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REMARK ON WOLF'S CONDITION FOR BOUNDARY REGULARITY OF NAVIER–STOKES EQUATIONS

Abstract. We prove Wolf's regularity condition up to the boundary for solutions to the Navier–Stokes equations satisfying non-slip boundary condition.

§1. Introduction

The note is inspired by an interesting result by J. Wolf, see [9]. It reads the following. Let a pair u and p be a suitable weak solution to the Navier–Stokes system in the parabolic cylinder $Q(z_0,R)=B(x_0,R)\times]t_0-R^2,t_0[$, where $B(x_0,R)$ is a ball of radius R centred at point $x_0\in\mathbb{R}^3$. There exists a positive constant ε such that if

$$\frac{1}{R^2}\int\limits_{Q(z_0,R)}|u(z)|^3dz<\varepsilon$$

then $u\in L_\infty(Q(z_0,R/2))$.

At the first glance, the condition and the result are independent of pressure p. But it is wrong impression as one can see from an elementary example in which $R=1$, $z_0=(0,0)$,

$$u(x,t)=c(t)\nabla h(x),\qquad p(x,t)=-c'(t)h(x)+\frac{1}{2}c^2(t)|\nabla h(x)|^2,$$

and h is a harmonic function. If there is no restriction on pressure, then the above assumption does not provide regularity. But of course there is an assumption on the pressure that is hidden in the definition of suitable weak solution. Let me recall it.

Definition 1.1. *A suitable weak solution u and p to the classical Navier–Stokes system in $Q(z_0,R)$ possess the following properties:*

$$u\in L_{2,\infty}(Q(z_0,R))\cap W^{1,0}_2(Q(z_0,R)),\qquad p\in L_{\frac32}(Q(z_0,R));$$

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$$\partial_t u + u \cdot \nabla u - \Delta u = -\nabla p, \qquad \operatorname{div} u = 0$$

in $Q(z_0, R)$ in the sense of distributions;

for a.a. $t \in]t_0 - R^2, t_0[$,

$$\int\limits_{B(x_0,R)} \varphi^2(x,t)|u(x,t)|^2 dx + 2 \int\limits_{t_0-R^2}^{t} \int\limits_{B(x_0,R)} \varphi^2 |\nabla u|^2 \, dx \, ds$$

$$\leqslant \int\limits_{t_0-R^2}^{t} \int\limits_{B(x_0,R)} |u|^2(\partial_t \varphi^2 + \Delta\varphi^2) + u \cdot \nabla\varphi^2(|u|^2 + 2p) \, dx \, ds$$

for all $\varphi \in C_0^\infty(B(x_0,R) \times]t_0 - R^2, t_0 + R^2[)$.

As we can see, p must have finite $L_{\frac{3}{2}}$-norm. The exponent $3/2$ is convenient but not a unique choice of a function class for the pressure. It would interesting to know how constants in the Wolf's condition depends on the pressure. For example, the classical Caffarelli–Kohn–Nirenberg condition, see [1], tells that there are two universal positive constants ε_1 and c_1 such that if

$$\frac{1}{R^2} \int\limits_{Q(z_0,R)} (|u(z)|^3 + |p - [p]_{B(x_0,R)}|^{\frac{3}{2}}) \, dz < \varepsilon_1$$

then

$$|u(z)| \leqslant \frac{c_1}{R}$$

for all $z = (x,t) \in Q(z_0, R/2)$. The latter condition is also invariant with respect the natural Navier–Stokes scaling $u(x,t) \to \lambda u(\lambda x, \lambda^2 t)$ and $p(x,t) \to \lambda^2 p(\lambda x, \lambda^2 t)$.

Now, we would like to reformulate Wolf's condition in the above scale invariant style.

