar surface $\mathbb{S}$ where $\mathbb{S} = \overline{\mathbb{C}}$ or $\mathbb{S}$ is a simply connected domain on $\overline{\mathbb{C}}$, was given by the author [194, Theorems 0.1 and 0.2]. More precisely, Theorem 0.1 in [194] establishes that in the case $\mathfrak{R} = \overline{\mathbb{C}}$, the extremal metric problem for a family $\mathcal{H}$ as above is equivalent to an extremal decomposition problem that deals with the maximum of a functional involving a linear combination of modules of doubly connected domains and the reduced modules of simply connected domains D_l with respect to some points $b_l \in D_l$. The extremal system of domains of this problem is defined by an associated quadratic differential having at the points b_l poles of second order with circled structure of trajectories.

The result of [194] have found a great number of applications (some of them are mentioned below).

In the works of several authors [148,156,238], the results in [39,194] were extended to a more general case where the family $\mathcal{H}$ consists of classes of four or more types and the associated quadratic differential has poles of second order with the radial or spiral structure of trajectories.

To give a complete statement of the problem in question, we need some definitions. To make the presentation simpler for understanding, we preserve the stile of the presentation from [39 I]. For brevity, we have not considered the case of the spiral structure of trajectories.

4.3. In the sequel, $\mathfrak{R}$ is a finite Riemann surface. Let

$$A = \{a_k\}_{k=1}^{n}, \quad B^{(0)} = \{b_l^{(0)}\}_{l=1}^{m}, \quad B = \{b_k\}_{k=1}^{r}$$

be some sets of distinct points on $\mathfrak{R}$ and on the boundary of $\mathfrak{R}$ if the latter is nonempty, where the points from $B^{(0)}$ and B belong to $\mathfrak{R}$ (one or two of these sets may be empty, but not all). We assume that a fixed local parameter is chosen in the vicinity of each point from $A \cup B^{(0)} \cup B$.

Let $\mathfrak{R}' = \mathfrak{R} \setminus \{A \cup B^{(0)} \cup B\}$. On $\mathfrak{R}'$, we consider homotopic classes of locally rectifiable Jordan curves of the following four types. The classes $H_1, \dots, H_{n_1}$ of the first type and the classes $H_{n_1+1}, \dots, H_{n_1+n_2}$ of the second type and domains associated with these classes (doubly-connected domains and quadrangles) are defined as in Theorem 4.1.

The third type consists of classes $H_{n_1+n_2+1}, \dots, H_{n_1+n_2+m}$ of closed curves, each of which consists of curves separating one of the points $b_l^{(0)} \in B$ from the other distinguished points on $\mathfrak{R}$ and from the boundary of $\mathfrak{R}$ if it exists, hence they are homotopic to the pointwise curve at the point $b_l^{(0)}$. A simply connected domain D on $\mathfrak{R}' \cup b_l^{(0)}, b_l^{(0)} \in D$, will be called associated with a class H of the third type if the family of closed Jordan curves separating the point $b_l^{(0)}$ from the boundary of D is contained in H.

Finally, if $B \neq \emptyset$, then *the fourth type* consists of classes $H_{n_1+n_2+m+s} = H_s^{(1)}$, $s = 1, \dots, p$, of arcs on R' with ends at not necessarily distinct points

$b_{k'(s)}, b_{k''(s)} \in B$. It is assumed that each one of the points $b_k \in B$ is an end of arcs belonging to one or several of the classes of the fourth type.

A bigon D on $\mathfrak{R}'$ having vertices at the points of the set B is called associated with a class H of the fourth type if the family of arcs in D connecting the vertices of D is contained in H. In this case, we assume that the domain D satisfies condition $(*)$ with respect to its vertices (see the definition of Sec.1.3).

According to which one of the four cases indicated above takes place, the module $M(D)$ of the doubly connected domain D for the class of curves separating its boundary components, the module $M^{(1)}(D)$ of the quadrangle D for the class of arcs connecting its opposite sides on the boundary of R, the reduce module $M(D, b_l^{(0)})$ of the simply connected domain D with respect to the point $b_l^{(0)} \in D$, or the reduce module $M(D, b_{k'}, b_{k''})$ of the bigon D with respect to its vertices $b_{k'}, b_{k''}$ will be called associated with the class H. The values of all these modules are defined by the choice of a fixed local parameter in the vicinity of each one of the points from $A \cup B^{(0)} \cup B$. We assume that all classes H_i are determined by systems of points $A, B^{(0)}$,and B in such a way that for each one of the domains D associated with one of these classes, the module of D associated with this class is bounded from above (and from below in the case of the reduced module of a bigon) by some constant that depends only on the position of the points from $A, B^{(0)}$, and B but not on the choice of the domain D.

By *an admissible system of domains D_i associated with a family $\mathcal{H}$ of classes H_i, $i = 1, \ldots, n_1 + n_2 + m + p$*, we mean a finite number of nonoverlapping domains on $\mathfrak{R}' \cup B^{(0)}$ such that each of them is associated with a certain class H_i and no two are associated with the same class. If for a certain class H_i of the first or the second type none of the domains indicated is associated with H_i, then the corresponding domain D_i is said to be degenerate, and by the module associated with such a class H_i we mean 0.

The family of all admissible systems of domains D_i, $i = 1, \ldots, n_1 + n_2 + m + p$, associated with the family $\mathcal{H}$, is denoted by $\mathcal{D}_{\mathfrak{R}}$.

Let

$$\alpha = \{\alpha_i\}_{i=1}^{n_1+n_2+m}, \quad h = \{h_s\}_{s=1}^{p}$$

be two given sets of positive numbers, and let

$$\alpha_k(h) = \sum_{s \in I_k} h_s,$$

where I_k is the set of all indices $s \in \{1, \dots, p\}$ such that the arcs from the class $H_{n_1+n_2+m+s} := H_s^{(1)}$ have limiting endpoints in one or two directions at the point $b_k^{(1)}$ (in the latter case, the corresponding index s occurs in I_k twice).

We assume that the interior angles ϕ_k of the bigons $D_s^{(1)}, s = 1, \dots, p$, at the vertices $b_k^{(1)}$ satisfy the condition

$$\phi_k = 2\pi \frac{h_k}{\alpha_k(h)}, \quad k = k'(s), k''(s).$$

The family of systems of domains in $\mathcal{D}_{\mathfrak{R}}$ that satisfy this condition is denoted by $\mathcal{D}_{\mathfrak{R}}(h)$.

For fixed systems α and h, we consider the following functional on the family $\mathcal{D}_{\mathfrak{R}}(h)$:

$$F_R(\alpha, h) = \sum_{i=1}^{n_1} \alpha_i^2 M(D_i) + \sum_{i=n_1+1}^{n_1+n_2} \alpha_i^2 M^{(1)}(D_i)$$
$$+ \sum_{l=1}^{m} \alpha_{n_1+n_2+l}^2 M(D_{n_1+n_2+l}, b_l^{(0)}) - \sum_{s=1}^{p} h_s^2 M(D_s^{(1)}, b_{k'(s)}, b_{k''(s)}). \quad (4.2)$$

Now we can state the theorem on the extremal decomposition in the family $\mathcal{D}_{\mathfrak{R}}(h)$. Below, by a critical trajectory of the quadratic differential $Q(z)dz^2$ we mean a trajectory that has its limiting endpoint at a zero or at a simple pole of this differential or passes through a point from the set A.

Theorem 4.2. *Let the above-formulated condition be fulfilled. Then there exists a meromorphic quadratic differential $Q(z)dz^2$ on $\mathfrak{R}$ uniquely determined by the following conditions.*

The differential $Q(z)dz^2$ has simple poles at the points $a_j \in A$ (possibly, not at all of these points), double poles at each one of the points $b_l^{(0)} \in B^{(0)}$ and $b_s \in B$, and has no other points on $\mathfrak{R}$.

Let Φ_R be the union of all critical trajectories and arcs of critical trajectories of $Q(z)dz^2$ lying on $\mathfrak{R}$, and let $\overline{\Phi}_{\mathfrak{R}}$ be the closure of $\Phi_{\mathfrak{R}}$. The inner closure $\hat{\Phi}_{\mathfrak{R}}$ of the set $\Phi_{\mathfrak{R}}$ is empty and $\mathfrak{R} \setminus \overline{\Phi}_{\mathfrak{R}}$ is the union of the domains $D_i^*, i = 1, \dots, n_1 + n_2 + m + p$, of the family $\mathcal{D}_{\mathfrak{R}}(h)$.

It is assumed that none of the domains D_i^*, $i = 1, \dots, n_1 + n_2$, *are degenerate. The lengths of the trajectories of* $Q(z)dz^2$ *in the domain* D_i^*, $i = 1, \dots, n_1$, *the closures of the arcs of the trajectories of* $Q(z)dz^2$ *in the domain* D_i^*, $i = n_1 + 1, \dots, n_1 + n_2$,, *and the trajectories of* $Q(z)dz^2$ *in the domain* D_i^*, $i = n_1 + n_2 + 1, \dots, n_1 + n_2 + m$, *are equal to* α_i. *The lengths in the* Q*-metric of the closures of the arcs of the orthogonal trajectories of* $Q(z)dz^2$ *in the domain* $D^*_{n_1+n_2+m+s}$, $s = 1, \dots, p$, *are equal to* h_s.

The system of domains $\{D_i^*\}_{i=1}^{n_1+n_2+m+p}$ *is the only system realizing the maximum of the functional* (4.2) *in the family* $\mathcal{D}_{\mathfrak{R}}(h)$.

Corollary 4.1. *From the metrical conditions of Theorem 4.1, we obtain differential equations for the functions* $g_i(z)$ *mapping the domains* D_i^* *onto a circular ring, a quadrangle, a disk, or a strip,respectively.*

In terms of a local parameter z such that $z(b_l^{(0)})$ (respectively, $z(b_k) = 0$), the function $Q(z)$ has the expansions

$$Q(z) = -\frac{\alpha_i^2}{4\pi^2} z^{-2} + \dots \qquad \text{if} \quad b_l^{(0)} \in B^{(0)};$$

$$Q(z) = \frac{\alpha_k(h)^2}{4\pi^2} z^{-2} + \dots \qquad \text{if} \quad b_k \in B.$$

Remark 4.1. Theorem 4.2 was first proved by Emel'yanov [148] (in the case $\mathfrak{R} = \overline{\mathbb{C}}$). In the paper of Emelyanov and the author[156], Theorem 4.2 was extended to the case where the family of domains in question contains biangles associated with classes of arcs asymptotically similar at the distinguished point on $\mathfrak{R}$ to logarithmic spirals of given slopes. Solynin [238] proved the theorem on extremal decomposition of $\mathfrak{R}$ in the family of domains of six types; along with the domains considered in Theorem 4.2, this family contains triangles with the vertices on $\mathfrak{R}$ and $\partial\mathfrak{R}$.

Return to Theorem 4.2. In the case $\mathfrak{R} = \overline{\mathbb{C}}$, the homotopy classes of the second type (consequently, the second sum in (4.1)) are absent and we have a simple analytic expression for the differential $Q(z)dz^2$. The family of domains $\mathcal{D}_{\overline{\mathbb{C}}}(h)$ and the functional $F_{\overline{\mathbb{C}}}(\alpha, h)$ are denoted simply by $\mathcal{D}(h)$ and $F(\alpha, h)$.

Theorem 4.3. *Let* $\mathfrak{R} = \overline{\mathbb{C}}$. *Suppose that the assumptions of Theorem* 4.2 *are fulfilled,* $n + 2(m + r) \leqslant 4$. *There exists a quadratic differential on* $\overline{\mathbb{C}}$

of the form

$$Q(z)dz^2 = P(z)\Big\{\prod_{k=1}^{n}(z-a_k)\prod_{l=1}^{m}(z-b_l^{(0)})^2\prod_{k=1}^{r}(z-b_k)^2\Big\}^{-1}dz^2 \qquad (4.3)$$

(where $P(z)$ is a polynomial of degree at most $n+2(m+r)-4$) that is uniquely determined by the conditions indicated in Theorem 5.1. The system of domains $D_i^*, i=1,\dots,n_1+m+p$, which form the set $\overline{\mathbb{C}}\setminus\overline{\Phi}$ for the differential $Q(z)dz^2$ is the only system realizing the maximum of the functional $F(\alpha,h)$ on the family $\mathcal{D}(h)$.

It is assumed that none of the domains D_i^*, $i=1,\dots,n_1$, are degenerated. Let $\zeta=g_i(z)$(respectively, $\zeta=g_{n_1+l}(z)$ and $\zeta=g_{n_1+m+s}(z)$ denote a conformal homeomorphism of the doubly-connected domain D_i^* onto the circular annulus $1<|\zeta|<M_i$ (respectively,of the simply connected domain D_{n_1+l} onto the disk $|\zeta|<R_l$, $g_{n_1+l}(b_l^{(0)})=0$, $g'_{n_1+l}(b_l^{(0)})=1$, and of the bigon $D^*_{n_1+m+s}$ onto the strip $-1/2<\operatorname{Im}\zeta<1/2$, $g_{n_1+m+s}(b_{k'(s)})=-\infty$, $g_{n_1+m+s}(b_{k"(s)})=+\infty$). In the domain D_i^*, $i=1,\dots,n_1+m$, we have

$$\alpha_i^2 d\zeta^2 = -4\pi^2 Q(z)dz^2,$$

and in the domain $D^*_{n_1+m+s}, s=1,\dots,p$, we have

$$h_s^2 d\zeta^2 = Q(z)dz^2.$$

For the maximum $F^*(\alpha,h)$ of the functional $F(\alpha,h)$ on $\mathcal{D}(h)$, we have

$$F^*(\alpha,h) = \sum_{i=1}^{n}\alpha_i^2 M(D_i^*) + \sum_{l=1}^{m}\alpha^2_{n_1+l}M(D^*_{n_1+l}, b_l^{(0)}) - \sum_{s=1}^{p}h_s^2 M(D_s^{(1)*}, b_{k'(s)}, b_{k"(s)}). \qquad (4.4)$$

Remark 4.1. In the case $\mathfrak{R}=\overline{\mathbb{C}}$ the relations of Corollary 4.1 give algebraic conditions for the polynomial $P(z)$ in (4.3). In simplest special cases, these conditions determine the polynomial $P(z)$ entirely.

If the set B is empty and the classes of fourth type are absent, then Theorem 4.3 was proved in [194]; in this case, the fourth sum in (4.4) is absent.

Theorem 4.3 completely characterizes the extremal system of domains and the mapping functions for a wide range of extremal decomposition problems. Some simple examples of the extremal problem solved with the help of Theorem 4.3 are given in Sec.5 of this survey.

