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PROOF OF SCHAUDER ESTIMATES FOR PARABOLIC INITIAL-BOUNDARY VALUE MODEL PROBLEMS VIA O. A. LADYZHENSKAYA'S FOURIER MULTIPLIERS THEOREM

Abstract. The paper is concerned with estimates of the Hölder norms of solutions of model parabolic initial-boundary value problems in a half-space. The proof is based on the theorem on the Fourier multipliers in anisotropic Hölder spaces due to O. A. Ladyzhenskaya and on K. K. Golovkin's theorem on equivalent norms in these spaces.

§1. Introduction and the scalar case.

O. A. Ladyzhenskaya [1] has proved the following theorem on the Fourier multipliers in the Hölder spaces of functions.

Theorem 1. *Let $C^{\alpha,\alpha\gamma}(\mathbb{R}^{n+1})$ be the space with the norm*

$$|u|_{C^{\alpha,\gamma\alpha}(\mathbb{R}^{n+1})} = \langle u\rangle^{(\alpha,\alpha\gamma)}_{\mathbb{R}^{n+1}} + \sup_{\mathbb{R}^{n+1}}|u(x,t)|, \tag{1.1}$$

where

$$\langle u\rangle^{(\alpha,\alpha\gamma)}_{\mathbb{R}^{n+1}} = \sup_{(X,Y)\in\mathbb{R}^{n+1}} \frac{|u(X)-u(Y)|}{\rho^\alpha(X-Y)},$$

$$\rho(X-Y)=\sum_{k=1}^n |x_k-y_k|+|t-\tau|^\gamma,\ X=(x,t),\ Y=(y,\tau),\ \gamma,\alpha\in(0,1). \tag{1.2}$$

Consider the convolution operator

$$v(x) = (m*u) = \int\limits_{\mathbb{R}^{n+1}} m(x-y,t-\tau)u(y,\tau)dy\,d\tau \tag{1.3}$$

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with

$$m(x,t) = \int\limits_{\mathbb{R}^{n+1}} e^{i(\xi\cdot x+\xi_0 t)}\widetilde{m}(\xi,\xi_0)d\xi d\xi_0 \equiv F^{-1}\widetilde{m}(\xi,\xi_0), \qquad \xi = (\xi_1,\dots,\xi_n) \in \mathbb{R}^n. \tag{1.4}$$

Assume that

$$\|F^{-1}\widetilde{\mathcal{M}}_j\|_{L_1(\mathbb{R}^{n+1})} \leqslant \mu, \tag{1.5}$$

where $j = \{0, \pm 1, \pm 2, \dots\}$,

$$\widetilde{\mathcal{M}}_j = \widetilde{m}(2^j\xi_1, .., 2^{j/\gamma}\xi_0)\chi(\xi,\xi_0), \tag{1.6}$$

and $\chi(\xi,\xi_0) = \theta((\rho(\xi,\xi_0))$, $\rho(\xi,\xi_0) = \sum\limits_{k=1}^{n} |\xi_k| + |\xi_0|^\gamma$, $\theta \in C_0^\infty(\mathbb{R}_+)$, $\theta : [0,\infty) \to [0,1]$, $\theta(\rho) = 1$ *for* $\rho \in [1/2, 1]$, $\theta(\rho) = 0$ *for* $\theta \in [0, 1/4] \cup [4, \infty)$. *Then*

$$\langle v\rangle^{(\alpha,\alpha\gamma)}_{\mathbb{R}^{n+1}} \leqslant c(n,\gamma,\alpha)\mu\langle u\rangle^{(\alpha,\alpha\gamma)}_{\mathbb{R}^{n+1}}. \tag{1.7}$$

This theorem has been used in [1, 2] for estimating the solution of the Cauchy problem for generalized Stokes equations.

The multipliers theorem in the Hölder spaces with a homogeneous metrics $\rho(\xi) = |\xi|$ (i.e., with $\gamma = 1$) is due to L. Hörmander [3].

We estimate the L_1-norm of $\mathcal{M}_j$ by the inequality

$$\|\mathcal{M}\|_{L_1(\mathbb{R}^{n+1})} \leqslant c\int\limits_0^\infty \dots \int\limits_0^\infty \|\Delta_1(h_1)\dots\Delta_n(h_n)\Delta_0(h_0)\widetilde{\mathcal{M}}\|_{L_2(\mathbb{R}^{n+1})} \frac{dh_1}{h_1^{3/2}} \dots \frac{dh_0}{h_0^{3/2}} \equiv c|||\widetilde{\mathcal{M}}||| \tag{1.8}$$

(see [4] and Section 3 of the present paper), where

$$\Delta_k(h_k)u(x,t) = u(x_1,\dots,x_k+h_k,\dots,x_n,t) - u(x,t),$$

$\Delta_0(h_0)u(x,t) = u(x,t+h_0) - u(x,t)$ and $h_0, h_1, \dots$ are the incremental steps. Estimate (1.8) is an analog of the Szász theorem concerning the uniform convergence of the Fourier series [5]. By (1.7) and (1.8),

$$\langle v\rangle^{(\alpha,\alpha\gamma)}_{\mathbb{R}^{n+1}} \leqslant c\sup_j |||\widetilde{\mathcal{M}}_j|||\langle u\rangle^{(\alpha,\alpha\gamma)}_{\mathbb{R}^{n+1}}. \tag{1.9}$$

In particular, $\sup\limits_j |||\widetilde{M}_j|||$ is finite, if the multiplier $\widetilde{m}$ is homogeneous, i.e.,

$$\widetilde{m}(\lambda\xi, \lambda^{1/\gamma}\xi_0) = \widetilde{m}(\xi,\xi_0), \quad \forall\lambda > 0,$$

and

$$D_{\xi_{i_1}},\dots D_{\xi_{i_k}}\widetilde{m}(\xi,\xi_0)\chi(\xi,\xi_0)\in L_2(\mathbb{R}^{n+1}). \tag{1.10}$$

In the present paper, we use Theorem 1 for deriving the Schauder estimates of solutions of $2b$- parabolic model initial-boundary value problems in a half-space. This simplifies the arguments in the paper [6], where potential-theoretic methods for obtaining such estimates are used. We work with the "parabolic" anisotropic spaces, which corresponds to $\gamma = 1/2b$. We assume that $\widetilde{m}$ is a function of $\xi \in \mathbb{R}^n$ and $s \in \mathbb{C}$ holomorphic with respect to s in the domain $\mathrm{Re}s > -\delta|\xi|^{2b}$, $\delta > 0$. Moreover, to ensure (1.10), we replace $\rho(\xi,\xi_0) = \sum\limits_{i=1}^{n}|\xi_i| + |\xi_0|^{1/2b}$ by a more regular function $\rho(\xi,\xi_0) = (\xi_0^2 + |\xi|^{4b})^{1/4b}$.

To clarify our arguments, we restrict ourselves in this section with the scalar case and consider model problems of the form

$$\begin{cases} L(\dfrac{\partial}{\partial x},\dfrac{\partial}{\partial t})u(x,t) = f(x,t), \quad x\in\mathbb{R}^n_+, \quad t\in(0,\infty),\\ B_q(\dfrac{\partial}{\partial x},\dfrac{\partial}{\partial t})u(x,t)|_{x_3=0} = \Phi_q(x',t), \quad q=1,\dots br,\\ \qquad\qquad x' = (x_1,\dots,x_{n-1})\in\mathbb{R}^{n-1},\\ u|_{t=0} = u_0(x), \quad \dots\dfrac{\partial^{r-1}}{\partial t^{r-1}}u\Big|_{t=0} = u_{r-1}(x), \quad x\in\mathbb{R}^n_+, \end{cases} \tag{1.11}$$

where L is a $2b$-parabolic operator in the sense of Petrovskii of order r with respect to t and of order $2br$ with respect to the spatial variables; the B_q are boundary operators of order $\sigma_q + 2br$ with arbitrary integral σ_q such that $\sigma_q + 2br \geqslant 0$. It is assumed that $L(i\xi,s)$ and $B_q(i\xi,s)$ are homogeneous in the following sense:

$$L(i\xi\lambda,s\lambda^{2b}) = \lambda^{2br}L(i\xi,s), \quad B_q(i\xi\lambda,s\lambda^{2b}) = \lambda^{\sigma_q+2br}B_q(i\xi,s), \quad \forall\lambda>0,$$

and, moreover, the complementing condition (Lopatinskii condition) is satisfied.

We recall that, according to the parabolicity condition, the roots of the polynomial $L(i\xi,s)$ with respect to s, $p_i(\xi)$, $i=1,\dots,r$, satisfy the inequality

$$\mathrm{Re}p_i \leqslant -\delta|\xi|^{2b}, \quad \forall\xi\in\mathbb{R}^n, \quad \delta>0.$$

It follows that for arbitrary $\xi' = (\xi_1,\dots,\xi_{n-1})\in\mathbb{R}^{n-1}$ and $s\in\mathbb{C}$ with $\mathrm{Re}s > -\delta|\xi'|^{2b}$ the polynomial $L(i\xi',s,i\tau)$ has br roots with respect to

τ with positive and br with negative imaginary part. The complementing condition means in particular that the homogeneous problem (1.36) has only a trivial solution.