Theorem 1.2. *Let u and p be a suitable weak solution to the Navier–Stokes equations in $Q(z_0, R)$. Given $M > 0$, there exist positive numbers $\varepsilon_\star = \varepsilon_*(M)$ and $c_* = c_*(M)$ such that if two conditions*

$$\frac{1}{R^2} \int\limits_{Q(z_0,R)} |u|^3 \, dx \, dt < \varepsilon_*(M)$$

and

$$\frac{1}{R^2} \int\limits_{Q(z_0,R)} |p - [p]_{B(x_0,R)}|^{\frac{3}{2}} \, dx \, dt < M$$

hold, then u is Hölder continuous is the closure of $Q(z_0,R/2)$. Moreover,

$$\sup_{z\in Q(z_0,R/2)}|u(z)|\leqslant\frac{c_*(M)}{R}.$$

Certainly, Theorem 1.2 implies Wolf's condition if we let

$$\varepsilon=\varepsilon_*\Big(1+\frac{1}{R^2}\int\limits_{Q(z_0,R)}|p-[p]_{B(x_0,R)}|^{\frac{3}{2}}\,dx\,dt\Big).$$

This type of theorems in the case of interior regularity appeared in [6], for further developments, see, for example, [3,8], and [2].

In the note, we shall study boundary regularity that perhaps cannot be treated by Wolf's method.

§2. Boundary Regularity

We shall study regularity up to a flat part of the boundary only. The following notation will be used in what follows: $B^+(x_0,R):=\{x=(x',x_3)\in\mathbb{R}^3:\ x\in B(x_0,R),\,x_{03}<x_3\}$, $B^+(r):=B^+(0,r)$, $B^+:=B^+(1)$.

Definition 2.1. *A suitable weak solution u and p to the classical Navier–Stokes system in $Q^+(z_0,R)=B^+(x_0,R)\times]t_0-R^2,t_0[$ possess the following properties:*

$$u\in L_{2,\infty}(Q^+(z_0,R))\cap W^{1,0}_2(Q^+(z_0,R)),\qquad p\in L_{\frac{3}{2}}(Q^+(z_0,R));$$

$$\partial_t u+u\cdot\nabla u-\Delta u=-\nabla p,\qquad \operatorname{div}u=0$$

in $Q^+(z_0,R)$ in the sense of distributions;

$$u(x',0,t)=0$$

for all $|x'-x'_0|<R$ and $t\in]t_0-R^2,t_0[$, where $x'=(x_1,x_2)$;

for a.a. $t\in]t_0-R^2,t_0[$,

$$\int\limits_{B^+(x_0,R)}\varphi^2(x,t)|u(x,t)|^2\,dx+2\int\limits_{t_0-R^2}^{t}\int\limits_{B^+(x_0,R)}\varphi^2|\nabla u|^2\,dx\,ds$$

$$\leqslant\int\limits_{t_0-R^2}^{t}\int\limits_{B^+(x_0,R)}|u|^2(\partial_t\varphi^2+\Delta\varphi^2)+u\cdot\nabla\varphi^2(|u|^2+2p)\,dx\,ds$$

for all $\varphi\in C^\infty_0(B(x_0,R)\times]t_0-R^2,t_0+R^2[)$.

Our aim is to show the following.

Theorem 2.2. *Let u and p be a suitable weak solution to the Navier–Stokes equations in $Q^+(z_0,R)$. Given $M>0$, there exist positive numbers $\varepsilon_\star=\varepsilon_*(M)$ and $c_*=c_*(M)$ such that if two conditions*

$$\frac{1}{R^2}\int\limits_{Q^+(z_0,R)}|u|^3\,dx\,dt<\varepsilon_*(M)$$

and

$$\frac{1}{R^2}\int\limits_{Q^+(z_0,R)}|p-[p]_{B^+(x_0,R)}|^{\frac{3}{2}}\,dx\,dt<M$$

hold, then u is Hölder continuous in the closure of $Q^+(z_0,R/2)$. Moreover,

$$\sup_{z\in Q^+(z_0,R/2)}|u(z)|\leqslant\frac{c_*(M)}{R}.$$

We start with the proof of the following auxiliary statement.