4.5. Dwell on a certain corollary to the previous theorem. Let $A = \{a_\nu\}_{\nu=1}^n$ and $B = \{b_k\}_{k=1}^m$ be given systems of distinct points on $\overline{\mathbb{C}}$, $n + m > 4$.We study the relation between two extremal problems. Let $\alpha = \{\alpha_k\}_{k=1}^m$ be a given system of positive numbers. The first problem consists of finding the maximum $\mathcal{M}_1^*(\alpha)$ of the functional

$$\mathcal{M}_1(\alpha) = \sum_{k=1}^m \alpha_k^2 M(D_k, b_k)$$

over the family $\mathcal{D}_1$ of all systems of nonoverlapping simply connected domains $\{D_k\}_{k=1}^m$ on $\overline{\mathbb{C}} \setminus A$, $b_k \in D_k$, $k = 1, \ldots, m$.

Now, let $\mathcal{H}$ be a family of homotopic classes $H_s, s = 1, \ldots, p$, of arcs on $\overline{\mathbb{C}}' = \overline{\mathbb{C}} \setminus \{A \cup B\}$, the limiting endpoints of which are the corresponding points $b_{k'(s)}, b_{k"(s)}$ of the set B.It is assumed that in the case where $b_{k'(s)} = b_{k"(s)} = b_{k(s)}$ the curves in H_s cannot be contracted on $\overline{C}'$ to the point $b_{k(s)}$. Let $h = \{h_s\}_{s=1}^p$ be a given system of positive numbers. The second problem consists of finding the maximum $\mathcal{M}_2^*(h)$ of the functional

$$\mathcal{M}_2(h) = -\sum_{s=}^m h_s^2 M^2(\tilde{D}_s, \tilde{b}_{k'(s)}, \tilde{b}_{k"(s)})$$

over the family $\mathcal{D}^{(2)}(h)$ of all admissible systems of domains $\{\tilde{D}_s\}_{s=1}^p$ associated with the family $\mathcal{H}$, where the domains $\tilde{D}_s, s = 1, \ldots, p$, satisfy condition $(*)$ and their interior angles at the boundary elements $\tilde{b}_k$ with supports at the points b_k are

$$\phi_k = 2\pi h_s / \sum_{t \in I_k} h_t, \quad k = k'(s), k"(s).$$

The following theorem due to Emel'yanov [149]establishes the relationship between these problems.

Theorem 4.4. *Let $\mathcal{H}$ be a family of homotopic classes $H_s, s = 1, \ldots, p$, of locally rectifiable arcs on $\overline{\mathbb{C}}$ of the form described above. Let $h = \{h_s\}_{s=1}^p$ be an arbitrary system of positive numbers, and let $\alpha = \alpha(h) = \{\alpha_k\}_{k=1}^m$, where*

$$\alpha_k = \sum_{s \in I_k} h_s, k = 1, \ldots, m.$$

Then

$$\mathcal{M}_1^*(\alpha(h) \leqslant -\mathcal{M}_2^*(h).$$

Let $\{\tilde{D}_s\}$ *be any system of domains in the family* $\mathcal{D}_2(h)$. *Then*

$$\mathcal{M}_1^*(\alpha(h)) \leqslant \sum_{s=1}^{p} h_s^2 M^{(2)}(\tilde{D}_s, \tilde{b}_{k'(s)}, \tilde{b}_{k"(s)}). \tag{4.5}$$

Equality in (4.5) *is attained only in the case where the domains* $\tilde{D}_s, s = 1, \dots, p$, *are bounded by the closures of orthogonal trajectories of the differential* $Q(z)dz^2$, *which determines the extremal system of domains for the problem on* $\mathcal{M}_1^*(\alpha(h))$.

Theorem 4.4 has a large number of applications (see,for instance, [?]). A more general result devoted to the "orthogonal" extremal decomposition problem is obtained by Solynin [238].

4.6. In applications of the method of modules, as a rule, the distinguished points on the surface $\mathbb{S}$ occurring in the definitions of the homotopy classes of curves are free parameters of the problem under study. In the results of the method of modules, these parameters acquire a clear geometric meaning, being the poles of the associated quadratic differential. The method of modules allows one to study the dependence of the maxima of the functionals occurring in extremal problems on the real parameters and the location of the distinguished points on $\mathbb{S}$.

We dwell on this question in the case of the functional of Theorem 4.3. The maximum $F^*_{\overline{\mathbb{C}}}(\alpha, h)$ mentioned in this theorem will be denoted by $\mathcal{M}(\alpha, h; A, B^{(0)}, B)$. For short, we denote these quantities by $\mathcal{M}(\alpha_i)$, $\mathcal{M}(a_k)$, etc., emphasizing the dependence of $\mathcal{M}$ on the parameter indicated. Let $Q(z)dz^2$ denote the differential (4.3).

The properties of the function M are described in the following theorem due to Emel'yanov [148]and Solynin [229, 238]).

Theorem 4.5. *Let the notation of Theorem* 4.3 *be used.* (1) *Let* $\alpha_i \in \alpha$ *or* $h_s \in h$. *Then*

$$\frac{\partial}{\partial \alpha_i}\mathcal{M}(\alpha_i) = 2\alpha_i M(D_i^*),$$

$$\frac{\partial}{\partial h_s}\mathcal{M}(h_s) = -2h_s M(D^*_{n_1+m+s}, b_{k'(s)}, b_{k"(s)}).$$

(2) *Let* $a_k \in A, a_k \neq \infty$. *Then*

$$\frac{\partial}{\partial a_k}\mathcal{M}(a_k) = \pi Q_k(a_k) \qquad \textit{where} \quad Q_k(z) = (z - a_k)Q(z).$$

(3) *Let* $b_l \in B^{(0)} \cup B, b_l \neq \infty$ *(we write* b_{m+k} *for* $b_k^{(1)}$*). Then*

$$\frac{\partial}{\partial b_l}\mathcal{M}(b_l) = \pi \tilde{Q}_l'(b_l), \qquad \textit{where} \quad \tilde{Q}_l(z) = (z - b_l)^2 Q(z).$$

Here $Q(z)dz^2$ *is the quadratic differential of Theorem* 4.3.

Note that assertion (2) of Theorem 4.5 has a simple geometric meaning: the gradient of the function $\mathcal{M}(a_k)$ at the point a_k^0 is directed along the tangent to the critical trajectory of the differential $Q(z)dz^2$ starting at the point a_k^0. This clarifies the role of Theorem 4.5 in the extremal problems in which it is required to establish some symmetry in the location of the poles of the associated quadratic differential.

5. THE METHOD OF MODULES OF CURVE FAMILIES. SOME ASPECTS OF APPLICATIONS OF THE METHOD

In this section, a brief account of results obtained with the help of the module method in various questions of geometric function theory is given.The module method combines very effective with variational and symmetrization methods, some results obtained by such combination are presented below. We restrict ourselves to the most easily formulated results.

As a rule, the modules method reduces to a geometrically explicit solution, giving complete information on the problem; however, obtaining an analytically implicit solution may turn out to be sufficiently complicated.

5.1. The early results of Jenkins.

In the early works of Jenkins, the approach based on consideration of the module of several classes of curves in combination with the symmetrization method of Pólya and Szegö was applied. In this way, Jenkins [20,22,32] solved a number of problems which where not amenable to other methods. In [20], the solution of the Gronwall problem consisting of finding the exact estimate of the modulus of a function in the class S with a fixed value of the modulus of the second coefficient c_2 in the expansion (1.1) was obtained.

In [32], theorems on the boundary distortion for univalent conformal mappings of multiply connected domains were established. The prototype of the results is a well-known Lowner's Lemma on the boundary distortion for a conformal mapping of the disk $|z| < 1$.

In the same way, some extremal problems in the class C of Bieberbach-Eilenberg functions were solved [22 I,II].

5.2. The initial results in extremal decomposition problems.

First extremal decomposition results are related to sums of reduced modulus.Let $n \geqslant 2$, and let $\mathbf{a} = \{a_1, \dots, a_n\}$ be a system of distinct points on $\overline{\mathbb{C}}$, $\boldsymbol{\alpha} = \{\alpha_1, \dots, \alpha_n\}$ be a system of positive numbers. Let $\mathcal{D}_n(\mathbf{a})$ be the family of all systems $\mathbb{D}_n = \{D_1, \dots, D_n\}$ of nonoverlapping simply connected domains on $\overline{\mathbb{C}}$, $a_k \in D_k$, $k = 1, \dots, n$. The maximum of the sum

$$\sum_{i=1}^{n} \alpha_i^2 M(D_i, a_i) \tag{5.1}$$

in the family $\mathcal{D}_n(\mathbf{a})$ will be denoted by

$$M(a_1, \dots, a_n; \alpha_1, \dots, \alpha_n), \quad M(a_1, \dots, a_n; 1, \dots, 1)$$

will be denoted by $M(a_1, \dots, a_n)$.

Lavrent'ev (1934) and Goluzin (1950) shoved that, in the family $\mathcal{D}_n(\mathbf{a})$, the exact inequalities hold:

$$\prod_{k=1}^{2} R(D_k, a_k) \leqslant |a_1 - a_2|^2,$$

$$\prod_{k=1}^{3} R(D_k, a_k) \leqslant \frac{64}{81\sqrt{3}} |(a_1 - a_2)(a_1 - a_3)(a_2 - a_3)|. \tag{5.2}$$

In 1952, Kolbina [185] obtained exact estimates for the sum

$$\alpha_1^2 M(D_1, a_1) + \alpha_2^2 M(D_2, a_2)$$

in the family of pairs of nonoverlapping domains D_1, D_2 on $\mathbb{C}, a_i \in D_i, i = 1, 2$, and for the sum (5.1) in the family $\mathcal{D}(\mathbf{a})$ in the case $n = 3$.The proof in [185] was one of the first applications of the Goluzin variational method. Jenkins [19] gave a significantly simpler proof by using of extremal metric considerations and showed a sharpening of results in [185].

In [41], Jenkins obtained a geometrically explicit solution of the problem on the maximum of the sum (5.1) for $n \geqslant 3$ with the help of the GCT. In the present time, this result is a direct corollary of Theorem 4.3.

In the case $n = 4$, an analytically implicit solution of the problem on $M(a_1, \dots, a_n)$ is obtained in [195] (see Sec. 5.7).

With the problem on the maximum of the sum (5.1) in the family $\mathcal{D}_n(\mathbf{a})$, an extremal decomposition problem in a family of systems of nonoverlapping bigons is immediately connected.

Let $n \geqslant 3$. Let $\mathbf{a} = \{a_1, \dots, a_n\}$ be a system of distinct points on the circle $|z| = 1$, enumerated in the order of increasing argument. Let $\mathcal{P}_n(\mathbf{a})$ be a family of systems of nonoverlapping bigons $P_k, k = 1, \dots, n$, on the z-sphere, where P_k has its vertices at the points a_k, a_{k+1}. It is assumed that the bigon $P_k, k = 1, \dots, n$, is associated with the class of arcs homotopic on $\overline{\mathbb{C}} \backslash \{a_1, \dots, a_n\}$ to the arc $\gamma_k = \{z : |z| = 1, \arg a_k < \arg z < \arg a_{k+1}\}$ and has at the vertices a_k, a_{k+1} the inner angles equal to π. Let $M(P_k, a_k, a_{k+1})$ be the reduced module of the bigon P_k with respect to the class of arcs connecting its sides.

The results of Lavrent'ev and Goluzin cited above are supplemented by the following simple theorem [208].

Let a_1, a_2 be distinct points of $\mathbb{C}$. In the family $\mathcal{P}_2(\mathbf{a})$ we have the inequality

$$\sum_{k=1}^{2} M(P_k, a_k, a_{k+1}) - \frac{2}{\pi} \log |a_1 - a_2|^2 \geqslant 0.$$

Let a_1, a_2, a_3 be distinct points of $\mathbb{C}$. In the family $\mathcal{P}_3(\mathbf{a})$ we have the inequality

$$\sum_{k=1}^{3} M(P_k, a_k, a_{k+1}) - \frac{2}{\pi} \log |(a_1 - a_2)(a_1 - a_3)(a_2 - a_3)| \geqslant \frac{2}{\pi} \log \frac{64}{81\sqrt{3}}. \tag{5.3}$$

As shown in [204 III], the minimum of the linear combination

$$\sum_{k=1}^{n} h_k^2 M(P_k, a_k, a_{k+1})$$

in the family $\mathcal{P}_n(\mathbf{a})$ in the case $n = 3$ for every nonnegative h_1, h_2, h_3 is equal to the maximum of the weight sum of reduced modules

$$\sum_{k=1}^{3} \alpha_k^2 M(D_k, a_k), \text{ where } \alpha_1 = h_1 + h_3, \alpha_2 = h_1 + h_2, \alpha_3 = h_2 + h_3,$$

of domains in the family $\mathcal{D}_3(\mathbf{a})$ of domains D_1, D_2, D_3 . In the case $n \geqslant 4$ the situation is different, this is observed already for $n = 4$ (see [204 III]).

Dwell on the extremal decomposition problem in the family of domains of distinct structure, many extremal problems in classes of conformal mappings are connected with this problem. Consider the functional

$$\alpha_1^2 M(D, \infty) + \alpha_2^2 M(D_2) \quad (\alpha_1 \geqslant 0, \alpha_2 \geqslant 0, \alpha_1 + \alpha_2 > 0) \tag{5.4}$$

defined on the family Δ of all pairs of nonoverlapping domains D_1, D_2 on $\overline{\mathbb{C}}' = \overline{\mathbb{C}} \setminus \{-1, 1, a\}, a \neq +1, -1$, where D_1 is a simply connected domain, $\infty \in D_1, D_2$ is a doubly-connected domains, separating the pairs of points $-1, 1$ and a, ∞ and belonging to a prescribed homotopic class. In various cases, solutions of this problem are given in [184, 163,196]. Let $\Delta^{(1)}, \Delta^{(2)}$ be two families of pairs of domains D_1, D_2 in Δ such that the domains D_2 are doubly- connected and are associated with the simplest homotopy classes $H^{(1)}$ and $H^{(2)}$ of closed Jordan curves on $\overline{\mathbb{C}}'$. (In the case where $\operatorname{Re} a \geqslant 0, \operatorname{Im} a > 0$, the classes $H^{(1)}$ and $H^{(2)}$ consist, respectively, of curves homotopic on $\overline{\mathbb{C}}$ to the slit along the segment $[-1, 1]$ and to the slit along the broken line with vertices $-1, ta$ and 1, where $t > 0$.) Let $\{D_1^{(j)}, D_2^{(j)}\}, j = 1, 2$, be the configuration providing the maximum $\mathcal{M}^{(j)} = \mathcal{M}^{(j)}(\alpha_1, \alpha_2, a)$ for the functional (5.4) over the family $\Delta^{(j)}$. Let $E(-1, 1, a)$ be the continuum of minimal capacity containing the points $-1, 1, a$.For $\alpha_2/\alpha_1 \leqslant \mu^{(j)}$, where $\mu^{(j)}$ depend on a and are defined in terms of conditions describing cap $E(-1, 1, a)$,the doubly-connected domains $D_2^{(j)}$ degenerate, namely, $D_1^{(j)} = \overline{\mathbb{C}} \setminus E(-1, 1, a)$, $D_2^{(j)} = \emptyset$. For $\alpha_1 = 0$, $D_1^{(j)} = \emptyset$ and the domain $D_2^{(j)}$ realizes the maximum of the conformal module in the family of doubly-connected domains on $\overline{\mathbb{C}}'$ associated with the class $H_j^{(2)}$; about the Chebotarev problem on the continuum of minimal capacity and the Teichmuller problem on the maximum of the conformal module we shall speak in the following sections.