It is proved in [6] that under some necessary compatibility conditions for the data u_j, Φ_q, f, the solution of (1.11) satisfies coercive Schauder type estimates

$$\langle u\rangle^{(l+2br,l/2b+r)}_{\mathbb{R}^n_{+T}} \leqslant c\Big(\langle f\rangle^{(l,l/2b)}_{\mathbb{R}^n_{+T}} + \sum_{k=0}^{r-1}\langle u_k\rangle^{(2b(r-k)+l)}_{\mathbb{R}^n_+} \tag{1.12}$$
$$+\sum_{q=1}^{br}\langle \Phi_q\rangle^{(l-\sigma_q,l/2b-\sigma_q/2b)}_{\mathbb{R}^{n-1}_T}\Big),$$

where $l > \max(0,\sigma_1,\dots,\sigma_{br})$, $\langle u\rangle^{(l_1,l_1/2b)}_{Q_T}$ and $\langle u\rangle^{(l)}_{\mathbb{R}^n_+}$ are principle parts of the norms in $C^{l_1,l_1/2b}(Q_T)$, and $C^l(\mathbb{R}^n_+)$, respectively, l and l_1 are not integers, $\mathbb{R}^n_{+T} = \mathbb{R}^n_+ \times (0,T)$, $\mathbb{R}^{n-1}_T = \mathbb{R}^{n-1}\times(0,T)$, $\mathbb{R}^n_+ = \{x_n > 0\}$, $T \leqslant \infty$, c is a constant independent of T.

According to K. K. Golovkin's results [7,8], the norm $\langle u\rangle^{(l,l/2b)}_{\mathbb{R}^n_{+,\infty}}$, $\mathbb{R}^n_{+,\infty} = \mathbb{R}^n_+\times\mathbb{R}_+$, can be defined as

$$\begin{aligned}\langle u\rangle^{(l,l/2b)}_{\mathbb{R}^n_{+,\infty}} &= \sum_{j=1}^n \sup_{h>0}\sup_{\mathbb{R}^n_{+,\infty}} h^{-l}|\Delta^p_j(h)u(x,t)| + \sup_{h_0>0}\sup_{\mathbb{R}^n_{+,\infty}} h_0^{-l/2b}|\Delta^p_0(h_0)u(x,t)|\\ &\equiv \sum_{j=1}^n\langle u\rangle^{(l)}_{x_j,\mathbb{R}^n_{+,\infty}} + \langle u\rangle^{(l/2b)}_{t,\mathbb{R}^n_{+,\infty}}\end{aligned} \tag{1.13}$$

with arbitrary $p > l$, $p_0 > l/2b$ (also for integral l), where

$$\Delta^p_j(h)u(x,t) = \sum_{k=0}^p (-1)^{p-k}C^k_p u(x+k\boldsymbol{e}_jh,t),\quad \boldsymbol{e}_j = (\delta_{jq})_{q=1,\dots,n},$$
$$\Delta^{p_0}_t(h_0)u(x,t) = \sum_{k=0}^{p_0}(-1)^{p_0-k}C^k_{p_0}u(x,t+kh_0)$$

are finite differences of the function u with respect to x_j and t. The above semi-norms are equivalent for arbitrary $p > l$ and $p_0 > l/2b$, respectively. Another semi-norm equivalent to (1.13) (and used more often) is

$$\langle u\rangle^{(l,l/2b)}_{\mathbb{R}^n_{+,\infty}} = \sum_{|j|=[l]}\langle D^j_x u\rangle^{(l-[l])}_{x,\mathbb{R}^n_{+,\infty}} + \langle D^{[l/2b]}_t u\rangle^{(l/2b-[l/2b])}_{t,\mathbb{R}^n_{+,\infty}}.$$

Moreover, the inequalities

$$\langle D_x^j D_t^k u\rangle^{(l-|j|-2bk,(l-|j|)/2b-k)}_{\mathbb{R}^n_{+,\infty}} \leqslant c\langle u\rangle^{(l,l/2b)}_{\mathbb{R}^n_{+,\infty}}, \quad |j|+2bk<l,$$

and

$$\langle D_x^j D_t^k u\rangle^{(\rho)}_{x_q,\mathbb{R}^n_{+,\infty}} \leqslant \varepsilon\langle u\rangle^{(l)}_{x_j,\mathbb{R}^n_{+,\infty}} + c(\varepsilon)\Big(\sum_{j\neq q}\langle u\rangle^{(l)}_{x_j,\mathbb{R}^n_{+,\infty}} + \langle u\rangle^{(l/2b)}_{t,\mathbb{R}^n_{+,\infty}}\Big), \ \ j<[l],$$

$$\langle D_x^j D_t^k u\rangle^{(\rho/2b)}_{t,\mathbb{R}^n_{+,\infty}} \leqslant \varepsilon_0\langle u\rangle^{(l/2b)}_{t,\mathbb{R}^n_{+,\infty}} + c(\varepsilon_0)\sum_{q=1}^{n}\langle u\rangle^{(l)}_{x_q,\mathbb{R}^n_{+,\infty}}, \ \ k<\Big[\frac{l}{2b}\Big], \tag{1.14}$$

hold with arbitrary $\varepsilon, \varepsilon_0 > 0$, if $\rho = l(1-\frac{|j|}{l}-\frac{2bk}{l}) > 0$ (see for instance [9, Lemma 2]). Finally, we mention "the inverse trace theorem" for the Hölder spaces. In principle, results of this type are well known. However, for the sake of completeness we present the respective result and the proof below.

Proposition 1. *If $u_k \in C^{2bm+l_1-2bk}(\mathbb{R}^n)$, $k=0,\dots m$, $l_1<2b$, then there exists a function $U(x,t)$ defined in $\mathbb{R}^n_\infty = \{x\in\mathbb{R}^n, t\geqslant 0\}$ such that*

$$D_t^k U|_{t=0} = u_k, \quad k=0,\dots,m \tag{1.15}$$

and

$$\langle U\rangle^{(2bm+l_1,m+l_1/2b)}_{\mathbb{R}^n_\infty} \leqslant c\sum_{k=0}^{m}\langle u_k\rangle^{(2bm+l_1-2bk)}_{\mathbb{R}^n}. \tag{1.16}$$

Proof. In the proof given below, the Hörmander multiplier theorem is applied to the following Cauchy problem for U:

$$\begin{cases} \Big(\dfrac{\partial}{\partial t} + (-1)^b\Delta^b\Big)^{m+1} U(x,t) = 0, \quad x\in\mathbb{R}^{n+1}, \quad t>0, \\ D_t^k U|_{t=0} = u_k, \quad k=0,\dots,m, \quad x\in\mathbb{R}^n. \end{cases} \tag{1.17}$$

The Fourier transform with respect to $x\in\mathbb{R}^n$ converts (1.17) into

$$(\frac{d}{dt}+|\xi|^{2b})^{m+1}\widetilde{U}(\xi,t)=0, \quad \frac{d^k}{dt^k}\widetilde{U}|_{t=0}=\widetilde{u}_k, \quad k=0,\dots,m, \quad \xi\in\mathbb{R}^n. \tag{1.18}$$

The general solution of the equation $(\frac{d}{dt}+|\xi|^{2b})^{m+1}\widetilde{U}=0$ has the form

$$\widetilde{U}(\xi,t) = (c_0+c_1t+\cdots+c_mt^m)e^{-t|\xi|^{2b}}.$$

The constants c_i are found from the initial conditions. After some calculations, we obtain

$$\widetilde{U}(\xi,t)=\sum_{k=0}^{m}\frac{1}{k!}\widetilde{u}_k t^k(1+P_{m-k}(\xi^{2b}t))e^{-\xi^{2b}t},$$

where P_j is a polynomial of degree j with constant coefficients and $P_{m-k}(0)=0$. Hence

$$Q(\xi)\widetilde{U}\equiv\widetilde{V}(\xi,t)=\sum_{k=0}^{m}\widetilde{v}_k(\xi)E_k(\xi,t), \tag{1.19}$$

where $\widetilde{v}_k(\xi) = \frac{1}{k!}i\xi_{q_{2bk+1}}\dots i\xi_{q_{2bm+[l_1]}}\widetilde{u}_k$, $Q(\xi) = i\xi_{q_1}\dots i\xi_{q_{2bm+[l_1]}}$, $q_i=1,\dots,n$,

$$E_k(\xi,t)=i\xi_{q_1}\dots i\xi_{q_{2bk}}t^k(1+P_{m-k}(\xi^{2b}t))e^{-|\xi|^{2b}t}.$$

The function E_k is a $C^\lambda(\mathbb{R}^n)$-multiplier for arbitrary $t>0$, because $E_k(2^j\xi,t)=E_k(\xi,2^{2bj}t)$ and

$$E_k(\xi,2^{2bj}t)\theta(|\xi|)$$

is uniformly bounded together with its derivatives with respect to ξ_j. Therefore,

$$\langle V(\cdot,t)\rangle_{\mathbb{R}^n}^{(\lambda)}\leqslant c\sum_{k=0}^{m}\langle v_k\rangle_{\mathbb{R}^n}^{(\lambda)},\quad \lambda=l_1-[l_1],\quad \forall t\geqslant 0,$$

by the Hörmander theorem, which implies

$$\langle U(\cdot,t)\rangle_{\mathbb{R}^n}^{(2bm+l_1)}\leqslant c\sum_{k=0}^{m}\langle u_k\rangle_{\mathbb{R}^n}^{(2b(m-k)+l_1)},\quad \forall t>0. \tag{1.20}$$

Now, we estimate the norm $\langle U\rangle_{t,\mathbb{R}^n_\infty}^{(m+l_1/2b)}$. We notice that $e^{-\xi^{2b}t}$ is the Fourier transform of the fundamental solution $G_0(x,t)$ of the equation $\left(\frac{\partial}{\partial t}+(-1)^b\Delta^b\right)u=0$. As it was shown by O. A. Ladyzhenskaya in [10] (see also [6]), G_0 is subject to the inequalities

$$|D_t^kD_x^jG_0(x,t)|\leqslant ct^{(-n-|j|)/2b-k}\exp\Big(-\frac{|x|^{\frac{2b}{2b-1}}}{t^{\frac{1}{2b-1}}}\Big). \tag{1.21}$$

The expressions $E_k(\xi,t)$ is the Fourier transfrom of the kernel $G(x,t)$ satisfying similar inequalities.