Proposition 2.3. *Let u and p be a suitable weak solution to the Navier–Stokes equations in $Q^+:=Q^+(0,1)$. Given $M>0$, there exist positive numbers $\varepsilon=\varepsilon(M)$ and $c=c(M)$ such that if two conditions*

$$\int\limits_{Q^+}|u|^3\,dx\,dt<\varepsilon(M)$$

and

$$\int\limits_{Q^+}|p-[p]_{B^+}|^{\frac{3}{2}}\,dx\,dt<M$$

hold, then $z=0$ is a regular point of u and therefore u is Hölder continuous in the closure of a parabolic vicinity of $z=0$. Moreover,

$$|u(0)|\leqslant c(M).$$

Proof. From the local energy inequality with a suitable choice of the cut-off function φ, it follows that

$$|u|^2_{2,Q^+(r)} := \sup_{-r^2<t<0} \int\limits_{B^+(r)} |u(x,t)|^2\,dx + \int\limits_{Q^+(r)} |\nabla u|^2\,dx\,dt$$
$$\leqslant c(r)\Big[\Big(\int\limits_{Q^+} |u|^3\,dz\Big)^{\frac{2}{3}} + \int\limits_{Q^+} |u|^3\,dz$$
$$+ \Big(\int\limits_{Q^+} |u|^3\,dz\Big)^{\frac{1}{3}}\Big(\int\limits_{Q^+} |p-[p]_{B^+}|^{\frac{3}{2}}\,dz\Big)^{\frac{2}{3}}\Big] \leqslant c(r)\,d(\varepsilon,M)$$

for any $r\in]0,1[$, where $Q^+(r) := Q^+(0,r)$ and

$$d(\varepsilon,M) := \varepsilon^{\frac{2}{3}} + \varepsilon + \varepsilon^{\frac{1}{3}}M.$$

Using standard multiplicative inequalities, one can show

$$\|u\cdot\nabla u\|_{\frac{9}{8},\frac{3}{2},Q^+(\frac{5}{6})} \leqslant c|u|^2_{2,Q^+(\frac{5}{6})} \leqslant c\,d(\varepsilon,M). \tag{2.1}$$

Next, as in paper [5], let us pick up a domain Ω with smooth boundary such that $B^+(4/5)\subset\Omega\subset B^+(5/6)$ and consider the following initial boundary value problem

$$\partial_t v^1 - \Delta v^1 + \nabla q^1 = -u\cdot\nabla u, \qquad \operatorname{div} v^1 = 0 \tag{2.2}$$

in $Q_0 = \Omega\times]-(5/6)^2,0[$ and

$$v^1 = 0 \tag{2.3}$$

on $\partial' Q_0 = (\Omega\times\{t=-(5/6)^2\})\cup(\partial\Omega\times[-(5/6)^2,0])$. For solutions to (2.2), (2.3), the following estimate is valid:

$$\|\partial_t v^1\|_{\frac{9}{8},\frac{3}{2},Q_0} + \|\nabla^2 v^1\|_{\frac{9}{8},\frac{3}{2},Q_0} + \|\nabla q^1\|_{\frac{9}{8},\frac{3}{2},Q_0} \leqslant c\|u\cdot\nabla u\|_{\frac{9}{8},\frac{3}{2},Q_0}. \tag{2.4}$$

Letting $v^2 = u - v^1$ and $q^2 = p - q^1$, we observe that the above introduced functions satisfy the following relations

$$\partial_t v^2 - \Delta v^2 + \nabla q^2 = 0, \qquad \operatorname{div} v^2 = 0$$

in $B^+(4/5)\times]-(4/5)^2,0[$, and

$$v^2(x',0,t) = 0$$

for $|x'| < 4/5$ and $t \in]-(4/5)^2, 0[$. According to [4] and [7], q^2 obeys the estimate

$$\begin{aligned}\|\nabla q^2\|_{9,\frac{3}{2},Q^+(3/4)} &\leqslant c(\|\nabla v^2\|_{\frac{3}{2},Q^+(4/5)} + \|v^2\|_{\frac{3}{2},Q^+(4/5)} \\ &+ \|q^2 - [q^2]_{B^+(4/5)}\|_{\frac{3}{2},Q^+(4/5)}) \leqslant c(\|\nabla u\|_{\frac{3}{2},Q^+(4/5)} + \|u\|_{\frac{3}{2},Q^+(4/5)} \\ &+ \|p - [p]_{B^+(4/5)}\|_{\frac{3}{2},Q^+(4/5)} + \|\nabla v^1\|_{\frac{3}{2},Q^+(4/5)} + \|v^1\|_{\frac{3}{2},Q^+(4/5)} \\ &+ \|q^1 - [q^1]_{B^+(4/5)}\|_{\frac{3}{2},Q^+(4/5)}).\end{aligned}$$