For any α_1, α_2 the quantity $M^{(j)}$ monotonically depends on a in the same way as does the cap $E(-1, 1, a)$ (see Sec.5.2).

5.3. Problem on the continuum of minimal capacity and related problems.

With the problems mentioned in the previous Section a problem indicated in the title is connected. Let $a_1, \ldots, a_n, n \geqslant 2$, be distinct points of

$\mathbb{C}$. By $E(a_1, \dots, a_n)$ we denote the continuum of minimal capacity containing the points $a_1, \dots, a_n$. The domain $D = \overline{\mathbb{C}} \setminus E(a_1, \dots, a_n)$ realizes the maximum of the reduced module $M(D, \infty)$ in the family of all simply connected domains on $\overline{\mathbb{C}} \setminus \{a_1, \dots, a_n\}$. Goluzin obtained a geometrically explicit solution of the problem on $E(a_1, \dots, a_n)$: he established an analytical expression for the associated quadratic differential and the condition defined its parameters. This result is a particular corollary to Theotem 4.3. The problem of obtaining of an analytically implicit solution of the problem on $E(a_,1, \dots, a_n)$ for arbitrary $a_1, \dots, a_n$in the case of large n is of considerable difficulty.

In the case $n = 3$, a complete solution of the problem is obtained. Theorem 4.3 implies the following result. The continuum $E(a_1, a_2, a_3)$ is the $\overline{\Phi}$-set for the quadratic differential

$$q(z)dz^2 = -\frac{z - c}{(z - a_1)(z - a_2)(z - a_3)} dz^2,$$

for which the zero $c = c(a_1, a_2, a_3)$ is defined by the condition of the connectivity of $\overline{\Phi}$-set for the differential considered. This geometrically explicit result yelds a description of geometric properties of the continuum $E(a_1, a_2, a_3)$ (see the paper of Pirl [217]), and an analytically implicit solution of the problem [194]. Namely, for the function $\zeta = g(z)$ mapping the domain $\overline{\mathbb{C}} \setminus \overline{\Phi}$ onto the disk $|\zeta| < 1$, we have the equation

$$q(z)dz^2 = -4\pi^2 d\zeta^2 / \zeta^2.$$

Therefore, $c = c(a_1, a_2, a_3)$ and cap $E(a_1, a_2, a_3)$ are found from a system of equations containing elliptic functions [194, Theorem 1.6]. In the symmetric case, the solution is simpler. The continuum $E(0, e^{i\psi}, e^{-i\psi}), 0 \leqslant \psi \leqslant \pi/2$, is the $\overline{\Phi}$-set for the quadratic differential

$$q(z, \psi) = -\frac{z - c(\psi)}{z(z - e^{i\psi})(z - e^{-\psi})} dz^2,$$

where $1 \geqslant c(\psi) \geqslant 0$. For $c = c(\psi)$ and $H(\psi) = \operatorname{cap} E(0, e^{i\psi}, e^{-i\psi})$, we have a simpler system of equations [194,Theorem 1.5].

Many extremal problems are connected with the problem on $E(a_1, a_2, a_3)$. We restrict ourselves to the following two examples. Let S_R be the class of functions $f \in S$ with real coefficients $c_2, c_3, \dots$ in the expansion (1.1). Let $K(S_R)$ be the Koebe set for this class of functions, i.e.,the exact domain covered by the image $f(U)$ of the disk U under every $f \in S_R$. The set $K(S_R)$ is easily found from results on $H(\psi) = \operatorname{cap} E(0, e^{i\psi}, e^{-i\psi})$. We have the following theorem [194].

The set $K(S_R)$ is bounded by the curve $w = r(\psi)e^{i\psi}$, $-\pi/2 \leqslant \psi \leqslant 3\pi/2$, where $r(\psi) = H(|\psi|)$ for $-\pi/2 \leqslant \psi \leqslant \pi/2, r(\psi) = H(|\pi - \psi|)$ for $\pi/2 \leqslant \psi \leqslant 3\pi/2$.

The set $K(S_R)$ was found first by Jenkins [48]by means of the General Coefficient Theorem. The set $K(S_R)$ is symmetric with respect to both coordinate axes. Let $w = r(\psi)e^{i\psi}, 0 \leqslant \psi \leqslant \pi/2$, be boundary points of $K(S_R)$. The value $r(\psi), 0 \leqslant \psi \leqslant \pi/2$, is defined by the following condition. Let the domain $D(\psi)$ realize the maximum of the reduced module $M(D(\psi), 0)$ in the family of all simply connected domains on $\overline{\mathbb{C}} \setminus \{0, r(\psi)e^{i\psi}, r(\psi e^{-i\psi})\}$. Then $r(\psi), 0 \leqslant \psi \leqslant \pi/2$, is determined by the condition $M(D(\psi), 0) = 1$. As was shown in [48], $D(\psi) = \overline{\mathbb{C}} \setminus \overline{\Phi}$, where $\overline{\Phi}$ is the union of closures of the critical trajectories of the quadratic differential

$$Q(w, \psi)dw^2 = \frac{r^2(\psi)}{a(\psi)} \frac{w - a(\psi)}{w^2(w - r(\psi)e^{i\psi})(w - r(\psi)e^{-i\psi})} dw^2, \qquad (5.5)$$

where $a(\psi)$ is uniquely determined by the condition of connectivity of the set Φ indicated.

For $0 < \psi < \pi/2 \quad , a(\psi) > 0$ and the critical trajectories of the differential (5.3)are the ray $w > a(\psi)$ and the trajectories T_1 and T_2 having respective limiting end points at $a(\psi), r(\psi)e^{i\psi}$ and $a(\psi), r(\psi)e^{-i\psi}$. Further, $a(0) = r(0), a(\pi/2) = \infty$, whence $r(0) = 1/4, r(\pi/2) = 1/2$.

The same description of the domain $D(\psi)$ follows immediately from Theorem 4.3. It is easily seen that the mapping $z \to r(\psi)/z$ maps the domain $D(\psi)$ into the exterior of the continuum $E(0, e^{i\psi}, e^{-i\psi})$ of the capacity $H(\psi)$, whence the boundary arc of the set $K(S, R)$ is determined by the condition $r(\psi) = H(\psi), 0 \leqslant \psi \leqslant \pi/2$.

From Theorem 4.5 and simple geometric properties of the continuum $E(a_1, a_2, a_3)$ (see,for instance, [194]), the following refinement of the result in [221] follows. This result is due to Emel'yanov [147] and Solynin [229].

Let a point a move along an arc of the ellipse with focuses $-1, 1$, so that $\arg a$ increases from 0 to $\pi/2$. Then $\operatorname{cap} E(-1, 1, a)$ strictly increases.

Let C be the Bieberbach–Eilenberg class, i.e.,the class of functions $f(z)$ regular in the disk $U = \{z : |z| < 1\}$ and such that $f(0)=0, f(z_1)f(z_2) \neq 1$

for $z_1, z_2 \in U$. Let $C(\lambda)$ be the subclass of functions $f(z) \in C$ with $|f'(0)| = \lambda, 0 < \lambda \leqslant 1$. In a similar way as above, the author found [197] that the Koebe set in the class $C(\lambda), 0 < \lambda \leqslant 1$, is bounded by the curve $w = R(\phi, \lambda)e^{i\psi}$, where $R(\phi, \lambda), 0 < R(\phi, \lambda) < 1$, is a solution of the equation

$$\operatorname{cap} E(-1, 1, 1/2[R(\psi, \lambda)e^{i\psi} + 1/R(\psi, \lambda)e^{-i\psi}]) = 1/(2\lambda).$$

5.4. The Teichmüller problem and the Vuorinen problem.

The Teichmuller problem can be formulated as foll ows.

Find the maximum of the conformal module in the family of doubly-connected domains on the z-sphere separating the point pairs $-1, 1$ and a, ∞.

We assume that $a \in I$, where $I = \{z : \operatorname{Re} z \geqslant 0, \operatorname{Im} z \geqslant 0\}, a \neq 1$. Let $M(a)$ be the desired maximum. In [194], the following theorem is proved.

$$\log M(a) = \pi \frac{\mathbf{K}'(k)}{\mathbf{K}(k)}, \quad k^2 = \frac{2}{a+1},$$

where the elliptic integrals $\mathbf{K}(k)$ and $\mathbf{K}'(k)$ are understood to be functions that are positive for $k^2 \in (0, 1)$ and defined for other k^2 by suitable analytic continuation (for the exact formulation see [194]).

An extremal domain $D(a)$ of this problem is bounded by the closures of critical trajectories of the quadratic differential

$$Q(z)dz^2 = \frac{e^{i\beta(a)}dz^2}{(z^2-1)(z-a)}, \tag{5.6}$$

where

$$\beta(a) = -\arg k^2 \mathbf{K}^2(k).$$

In the case $a \notin [0, 1)$ the domain $D(a)$ is unique, in the case $a \in [0, 1)$ the extremal domains are $D(a)$ and the domain $\overline{D}_(a)$ symmetric to $D(a)$ with respect to the real axis.

This result is obtained from Theorem 4.3 and properties of the elliptic modular functions, these properties determine the choice of the homotopy class of curves with which the extremal domain is associated.

In the cases $a > 1$ and $a = ih, h \geqslant 0$, the domain $D(a)$ is symmetric with respect to both coordinate axes. In the first case, the boundary components of $D(a)$ are the segment $[-1, 1]$ and the ray $z \geqslant a$, in the second case the arc $\{z : |z - a| = (1 + h^2)^{1/2}, \operatorname{Im} z \leqslant 0\}$ and the ray $z = at, t \geqslant 1$.

We have the following property of $M(a)$ [239]. Let E be the ellipse with the focuses $-1, 1$.

If a point a moves along an arc of the ellipse E in such a way that arg a increases, remaining in I, then $M(a)$ strictly increases.

This assertion easily follows from Theorem 4.4. Indeed

$$\arg \operatorname{grad} M(a) = \arg \frac{a^2-1}{e^{i\beta(a)}}$$

and from the expression for $\beta(a)$ it follows that

$$0 < \arg \operatorname{grad} M(a) - \arg \sqrt{a^2-1} < \pi/2,$$

$\arg \sqrt{a^2-1}$ is the argument of the normal to the ellipse E at the point a.

A hyperbolic analog of the Teichmuller problem is the Vuorinen problem. It can be formulated by the following way.

Let $1 < R < \infty$. As a model of the hyperbolic plane let us take the disk $U_R = \{z : |z| < R\}$ with the metric defined by the line element $ds = |dz|/\sqrt{1 - R^{-2}|z|^2}$. Let $C_R = \{z : |z| = R\}, I_R = \{z : z \in U_R, \operatorname{Re} z \geqslant 0, \operatorname{Im} z \geqslant 0\}$.

Let $a \in I_R, a \neq 1$. Let $D_R(a)$ be the family of all doubly-connected domains in the disk U_R, separating the points $-1, 1$ from the point a and the circle C_R. Find the maximum $M_R(a)$ of the conformal module in the family $D_R(a)$ and the domains, realizing this maximum, and investigate the properties of $M_R(a)$ as a function of a.

A solution of this problem is obtained by Emel'yanov and the author [157] and is as follows. Theorem 4.3 establishes a solution of this problem formulated in terms of hyperelliptic functions [157]. In the cases $a \in (1, R)$ and $a = ih, h \in (0, R)$, the extremal configurations are symmetric and $M_R(a)$ is expressed in explicit form by the elliptic integrals.

Let E_R be a hyperbolic ellipse with focuses $-1, 1$ and H_R be a confocal hyperbolic hyperbola. The following result [157] establishes the role of symmetric configurations indicated above in the problem under consideration.

The functional $M_R(a)$ strictly increases if the point a moves along an arc of a hyperbolic ellipse E_R belonging to I_R and if the point a moves along an arc of a hyperbolic hyperbola H_R belonging to the same set, so that $\operatorname{Im} a$ *increases.*

The proof is obtained in [157] by means of detailed analysis of geometric properties of trajectories of associated quadratic differentials for the given problem and Theorem 4.4.

The extremal configurations of some problems on extremal decompositions of the disk with three distinguished points have the same properties as the extremal configurations in the Vuorinen problem. Two such problems was considered by Emel'yanov [155]. Dwell on one of them.

Let $p \in U$, $\operatorname{Re} p > 0$, $\operatorname{Im} p > 0$, $0 < x < 1$. Find domains D_1, D_2 in the disk U, realizing maximum $\mathcal{M}(p)$ of the sum

$$M(D_1) + \alpha^2 M(D_2), \quad \alpha > 0,$$

in the family of all pairs $\mathbb{D}$ of nonoverlapping doubly-connected domains D_1, D_2, where the domain D_1 separates the points $-x, x$ from p, the domain D_2 separates the points $p, -x, x$ from the circle $|z| = 1$.

There are the numbers α_- and $\alpha_+, \alpha_- < 1 < \alpha_+$, for which respectively the domain and the domain degenerate, for $\alpha = 1$ the point p is not singular and the domains $D_1(1), D_2(1)$ are joined into one domain, which is $U \setminus [-x, x]$. The domain $D_1(\alpha_-)$ is the extremal domain of the Vuorinen problem. In [155], the following theorem is proved.

The functional $\mathcal{M}(p)$ strictly increases if the point p moves along an arc of the hyperbolic ellipse with focuses $-x, x$ in the direction to the imaginary axes.

Let $p \in E, \operatorname{Re} p > 0, \operatorname{Im} p > 0$, and let p_0 and $p_1 = i\sqrt{(p_0^2 - x^2)/(1 - p_0^2)}$ be the points of intersection of the ellipse E with the coordinate axe. By the last theorem,

$$\mathcal{M}(p_0, \alpha) < \mathcal{M}(p, \alpha) < \mathcal{M}(p_1, \alpha).$$

The values $\mathcal{M}(p, \alpha)$ and $\mathcal{M}(p_1, \alpha)$ are easily determinated by the Q-lengths of orthogonal trajectories and their arcs of the associated quadratic differential for a given problem.

5.5. Extremal problems in the classes of univalent functions.

Many extremal problems in the basic classes of univalent mappings are connected with simple problems of the extremal decomposition; about such extremal decomposition problems we shall speak in Sec. 5.2. The examples of results obtained owing to the indicated connection are the results on the maximum and minimum of $|f(z_0)|$ in the class $R(\lambda)$ obtained by Gavrilyuk amd Solynin [163]; a result on the region of values $f(z_0)$ in the class S_R, say, $\Delta(S_R, z_0)$, is due to Fedorov [160].