We consider the convolution $W(x,t)=\int\limits_{\mathbb{R}^n} G(x-y,t)\varphi(y)\,dy$. Since

$$\int\limits_{\mathbb{R}^n} G_\tau(x-y,\tau)(y_{i_1}-x_{i_1})\dots(y_{i_q}-x_{i_q})\,dy=0,\quad q<2b,$$

we have

$$W(x,t+h)-W(x,t)=\int\limits_t^{t+h} G_\tau(x-y,\tau)(\varphi(y)-T[\varphi])\,dy,$$

where

$$T[\varphi]=\varphi(x)+\sum_{k_1=1}^n\frac{\partial\varphi(x)}{\partial x_{k_1}}(y_{k_1}-x_{k_1})+\frac{1}{2!}\sum_{k_1,k_2=1}^n\frac{\partial^2\varphi(x)}{\partial x_{k_1}\partial x_{k_2}}(y_{k_1}-x_{k_1})(y_{k_2}-x_{k_2})+\dots$$

$$+\frac{1}{[l_1]!}\sum_{k_1,k_2,\dots,k_{[l_1]}=1}^n\frac{\partial^{[l_1]}\varphi(x)}{\partial x_{k_1}\partial x_{k_2}\dots\partial x_{k_{[l_1]}}}(y_{k_1}-x_{k_1})(y_{k_2}-x_{k_2})\dots(y_{k_{[l_1]}}-x_{k_{[l_1]}}).$$

This implies that

$$|W(x,t+h)-W(x,t)|\leqslant c\int\limits_t^{t+h}\int\limits_{\mathbb{R}^n}\frac{|x-y|^{l_1}\,d\tau dy}{(\tau+|x-y|^{2b})^{(n+2b)/2b}}\langle\varphi\rangle_{\mathbb{R}^n}^{(l_1)}\leqslant ch^{l_1/2b}\langle\varphi\rangle_{\mathbb{R}^n}^{(l_1)}. \tag{1.22}$$

Since $\left(\frac{\partial}{\partial t}\right)^m\widetilde{U}$ can be represented as a linear combination of $|\xi|^{2b(m-k)}\widetilde{u}_kE'_k$, where E'_k is a function of the same type as E_k, we use (1.22), and arrive at the estimate

$$\langle U\rangle_{t,\mathbb{R}^n_\infty}^{(m+l_1/2b)}\leqslant c\sum_{i=0}^m\langle u_k\rangle_{\mathbb{R}^n}^{(2b(m-k)+l_1)}.$$

Together with (1.20), this inequality proves (1.16). □

Now, we return to the estimate of the convolution integral (1.3).

Proposition 2. *If* $\widetilde{m}(\lambda\xi,\lambda^{2b}\xi_0)=\widetilde{m}(\xi,\xi_0)$ *and* (1.10) *holds, then*

$$\langle v\rangle_{\mathbb{R}^{n+1}}^{(l,l/2b)}\leqslant c\langle u\rangle_{\mathbb{R}^{n+1}}^{(l,l/2b)} \tag{1.23}$$

for arbitrary non-integral l. *If* $\widetilde{m}(\lambda\xi,\lambda^{2b}\xi_0)=\lambda^{-\omega}\widetilde{m}(\xi,\xi_0)$, *with* $\omega\geqslant1$ *and* $|D_{\xi_{k_1}}\dots D_{\xi_{k_d}}D^k_{\xi_0}\widetilde{m}|\leqslant c\rho^{-\omega-d-2bk}(\xi,\xi_0)$, $\quad k=0,1,$ *then*

$$\langle v\rangle_{\mathbb{R}^{n+1}}^{(l+\omega,(l+\omega)/2b)}\leqslant c\langle u\rangle_{\mathbb{R}^{n+1}}^{(l,l/2b)}. \tag{1.24}$$

Proof. We apply (1.9) (setting $\gamma = 1/2b$) to $\Delta_j^p(h)v(x,t)$ with arbitrarily fixed $h > 0$ and $p > l$ and assuming that $\alpha < l$. This gives

$$\begin{aligned} &h^{-l+\alpha} \sup_{h_1>0} h_1^{-\alpha} \sup_{\mathbb{R}^{n+1}} |\Delta_j^p(h)\Delta_j^{p_1}(h_1)v(x,t)| \\ &\leqslant c\Big(h^{-l+\alpha} \sum_{k=1}^{n} \sup_{h_1>0} h_1^{-\alpha} \sup_{\mathbb{R}^{n+1}} |\Delta_j^p(h)\Delta_k^{p_1}(h_1)u(x,t)| \\ &+ h^{-l+\alpha} \sup_{h_2>0} h_2^{-\alpha/2b}| \sup_{\mathbb{R}^{n+1}} |\Delta_j^p(h)\Delta_t^{p_2}(h_2)u(x,t)|\Big), \end{aligned} \tag{1.25}$$

where $p_1 > l$, $p_2 > l/2b$. It is clear that (1.25) implies

$$\begin{aligned} &h^{-l} \sup_{\mathbb{R}^{n+1}} |\Delta_j^{p+p_1}(h)v(x,t)| \\ &\leqslant c\Big(\sup_{h>0} h^{-l} \sup_{\mathbb{R}^{n+1}} |\Delta_j^p(h)u(x,t)| + \sum_{k=1}^{n} \sup_{h_1>0} h_1^{-l} \sup_{\mathbb{R}^{n+1}} |\Delta_j^{p_1}(h_1)u(x,t)| \\ &+ \sup_{h_2>0} h_2^{-l/2b} \sup_{\mathbb{R}^{n+1}} |\Delta_t^{p_2}(h_2)u(x,t)|\Big), \end{aligned} \tag{1.26}$$

and, consequently,

$$\langle v\rangle_{x_j,\mathbb{R}^{n+1}}^{(l)} \leqslant c\langle u\rangle_{\mathbb{R}^{n+1}}^{(l,l/2b)}, \quad j = 1, \dots, n.$$

The inequality

$$\langle v\rangle_{t,\mathbb{R}^{n+1}}^{(l/2b)} \leqslant c\langle u\rangle_{\mathbb{R}^{n+1}}^{(l,l/2b)}$$

is established in the same way, which completes the proof of (1.23).

To prove (1.24), we consider the equations

$$\begin{aligned} &(e^{i\xi_q h} - 1)^p\widetilde{v} = (e^{i\xi_q h} - 1)^p\widetilde{m}(\xi,\xi_0)\widetilde{u}, \quad p > l + \omega, \quad q = 1, \dots, n, \\ &(e^{i\xi_0 h_0} - 1)^{p_0}\widetilde{v} = (e^{i\xi_0 h_0} - 1)^{p_0}\widetilde{m}(\xi,\xi_0)\widetilde{u}, \quad p_0 > (l + \omega)/2b \end{aligned} \tag{1.27}$$

and estimate the norms (1.8) of the multipliers

$$\widetilde{\mathcal{M}}_j = \chi(\xi,\xi_0)(e^{i2^j\xi_q h} - 1)^p\widetilde{m}(2^j\xi', 2^{2bj}\xi_0)$$

and

$$\widetilde{\mathcal{M}}_{j_0} = \chi(\xi,\xi_0)(e^{i2^{2bj}\xi_0 h_0} - 1)^{p_0}\widetilde{m}(2^j\xi', 2^{2bj}\xi_0).$$

We can show that these norms are controlled by ch^ω and $ch_0^{\omega/2b}$, respectively. We have

$$
\begin{aligned}
&|\widetilde{m}(2^j\xi, 2^{2jb}\xi_0)||e^{i2^j\xi_q h}-1|^p \leqslant c2^{-j\omega}(|\xi_0|+|\xi|^{2b})^{-\omega/2b}(2^j h|\xi_q|)^\omega \leqslant ch^\omega,\\
&|(D_{\xi_{k_1}}\dots D_{\xi_{k_d}} D_{\xi_0}^k \widetilde{m}(2^j\xi, 2^{2bj}\xi_0))(e^{i2^j\xi_q h}-1)^p| \leqslant ch^\omega \frac{|\xi_q|^\omega}{(|\xi_0|+|\xi|^{2b})^{(\omega+d_1)/2b}},\\
&|(D_{\xi_{k_1}}\dots D_{\xi_{k_d}} D_{\xi_0}^k \widetilde{m}(2^j\xi, 2^{2bj}\xi_0))D_{\xi_q}(e^{i2^j\xi_q h}-1)^p| \leqslant ch^\omega \frac{|\xi_q|^{\omega-1}}{(|\xi_0+|\xi|^{2b})^{(\omega+d_1)/2b}},
\end{aligned}
\tag{1.28}
$$

where $k=0,1$, $d_1=d+2kb$. In the expression (1.8), we split each integral with respect to h_k I into two parts: from 0 to 1 and from 1 to ∞, and in the first integral we use the relation

$$
\Delta_k(h_k)\widetilde{\mathcal{M}}_j(\xi) = h_k \int_0^1 D_{\xi_k}\widetilde{\mathcal{M}}_j(\xi_1,\dots\xi_k+sh_k,\dots\xi_0)\,ds
$$

for $\widetilde{\mathcal{M}}_j = \widetilde{m}(2^j\xi, 2^{2jb}\xi_0)(e^{i2^j\xi_q h}-1)^p\chi(\xi,\xi_0)$. In this way, we estimate $|||\widetilde{\mathcal{M}}_j|||$ by the sum of the terms

$$
\begin{aligned}
&c\int_0^1 \frac{h_{k_1}\,dh_{k_1}}{h_{k_1}^{3/2}}\dots\int_0^1 \frac{h_{k_m}\,dh_{k_m}}{h_{k_m}^{3/2}}\int_1^\infty \frac{dh_{k_{m+1}}}{h_{k_{m+1}}^{3/2}}\\
&\dots\int_1^\infty \|D_{k_1}\dots D_{k_m}\widetilde{\mathcal{M}}_j\|_{L_2(\mathbb{R}^{n+1})}\frac{dh_{k_{n+1}}}{h_{k_{n+1}}^{3/2}}\\
&\leqslant c\|D_{k_1}\dots D_{k_m}\widetilde{\mathcal{M}}_j\|_{L_2(\mathbb{R}^{n+1})},
\end{aligned}
\tag{1.29}
$$

where $k_p\neq k_q$ for $p\neq q$, $k_i=0,1,\dots,n$, $D_k=\frac{\partial}{\partial\xi_k}$. In view of (1.28), each term is controlled by ch^ω. Hence

$$
\langle\Delta_q^p(h)v\rangle_{\mathbb{R}^{n+1}}^{(l,l/2b)} \leqslant ch^\omega\langle u\rangle_{\mathbb{R}^{n+1}}^{(l,l/2b)}, \quad q=1,\dots,n,
$$

which implies

$$
\langle v\rangle_{x_q,\mathbb{R}^{n+1}}^{(l+\omega)} \leqslant c\langle u\rangle_{\mathbb{R}^{n-1}}^{(l,l/2b)}, \quad q=1,\dots,n.
$$

The norm of $\widetilde{\mathcal{M}}_{j_0} = \widetilde{m}(2^j\xi, 2^{2jb}\xi_0)(e^{i2^{bj}\xi_0 h_0}-1)^{p_0}\chi(\xi,\xi_0)$ is estimated in the same way, and as a result we obtain $\langle\Delta_t^{p_0}(h_0)v\rangle_{\mathbb{R}^{n+1}}^{(l,l/2b)} \leqslant ch_0^{\frac{\omega}{2b}}\langle u\rangle_{\mathbb{R}^{n+1}}^{(l.l/2b)}$. This completes the proof of (1.24) and of the proposition. □

Now we proceed to the proof of the estimate (1.12), which is the main result of the section.

Theorem 2. *Assume that*

$$f \in C^{l,l/2b}(\mathbb{R}^n_{+,\infty}),\ u_q \in C^{2b(r-q)+l}(\mathbb{R}^n_+),\ \Phi_k \in C^{-\sigma_k+l,-\sigma_k/2b+l/2b}(\mathbb{R}^{n-1}_\infty)$$

with a positive non-integer $l > \max(0, \sigma_1, \dots, \sigma_{br})$ *and the compatibility conditions*

$$\Big(\frac{\partial}{\partial t}\Big)^k B_q u - \frac{\partial^k \Phi_q}{\partial t^k}\Big|_{t=0} = 0, \quad k = 0, \dots, \Big[\frac{l-\sigma_q}{2b}\Big] \tag{1.30}$$

are satisfied, where $\frac{\partial^k u}{\partial t^k}|_{t=0}$ *are found from the initial conditions if* $k \leqslant r-1$, *and from* $\frac{\partial^{k-r}}{\partial t^{k-r}}(Lu - f)|_{t=0} = 0$ *if* $r \leqslant k < r + l/2b$. *Then the solution of* (1.11) *satisfies* (1.12) *with* $T = \infty$.

Proof. In view of Proposition 1 (with $m = r + [l/2b]$, $l_1 = l - 2b[l/2b]$), the theorem is easily reduced to the case $u_k \equiv \frac{\partial^k u}{\partial t^k}|_{t=0} = 0$, $k = 0, \dots, r + \big[\frac{l}{2b}\big]$. By the definition of u_k,

$$\sum_{k=r}^{r+[l/2b]} \langle u_k\rangle^{(2b(r-k)+l)}_{\mathbb{R}^n_+} \leqslant c\Big(\sum_{k=0}^{r-1} \langle u_k\rangle^{(2b(r-k)+l)}_{\mathbb{R}^n_+} + \sum_{k=0}^{[l/2b]} \langle \frac{\partial^k f}{\partial t^k}\Big|_{t=0}\rangle^{(l-2bk)}_{\mathbb{R}^n_+}\Big).$$

We extend u_k into $\mathbb{R}^n$ with preservation of class and construct the function U, as in Proposition 1. Since $\frac{\partial^i}{\partial t^i}(LU - f)\big|_{t=0} = 0$ for $i = 0, \dots, [l/2b]$ and $\frac{\partial^j}{\partial t^j}(B_q U - \Phi_q)\big|_{t=0} = 0$ for $j = 0, \dots, [l/2b - \sigma_q/2b]$, we obtain for $v = u - U$ the initial-boundary value problem

$$\begin{cases} L\Big(\dfrac{\partial}{\partial x}, \dfrac{\partial}{\partial t}\Big) v = g(x,t), \quad x \in \mathbb{R}^n_+, \quad t > 0, \\ \dfrac{\partial^k v}{\partial t^k}\Big|_{t=0} = 0, \quad k = 0, , , m-1, \quad x \in \mathbb{R}^n_+, \\ B_q\Big(\dfrac{\partial}{\partial x}, \dfrac{\partial}{\partial t}\Big) v = \Psi_q(x', t), \\ x' = (x_1, \dots, x_{n-1}) \in \mathbb{R}^{n-1}, \quad t > 0, \quad q = 1, \dots, m, \end{cases} \tag{1.31}$$

where $g = f - LU$, $\Psi_q = \Phi_q - B_q U|_{t=0}$ are functions satisfying the conditions

$$\Big(\frac{\partial}{\partial t}\Big)^k g\Big|_{t=0} = 0, \quad k = 0, \dots, [l/2b], \quad \Big(\frac{\partial}{\partial t}\Big)^j \Psi_q\Big|_{t=0} = 0,$$
$$j = 0, \dots, [l/2b - \sigma_q/2b], \quad q = 1, \dots, br.$$

We construct the solution of (1.31) in the form $v = w + z$, where w is a solution of the Cauchy problem

$$L\Big(\frac{\partial}{\partial x}, \frac{\partial}{\partial t}\Big) w = g^*(x,t), \quad x \in \mathbb{R}^n, \quad t > 0, \qquad (1.32)$$
$$\frac{\partial^k w}{\partial t^k}\Big|_{t=0} = 0, \quad k = 0, \dots, r-1, \quad x \in \mathbb{R}^n,$$

and g^* is the extension of g into $\mathbb{R}^n_\infty$ with preservation of class, i.e., such that

$$\langle g^* \rangle^{(l,l/2b)}_{\mathbb{R}^n_\infty} \leqslant c \langle g \rangle^{(l,l/2b)}_{\mathbb{R}^n_{+,\infty}},$$

$g^*(x,t) = 0$ for $t < 0$. The function z is defined as a solution of

$$\begin{cases} L\Big(\dfrac{\partial}{\partial x}, \dfrac{\partial}{\partial t}\Big) z(x,t) = 0, \\ x \in \mathbb{R}^n_+, \quad t > 0, \quad \dfrac{\partial^k z}{\partial t^k}\Big|_{t=0} = 0, \quad k = 0, \dots, r-1, \quad x \in \mathbb{R}^n_+, \\ B_q\Big(\dfrac{\partial}{\partial x}, \dfrac{\partial}{\partial t}\Big) z = \phi_q(x',t), \\ x' = (x_1, \dots, x_{n-1}) \in \mathbb{R}^{n-1}, \quad t > 0, \quad q = 1, \dots, br, \end{cases} \qquad (1.33)$$

with $\phi_q = \Psi_q - B_q w|_{x_n=0}$.

To estimate $w(x,t)$, we perform the Fourier-Laplace transformation

$$\mathcal{F}w \equiv \widetilde{w}(\xi,s) = \int_0^\infty e^{-st} dt \int_{\mathbb{R}^n} e^{-i\xi\cdot x} u(x,t)\, dx$$

in (1.32) with $\mathrm{Re}s = a \geqslant 0$. Then (1.32) is converted into

$$\widetilde{w} = \frac{\widetilde{g}^*}{L(i\xi, s)}. \qquad (1.34)$$

For a large class of g^*, the function (1.34) is holomorphic with respect to s for $\mathrm{Re}s > 0$, hence $w(x,t)$ vanishes for $t \leqslant 0$. We set $a = 0$ in (1.34) and make use of Proposition 2. Since $L(\xi, i\xi_0)$ satisfies the assumptions of Proposition 2 with $\omega = 2br$, we obtain

$$\begin{aligned} \langle w \rangle^{(2br+l, r+l/2b)}_{\mathbb{R}^n_\infty} &\leqslant c \langle g^* \rangle^{(l,l/2b)}_{\mathbb{R}^n_\infty} \leqslant c \langle g \rangle^{(l,l/2b)}_{\mathbb{R}^n_{+,\infty}} \\ &\leqslant c\Big(\langle f \rangle^{(l,l/2b)}_{\mathbb{R}^n_{+,\infty}} + \sum_{k=0}^{r-1} \langle u_k \rangle^{(2b(r-k)+l)}_{\mathbb{R}^n_+} \Big) \end{aligned} \qquad (1.35)$$

We pass to the estimate of z. The Fourier-Laplace transform with respect to (x', t) converts (1.33) into

$$\begin{cases} L\Big(i\xi', s, \dfrac{d}{dx_n}\Big)\widetilde{z}(\xi', s, x_n) = 0, \quad x_n > 0, \\ B_q\Big(i\xi', s, \dfrac{d}{dx_n}\Big)\widetilde{z}|_{x_n=0} = \widetilde{\phi}_q(i\xi', s), \quad q = 1, \dots, br, \quad \underset{x_n\to\infty}{z \to 0}, \end{cases} \tag{1.36}$$