Dwell on the last result.A geometrically explicit result in the problem on $\Delta(S_R, z_0)$ was obtained by Jenkins [48] with the help of the GCT. The last problem was considered later by Chernikov [130], who used the Goluzin variational method. As is well known, $\Delta(S_R, z_0)$ is contained in $\Delta(T, z_0)$, where $\Delta(T, z_0)$ is the region of values $f(z_0)$ in the class T of typical real functions, and part of the boundary of $\Delta(S_R, z_0)$ belongs to the boundary of $\Delta(T, z_0)$. Finding the remaining part of the boundary of $\Delta(S_R, z_0)$ turned out to be difficult. Fedorov [160] succeeded in investigating the boundary of $\Delta(S_R, z_0)$ and obtained in this way a complete solution of the problem. The proof in [160] is based on the simultaneous consideration of two extremal decomposition problems: the above problem on $M^{(j)}$ for pure imaginary values of a (see Sec. 5.2) and the problem on the maximum of the sum (5.4)over another family of pairs of domains (the latter problem is connected with the problem on a continuum of minimal capacity, which was solved in [159]).

A number of problems on regions of values of functional systems on the classes of univalent functions are studied in the book of A.Vasil'ev [250]. In [250], the set of values of the system

$$\{|f(z_1)|, |f(z_2|\}, \quad 0 < z_1 < z_2 < 1,$$

in the class S_R is found. The lower bound of this set is easily established by considering of the reduced module of the bigon $U \setminus \{[-1, r_1] \cap [r_2, 1]\}$ with both vertices at $z = 0$ and its image for the extremal mapping. The upper bound is found with the help of considering the problem on the maximum of the sum (5.4) in the case $a \geqslant 1$ and the problem of the maximum of the corresponding functional in the family of pairs of domains, defined on the sphere with distinguished points $-1, 1, a_1, c_1, \infty$, where $0 < a_1 < c_1 < \infty$.

By the same module approach the regions of values of some functional systems in the class $S^{(M)}$ of bounded functions from the class S and in the Montel class of functions $f(z) = a_1 z + a_2 z^2 + \dots$, regular and univalent in U and satisfying the condition $f(\omega) = \omega, \quad 0 < \omega < 1$, are found [250].

Jenkins [41] introduced the class $\Sigma(r)$ of functions $f(z)$ from the class Σ, which map $|z| > 1$ onto a domain whose complement contains the domain with the inner conformal radius with respect to the origin at least r, $0 < r < 1$. In [41, 50], exact estimates for $|\alpha_0|$ and $|\alpha_1|$ in the class $\Sigma(r)$ were obtained. The class $\Sigma(r)$ is particularly convenient for applying the module method. By this method Gavrilyuk and Solynin [163] solved some extremal problem in this class. Solynin [233] obtained an exact estimate for the diameter of level curves, i.e., for the functional

$$|f(z_1) - f(z_2)|, \quad |z_1| = |z_2| = \rho > 1,$$

in the class $\Sigma(r)$.

In the class Σ, a distortion theorem is known ($z = \rho e^{i\theta}$):

$$\frac{(1-\rho^{-2})^2}{4\rho^2(1+\rho^{-2})^2} \leqslant \frac{|f'(z)f'(-z)|}{|f(z)-f(-z)|^2} \leqslant \frac{(1+\rho^{-2})^2}{4\rho^2(1-\rho^{-2})^2}. \tag{5.7}$$

Extending this result, Suita [243] obtained the inequalities

$$\frac{(1-\rho^{-3})}{3\sqrt{3}\rho^3(1+\rho^{-3})^3} \leqslant \prod_{k=1}^{3} \frac{|f'(z\omega^{k-1})|}{f(z\omega^{k-1}) - f(z\omega^k)|} \leqslant \frac{(1+\rho^{-3})^3}{3\sqrt{3}r^3(1-\rho^{-3})^3}, \tag{5.8}$$

where $z = \rho e^{i\theta}, \rho > 1$ and $\omega = e^{2i\pi/3}$, and showed all equality cases. Suita obtained inequalities (5.6) with the help of the GCT of Jenkins. However the inequalities (5.6) are simple corollaries to the Goluzin inequality (see the second inequality in (5.2)) and inequality (5.3).

In [200, 208], the upper and the lower estimates for the functional in (5.7) in the class $\Sigma(r)$ are obtained. Under limit passage for $\rho \to \infty$, from the indicated result in [233] and the result in [200] the maximum of $|\alpha_1|$ in the class $\Sigma(r)$ is found. Earlier the indicated maximum was obtained in [50] by means of the GCT (in the Extended Form). Note that the extension of inequalities (5.8) to the class $\Sigma(r)$ leads to an estimate of $|\alpha_2|$ in the class $\Sigma(r)$.

5.6. Harmonic measures and triad modules.

There are various connections between harmonic measures and modules. Of importance is the relationship between harmonic measures of a certain type and triad modules.

A number of Jenkins' results [42, 80, 103 I, II, III; 120] and related works of various authors were devoted to problems concerning harmonic measures.

By a triad (P, α, D), the configuration consisting of a simply connected domain D of hyperbolic type, an open border arc α of D, and a point P interior to D is meant (the triads of Jenkins). We denote the harmonic measure of α taken in P with respect to D by $\omega(P, \alpha, D)$. By the module $M(P, \alpha, D)$ we mean the module of the class of locally rectifiable open arcs in $D \setminus \{P\}$ running from α back to α and separating P from the closed border arc α^* complementary to α. This module is called a triad module; this term was introduced by Jenkins [42]. There is a strictly monotone increasing function that relates $\omega(P, \alpha, D)$ to $M(P, \alpha, D)$.

In [103 I] the following simple property of a triad module is noted. Let $U = \{|z| < 1\}$ and let α be the arc on the unit circle from $e^{-i\beta/2}$ to $e^{i\beta/2}$ (in the positive sense), $0 < \beta < 2\pi$. Consider the triad $(0, \alpha, U)$, and let $M(0, \alpha, U)$ be its triad module. The quadratic differential

$$\frac{dz^2}{z(z - e^{i\beta/2})(z - e^{-i\beta/2})}$$

determines the extremal configuration of the Mori problem concerning the maximum of the module in the family of doubly-connected domains separating pairs of points $0, \infty$ and $e^{i\beta/2}, e^{i(2\pi-\beta/2)}$ (see Sec. 5.4). From the definition of the triad module it follows that the desired triad module is equal to twice of the mentioned maximum. Hence for $M(0, \alpha, U)$ we have a relation in terms of the elliptic integrals:

$$M(0, \alpha, U) = \frac{1}{2}\mathbf{K}'(\cos\beta/4)/\mathbf{K}(\cos\beta/4).$$

Dwell on the result in [80]. Let $U = \{|z| < 1\}$, and let α be a half-open arc in U with end points $\zeta \in U$ and 1. Let $G = U \setminus \alpha$. Gaier [162] considered the problem of estimating from below the harmonic measure $\omega(0, \alpha, G)$ of the arc α with respect to G at the origin, and he gave an explicit but not sharp estimate for this quantity. Jenkins showed that the problem is most naturally stated in terms with topology determination, and he first solved the problem when the change in the argument on the arc α from 1 to ζ has an assigned value. Shortly, this solution is stated as follows.

Let α be an arc in $\overline{U} \setminus \{0\}$ with end points $\zeta \in U$ and 1. Further we assume that the change of argument $\Delta_\alpha(\arg z)$ has the assigned value δ. Then

$$\omega(0, \alpha, \Gamma_\alpha) \geqslant \omega(0, \alpha^*, G_{\alpha^*}),$$

where α^* is a competing arc uniquely determined as follows. There is a unique point $e^{i\chi}, \chi$ is real,such that the quadratic differential

$$Q(z)dz^2 = c(z - e^{i\chi})[z(z - \zeta)(z - \overline{\zeta}^{-1})(z - 1)]^{-1}dz^2$$

with the constant $c(\neq 0)$ is real in the unit circumference and α^* consists of a trajectory of $Q(z)dz^2$ on $|z| = 1$ from 1 to $e^{i\chi}$ and a trajectory in D

from $e^{i\chi}$ to ζ together with their end points. Equality may occur only if α coincides with α^*.

Fuchs (see [127]) raised the problem of finding the greatest lower bound of the harmonic measure at the origin of a set in $|z| \leqslant 1$ which meets every radius. This problem has been investigated by Marshal and Sundberg [213]. For a continuum, a geometric explicit solution of this problem is obtained by Jenkins [103 I]. In this paper it is shown that the result in [80] cited above readily gives a characterization of the extremal in the problem of Fuchs. Solynin [236] extended this result, considering the above continuum whose index about the origin is a half integer $n/2$; he obtained an analytically implicit solution. Jenkins [120] simplified Solynin's proof and gave a geometrically explicit identification of the extremal configuration.

A new approach to the problem of Fuchs provides the Jenkins result [116] devoted to the n-fold symmetrization. In [120], Jenkins simplifies the proof in [236] and gives a geometrically explicit characterization of the extremal configuration.

In this connection, see also the papers of Liao [211], the Jenkins references [223, Sec. 9], Solynin's article [236].

5.7. Problems with free poles of quadratic differential.

Let E be a continuum on $\mathbb{C}$, and let $D_n(E), n \geqslant 2$, be the nth diameter of E:

$$d_n(E) = \{\max_{c_k, c_l \in E} \prod_{1 \leqslant k < k \leqslant n} |c_k - c_l|\}^{2/[n(n-1)]}.$$

The problem on the maximum of $d_n(E)$ in the family of all continua E of the unit capacity is an example of the problem with free poles of associated quadratic differential: Goluzin showed that an extremal continuum of this problem is the Φ-set for the quadratic differential

$$Q(z)dz^2 = -\sum_{1 \leqslant k < l \leqslant n} \frac{1}{(z - c_k)(z - c_l)} dz^2;$$

here c_k, i.e., the Fekete points on E, are unknown parameters of the given problem, Reich and Schiffer [221] shoved that the each extremal continuum of the problem under considerations possesses this property. The extremal configuration is unique up a linear transformation and it is the continuum of minimal capacity for its Fekete points. The problem is solved for $n = 2, 3, 4$. By the Faber theorem, in the case $n = 2$ an extremal continuum is the segment $E_2^* = [-2,2]$, Goluzin showed that in the case $n = 3$, this continuum is $E_3^* = \cup_{k=1}^3 [0, 4^{1/3}\omega^{k-1}, \omega = e^{2\pi/3}$. The problem on $d_4(E)$ is solved by the author [194]. The extremal continuum E_4^* is symmetric with respect to both coordinate axes and it is connected with a suitable continuum of minimal capacity by the condition

$$E_4^* = \{z : H(\alpha_))z^2 \in E(0, e^{i\alpha_0}, e^{-i\alpha_0})\},$$

where $\alpha_0, 0 < \alpha_0 < \pi/2$, is the solution of the equation

$$c(\alpha) = \frac{1}{3}\cos\alpha$$

(see notation in Sec.(5.3)). In the proof in [194], various methods of investigation were used.

Let $\mathbf{a} = \{a_1, \ldots, a_n\}$ be a system of distinct points on $\overline{\mathbb{C}}$,and let $\boldsymbol{\alpha} = \{\alpha_1, \ldots, \alpha_n\}$ be a system of positive numbers. As above, let $\mathcal{D}_n(\mathbf{a})$ be the family of all systems $\mathbb{D} = \{D_1, \ldots, D_n\}$ of nonoverlapping simple connected domains on $\overline{\mathbb{C}}, a_k \in D_k, k = 1, \ldots, n$. The first results of the module method in the problems of the maximum $M(a_1, \ldots, a_n; \alpha_1, \ldots, \alpha_n)$ of the sum (5.1) over the family $\mathcal{D}_n(\mathbf{a})$ were mentioned in Sec.5.2. The indicated maximum will be denoted by $M(\mathbf{a}, \boldsymbol{\alpha}), M(\mathbf{a}, \mathbf{1})$ will be denoted by $M(\mathbf{a})$. The problem on $M(\mathbf{a})$ will be called the problem A_n.

The problem on the maximum of the conformal invariant

$$2\pi \sum_{k=1}^{n} M(D_k, a_k) - \frac{2}{n-1} \sum_{1 \leqslant k < l \leqslant n} \log |a_k - a_l|$$

in the family $\mathcal{D}(\mathbf{a})$ with respect to every point system $\mathbf{a} = \{a_1, \ldots, a_n\}$ will be called the problem B_n.

In the cases $n = 2, 3$, the problems A_n and B_n are equivalent, and we have the results of Lavrent'ev, Golusin, and Kolbina (for their proofs, see comments of Jenkins [19]).

Theorem 4.3 has led to a complete solution of the problems A_n and B_n for $n = 4$ [195]. The maximum in the problem A_4 is expressed in terms of the problem on $E(-1, 1, a)$, where a is expressed by the cross-ratio of the quadrangle of points under consideration. It allowed one to find the largest value of the above maximum for all values of a, and thus also to solve the problem B_4; see [195] and the paper of Fedorov [158], as well.

For $n \geqslant 5$, the problems A_n and B_n remain unsolved. Under the additional assumption that the systems of points $\{a_1, \dots, a_5\}$ are symmetric with respect to a circle or a line, the maximum I_5^* was found by the author [199] and Dubinin [138]; in this case, the extremal system is $\{1, \omega, \omega^2, 0, \infty\}$, where $\omega^3 = 1$. It is plausible that this system of points is also extremal for the problem B_5 in the general case.

An investigation of some candidates for the extremal configurations of the problem B_n is given in [206].

Let now $\mathbf{a} = \{a_1, \dots, a_n\}$ be a system of distinct points of the disk U, and let $\boldsymbol{\alpha} = \{\alpha_1, \dots, \alpha_n\}$ be a system of positive numbers. Let $\mathcal{D}_U(\mathbf{a})$ be the family of all systems $\mathbb{D} = \{D_1, \dots, D_n\}$ of nonoverlapping simply connected domains in $U, a_k \in D_k, k = 1, \dots, n$. The problem on the maximum of the sum (5.4) for all systems $\mathbf{a}$ of points $a_k \in U$ and systems of domains D_k of the family $\mathcal{D}_U(\mathbf{a})$ will be called the problem K_n. The problem of the maximum of the functional

$$J_n = 2\pi \sum_{k=1}^{n} M(D_k, a_k) - \Big\{ \frac{2}{3(n-1)} \sum_{1 \leqslant k < l \leqslant n} \log(|a_k - a_l||1 - \overline{a}_k a_l|) + \frac{1}{3} \sum_{k=1}^{n} \log(1 - |a_k|^2) \Big\}$$

in the family $D_U(\mathbf{a})$ with respect to every system $\mathbf{a} = \{a_1, \dots, a_n\}$ of the points in U will be called the problem L_n.

The difficulty in solving Problems B_n, K_n, and L_n for sufficiently large n is connected with the presence of various admissible configurations, satisfying the necessary conditions, but not realizing the desired maximum. Therefore, it is of interest to establish additional conditions that the extremal configurations must satisfy. These conditions are given by the following theorem, due to Kuznetsov [190, 193].

The associated quadratic differential in Problem B_n does not have multiple zeros. The bound every from domains of the extremal system in Problems B_n and $K_n$is a closed Jordan curve.

Geometrically, this theorem shows that the domains indicated do not have interior slits and "holes".

An addition to the previous theorem, see [193].

In the case $n = 2$, the maximum of the sum (5.4) in the family $\mathcal{D}_U(\mathbf{a})$ was found by Kufarev and Falles [189]. Using their results cited above, Kuznetsov [193] obtained a simple solution of the problem K_2 for any α_1, α_2. The solution of the problem K_3 in the case $\boldsymbol{\alpha} = \mathbf{1}$ was obtained by Kostyuchenko [186]. In addition to theoretic function reasonings, this solution was needed some computer calculation.

5.8. Problems in the presence of a symmetric conditions.

As it was indicated above, the extremal decompositions problems in the case of large number of free parameters are of considerable difficulty. Therefore it is natural to consider these problems for the condition that the disposition of the points a_k satisfies come additional conditions. First result in this direction belongs to Dubinin [134], and it is given by the following theorem.

Let $\mathbf{a} = \{a_1, \ldots, a_n\}, n \geqslant 2$, *be a system of the points of* C. *In the family* $\mathcal{D}(\mathbf{a})$

$$\mathcal{M}(a_1, \ldots, a_n) \leqslant \frac{n}{2\pi} \log \frac{4}{n}. \tag{5.9}$$

Equality in (5.9) *occurs only in the case where the points* $a_1, \ldots, a_n$ *are uniformly distributed on* C.

This result is supplemented by the following theorem [204 III] in the family $\mathcal{P}_n(\mathbf{a})$ (see Sec.(5.2)).

Let $\mathbf{a} = \{a_1, \ldots, a_n\}$ *be a system of points on* $C, n \geqslant 3$. *In the family* $\mathcal{P}_n(\mathbf{a})$ *we have the inequality*

$$\max_{\mathbf{a}} \min_{\mathcal{P}_n(\mathbf{a})} \sum_{k=1}^{n} M(P_k, a_k, a_{k+1}) \leqslant \frac{2n}{\pi} \log \frac{4}{n}. \tag{5.10}$$

Equality in (5.10) *occurs only in the same case as in* (5.9).

Dubinin [134] showed that the maximum of every of the functionals $\mathcal{M}(0, a_1, \ldots, a_n)$ and $\mathcal{M}(0, a_1, \ldots, a_n, \infty; \alpha, 1, \ldots, \alpha), \alpha^2 = 1/2$, where $a_1, \ldots, a_n$ are points of C, occurs only in the case of indicated symmetric disposition of the points $a_1, \ldots, a_n$. This gave simple expressions for desired maxima.

The author [203, 202 I] established, that the mentioned property is satisfied for the functional $\mathcal{M}(0, a_1, \ldots, a_n; \alpha, 1, \ldots, 1)$ for $0 < \alpha \leqslant 1$ and for the second of the indicated functionals for $\alpha^2 \leqslant \frac{n^2}{8}$.

In the proof, Dubinin used the method of separating transformation of condensers and domains, the author used Theorem 4.4, which establishes the connection between two extremal decomposition problems.

Concerning the condition $\alpha^2 \leqslant n^2/8$ in the problem on the functional $\mathcal{M}(0, a_1, \dots, a_n; \alpha, 1, \dots, 1, \alpha)$, the following theorem is proved [202, I].

The maximum $\mathcal{M}(0, e^{i\beta}, e^{-i\beta}, \infty)$, where $0 \leqslant \beta \leqslant \pi/2$, is attained for $\beta = \beta_0$, where $\beta_0, \pi/6 < \beta_0 < \pi/2$, is the solution of the equation $c(\beta) = 1/2, c(\beta)$ is the zero of the quadratic differential, defining the continuum of minimal capacity $E(0, e^{i\beta}, e^{-i\beta})$, $0 \leqslant \beta \leqslant \pi/2$.

A number of extremal decomposition problems in the presence of certain symmetry in the condition of the problem under consideration is solved in [149, 204–207].

5.9. Problems for which associated quadratic differential is a complete square.

In decomposition problems cited in the title the module method, as a rule,immediately yields a complte solution. We shall indicate some such results, following the presentation in Emel'yanov's article [151].

Let $P = \{a_1, \dots, a_n\}$ be a system of distinguished points on $\overline{\mathbb{C}}$ and let $\overline{\mathbb{C}}' = \overline{\mathbb{C}} \setminus P$. Let $\mathcal{P}$ be the family of all systems $\mathbb{D} = \{D_1, \dots, D_{n_1}, D_1^{(1)}, \dots, D_{n_2}^{(1)}\}$ of nonoverlapping domains, where D_j is a simply connected domain on $\overline{\mathbb{C}}' \cup \{a_j\}$ such that $a_j \in D_j$ for $j = 1, \dots, n_1 (n_1 \leqslant n), D_k^{(1)}, k = 1, \dots, n_2$, is a doubly connected domain on $\overline{\mathbb{C}}'$. It is assumed that the family $\mathcal{P}$ is associated with a given family of homotopy classes of closed Jordan curves on $\overline{\mathbb{C}}$. Let $\boldsymbol{\alpha} = \{\alpha_1, \dots, \alpha_{n_1}\}$ and $\mathbf{l} = \{l_1, \dots, l_{n_2}\}$ be two given systems of positive numbers.

Let $\mathbb{D}^* = \{D_1^*, \dots, D_{n_2}^{(1)*}\}$ be the system of domains realizing the maximum value of the functional

$$F(\mathbb{D}) = \sum_{j=1}^{n_1} \alpha_j^2 M(D_j, a_j) + \sum_{k=1}^{n_2} l_k^2 M(D_k^{(1)})$$

in the family $\mathcal{P}$. Here $M(D_k^{(1)})$ denotes the module of the domain $D_k^{(1)}$ with respect to the family of curves separating its boundary components.

It is known that the extremal system $\mathbb{D}^*$ is unique, and where exists a unique quadratic differential associated with the problem such that his

Φ-set decomposes $\overline{\mathbb{C}}$ onto domains forming the system $\mathbb{D}^*$. This differential is of the form

$$Q(w)dw^2 = \sum_{j=1}^{n}\Big(\frac{A_j}{(w-a_j)^2} + \frac{\lambda_j}{(w-a_j)}\Big)dw^2,$$

where $A_j = -\alpha_j^2/4\pi^2$ if $j \leqslant n_1$ and $A_j = 0$ otherwise.

Let

$$\mathcal{M}(P) = F(\mathbb{D}^*),$$

and consider the functional

$$J(P) = \mathcal{M}(P) + \frac{1}{4\pi}\sum_{j,k,j\neq k}\mu_{k,j}\log|a_k - a_j|^2, \quad \mu_{k,j} = \mu_{j,k}, \tag{5.12}$$

where the $\mu_{k,j}$ are some real numbers. We assume that the condition

$$\sum_{k,k\neq j}\mu_{k,j} = -\alpha_j^2, \quad j = 1,\dots,n,$$

holds, where we set $\alpha_j = 0$ for $n_1 < n \leqslant n_2$. In this case, the functional $J(P)$ is conformally invariant.

If the functional (5.12) is bounded from above then exists an extremal systems of points $P^* = \{a_1^*,\dots,a_n^*\}$ and an associated quadratic differential $Q_{P^*}(w)dw^2$.

The following preposition is valid [151].

Let $P^* = \{a_1^*,\dots,a_n^*\}$ be an extremal system of points for the functional (5.12). *Then the quadratic differential $Q_{P^*}(w)dw^2$ has the form*

$$Q_{P^*}(w)dw^2 = \frac{1}{4\pi^2}\sum_{k,j=1,k<j}^{n}\mu_{k,j}\Big(\frac{1}{w-a_k^*} - \frac{1}{w-a_j^*}\Big)^2.$$

If $P^* = \{a_1^*,\dots,a_n^*,\infty\}$,then

$$Q_{P^*}(w)dw^2 = \frac{1}{4\pi^2}\Big(\sum_{k,j=1,k<j}^{n}\mu_{k,j}\Big(\frac{1}{w-a_k^*} - \frac{1}{w-a_j^*}\Big)^2 + \sum_{j=1}^{n}\mu_{j,n+1}\frac{1}{(w-a_j^*)^2}\Big)dw^2.$$

By means of obtained expressions for $Q_{P^*}(w)dw^2$, in [151], the inequalities in the class Σ are obtained which are generalizations of the inequalities of Golusin and Grunsfy, respectively.

Dwell on another results in [151]. The following lemma is valid.

Let $P = \{a_1, \dots, a_n, \infty\}$ be a set of distinguished points on $\overline{\mathbb{C}}$, let $F(\mathbb{D})$ be a functional of the form (5.11) of the extremal decomposition problem corresponding to the set P, and let $Q_P(z)dz^2$ be the associated quadratic differential. If

$$Q_P(z)dz^2 = -\frac{1}{4\pi^2}\Big(\sum_{j=1}^{n} \frac{\alpha_j}{z-a_j}\Big)^2,$$

then

$$\mathcal{M}(P) = F(\mathbb{D}^*) = -\frac{1}{2\pi} \sum_{p,q=1, p\neq q}^{n} \alpha_p \alpha_q \log |a_p - a_q|.$$

As it is noted in [151], this lemma follows of [143, Theorem 1]. In [151], a not complicated proof of this lemma by the module method is given.

Now let $P = \{a_1, a_2, a_3, \infty\}$ be a system of distinct points of $\overline{\mathbb{C}}$. Consider the problem of the extremal decomposition of the z-sphere in the family of all nonoverlapping simply connected domains $D_j, a_j \in D_j, j = 1, \dots, 4 (a_4 = \infty)$, with the functional

$$F(\mathbb{D}) = M(D_1, a_1) + M(D_2, a_2) + M(D_3, a_3) + 9M(D_4, \infty).$$

Let $\mathbb{D}^* = \{D_1^*, \dots, D_4^*\}$ be the extremal systems of domain, $\mathcal{M}(P) = F(\mathbb{D}^*)$. Set

$$J(P) = \mathcal{M}(P) + \frac{1}{2\pi}(\log |a_2 - a_1|^2 + \log |a_3 - a_1|^2 + \log |a_1 - a_3|^2).$$

The functional $J(P)$ is bounded from above [151, Theorem 1]. By Theorem 4.3, an associated quadratic differential of this problem is a complete square. We have the following theorem [151].

Let $T = T_0 \cup T^+ \cup T^-$, where $T_0 = [-2, 1]$,

$$T^+ = \{z : |z - \omega| = \sqrt{3},\ z \geqslant 0\}, \quad T^- = \{\overline{z} : z \in T^+\};$$

here $\omega = e^{2\pi/3}$. The maximum of the functional $J(P)$ is equal to 0 and it is attained at every systems of points

$$P_a = \{\omega, \overline{\omega}, a, \infty\}, \quad a \in T,$$

and also at systems of points obtained from the indicated systems of points by linear-fractional transformations and only at such systems of points.

Now we consider the decomposition problem of the w-sphere into simply connected domains $D_j, j = 1, \ldots, 4$, such that $a_j \in D_j$ and a doubly-connected domain D. The domain D has to separate two pairs of points a_1, a_3 and a_2, ∞ and is associated with the class closed Jordan curves homotopic on $\mathbb{C} \setminus \{a_1, a_2, a_3\}$ to the slit along the segment $[a_1, a_3]$. Set

$$\tilde{F}(\mathbb{D}) = M(D_1, a_1) + M(D_2, a_2) + M(D_3, a_3) + 9M(D_4, \infty) + 4M(D),$$
$$\tilde{\mathcal{M}}(P) = \tilde{F}(\mathbb{D}^*).$$

Let the functional $\tilde{J}(P)$ be defined similarly to (5.13), i.e.,

$$\tilde{J}(P) = \tilde{\mathcal{M}}(P) + \frac{1}{2\pi}(\log|a_2 - a_1|^2 + \log|a_3 - a_2|^2 + \log|a_1 - a_3|^2. \quad (5.14)$$

The functional $\tilde{J}(P)$ is bounded from above. Let $\Omega_{1,3}$ denote the component of $\overline{\mathbb{C}} \setminus T$ containing the point $w = \infty$. The following statement is an easy consequence of the last theorem.

The maximum value of the functional $\tilde{J}(P)$ is equal to 0. The functional $\tilde{J}(P)$ achieves its maximal value at systems of points obtained from the system $P_a = \{\omega, \overline{\omega}, a, \infty\}$, where $a \in \Omega_{1,3}$, by linear-fractional transformations and only at such system s of points.

By means of the last theorem and its corollary, an inequality for a combination of initial coefficients of the expansion of a function $f(z) \in \Sigma$ is obtained in [151]. In contrary to the analog inequality obtaining with the use of the GCT of Jenkins, this inequality is valid without any restrictions on the first coefficients of this expansion.

Some extremal decomposition problems in multiply connected domains for which the associated quadratic differentials are complete squares are considered by Emel'yanov in [154].

5.10. Solution of some isoperimetric problems.

By means of the method of modules, a solution of a number of isoperimetric problems was obtained. We shall indicate some application of the notion of the reduced module of a triangle introduce by Solynin [230, 238]. In [218], Pólya and Szegö posed the problem of finding the maximum of the conformal radius $R(\Delta_n, 0)$ over the family of all n-gons $\Delta_n (0 \in \Delta_n)$ of a given area. For $n = 3, 4$, this problem was solved in [218] with the help of the Steiner symmetrization;for $n > 4$, the proof in [218] fails. Solynin [230] obtained a solution of this isoperimetric problem for all n simultaneously;

the maximum is attained only in the case where Δ_n is a regular n-gon with center at the origin. In [234], some inequalities between geometric and functional characteristics of n-gons, such as the perimeter, diameter, inner radius, transfinite diameter, torsion rigidly, and electrostatic capacity, were established. The proof uses the notion of dissymmetrization introduced by Dubinin [132].

Some difficult isoperimetric problems for n-gons were solved by Solynin and Zalgaller [240,241]. In the first paper, the authors proved that among all n-gons Δ_n with fixed area, the regular n-gon, and only this one, has minimal logarithmic capacity. This result was conjectured by Pólya and Szegö [218] (and was proved by them for $n = 3, 4$). Let $C\Delta_n$ be the unbounded component of $\overline{\mathbb{C}} \setminus \Delta_n$. The proof uses the connection between the reduced module of $C\Delta_n$ and the reduced modules of the trilaterals associated with a special triangulation of an n-bigon Δ_n; this approach was developed by Solynin in an earlier work [230]).

In [241] the authors prove various isometric inequalities for a curvilinear polygon with n sides, each of which is a smooth arc of curvature at most k. The proof relies on the method of dissymmetrization and on a special purely geometric theorem for the polygons under consideration.

5.11. Concluding remarks.

In the theory of the module method, presented in Sec.4, the associated quadratic differentials have poles of order not exceeding 2. The open question is to extend this theory to quadratic differentials with poles of higher order.