The solution of Problem (1.36) is given by the formula

$$\widetilde{z}(\xi', s, x_n) = \frac{1}{2\pi i}\sum_{q,k=1}^{br} b^{qk}(\xi', s)\int\limits_{\gamma^+}\frac{M^+_{br-q}(\xi', s, \tau)}{M^+(\xi', s, \tau)}e^{i\tau x_n}d\tau\widetilde{\phi}_k \tag{1.37}$$

$$\equiv \sum_{k=1}^{br}\widetilde{m}_k(\xi', s, x_n)\widetilde{\phi}_k\xi', s)$$

(see [11]), where γ^+ is the contour enclosing all the roots τ_j^+ of $L(i\xi', s, i\tau)$ with positive imaginary part,

$$M^+(\xi', s, \tau) = \Pi_{j=1}^{br}(\tau - \tau_j^+(\xi', s)) = \sum_{k=0}^{br} a_k(\xi', s)\tau^{br-k}$$

and $M_p^+(\xi', s, \tau) = \sum\limits_{k=0}^{p} a_k(\xi', s)\tau^{p-k}$. We introduce the $br \times br$ matrix $\mathcal{B}(\xi', s)$ which entries b_{qk} are coefficients in the decomposition

$$B_q'(\xi', s, \tau) = \sum_{k=1}^{br} b_{qk}(\xi', s)\tau^{k-1} \equiv B_q(i\xi', s, i\tau)(\text{mod } M^+),$$

where B_q' are the remainders arising from the division of $B_q(i\xi', s, i\tau)$ by M^+. Finally, b^{qk} are the elements of $\mathcal{B}^{-1}$. According to the complementing condition, the polynomials $B_q(i\xi', s, i\tau)$ should be linearly independent modulo the polynomial $M^+(\xi', s, \tau)$, i.e., $\det\mathcal{B} \neq 0$ for arbitrary $\xi' \in \mathbb{R}^{n-1}$ and $s \in \mathbb{C}$ such that $\mathrm{Re} s > -\delta|\xi'|^{2b}$.

The roots $\tau_j^+(\xi', s)$ are homogeneous: $\tau_j^+(\lambda\xi', \lambda^{2b}s) = \lambda\tau_j^+(\xi', s)$ and

$$c_1(|s| + |\xi|^{2b})^{1/2b} \leqslant \mathrm{Im}\tau_j^+(\xi', s) \leqslant |\tau_j^+(\xi, s)| \leqslant c_2(|s| + |\xi|^{2b})^{1/2b}.$$

Hence the contour γ^+ can be chosen in such a way that

$$\frac{1}{2}c_1(|s| + |\xi|^{2b})^{1/2b} \leqslant \mathrm{Im}\tau \leqslant |\tau| \leqslant \frac{3}{2}c_2(|s| + |\xi|^{2b})^{1/2b}$$

for $\tau \in \gamma^+$. Since $b_{qk}(\lambda\xi', \lambda^{2b}s) = \lambda^{\sigma_q+2br-k+1} b_{qk}(\xi', s)$ (and if $k-1 > \sigma_q + 2br$, then $b_{qk} = 0$), we have

$$|\det \mathcal{B}| \geqslant c(|s| + |\xi'|^{2b})^{\sum_{q=1}^{br}(\sigma_q+2br-(q-1))/2b} \tag{1.38}$$

with $\sum_{q=1}^{br}(\sigma_q + 2br - (q-1)) > 0$. It follows that

$$b^{jk}(\lambda\xi', \lambda^{2b}s) = \lambda^{-\sigma_k-2br+(j-1)} b^{jk}(\xi', s)$$
$$|b^{jk}(\xi', s)| \leqslant c(|s| + |\xi'|^{2b})^{(-\sigma_k-2br+(j-1))/2b}.$$

Moreover, we have

$$a_{pj}(\lambda\xi', \lambda^{2b}s) = \lambda^p a_{pj}(\xi', s), \quad |a_{pj}(\xi', s)| \leqslant c(|s| + |\xi|^{2b})^{p/2b},$$
$$\widetilde{m}_k(\lambda\xi', \lambda^{2b}s, x_n) = \lambda^{-\sigma_k-2br}\widetilde{m}_k(\xi', s, \lambda x_n), \tag{1.39}$$
$$|\widetilde{m}_k(\xi', s, x_n)| \leqslant c(|s| + |\xi'|^{2b})^{(-\sigma_k-2br)/2b}.$$

The coefficients $a_k(\xi', s)$ of M^+ depend on symmetric functions of the roots τ_j^+ and are smooth functions of ξ' and s (see [12]); the functions b^{qk} are smooth as well. Hence

$$|D_\xi^j D_s^{j_0} \widetilde{m}_{qk}(\xi', s, x_n)| \leqslant c(|s| + |\xi|^{2b})^{(-2br-\sigma_k-2bj_0-|j|)/2b}, \tag{1.40}$$

and for arbitrary $x_n > 0$.

Since the functions (1.37) are analytic in s for $\mathrm{Re}\, s > 0$, the corresponding functions $z(x,t)$ vanish for $t < 0$. We set $a = 0$ and notice that the multipliers $\widetilde{m}_k$ satisfy the assumptions of Proposition 2 with $\omega_k = 2br + \sigma_k \geqslant 0$. Therefore,

$$\langle z(\cdot, x_n)\rangle^{(2br+l, r+l/2b)}_{\mathbb{R}^{n-1}_\infty} \leqslant c\sum_{k=1}^{br} \langle \phi_k\rangle^{(l-\sigma_k, l/2b-\sigma_k/2b)}_{\mathbb{R}^{n-1}_\infty}, \quad \forall x_n > 0. \tag{1.41}$$

It remains to estimate $N_0 \equiv \langle D^{2br+[l]}_{x_n} z\rangle^{(l-]l])}_{x_n, \mathbb{R}^n_{+,\infty}}$. We use the equation $Lz = 0$ and the interpolation inequalities. In view of the above equation, N_0 does not exceed the linear combination of the norms

$$N(j,k) \equiv \langle D^j_x D^k_t z\rangle^{(l-[l])}_{x_n, \mathbb{R}^n_{+,\infty}},$$

where $|j| + 2bk = 2br + [l]$, $j_n < 2br + [l]$. In turn, every such norm is controlled by

$$\varepsilon N_0 + c(\varepsilon) \sup_{x_n > 0} \langle z_q(\cdot, x_n)\rangle^{(2br+l, r+l/2b)}_{\mathbb{R}^{n-1}_\infty},$$

in view of (1.14). Taking ε sufficiently small and making use of (1.41), after simple calculations we arrive at

$$\langle z\rangle^{(2br+l,r+l/2b)}_{\mathbb{R}^n_{+,\infty}} \leqslant c\sum_{k=1}^{br}\langle\phi_k\rangle^{(l-\sigma_k,l/2b-\sigma_k/2b)}_{\mathbb{R}^{n-1}_\infty}. \tag{1.42}$$

The estimate (1.12) with $T=\infty$ follows from this inequality and from (1.35) and (1.16). $\square$

Remark. Since $\widetilde{w}$ and $\widetilde{z}$ given by (1.34) and (1.37) are holomorphic functions of s for Re $s>0$, the inequality (1.12) holds for arbitrary $T\leqslant\infty$ (see details in [13]).

§2. Model problems for parabolic systems.

We pass to a short discussion of model problems for systems which are parabolic in the sense of I. G. Petrovskii [14]. We write these systems in the form

$$\mathcal{L}(\frac{\partial}{\partial x},\frac{\partial}{\partial t})\boldsymbol{u}(x,t)=\boldsymbol{f}(x,t)=(f_1,\dots,f_m)^T, \tag{2.1}$$

where $\mathcal{L}$ is a $m\times m$ matrix, which elements $l_{kj}(\frac{\partial}{\partial x},\frac{\partial}{\partial t})$ are differential operators of order $2br_j$ with respect to x_i and of the order r_j with respect to t. They satisfy the homogeneity condition

$$l_{kj}(i\lambda\xi,\lambda^{2b}s)=\lambda^{2br_j}l_{kj}(i\xi,s).$$

The polynomial $L(i\xi,s)=\det\mathcal{L}(i\xi,s)$ is assumed to satisfy the same parabolicity condition as in Section 1. Moreover, we assume that

$$l_{kj}(i\xi,s)=\delta_{kj}s^{r_k}+l'_{kj}(i\xi,s),$$

where l'_{kj} do not contain s^{r_j}. In other words, the system (2.1) has the form

$$\frac{\partial^{r_k}u_k}{\partial t^{r_k}}+\sum_{j=1}^{k}l'_{kj}\Big(\frac{\partial}{\partial x},\frac{\partial}{\partial t}\Big)u_j=f_k(x,t),\quad k=1,\dots,m,$$

where $l'_{kj}u_j$ do not contain higher order derivatives $\frac{\partial^{r_j}u_j}{\partial t^{r_j}}$. From the parabolicity condition it follows that for arbitrary $\xi'\in\mathbb{R}^{n-1}$ and $s\in\mathbb{C}$ such that

$$\mathrm{Re}s>-\delta|\xi'|^{2b} \tag{2.2}$$

the polynomial $L(i\xi',s,i\tau)$ has br roots $\tau_i^+(\xi',s)$ and br roots $\tau_k^-(\xi',s)$ with positive and negative imaginary part, respectively, where $r=\sum\limits_{i=1}^{m}r_i$.