In a number of cases, extremal decomposition problems in which the associated quadratic differentials have poles of higher orders can be reduced to problems in which the quadratic differentials have poles of orders $\leqslant 2$. Such a reduction is based on the fact that a quadratic differential with a pole f order $n \geqslant 4$, say, at the point $z = 0$, can be approximated by a quadratic differential with $n - 2$ poles of second order that are symmetrically located on the circle $|z| = \epsilon$ (in the case $n = 3$, also there is a simple pole at the origin). This approach was used in [200, 208] for the estimate of the coefficient α_1 in the class $\Sigma(r)$.

Another approach consist of the introduction of reduced modules of domains similar to the end and strip domains of a quadratic differentials with poles of high order, and of the consideration of a decomposition problem in a family of domains containing the domains indicated. The preliminary result in this direction was obtained in [209]. This result can be regarded as an analog of the Jenkins GCT.

6. OTHER RESULTS OF J.A.JENKINS

Jenkins is the author of more than 130 of scientific papers.These papers are devoted to various questions of GFT. They are prominant applications of the method of the extremal metric and related approaches.

On the general Coefficient Theorem, its extensions and generalizations, we said in Sec. 3. The general principle stated in [39 I, II] and its developments were presented in Sec. 4. Dwell briefly on other results of Jenkins.

6.1. Jenkins made a large contribution to the development of the method of symmetrization.

Let us give one of the results of Jenkins in the symmetrization method of Pólya and Szegö.

Let D be a doubly-connected domain in the w-sphere, and let P be a point and λ be a ray with end point at P. Let D^* be the domains associated with D by the circular symmetrization determined by P and λ (see definitions in [41]). Let $M(D)$ and $M(D*)$ be the modules of D and D^*, in each case for the class of curves separating the boundary components. Then (Pólya and Szegö)

$$M(D) \leqslant M(D^*). \tag{6.1}$$

Jenkins obtained the following uniqueness result [44].

Let the circular symmetrization be defined by the origin and the positive real axis. Equality in (6.1) occurs only if D^* is obtained from D by a rigid rotation about the point $w = 0$.

Analogous results are valid for the symmetrization of quadrangles.

As it was be noted (see Sec. 5.1) in early papers of Jenkins [22 ?,20,32], the efficiency of the combination of the general extremal metric principle with the symmetrization method of Pólya and Szegö was demonstrated with examples of the solution of difficult problems.

6.2. Some results of the GFT express the fact that a given set has a minimal capacity (or possesses an analogous extremal property) in some family of planar sets satisfying one or another geometric condition. The examples are many of covering theorems. Jenkins [11] obtained the following result by a symmetrization argument.

Let $f \in S$ and let $L(f, r)$ denote the Lebesgue length measure of the set of values on $|w| = r$ not covered by the image of $|z| < 1$ under the mapping $w = f(z)$. For $1/4 < r < 1$, we have the sharp bound

$$L(f, r) \leqslant 2r \arccos(8r^{1/2} - 8r - 1)$$

with equality only for functions given explicitly in [11].

This work has had a number of continuations.In this connection, see the recent paper of Dubinin [140].

6.3. In [65], Jenkins gave a simple proof of a criterion for a closed set E to have minimal capacity in a given class of admissible sets and obtained the corresponding uniqueness assertion. This criterion is formulated as conditions on E and on the comparison sets that are expressed in terms of the topology of the orthogonal trajectories to the level curves of the Green function for the domain $D = \overline{\mathbb{C}} \setminus E$. Conceptually, the paper of Tamrazow [205] on covering of curves under conformal mapping is close to this paper.

The paper [71] is devoted to geometric questions related to capacity. The results of this paper, very simple in formulation, led to significant refinements in a number of previously known results of the covering theorem type in the classes Σ and S.

6.4. A number of Jenkins' results [42,80,103I,II,III;120] were devoted to problems concerning harmonic measures. Some of these results were sited in Sec. 5.6.

6.5. The method of the extremal metric has various forms. Using a form of this method close to the area method, Jenkins [63] proved the Special Coefficient Theorem. This result does not contain many of the most interesting applications of the General Coefficient Theorem, but it makes possible to consider a number of other problems.

In [61], Jenkins applied a modified form of the methods of the extremal metric to obtain generalizations of the usual span theorems for multiply connected domains. In this way, it was possible to prove for the first time theorems of this kind for functions regular in a domain (previously, such problems had been considered only for functions having given singularities).

6.6. A number of Jenkins' papers is devoted to the theory of Riemann surfaces, results on the boundary correspondence, applications of the method of the extremal metric to nonunivalent functions, and other questions. Many of his papers dealt with the theory of quasiconformal mappings.

A short account by Jenkins [100] and his fundamental survey article [123] have been devoted to the method of the extremal metric in its various aspects. These publications reflected many results of Jenkins.

PUBLICATIONS OF JAMES A. JENKINS

1. *Some problems in conformal mapping.* Thesis (Ph.D.) Harvard University. 1948.
2. *Some problem in conformal mapping.* — Trans. Amer. Math. Soc. **67** (1949), 327–350.
3. *Positive quadratic differentials in triply-connected domains.* — Ann. Math. (2) **53** (1951), 1–3.
4. *Hyperelliptic trajectories (jointly with D. C. Spencer).* — Ann. Math. (2) **53** (1951), 4–35.
5. *On an inequality of Goluzin.* — Amer. J. Math. **73** (1951), 181–185.
6. *On the topological theory of functions.* — Canad. J. Math. **3** (1951), 276–289.
7. *On a theorem of Spencer.* — J. London Math. Soc. **26** (1951), 313–316.
8. *Generalization of a theorem of Mandelbrojt.* — Amer. J. Math. **73** (1951), 807–812.
9. *Remarks on "Some problems in conformal mapping."* — Proc. Amer. Math. Soc. **3** (1952), 147–151.
10. *Contour equivalent pseudoharmonic functions and pseudoconjugates* (jointly with M. Morse). — Amer. J. Math. **74** (1952), 23–51.
11. *On values omitted by univalent functions.* — Amer. J. Math. **75** (1953), 406–408.
12. *Various remarks on univalent functions.* — Proc. Amer. Math. Soc. **4** (1953), 595–599.
13. *A symmetrization results for some conformal invariants.* — Amer. J. Math. **75** (1953), 510–522.
14. *Some results related to extremal length.* Contributions to the theory of Riemannn surfaces, pp. 87–94. Annals of Mathematics Studies, No. 30. Princeton University Press, Princeton, N.J., 1953.

15. *Topological methods on Riemann surfaces. Pseudoharmonic functions* (jointly with Marston Morse). Contributions to the theory of Riemann surfaces, pp. 111–139. Annals of Mathematics Studies, No. 30. Princeton University Press, Princeton, N.J., 1953.
16. *Another remarks on "Some problems in conformal mapping."* — Proc. Amer. Math. Soc. **4** (1953), 978–981.
17. *Conjgate nets,conformal structure, and interior transformations on open Riemann surfaces* (jointly with M.Morse) — Proc. Nat. Acad. Sci. U.S.A. **39** (1953), 1261–1268.
18. *Curve families F^* locally the level curves of a pseudoharmonic function* (jointly with M.Morse). — Acta Math. **91** (1954), 1–42.
19. *A recent note of Kolbina.* — Duke Math. J. **21** (1954), 155–162.
20. *On a problem of Gronvall.* — Ann. Math.(2) **59** (1954), 490–504.
21. *On the local structure of the trajectories of a quadratic differential.* — Proc. Amer. Math. Soc. **5** (1954), 357–362.
22. *On Bieberbach–Eilenberg functions.* I, II, III. — Trans. Amer. Math. Soc. **76** (1954), 389–396; **78** (1955), 510–515; **119** (1965), 195–215.
23. *A general coefficient theorem.* — Trans. Amer. Math. Soc. **77** (1954), 262–280.
24. *Some uniqueness in the theory of symmetrization.* I, II. — Ann. Math. (2) **61** (1955), 106–115; **75** (1962), 223–230.
25. *On circumferentially mean p-valent functions.* — Trans. Amer. Math. Soc. **79** (1955), 423–428.
26. *On circularly symmetric functions.* — Proc. Amer. Math. Soc. **6** (1955), 620–624.
27. *On a lemma of R. Huron.* — J. London Math. Soc. **30** (1955), 382–384.
28. *Sur quelques aspects globaux du theoreme de Picard.* — Ann. Sci. Ecole Norm. Sup. (3) **72** (1955), 151–161.
29. *On a problem of Luzin.* — Michigan Math. J. **3** (1955–1956), 187–189.
30. *On explicit bound s in Schottky's theorem.* — Canad. J. Math.**7** (1955), 76–82.
31. *Conjugate nets on an open Riemann surface* (jointly with M. Morse). — Lectures on functions of a complex variable, pp. 123–185. The University of Michigan Press, Ann Arbor, 1955.

32. *Some theorem on boundary distortion.* — Trans. Amer. Math. Soc. **81** (1956), 477–500.
33. *On quasiconformal mappings.* — Rational Mech. Anal. **5** (1956), 343–352.
34. *On a result on Keogh.* — J. London Math. Soc.**31** (1956), 391–399.
35. *On emplicit bounds in Landau's theorem.* I, II. — Canad. J. Math.**8** (1956), 423–425; **33** (1981), 559–562.
36. *Some new canonical mappings for multiply-connected domains.* — Ann. Math. (2) **65** (1957), 119–196.
37. *A new criterion for quasiconformal mapping.* — Ann. Math. (2) **65** (1957), 208–214.
38. *On a conjecture of Spencer.* — Ann. Math. (2) **65** (1957), 405–410.
39. *On the existence of certain general extremal metrics.* I, II. — Ann. Math. (2) **66** (1957), 440–453; Tohoku Math. J.(2) **45** (1993), No.2, 249–257.
40. *On a canonical conformal mapping of J. L. Walsh.* — Trans. Amer. Math. Soc.**88** (1958), 207–213.
41. *Univalent functions and conformal mapping.* — Ergebnisse der Mathematik und ihrer Grenzgebiete. Neue Folge, Heft 18. Reihe: Moderne Functionentheory. Springer-Verlag, Berlin–Gottingen–Heidelberg, 1958. Secong print. corrected, 1965. Рус. перев. 1-го изд. Дж. Дженкинс, Однолистные функции и конформные отображения. М., 1962.
42. *On the Denjoy conjecture.* — Canad. Math. J.**10** (1958), 627–631.
43. *Book Review: Introduction to Riemann surfaces.* — Bull.Amer. Math. Soc.**64** (1958), No.6, 382–385.
44. *A counter-example to a statement on conformal mapping of Riemann surfaces.* — Proc. Amer. Math. Soc.**10** (1959), 423–424.
45. *On the type problem.* — Canad. J. Math. **11** (1959), 427–431.
46. *Book Review: Multivalent functions.* — Bull. Amer. Math. Soc.**65** (1959), No. 31, 63–166.
47. *On weighted distortion in conformal mapping.* I, II. — Illinois J. Math.**4** (1960), 28–37; Bull. London Math. Soc.**30** (1998), No. 2, 151–158.
48. *On univalent functions with real coefficients.* — Ann. Math. (2) **71** (1960), 1–15.

49. *On certain coefficients of univalent functions.* I, II. — 1960 Analytic functions, pp. 159–164. Princeton Univ. Press, Princeton, N.J.; Trans. Amer. Math. Soc.**96** (1960), 534–545.
50. *An extension of the general coefficient theorem.* — Trans. Amer. Math. Soc.**95** (1960), 387–407.
51. *On the global structure of the trajectories 0f a positive quadratic differential.* — Illinois J. Math. **4** (1960), 405–412.
52. *The general coefficient theorem and its applications.* Contributions to function theory (Internat. Colloq. Function Theory, Bombay,1960), pp. 211–218. Tata Institute of Fundamental Research, Bombay.
53. *Some problems for typically real functions.* — Canad. J. Math. **13** (1961), 299–304.
54. *On the schlicht Bloch constant.* I, II. — J. Math. Mech. **10** (1961), 729–734; Indiana Univ. Math. J. **47** (1998), 67–92.
55. *On hyperbolic surfaces in three-dimensional Euclidian space.* — Michigan Math. J. **8** (1961), 1–5.
56. *On a conjecture of Goodman concerning meromorphic univalent functions.* — Michigan Math. J. **9** (1962), 25–27.
57. *The general coefficient theorem and certain applications.* — Bull Amer. Math. Soc. **68** (1962), 1–9.
58. *Remarks on the paper "Generalization of a theorem of Mandelbrojt".* — Amer. J. Math.**84** (1962), 383–385.
59. *On a paper of Reich concerning minimal slit domains.* — Proc. Amer. Math. So. **13** (1962), 358–360.
60. *An addendum to the general coefficient theorem.* — Trans. Amer. Math. Soc. **107** (1963), 125–128.
61. *On some span theorem.* — Illinois J. Math. **7** (1963), 104–117.
62. *A normalization in the general coefficient theorem.* 1963 Proc. Internat. Congr. Mathematicians (Stockholm, 1962) pp. 347–350 Inst. Vittag-Leffler, Djursholm.
63. *Some area theorems and a special coefficient theorem.* — Illinois J. Math. **8** (1964), 80–99.
64. *On a result of Nehary.* — Proc. Amer. Math. Soc. **17** (1966), 62–66.
65. *On certain problems of minimal capacity.* — Illinois Math. J. **10** (1966), 460–465.

66. *On certain extremal problems for the coefficients of univalent functions.* — J. Analyse Math. **18** (1967), 173–184.
67. *On local maximality for the coefficient* a_6 (jointly with M. Ozawa). — Nagoya Math. J. **30** (1967), 71–78.
68. *On local maximality for the coefficient* a_8 (jointly with M. Ozawa). — Illinois J. Math. **11** (1967), 596–602.
69. *On an inequality considered by Robertson.* — Proc. Amer. Math. Soc. **19** (1968), 549–550.
70. *A uniqueness result in conformal mapping.* I, II. — Proc. Amer. Math. Soc. **22** (1969), 324–325; **85** (1982), No. 2, 231–232.
71. *On certain geometrical problems associated with capacity.* — Math. Nachr. **39** (1969), 349–356.
72. *Book Review: Value distribution theory.* — Bull. Amer. Math. Soc. **75** (1969), 917–923.
73. *On the growth of slowly increasing unbounded harmonic functions* (jointly with Kotaro Oikawa). — Acta Math. **124** (1970), 37–63.
74. *On the Pragmen–Lindelof theorem, the Denjoi conjecture, and related results.* 1970. Mathematical Essays Dedicated to A. J. Mancyntare, pp. 183–200 Ohio Univ.Press, Attens, Ohio.
75. *A remark on p-valent functions* (jointly with Kotaro Oikawa). — Austral. Math. Soc. **12** (1971), 397–404.
76. *A topological three pole theorem.* — Indian Univ. Math. J. **21** (1971/72), 1013–1018.
77. *On results of Ahlfors and Hayman.* — Illinois J. Math. (jointly with Kotaro Oikawa) **15** (1971), 664–671.
78. *A remark on "pairs" of regular functions.* — Proc. Amer. Math. Soc. **31** (1972), 119–121.
79. *On regular functions on Riemann surfaces* (jointly with Nobuyki Suita). I, II. — Illinois Math. J.**17** (1973), 563–570; **19** (1975), 122–126.
80. *On a problem concerning harmonic measure.* — Math. Z. **135** (1973/74), 279–283.
81. *On analytic self-mappings of Riemann surfaces* (jointly with Nobuyuki Suita). I, II. — Math. Ann.**202** (1973), 37–56; **209** (1974), 109–115.
82. *Some remarks on Weierstrass points.* — Proc. Amer. Math. Soc.**44** (1974), 121–122.

83. *On a lemma of Tord Hall. Collections of articles in celebration of the sixtieth birthday of Ky Fan.* — Bull. Inst. Math. Acad. Sinica **2** (1974), 371–375.
84. *On a result of M.Heins.* — Proc. Edinburgh Math. Soc. (2) **19** (1974/75), No. 4, 371–373.
85. *A Hilbert space method in the theory of schlicht functions.* — Proceedings of the Symposium on Complex Analysis (Univ. Kent, Canterbury, 1973), pp. 75–79. London Math. Soc. Lecture Note Se., No. 12, Cambrodge Univ. Press, London, 1974.
86. *A general form of the annulus theorem.* — Proceedings of the Symposium on Complex Analysis (Univ. Kent. Kanterbury, 1973), pp. 91–92. London Math. Soc. Lecture Notes Ser., No. 12, Cambridge Univ. Press, London, 1974.
87. *On results of R. Nevanlinna and Ahlfors.* — Bull. London Math. Soc. **7** (1975), 81–83.
88. *On the sharp form of the three-circles theorem.* — Kodai Math. Sem. Rep. **27** (1976), No. 1-2, 155–158.
89. *On Ahlfors' "second fundamental inequality"* (jointly with Kotaro Oikawa). — Proc. Amer. Math. Soc. **62** (1977), No. 2, 266–270.
90. *Conformality and semi-conformality at the boundary* (jointly with Kotaro Oikawa) — J. Reine Angew. Math. **291** (1977), 92–117.
91. *A Loewner approach to a coefficient inequality for bounded univalent functions* (jointly with Duana W.de Temple). — Proc. Amer. Math. Soc. **65** (1977), No. 1, 125–126.
92. *On a theorem of Caratheodory and Fejer* (jointly with Kotaro Oikawa). — Bull. London Math. Soc. **9** (1977), No. 2, 165–167.
93. *On a quadratic differentials whose trajectory structure consists of ring domains.* Complex analysis. (Proc. S.U.N.Y. Conf., Brockport. N.Y., 1976), pp. 65–71. Lecture Notes in Pure and Appl. Math., vol. 3,Dekker, New York, 1978.
94. *On the representation and compactification of a Riemann surfaces* (jointly with Nobuyuki Suita). — Bull. Inst. Math. Acad. Sinica **6** (1978), No. 2, part 2, 423–427.
95. *On the Pick-Nevanlinna problem* (jointly with Nobuyuki Suita). — Kodai Math. J.**2** (1979), No. 1, 82–102.
96. *On a problem of A. A. Goldberg.* — Ann. Univ. Mariae Curie-Sklodowska Sect. A 36/37 (1982/83), 83–86 (1985).

97. *On univalent functions omitting two values.* — Complex Variables Theory Appl. **3** (1984), Nos. 1–3, 169–172.
98. *On the boundary behavior of functions for which the Riemann image has finite spherical area* (jointly with Kotaro Oikawa). — Kodai Math. J **8** (1985), No. 3, 317–321.
99. *On a result of Ostrowski* (jointly with Kotaro Oikawa). — Complex Variables Theory Appl. **5** (1986), Nos. 2–4, 155–163.
100. *The method of the extremal metric.* The Bieberbach conjecture West Lafayette, Ind., 1985, 95–104, Math. Surveys Monogr., 21, Amer. Math. Soc., Providence, RI, 1986.
101. *On meromophic univalent function.* — Complex Variables Theory Appl, **7** (1986), Nos. 1–3, 83–87.
102. *On Ahlfors' spiral generalization of the Denjoy conjecture.* — Indiana Univ. Math. J. **36** (1987), No. 1, 41–44.
103. *Some estimates for harmonic measure.* I, II, III. — Complex Analysis, I (College Park, Md., 1985–86), 210–214, Lecture Notes in Math., 1275, Springer, Berlin, 1987; Proc. Amer. Math. Soc. **111** (1991), No. 2, 441–442; **119** (1993), No.1, 199–201.
104. *On univalent functions with integral coefficients.* — Complex Variables Theory Appl. **9** (1987), Nos. 2–3, 221–226.
105. *On analytic maps of plane domains.* — Kodai Math. J. **11** (1988), No. 1, 38–43.
106. *On boundary correspondence for domains on the sphere.* — Complex Analysis, 115–119, Birkhauser, Basel, 1988.
107. *On quasiconformal mappings with given boundary values.* — Indiana Univ. Math. **37** (1988), No. 4, 929–934.
108. *Helmut Grunsky.* Jaresber. Deutsch. Math.-Verein. **91** (1989), No. 4, 159–167.
109. *On analytic paths. Constantin Caratheodory: an international tribune*, Vol. I, II548–553, World Sci. Publ., Teaneck, NJ, 1991.
110. *On quasiconformal mapping of L-strips.* — Analyse complexe (Bucharest, 1989). Rev. Roumaine Math. Pures Appl. **36** (1991), Nos. 7–8, 377–382.
111. *A criterion associated with the schlicht Bloch constant.* — Kodai Math. J. **15** (1992), No. 1, 79–81.
112. *On a result of Beurling.* — Indiana Univ. Math. J. **41** (1992), 1070–1080.

113. *On the local behavior of certain homeomorphism.* I, II (jointly with Melkana Brakalova). — Kodai Math. J. **17** (1994), 201–213; Zap. Nauchn. Semin. POMI **237** (1997), 11–20.
114. *On metric defined by modules.* — Pacific J. Math. **167** (1995), 289–292.
115. *On comb domains.* — Proc. Amer. Math. Soc. **124** (1996), No. 1, 187–191.
116. *On n-fold symmetrization.* — J. Anal. Math. **69** (1996), 67–71.
117. *Lars Valerian Ahlfors* (jointly with C. Earle, D. Hejhal, T. Jorgensen, A. Marden). — Notices Amer. Math. Soc. **45** (1998), No. 2, 248–255.
118. *On solutions of the Beltrami equation* (jointly with Melkana Brakalova). I, II. — J. Anal. Math. **76** (1998), 67–92; Publ. Inst. Math. (Beograd) (N.S.) **75(89)** (2004), 3–8.
119. *On the mixed problem for extremal decompositions.* — Indiana Univ. Math. J. **49** (2000), 891–896.
120. *On a problem of Solynin.* — Math. Nachr. **230** (2001), 93–98.
121. *On two point distortion theorems for bounded univalent regular functions.* — Kodai Math. J. **24** (2001), 329–338.
122. *On a paper of L.Carleson: "On mappings, conformal at the boundary"* (jointly with Melkana Brakalova). — J. Analyse Math. **19** (1967), 1–13; Ann. Acad. Sci. Fenn. Math. **27** (2002), No. 2, 485–490.
123. *The method of the extremal metric.* Handbook of complex analysis: geometric function theory, Vol. 1,393–456, North-Holland, Amsterdam, 2002.

REFERENCES

124. L. V. Ahlfors, *Untersuchungen zur Theorie der konformen Abbildung und der ganzen Funktionen.* — Acta Soc. Scient. Fenn., N.S. A I, No. 9 (1930), 1–40.

125 L. V. Ahlfors, A.Beurling, *Invariants conformes et problemes extremaux.* — Dixieme Congres des Mathematiciens Scandinaves, 1946, 341–351.

126. К. И. Бабенко, *О строении области коэффициентов однолистных функций класса S.* — Докл. АН СССР **194** (1970), No. 2, 247–249.

127. K. Barth, D. Brannan, W. Hayman, *Research problems in complex analysis.* — Bull. Amer. Math. Soc., **16** (1984), 490–517.

128. L. de Branges, *A proof of the Bieberbach conjecture.* LOMI Preprints E-5-84, Leningrad, 1984.
129. L. de Branges, *A proof of the Bieberbach conjecture.* — Acta Math. **154** (1985), 137–152.
130. В. В. Черников, *Однолистные функции с вещественными коэффициентами.* — Уч. зап. Томск. ун-та **36** (1960), 3–12.
131. A. Douady, J. Hubbard, *On the density of Strebel differentials.* — Invent. Math. **30** (1975), 175–179.
132. В. Н. Дубинин, *Об изменении гармонической меры при симметризации.* — Мат. сб. **124** (1984), No. 2, 272–279.
133. В. Н. Дубинин, Преобразование функций и принцип Дирихле. - Мат.заметки, **38** (1985), 1,49-55.
134. В. Н. Дубинин, *Разделяющее преобразование областей и задачи об экстремальном разбиении.* — Зап. научн. семин. ПОМИ **168** (1988), 48–66.
135. В. Н. Дубинин, *Симметризация в геометрической теории функций комплексного переменного.* — Успехи мат. наук **49** (1994), No. 5, 13–76.
136. В. Н. Дубинин, *Асимптотика модуля вырождающегося конденсатора и некоторые её применения.* — Зап. научн. семин. ПОМИ **237** (1997), 56–73.
137. В. Н. Дубинин, *Обобщенные конденсаторы и асимптотика их емкостей при вырождении некоторых пластин.* — Зап. научн. семин. ПОМИ **302** (2003), 38–51.
138. В. Н. Дубинин, *Емкости конденсаторов и симметризация в геометрической теории функций комплексного переменного.* Владивосток, 2009.
139. В. Н. Дубинин, *Методы геометрической теории функций в классических и современных задачах для полиномов.* — Успехи мат. наук **67** (2012), No. 4, 3–88.
140. В. Н. Дубинин, *К теореме Дженкинса о покрытии окружностей голоморфными в круге функциями.* — Зап. научн. семин. ПОМИ **418** (2013), 69–73.
141. В. Н. Дубинин, Д. Б. Карп, В. А. Шлык, *Избранные задачи геометрической теории функций и теории потенциала.* — Владивост. мат. ж. **8** (2008), No. 1, 46–95.
142. В. Н. Дубинин, Д. А. Кириллова, *К задачам об экстремальном разбиении.* — Зап. научн. семин. ПОМИ **357** (2008), 54–74.

143. В. Н. Дубинин, Л. В. Ковалёв, *Приведенный модуль комплексной сферы.* — Зап. научн. семин. ПОМИ **254** (1998), 76–94.
144. В. Н. Дубинин, Е. В. Костюченко, *Экстремальные задачи теории функций, связанные с m-кратной симметрией.* — Зап. научн. семин. ПОМИ **276** (2001), 3–11.
145. P. L. Duren, M. Schiffer, *The theory of the second variation in extremal problems for univalent functions.* — J. Anal. Math. **10** (1962/1963), 193–252.
146. P. L. Duren, M. Schiffer, *Univalent functions which map into regions of given transfinite diameter.* — Trans. Amer. Math. Soc. **323** (1991), No. 1, 413–428.
147. Е. Г. Емельянов, *Некоторые свойства модулей семейств кривых.* — Зап. научн. семин. ЛОМИ **144** (1985), 72–82.
148. Е. Г. Емельянов, *К задачам об экстремальном разбиении.* — Зап. научн. семин. ЛОМИ **154** (1986), 76–89.
149. Е. Г. Емельянов, *О связи двух задач об экстремальном разбиении.* — Зап. научн. семин. ЛОМИ **160** (1987), 91–98.
150. Е. Г. Емельянов, *Задачи об экстремальном разбиении в пространствах римановых поверхностей.* — Зап. научн. семин. ПОМИ **263** (2000), 84–104.
151. Е. Г. Емельянов, *Конформно-инвариантные функционалы на римановой сфере.* — Зап. научн. семин. ПОМИ **276** (2001), 134–154
152. Е. Г. Емельянов, *К задаче о максимуме произведения степеней конформных радиусов неналегающих областей.* — Зап. научн. семин. ПОМИ **386** (2002), 103–114.
153. Е. Г. Емельянов, *О приведенном модуле полуполосы.* - Зап. научн. семин. ПОМИ **314** (2004), 76–82.
154. Е. Г. Емельянов, *О квадратичных дифференциалах в многосвязных областях, являющихся полными квадратами.* I, II. — Зап. научн. семин. ПОМИ **337** (2006), 113–133; **350** (2007), 40–51.
155. Е. Г. Емельянов, *Касательные свойства траекторий некоторых двух квадратичных дифференциалов.* — Зап. научн. семин. ПОМИ **418** (2013), 90–104
156. Е. Г. Емельянов, Г. В. Кузьмина, *Теоремы об экстремальном разбиении в семействе систем областей различных типов.* — Зап. научн. семин. ПОМИ **237** (1997), 74–104.

157. Е. Г. Емельянов, Г. В. Кузьмина, *Задача Вуоринена о максимуме конформного модуля.* — Зап. научн. семин. ПОМИ **404** (2012), 120–134.
158. С. И. Федоров, *О максимуме одного конформного инварианта в задаче о неналегающих областях.* — Зап. научн. семин. ЛОМИ **112** (1981), 171–183.
159. С. И. Федоров, *О вариационной проблеме Чеботарева в теории емкости плоских множеств и теоремах покрытия для однолистных конформных отображений.* — Мат. сб. **124** (1984), No. 1, 121–139.
160. С. И. Федоров, *Модули некоторых семейств кривых и множество значений $f(z^0)$ в классе однолистных функций с вещественными коэффициентами.* — Зап. научн. семин. ПОМИ **139** (1984), 156–167.
161. O. M. Fomenko, G. V. Kuz'mina, *The last 100 days of the Bieberbach conjecture.* — Math. Intellgencer **8** (1986), No. 1, 40–47.
162. D. Gaier, *Estimates of conformal mappings near the boundary.* — Indiana Univ. Math. J. **21** (1972), 581–595.
163. М. Н. Гаврилюк, А. Ю. Солынин, *Применение проблем модуля к некоторым экстремальным задачам.* — Кубан. ун-т, Краснодар, 1983 (Рукопись деп. в ВИНИТИ, 3072-83 Деп.)
164. Г. М. Голузин, *Геометрическая теория функций комплексного переменного.* М., 1952; 2-ое изд., М., 1966.
165. В. В. Горяйнов, *О параметрическом представлении однолистных функций.* — Докл. АН СССР **245** (1979), 378–371.
166. В. В. Горяйнов, *Параметрический метод и экстремальные конформные отображения.* — Докл.АН СССР **271** (1983), 1052–1055.
167. В. В. Горяйнов, *Полугруппы конформных отображений.* — Мат. сб. (Н.с.) **129** (1986), No. 4, 451–472.
168. В. В. Горяйнов, *Общая теорема единственности и геометрия экстремальных конформных отображений в задачах искажения и вращения.* — Изв. высш. учебн. завед. **10** (1986), 40–47.
169. H. Grötzsch, *Über einige Extremalprobleme der konformen Abbildung.* I, II. — Ber. Verh. sachs. Akad. Wiss. Leipzig, Math.-phys. kl. **80** (1928), 367–376; 497–502.
170. H. Grötzsch, *Über die Verzerrung bei schlichten nichtkonformen Abbildungen und uber eine damit zusammenhangende Erweiterung*

des Picardschen Satzes. — Ber. Verh. sachs. Akad. Wiss. Leipzig, Math.-phys. Kl. **80** (1928), 503–507.