The Cauchy problem and the model initial-boundary value problem in a half-space for system (2.1) are stated as follows:

$$\begin{cases} \mathcal{L}\Big(\dfrac{\partial}{\partial x},\dfrac{\partial}{\partial t}\Big)\boldsymbol{u}(x,t)=\boldsymbol{f}(x,t), \quad x\in\mathbb{R}^n, \quad t>0,\\ \dfrac{\partial^j u_k}{\partial t^j}|_{t=0}=\boldsymbol{\varphi}_{kj}(x), \quad x\in\mathbb{R}^n, \quad k=1,2,\dots,m, \quad j=0,\dots,r_k-1, \end{cases} \tag{2.3}$$

$$\begin{cases} \mathcal{L}\Big(\dfrac{\partial}{\partial x},\dfrac{\partial}{\partial t}\Big)\boldsymbol{u}(x,t)=\boldsymbol{f}(x,t), \quad x\in\mathbb{R}^n_+, \quad t>0,\\ \dfrac{\partial^j u_k}{\partial t^j}|_{t=0}=\boldsymbol{\varphi}_{kj}(x), \quad x\in\mathbb{R}^n_+, \quad k=1,2,\dots,m, \quad j=0,\dots,r_k-1,\\ \mathcal{B}\Big(\dfrac{\partial}{\partial x},\dfrac{\partial}{\partial t}\Big)\boldsymbol{u}|_{x_n=0}=\boldsymbol{\Phi}(x',t), \quad x'\in\mathbb{R}^{n-1}, \quad t>0. \end{cases} \tag{2.4}$$

The elements of the matrix $\mathcal{B}=\big(B_{qj}(\frac{\partial}{\partial x'},\frac{\partial}{\partial t})\big)_{q=1,\dots,br,j=1,\dots,n}$ are differential operators of order $2br_j+\sigma_q$; we assume that $2br_j+\sigma_q\geqslant 0$. They are also homogeneous:

$$B_{qj}(i\lambda\xi',\lambda^{2b}s)=\lambda^{\sigma_q+2br_j}B_{qj}(i\xi',s),$$

and the following complementing condition is satisfied: the rows of the matrix $\mathcal{B}(i\xi',i\tau,s)\widehat{\mathcal{L}}(i\xi',i\tau,s)$ where $\widehat{\mathcal{L}}=(\det\mathcal{L})\mathcal{L}^{-1}$ is the co-factor matrix of $\mathcal{L}$ are linearly independent modulo the polynomial $M^+(\xi',s,\tau)=\Pi_{j=1}^{br}(\tau-\tau_j^+(\xi',s))$ for arbitrary $\xi'\in\mathbb{R}^{n-1}$ and $s\in\mathbb{C}$ satisfying (2.2). The equivalent formulation of this condition is as follows: let $\mathcal{A}=\mathcal{B}\widehat{\mathcal{L}}(\mathrm{mod} M^+)$ be the $(br\times m)$-matrix the elements $A_{qj}(\xi,s,\tau)=\sum\limits_{m=1}^{br}a_{qj}^{(m)}(\xi',s)\tau^{m-1}$ of which are the remainders resulting from the division of the elements of $\mathcal{B}\widehat{\mathcal{L}}$ by M^+. Then the rank of the matrix

$$\mathfrak{A}=\begin{pmatrix} a_{11}^{(1)} & \dots & a_{11}^{(br)} & a_{12}^{(1)} & \dots & a_{12}^{(br)} & \dots & a_{1m}^{(1)} & \dots & a_{1m}^{(br)}\\ a_{21}^{(1)} & \dots & a_{21}^{(br)} & a_{22}^{(1)} & \dots & a_{22}^{(br)} & \dots & a_{2m}^{(1)} & \dots & a_{2m}^{(br)}\\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots\\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots\\ a_{br1}^{(1)} & \dots & a_{br1}^{(br)} & a_{br2}^{(1)} & \dots & a_{br2}^{(br)} & \dots & a_{br1}^{(1)} & \dots & a_{brm}^{(br)} \end{pmatrix}$$

is equal to br for arbitrary $\xi'\in\mathbb{R}^{n-1}$ and $s\in\mathbb{C}$ satisfying (2.2).

We proceed with an analog of Theorem 2.

Theorem 3. *Assume that* $f_j \in C^{l,l/2b}(\mathbb{R}^n_{+,\infty})$,

$$\varphi_{jk} \in C^{2b(r_j-k)+l,r_j-k+l/2b}(\mathbb{R}^n_+), \quad \phi_q \in C^{l-\sigma_q,l/2b-\sigma_q/2b}(\mathbb{R}^{n-1}_\infty),$$

where

$$j = 1,\dots,m, \quad k = 0,\dots,r_j-1, \quad q = 1,\dots,br, \quad l > \max(0,\sigma_1,\dots,\sigma_{br}).$$

Let the compatibility conditions

$$(\frac{\partial}{\partial t})^p \Big(\sum_{j=1}^m B_{qj}u_j - \Phi_q\Big)\Big|_{t=0} = 0, \quad p = 0,\dots,[\frac{l-\sigma_q}{2b}]$$

be satisfied, where $\frac{\partial^k}{\partial t^k}u_j|_{t=0} = \varphi_{jk}(x)$ *are found from the initial conditions if* $k < r_j$*, and from*

$$\Big(\frac{\partial}{\partial t}\Big)^p \Big(\sum_{j=1}^m l_{ij}u_j - f_i\Big)\Big|_{t=0} = 0, \quad p = 0,\dots,[l/2b],$$

if $k = r_j,\dots,r_j+[l/2b]$*. Then the solution of* (2.4) *satisfies the inequality*

$$\begin{aligned} &\sum_{i=1}^m \langle u_i\rangle^{(2br_i+l,r_i+l/2b)}_{\mathbb{R}^n_{+,\infty}} \\ &\leqslant c\Big(\sum_{i=1}^m \langle f_i\rangle^{(l,l/2b)}_{\mathbb{R}^n_{+,\infty}} + \sum_{i=1}^m \sum_{k=0}^{br_i-1} \langle\varphi_{ik}\rangle^{(2b(r_i-k)+l,r_i-k+l/2b)}_{\mathbb{R}^n_+} \\ &+ \sum_{q=1}^{br} \langle\Phi_q\rangle^{(l-\sigma_q,(l-\sigma_q)/2b)}_{\mathbb{R}^{n-1}_{+,\infty}}\Big). \end{aligned} \tag{2.5}$$

Sketch of the proof. As in Theorem 2, we reduce (2.4) to a similar problem with $\varphi_{jk} = 0$ by constructing auxiliary functions U_i such that $\big(\frac{\partial}{\partial t}\big)^k U_i|_{t=0} = \varphi_{ik}(x)$, $k = 0,\dots,r_i+[\frac{l}{2b}]$, and

$$\sum_{i=1}^m \langle U_i\rangle^{(2br_i+l,r+l/2b)}_{\mathbb{R}^n_{+,\infty}} \leqslant c\Big(\sum_{i=1}^m \sum_{k=0}^{r_i-1} \langle\varphi_{ik}\rangle^{(2b(r_i-k)+l)}_{\mathbb{R}^n_+} + \sum_{k=0}^{[\frac{l}{2b}]} \langle\frac{\partial^k \mathbf{f}}{\partial t^k}\big|_{t=0}\rangle^{(l-2bk)}_{\mathbb{R}^n_+}\Big).$$

For $\boldsymbol{v}=\boldsymbol{u}-\boldsymbol{U}$, we obtain the problem

$$\begin{cases} \mathcal{L}\Big(\dfrac{\partial}{\partial x},\dfrac{\partial}{\partial t}\Big)\boldsymbol{v}=\boldsymbol{g}=\boldsymbol{f}-\mathcal{L}\boldsymbol{U}, \\ \Big(\dfrac{\partial}{\partial t}\Big)^k v_i|_{t=0}=0, \quad i=1,\dots,m, \quad k=0,\dots,r_i-1, \\ \mathcal{B}\Big(\dfrac{\partial}{\partial x},\dfrac{\partial}{\partial t}\Big)\boldsymbol{v}|_{x_n=0}=\boldsymbol{\Psi}(x',t)=\boldsymbol{\Phi}-\mathcal{B}\boldsymbol{U}|_{x_n=0}. \end{cases} \tag{2.6}$$

The functions g_i, Ψ_q satisfy additional conditions

$$(\frac{\partial}{\partial t})^k g_i|_{t=0}=0, \quad i=1,\dots,m, \quad k=0,\dots,[l/2b],$$

$$(\frac{\partial}{\partial t})^k \Psi_q|_{t=0}=0, \quad q=1,\dots,br, \quad k=0,\dots,[(l-\sigma_q)/2b].$$

As in Theorem 2, (2.6) is reduced to the Cauchy problem

$$\begin{cases} \mathcal{L}(\dfrac{\partial}{\partial x},\dfrac{\partial}{\partial t})\boldsymbol{w}=g^*(x,t), \quad x\in\mathbb{R}^n, \\ (\dfrac{\partial}{\partial t})^k w_i|_{t=0}=0, \quad i=1,\dots,m, \quad k=0,\dots,r_i-1, \end{cases} \tag{2.7}$$

where g^* is the extension of g, and to the initial-boundary value problem

$$\begin{cases} \mathcal{L}(\dfrac{\partial}{\partial x},\dfrac{\partial}{\partial t})\boldsymbol{z}=0, \quad x\in\mathbb{R}^n, \\ (\dfrac{\partial}{\partial t})^k z_i|_{t=0}=0, \quad i=1,\dots,m, \quad k=0,\dots,r_i-1, \\ \mathcal{B}(\dfrac{\partial}{\partial x},\dfrac{\partial}{\partial t})\boldsymbol{z}|_{x_n=0}=\boldsymbol{\phi}(x',t)=\widetilde{\Psi}-\mathcal{B}\boldsymbol{w}|_{x_n=0}. \end{cases} \tag{2.8}$$

Using the Fourier–Laplace transform, solution of (2.7) is obtained in the form

$$\widetilde{\boldsymbol{w}}=\frac{\widehat{\mathcal{L}}(i\xi,s)\widetilde{\boldsymbol{g}}^*}{L(i\xi,s)},$$

and it is easy to show (as in Theorem 2), that

$$\sum_{i=1}^m\langle w_i\rangle_{\mathbb{R}^n_\infty}^{(2br_i+l,r_i+l/2b)}\leqslant c\sum_{i=1}^m\langle g_i^*\rangle_{\mathbb{R}^n_\infty}^{(l,l/2b)}\leqslant c\sum_{i=1}^m\langle g_i\rangle_{\mathbb{R}^n_{+,\infty}}^{(l,l/2b)}. \tag{2.9}$$

We proceed with problem (2.8). Making the Fourier transform with respect to x' and the Laplace transform in t, we reduce (2.8) to

$$\begin{cases} \mathcal{L}(i\xi', s, \dfrac{d}{dx_n})\widetilde{z} = 0, \quad x_n > 0, \\ \mathcal{B}(i\xi', s, \dfrac{d}{dx_n})\widetilde{z}|_{x_n=0} = \widetilde{\phi}(\xi', s), \quad \underset{x_n\to\infty}{\widetilde{z} \to 0}. \end{cases} \tag{2.10}$$

The explicit formula for the solution of this problem is

$$\begin{aligned} \widetilde{z}_j(\xi', s, x_n) &= \sum_{k=1}^{br} \frac{1}{2\pi i} \sum_{p=1}^{m} \widehat{L}_{jp}(i\xi', s, \frac{d}{dx_n}) \\ &\int_{\gamma^+} N_{pk}(\xi', s, \tau) \frac{e^{i\tau x_n}}{M^+(\xi', s, \tau)} d\tau \widetilde{\phi}_k(\xi', s) \\ &\equiv \sum_{k=1}^{br} \widetilde{m}_{jk}(\xi', s, x_n)\widetilde{\phi}_k(\xi', s) \end{aligned} \tag{2.11}$$

(see [6]), where $j = 1, \ldots, m$, γ^+ is the contour enclosing all the roots τ_j^+, $N_{pk} = \sum\limits_{h=1}^{br} a_h^{(pk)} M^+_{br-h}(\xi', s, \tau)$, and $a_h^{(pk)}$ are elements of the matrix

$$\mathfrak{A}^{-1} = \begin{pmatrix} a_1^{(11)} & \ldots & a_{br}^{(11)} & a_1^{(12)} & \ldots & a_{br}^{(12)} & \ldots & a_1^{(1m)} & \ldots & a_{br}^{(1m)} \\ a_1^{(21)} & \ldots & a_{br}^{(21)} & a_1^{(22)} & \ldots & a_{br}^{(22)} & \ldots & a_1^{(2m)} & \ldots & a_{br}^{(2m)} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ a_1^{(br1)} & \ldots & a_{br}^{(br1)} & a_1^{(br2)} & \ldots & a_{br}^{(br2)} & \ldots & a_1^{(brm)} & \ldots & a_{br}^{(brm)}. \end{pmatrix}^T . \tag{2.12}$$

whic is the right inverse to $\mathfrak{A}$. The existence of $\mathfrak{A}^{-1}$ follows from the complementing condition but in general $\mathfrak{A}^{-1}$ is not unique. Since

$$\frac{1}{2\pi i} \int_{\gamma^+} \frac{\tau^{j-1} M^+_{br-h}}{M^+} d\tau = \delta_{jh},$$

we have (as in the scalar case)

$$\sum_{j=1}^{m} B_{qj}(i\xi', s, \frac{d}{dx_n})\widetilde{z}_j(\xi', s, x_n)|_{x_n=0}$$
$$= \sum_{p=1}^{m}\sum_{k=1}^{br} \frac{1}{2\pi i}\int_{\gamma^+} A_{qp}(\xi', s, \tau)N_{pk}(\xi', s, \tau)\frac{e^{\tau x_n}}{M^+}d\tau|_{x_n=0}\widetilde{\phi}_k(\xi', s)$$
$$= \sum_{i=1}^{br}\sum_{j=1}^{m} a_{qj}^{(i)}a_i^{(jk)}\widetilde{\phi}_k(\xi', s) = (\mathfrak{A}\mathfrak{A}^{-1}\widetilde{\phi}(\xi', s))_q = \widetilde{\phi}_q(\xi', s).$$

Now we turn to the construction of $\mathfrak{A}^{-1}$. Let $\mathfrak{A}_i$ be distinct matrices of order br composed of columns of $\mathfrak{A}$ and let $\det\mathfrak{A}_i = \Delta_i(\xi', s)$. Since

$$a_{qj}^{(i)}(\lambda\xi', \lambda^{2b}s) = \lambda^{\sigma_q+2br-i+1}a_{qj}^{(i)}(\xi', s), \tag{2.13}$$

the determinant Δ_i is homogeneous: $\Delta_i(\lambda\xi', \lambda^{2b}s) = \lambda^{k(i)}\Delta_i(\xi', s)$. Let $\varkappa(i)$ be integers such that $\varkappa(i)k(i) = K$ is independent of i. In view of the complementing condition, the expression

$$D(\xi', s) = \sum_i \Delta_i^{\varkappa(i)}(\xi', s)\overline{\Delta}_i^{\varkappa(i)}(\xi', s)$$

is subject to

$$|D(\xi', s)| \geqslant c(|s| + |\xi'|^{2b})^{K/b} > 0$$

for arbitrary non-zero (ξ', s) satisfying (2.2). The right inverse matrix $\mathfrak{A}^{-1}$ can be defined by

$$\mathfrak{A}^{-1} = \sum_i \frac{\Delta_i^{\varkappa(i)-1}(\xi', s)\overline{\Delta}_i^{\varkappa(i)}(\xi', s)}{D(\xi', s)}\mathfrak{B}_i(\xi', s), \tag{2.14}$$

where $\mathfrak{B}_i$ is the matrix of dimensions $brm \times br$ consisting of purely zero rows and rows of the co-factor matrix $\widehat{\mathfrak{A}}_i$ in such a way that

$$\mathfrak{A}\mathfrak{B}_i = \Delta_i I_{br}$$

(I_{br} is the identity matrix of order br). Clearly, $\mathfrak{A}^{-1}$ can be written in the form (2.12). From (2.13) and (2.14) it follows that

$$a_i^{(jk)}(\lambda\xi', \lambda^{2b}s) = \lambda^{-\sigma_q-2br+i-1}a_i^{(jk)}(\xi', s),$$

and

$$\widetilde{m}_{jk}(\lambda\xi', \lambda^{2b}s, x_n) = \lambda^{-\sigma_k-2br_j}\widetilde{m}_{jk}(\xi', s, \lambda x_n).$$

We also need to show that the functions $\widetilde{m}_{jk}$ are holomorphic with respect to s in the domain defined by (2.2). These functions solve the problem (2.10) with $\widetilde{\phi}_q = \delta_{qk}$. We fix arbitrary s_0 satisfying (2.2) and a small neighborhood $\mathcal{U}$ of s_0. The matrix

$$\mathfrak{A}^{-1} = \sum_i \frac{\Delta_i^{\varkappa(i)-1}(\xi', s)\overline{\Delta}_i^{\varkappa(i)}(\xi', s_0)}{D(\xi', s, s_0)} \mathfrak{B}_i(\xi', s)$$

with

$$D(\xi', s, s_0) = \sum_i \Delta_i^{\varkappa(i)}(\xi', s)\overline{\Delta}_i^{\varkappa(i)}(\xi', s_0)$$

is a right inverse of $\mathfrak{A}(\xi', s)$, provided that $s \in \mathcal{U}$ is close to s_0. It is a holomorphic function of $s \in \mathcal{U}$, and so are $\widetilde{m}_{qj}$, because of uniqueness of the solution of (2.10).

We set $a = 0$ and make use of Proposition 2. Since $2br_j + \sigma_k \geqslant 0$, we have

$$\langle z_j \rangle^{(2br_j+l, r_j+l/2b)}_{\mathbb{R}^{n-1}_\infty} \leqslant c \sum_{k=1}^{br_j} \langle \phi_k \rangle^{(l-\sigma_k, (l-\sigma_k)/2b)}_{\mathbb{R}^{n-1}_\infty}.$$

for arbitrary $x_n > 0$. Now, using the equation $\mathcal{L}\boldsymbol{z} = 0$ and the interpolation inequality (1.14), we prove (as in Section 1) that

$$\sum_{j=1}^m \langle z_j \rangle^{(2br_j+l, r_j+l/2b)}_{\mathbb{R}^n_{+,\infty}} \leqslant c \sum_{k=1}^{br} \langle \phi_k \rangle^{(l-\sigma_k, (l-\sigma_k)/2b)}_{\mathbb{R}^{n-1}_\infty}$$

This completes the proof of Theorem 3.

§3. Proof of inequality (1.8).