171. H. Grötzsch, *Über die Verzerrung bei schlichten konformen Abbildung mehrfach zusammenhangender schlishter Bereiche.* I, II, III. — Ber. Verh. sachs. Akad. Wiss. Leipzig, Math.-phys. Kl. **81** (1929), 38–50, 217–221; **83** (1931), 283–297.
172. H. Grötzsch, *Über konforme Abbildung unendlich vielfach zusammenhangender schlichter Bereiche mit endlichvielen Haufungsrandkomponenten.* — Ber. Verh. sachs. Akad. Wiss. Leipzig, Math.-phys. Kl. **81** (1929), 51–86.
173. H. Grötzsch, *Zum Parallelschlitztheorem der konformen Abbildung schlichter unendlich-vielfach zusammenhangender Bereich.* — Ber. Verh. sachs. Akad. Wiss. Leipzig, Math.-phys. Kl. **83** (1931), 185–200.
174. H. Grötzsch, *Über die Verschiebung bei schlichter konformer Abbildung schlichter Bereiche.* I, II. — Ber. Verh. sachs. Akad. Wiss. Leipzig, Math.-phys. Kl. **83** (1931), 254–279; **84** (1932), 269–278.
175. H. Grötzsch, *Über das Parallelschlitztheorem der konformen Abbildung schlichter Bereiche.* — Ber. Verh. sachs. Akad. Wiss. Leipzig, Math.-phys. Kl. **84** (1932), 15–36.
176. H. Grötzsch, *Über zwei Verschiebungsprobleme der konformen Abbildung.* — Sitzgsber. preus. Akad. Wiss., Phys.-math. Kl. (1933), 87–100.
177. H. Grötzsch, *Über die Geometrie der schlichten komformen Abbildung.* I, II, III. — Sitzgsber. preus. Akad. Wiss., Phys.-math. Kl. (1933), 654–671, 893–908; (1934), 434–444.
178. H. Grötzsch, *Verallgemeinerung eines Bieberbachschen Satzes.* — Jber. Deutsch. Math.-Ver. **44** (1934), 143–145.
179. H. Grötzsch,*Über Flachensatze der konformen Abbildung.* — Jber. Deutsch. Math.-Ver. **44** (1934), 266–269.
180. H. Grötzsch, *Einige Bemerkungen zur schlichten konformen Abbildung.* — Jber. Deutsch. Math.-Ver. **44** (1934), 270–275.
181. H. Grunsky, *Neue Abschatzungen zur konformer Abbilding ein- und mehrfach zusammenhangender Bereiche.* — Schr. Math. Seminars Inst. angew. Math. Univ. Berlin **1** (1932), 93–140.
182. В. Я. Гутлянский, *Параметрическое представление однолистных функций.* — Докл. АН СССР **194** (1970), No. 4,750–753.
183. В. К. Хейман, *Многолистные функции.* М., 1960.

184. H. Huckemann, *On extremal decompositions of the unit disk.* — J. Anal. Math. **19** (1967), 173–202.
185. Л. И. Колбина, *Некоторые экстремальные задачи в конформном отображении.* — Докл. АН СССР **84** (1952), 865–868.
186. Е. В. Костюченко, *Решение одной задачи об экстремальном разбиении.* — Дальневост. матем. журн. **2** (2001), No. 1, 3–15.
187. Y. Kubota, *A coefficient inequality for certain meromorphic functions.* — Kodai Math. Sem. Rep. **26** (1974/1975), No. 1, 85–94.
188. Y. Kubota, *On the fourth coefficient of meromorphic univalent functions.* — Kodai Matn. Sem. Rep. **26** (1974/1975), Nos. 2-3, 267–288.
189. П. П. Куфарев, Ф. Э. Фалес, *Об одной экстремальной задача для дополнительных областей.* — Уч. записки Томск. ун-та **17** (1952), 25–35.
190. В. О. Кузнецов, *О свойствах ассоциированных квадратичных дифференциалов в некоторых экстремальных задачах.* — Зап. научн. семин. ПОМИ **168** (1988), 85–97.
191. В. О. Кузнецов, *К задаче о произведении конформных радиусов неналегающих областей.* — Зап. научн. семин. ПОМИ **212** (1994), 114–128.
192. В. О. Кузнецов, *О максимуме произведения конформных радиусов неналегающих областей в круге.* — Зап. научн. семин. ПОМИ **237** (1997), 105–118.
193. В. О. Кузнецов, *Некоторые геометрические свойства экстремальных разбиений.* — Зап. научн. семин. ПОМИ **418** (2014), 121–135.
194. Г. В. Кузьмина, *Модули семейств кривых и квадратичные дифференциалы.* — Труды Мат. ин-та им. В. А. Стеклова АН СССР **139** (1980), 1–243.
195. Г. В. Кузьмина, *К задаче о максимуме произведения конформных радиусов неналегающих областей.* — Зап. научн. семин. ЛОМИ **100** (1980), 131–145.
196. Г. В. Кузьмина, *Об одной проблеме модуля для семейств кривых.* Препринты ЛОМИ, Р-6-83, Ленинград, 1983.
197. Г. В. Кузьмина, *Теоремы покрытия в классе функций Бибербаха–Эйленберга.* — Зап. научн. семин. ЛОМИ **112** (1981), 143–158.

198. Г. В. Кузьмина, *Об экстремальных свойствах квадратичных дифференциалов с полосообразными областями в структуре траекторий.* — Зап. научн. семин. ЛОМИ **154** (1986), 110–129.
199. Г. В. Кузьмина, *К вопросу об экстремальном разбиении римановой сферы.* — Зап. научн. семин. ПОМИ **185** (1990), 72–95.
200. Г. В. Кузьмина, *Задачи об экстремальном разбиении и оценки коэффициентов в классе* $\Sigma(r)$. — Зап. научн. семин. ПОМИ **196** (1991), 101–116.
201. Г. В. Кузьмина, *Методы геометрической теории функций.* — Алгебра и анализ **9** (1997), вып. 3, 41–103; вып. 5, 1–50.
202. Г. В. Кузьмина, *Задачи об экстремальном разбиении римановой сферы.* I, II, III. — Зап. научн. семин. ПОМИ **276** (2001), 253–275; **286** (2002), 126–147; **314** (2004), 124–141.
203. Г. В. Кузьмина, *Метод экстремальной метрики в задачах о максимуме произведения степеней конформных радиусов неналегающих областей при наличии свободных параметров.* — Зап. научн. семин. ПОМИ **302** (2003), 52–67.
204. Г. В. Кузьмина, *О симметричных конфигурациях в задачах об экстремальном разбиении.* I, II, III. — Зап. научн. семин. ПОМИ **350** (2007), 160–172; **357** (2008), 158–178; **371** (2009), 117–136.
205. Г. В. Кузьмина, *Метод экстремальной метрики в задаче о максимуме конформного инварианта.* — Зап. нучн. семин. ПОМИ **383** (2010), 126–143.
206. Г. В. Кузьмина, *Задачи о максимуме одного конформного инварианта при наличии высокой степени симметрии.* — Зап. научн. семин. ПОМИ **392** (2011), 146–158.
207. Г. В. Кузьмина, *Квадратичные дифференциалы с полосообразными областями в структуре траекторий в задачах об экстремальном разбиении.* — Зап. научн. семин. ПОМИ **404** (2012), 199–213.
208. Г. В. Кузьмина, *Метод модулей и экстремальные задачи в классе* $\Sigma(r)$. — Зап. научн. семин. ПОМИ **418** (2013), 136–152.
209. Г. В. Кузьмина, *Метод модулей в общей задаче об экстремальном разбиении.* — Зап. научн. семин. ПОМИ **440** (2115), 179–186.
210. Н. А. Лебедев, *Принцип площадей в теории однолистных функций.* М., 1975.

211. L. Liao, *Certain extremal problems concerning module and harmonic measure.* — J. Anal. Math. **40** (1981), 1–42.
212. K. Loewner, *Untersuchungen uber schlichte konforme Abbildungen des Einheitskreises.* I. – Math. Ann.**89** (1923), 103–121.
213. D. Marshal, C, Sundberg, *Harmonic measure and radial projection.* — Trans. Amer. Math. Soc. **316** (1989), 81–95.
214. A. Obrock, *Teichmuller inequalities without coefficient normalization.* — Trans. Amer. Math. Soc. **159** (1971), 391–416.
215. A. Pfluger, *The convexity of certain sections of the nth coefficient body for schlicht functions.* — Некоторые проблемы математики и механики, Л., 1970, 233–241.
216. D. Phelps, *On a coefficient problem in univalent functions.* — Trans. Amer. Math. Soc. **143** (1969), 475–485.
217. U. Pirl, *Über die geometrische Gestalt einesExtremalkontinuums aus der Theorie der konformen Abbildung.* — Math. Nachr. **39** (1969), No.4–6, 297–312.
218. G. Polia, Szego, *Isoperimetric Inequalities in Mathematical Phisics.* — Ann. Math. Stud., vol. 27, Princeton University Press, Princeton, 1951.
219. Ch. Pommerenke, *Über die Subordination analytischer Funktionen.* — J. reine angew. math. **218** (1965), 150–173.
220. Ch. Pommerenke, *Univalent Functions.* (With a chapter on quadratic differentials by G. Jensen.) — Math. Lehrbucher, Bd. 25, Vandenhoeck and Ruprecht, Gottingen, 1975. M., 1975.
221. E. Reich, M. Schiffer, *Estimates for transfinite diameter of a continuum.* — Math. Z. **85** (1964), 91–106.
222. H. Renelt, *Konstruktion gewisser quadratischer Differentiale mit Hilfe von Dirichketintegralen.* — Math Nachr. **73** (1970), 125–142.
223. A. C. Schaeffer, D. C. Spenser, *The coefficients of schlicht functions.* I, II. — Duke Math. J. **10** (1943), 611–635; **12** (1945), 107–125.
224. A. C. Schaeffer, D. C. Spencer, *A variational method in conformal mapping.* — Duke Math. J. **14** (1947), 949–966.
225. A. C. Schaeffer, D. C. Spenser, *Coefficient regions for schlicht functions.* — Amer. Math. Soc. Colloq. Publ. **35**, Amer. Math. Soc., New York, 1950.
226. M. Schiffer, *A method of variation within the family of simple functions.* — Proc. London Math. Soc. **44** (2) (1938), 432–449.

227. M. Schiffer, *On the coefficients of simple functions.* — Proc. London Math. Soc. **44** (2) (1938), 450–452.
228. M. Schiffer, *Variation of the Green function and the theory of the p-valued functions.* — Amer. J. Math.**65** (1943), 341–360.
229. А. Ю. Солынин, *Зависимость проблемы модуля для семейства нескольких классов кривых от параметра.* — Зап. научн. семин. ЛОМИ **144** (1985), 136–145.
230. А. Ю. Солынин, *Решение одной изопараметрической задачи Полиа–Сеге.* — Зап. научн. семин. ПОМИ **168** (1988), 140–153.
231. А. Ю. Солынин,*О разделении континуумов окружностями.* — Зап. научн. семин. ПОМИ **168** (1988),154–157.
232. А. Ю. Солынин, *Модули двусвязных областей и конформно-инвариантные метрики.* — Зап. научн. семин. ПОМИ **196** (1991), 122–131.
233. А. Ю. Солынин, *Некоторые экстремальные задачи в классе* $\Sigma(r)$. — Зап. научн. семин. ПОМИ **196** (1991), 138–153.
234. А. Ю. Солынин, *Изопериметрические неравества для полигонов и диссимметризация.* — Алгебра и анализ **4** (1992), 210–234.
235. А. Ю. Солынин,*Геометрические свойства экстремальных разбиений и оценки модулей семейств кривых в круговом кольце.* — Зап. научн. семин. ПОМИ **204** (1993), 93–114.
236. A. Yu. Solynin, *Extremal problems for conformal modules and estimates for harmonic measure.* — J. Anal. Math. **74** (1998), 1–50.
237. А. Ю. Солынин, *Минимизация конформного радиуса при круговом сужении области.* — Зап. научн. семин. ПОМИ **254** (1998), 145–164.
238. А. Ю. Солынин, *Модули и экстремально-метрические проблемы.* — Алгебра и анализ **11** (1999), вып. 1, 3–86.
239. A. Yu. Solynin, M. Vuorinen, *Extremal problems and symmetrization for plane ring domains.* — Trans. Amer. Math. Soc. **348** (1999), 4095–4112.
240. A. Yu. Solynin, V. A. Zalgaller, *An isoperimetric inequality for logarithmic capacity of polygons.* — Ann. Math. (2) **159** (2004), No. 1, 277–303.
241. A. Yu. Solynin, V. A. Zalgaller, *The inradius, the fitst eigenvalue, and the torsional rigidity of curvilinear polygons.* — Bull. London Math. Soc. **42** (2010), No. 5, 765–783.

242. K. Strebel, *Quadratic differentials*, Ergeb. Math. Grenzgeb. (3), Bd.5, Springer-Verlag, Berlin-New York, 1984.
243. N. Suita, *A distortion theorem of univalent functions related to symmetric three points.* — Kodai Math. Semin. Rep. **14** (1962), 26–30.
244. П. М. Тамразов, *Теоремы покрытия линий при конформном отображении.* — Мат. сб. **66** (1965), No. 4, 502–524.
245. П. М. Тамразов, *К общей теореме о коэффициентах.* — Мат. сб. **72** (1967), No. 1, 59–71.
246. П. М. Тамразов, *Экстремальные конформные отображения и полюсы квадратичных дифференциалов.* — Изв. АН СССР. Сер. мат. **32** (1968), No. 5, 1033–1043.
247. O. Teichmuller, *Untersuchungen uber konforme und quasikonforme* Abbildung. Dtsch. Math. **3** (1938), 621–678.
248. O. Teichmuller, *Ungleichungen zwischen den Koeffizienten schlichter Funktionen.* — Sitzgsber. preus. Akad. Wiss., Phys.-math. Kl. (1938), 363–375.
249. O. Teichmuller, *Extremale quasikonforme Abbildungen und quadratische Differentiale.* — Abh. preus. Akad. Wiss., Math.-naturwiss. Kl. **22** (1939).
250. A. Vasil'ev, *Moduli of families of curves for conformal and quasiconformal mappings.* Lect. Notes Math., Springer, 2002.

С.-Петербургское отделение
Математического института
им. В. А. Стеклова РАН
Фонтанка 27, 191023
С.-Петербург, Россия
E-mail: kuzmina@pdmi.ras.ru

Поступило 5 мая 2016 г.