We prove (1.8) for functions defined in $\mathbb{R}^n$. Since

$$\mathbb{R}^n = \bigcup_{k_1, \dots k_n} R(\boldsymbol{k}),$$

where $k_i = 0, \pm 1, \pm 2, \dots,$

$$R(\boldsymbol{k}) = \{x \in \mathbb{R}^n : \quad 2^{k_i - 1} \leqslant |x_i| < 2^{k_i}\}, \quad i = 1, .., n$$

and $|R(\boldsymbol{k})| = \operatorname{mes} R(\boldsymbol{k}) = 2^n \Pi_{i=1}^n (2^{k_i} - 2^{k_i - 1}) = \Pi_{i=1}^n 2^{k_i}$, we have

$$\int_{\mathbb{R}^n} |\mathcal{M}|\, dx = \sum_{k_1, \dots, k_n} \int_{R(\boldsymbol{k})} |\mathcal{M}(x)|\, dx \leqslant \sum_{k_1, \dots, k_n} 2^{\frac{1}{2}(k_1 + \dots k_n)} \Big(\int_{R(\boldsymbol{k})} |\mathcal{M}|^2 dx \Big)^{1/2}.$$

If $2^{k_i-1} \leqslant |x_i| < 2^{k_i}$, then $|x_i|/2^{k_i+1} \in [1/4, 1/2)$ and $|\sin(\pi x_i/2^{k_i+1})| \in [1/\sqrt{2}, 1)$. It follows that

$$1 \leqslant 2^n \sin^2 \frac{\pi x_1}{2^{k_1+1}} \dots \sin^2 \frac{\pi x_n}{2^{k_n+1}}$$

for $x \in R(\boldsymbol{k})$ and in view of the Parceval formula

$$\begin{aligned} \int\limits_{R(\boldsymbol{k})} |\mathcal{M}|^2\, dx &\leqslant 2^n \int\limits_{\mathbb{R}^n} |\mathcal{M}(x)|^2 \Pi_{i=1}^n \sin^2 \frac{\pi x_i}{2^{k_i+1}}\, dx \\ &= \pi^n \int\limits_{\mathbb{R}^n} |\Delta_1(\frac{\pi}{2^{k_1}}) \dots \Delta_n(\frac{\pi}{2^{k_n}}) \widetilde{\mathcal{M}}(\xi)|^2 d\xi. \end{aligned}$$

This implies

$$\begin{aligned} &\int\limits_{\mathbb{R}^n} |\mathcal{M}(x)| dx \leqslant \pi^{n/2} \sum_{k_1,\dots,k_n} 2^{\frac{1}{2}(k_1+\dots k_n)} \\ &\sup_{0<t_1<\pi/2^{k_1}} \dots \sup_{0<t_n<\pi/2^{k_n}} \|\Delta_1(t_1)\dots\Delta_n(t_n)\widetilde{\mathcal{M}}\|_{\mathbb{R}^n} \\ &\leqslant c(n) \sum_{k,\dots,k_n=-\infty}^{\infty} \int\limits_{\pi/2^{k_1}}^{\pi/2^{k_1-1}} \frac{dh_1}{h_1^{3/2}} \dots \int\limits_{\pi/2^{k_n}}^{\pi/2^{k_n-1}} \sup_{0<t_i<h_i} \|\Delta_1(t_1)\dots\Delta_n(t_n)\widetilde{\mathcal{M}}\|_{\mathbb{R}^n} \frac{dh_n}{h_n^{3/2}} \\ &= c_1(n) \int\limits_0^{\infty} \frac{dh_1}{h_1^{3/2}} \dots \int\limits_0^{\infty} \sup_{0<t_1<h_1} \dots \sup_{0<t_n<h_n} \|\Delta_1(t_1)\dots\Delta_n(t_n)\widetilde{\mathcal{M}}\|_{\mathbb{R}^n} \frac{dh_n}{h_n^{3/2}} \\ &\equiv c_1(n) J_0, \end{aligned} \tag{3.1}$$

where $\|\cdot\|_{\mathbb{R}^n}$ denotes the norm in $L_2(\mathbb{R}^n)$.

It remains to prove that the last integral is controlled by $|||\widetilde{\mathcal{M}}|||$. For $n = 1$ this is shown in [7] in the following way. Let $u = \widetilde{\mathcal{M}}$. From the identity

$$u(\xi + h) - u(\xi) = (u(\xi + h) - u(\xi + \eta)) + (u(\xi + \eta) - u(\xi)),$$

it follows that

$$|u(\xi + h) - u(\xi)| \leqslant \frac{1}{h} \int\limits_0^h (|u(\xi + h) - u(\xi + \eta)| + |u(\xi + \eta) - u(\xi)|) d\eta,$$

$$\|\Delta(h)u\| \leqslant \frac{1}{h}\int_0^h (\|\Delta(h-\eta)u\| + \|\Delta(\eta)u\|)d\eta \leqslant \frac{2}{h}\int_0^h \|\Delta(\eta)u\|d\eta, \quad (3.2)$$

where $\|\cdot\|$ is the norm in $L_2(\mathbb{R})$. Using (3.2) we obtain the estimate

$$\begin{aligned}\sup_{h/2<\eta<h} \|\Delta(\eta)u\| &\leqslant \sup_{h/2<\eta<h} \frac{2}{\eta}\int_0^\eta \|\Delta_\tau u\|d\tau \\ &\leqslant \frac{4}{h}\int_0^h \|\Delta_\tau u\|d\tau = 4\int_0^1 \|\Delta(h\theta)u\|d\theta,\end{aligned}$$

which implies

$$\int_0^\infty \sup_{h/2<\eta<h} \|\Delta(\eta)u\|\frac{dh}{h^{3/2}} \leqslant 4\int_0^1 \theta^{1/2}d\theta \int_0^\infty \|\Delta(h_1)u\|\frac{dh_1}{h_1^{3/2}}.$$

Hence

$$\begin{aligned}\int_0^\infty \sup_{0<\eta<h} \|\Delta(\eta)u\|\frac{dh}{h^{3/2}} &\leqslant \int_0^\infty \sup_{0<\eta<h/2} \|\Delta(\eta)u\|\frac{dh}{h^{3/2}} \\ &+ \int_0^\infty \sup_{h/2<\eta<h} \|\Delta(\eta)u\|\frac{dh}{h^{3/2}} \\ &\leqslant \frac{1}{\sqrt{2}}\int_0^\infty \sup_{0<\eta<h} \|\Delta(\eta)u\|\frac{dh}{h^{3/2}} + c\int_0^\infty \|\Delta(h)u\|\frac{dh}{h^{3/2}},\end{aligned} \quad (3.3)$$

and

$$\int_0^\infty \sup_{0<\eta<h} \|\Delta(\eta)u\|\frac{dh}{h^{3/2}} \leqslant \frac{c\sqrt{2}}{\sqrt{2}-1}\int_0^\infty \|\Delta(h)u\|\frac{dh}{h^{3/2}} = \frac{c\sqrt{2}}{\sqrt{2}-1}|||\widetilde{\mathcal{M}}|||. \quad (3.4)$$

In the case $n = 2$, we start with the inequality similar to (3.3): $J_0 \leqslant J_1 + J_2 + J_3 + J_4$, where

$$J_0 = \int_0^\infty \int_0^\infty \sup_{0 \leqslant \eta_1 \leqslant h_1} \sup_{0 \leqslant \eta_2 \leqslant h_2} \|\Delta_1(\eta_1)\Delta_2(\eta_2)u\|_{\mathbb{R}^2} \frac{dh_1 dh_2}{h_1^{3/2} h_2^{3/2}},$$

$$J_1 = \int_0^\infty \int_0^\infty \sup_{0 \leqslant \eta_1 \leqslant h_1/2} \sup_{0 \leqslant \eta_2 \leqslant h_2/2} \|\Delta_1(\eta_1)\Delta_2(\eta_2)u\|_{\mathbb{R}^2} \frac{dh_1 dh_2}{h_1^{3/2} h_2^{3/2}},$$

$$J_2 = \int_0^\infty \int_0^\infty \sup_{0 \leqslant \eta_1 \leqslant h_1/2} \sup_{h_2/2 \leqslant \eta_2 \leqslant h_2} \|\Delta_1(\eta_1)\Delta_2(\eta_2)u\|_{\mathbb{R}^2} \frac{dh_1 dh_2}{h_1^{3/2} h_2^{3/2}}$$

$$J_3 = \int_0^\infty \int_0^\infty \sup_{h_1/2 \leqslant \eta_1 \leqslant h_1} \sup_{0 \leqslant \eta_2 \leqslant h_2/2} \|\Delta_1(\eta_1)\Delta_2(\eta_2)u\|_{\mathbb{R}^2} \frac{dh_1 dh_2}{h_1^{3/2} h_2^{3/2}}$$

$$J_4 = \int_0^\infty \int_0^\infty \sup_{h_1/2 \leqslant \eta_1 \leqslant h_1} \sup_{h_2/2 \leqslant \eta_2 \leqslant h_2} \|\Delta_1(\eta_1)\Delta_2(\eta_2)u\|_{\mathbb{R}^2} \frac{dh_1 dh_2}{h_1^{3/2} h_2^{3/2}}.$$

It is clear that $J_1 = 2^{-1} J_0$, $J_4 \leqslant c_1 |||\widetilde{\mathcal{M}}|||$,

$$J_2 \leqslant c_2 \int_0^\infty \int_0^\infty \sup_{0 \leqslant \eta_1 \leqslant h_1} \|\Delta_1(\eta_1)\Delta_2(h_2)u\|_{\mathbb{R}^2} \frac{dh_1 dh_2}{h_1^{3/2} h_2^{3/2}} \equiv J_{20},$$

$$J_3 \leqslant c_2 \int_0^\infty \int_0^\infty \sup_{0 \leqslant \eta_2 \leqslant h_2} \|\Delta_1(h_1)\Delta_2(\eta_2)u\|_{\mathbb{R}^2} \frac{dh_1 dh_2}{h_1^{3/2} h_2^{3/2}} \equiv J_{30}.$$

The integrals J_{20} and J_{30} can be estimated by $|||\mathcal{M}|||$ in the same way as

$$\int_0^\infty \sup_{0 < \eta < h} \|\Delta(\eta)u\| \frac{dh}{h^{3/2}}$$

in the one-dimensional case. Hence $J_0 \leqslant J_0/2 + c_3 |||\widetilde{\mathcal{M}}|||$ and $J_0 \leqslant 2c_3 |||\widetilde{\mathcal{M}}|||$.

Following the same scheme we can prove that $J_0 \leqslant c|||\widetilde{\mathcal{M}}|||$ for arbitrary n.

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