Assuming $0 < \varepsilon < 1$, we find elementary bounds:

$$\begin{aligned}\|\nabla u\|_{\frac{3}{2},Q^+(4/5)} + \|u\|_{\frac{3}{2},Q^+(4/5)} &\leqslant c|u|_{2,Q^+} \leqslant c(M), \\ \|p - [p]_{B^+(4/5)}\|_{\frac{3}{2},Q^+(4/5)} &\leqslant c\|p - [p]_{B^+}\|_{\frac{3}{2},Q^+} \leqslant cM.\end{aligned}$$

Next, from (2.1), (2.4), and the elliptic embedding, it follows that:

$$\|\nabla v^1\|_{\frac{3}{2},Q^+(4/5)} + \|v^1\|_{\frac{3}{2},Q^+(4/5)} \leqslant c\|\nabla^2 v^1\|_{\frac{9}{8},\frac{3}{2},Q_0} \leqslant c(M)$$

and

$$\|q^1 - [q^1]_{B^+(4/5)}\|_{\frac{3}{2},Q^+(4/5)}) \leqslant \|\nabla q^1\|_{\frac{9}{8},\frac{3}{2},Q^+(4/5)} \leqslant c(M).$$

So, finally, we find

$$\|\nabla q^2\|_{9,\frac{3}{2},Q^+(3/4)} \leqslant c(M). \tag{2.5}$$

It is worthy to notice that the right hand side is independent of ε.

We then have for $0 < r < 2/3$

$$\begin{aligned}I_\varepsilon(r) &:= \frac{1}{r^2}\int\limits_{Q^+(r)} (|u|^3 + |p - [p]_{B^+(r)}|^{\frac{3}{2}})\,dz \\ &\leqslant c\frac{1}{r^2}\Big(\varepsilon + \int\limits_{Q^+(r)} (|q^1 - [q^1]_{B^+(r)}|^{\frac{3}{2}} + |q^2 - [q^2]_{B^+(r)}|^{\frac{3}{2}})\,dz\Big).\end{aligned}$$

By Poincare inequality,

$$\begin{aligned}\frac{1}{r^2}\int\limits_{Q^+(r)} |q^1 - [q^1]_{B^+(r)}|^{\frac{3}{2}}\,dz &\leqslant c\frac{1}{r^{\frac{3}{2}}}\int\limits_{-r^2}^{0}\Big(\int\limits_{B^+(r)} |\nabla q^1|^{\frac{9}{8}}\,dx\Big)^{\frac{4}{3}}\,dt \\ &\leqslant c\frac{1}{r^{\frac{3}{2}}}\|\nabla q^1\|^{\frac{3}{2}}_{\frac{9}{8},\frac{3}{2},Q_0} \leqslant c\frac{1}{r^{\frac{3}{2}}}\,d(\varepsilon, M).\end{aligned}$$

For the second part of the pressure, we have the same arguments plus estimate (2.5)

$$\frac{1}{r^2}\int\limits_{Q^+(r)}|q^2-[q^2]_{B^+(r)}|^{\frac{3}{2}}\,dz\leqslant cr^2\int\limits_{-r^2}^{0}\Big(\int\limits_{B^+(r)}|\nabla q^2|^9\,dx\Big)^{\frac{1}{6}}\,dt$$
$$\leqslant cr^2\|\nabla q^2\|_{9,\frac{3}{2},Q^+(3/4)}\leqslant c(M)r^2.$$

So, we find

$$I_\varepsilon(r)\leqslant c\Big(\frac{1}{r^2}\varepsilon+\frac{1}{r^{\frac{3}{2}}}\,d(\varepsilon,M)\Big)+c(M)r^2 \tag{2.6}$$

for any $0<\varepsilon<1$ and for any $0<r<2/3$.

Let us state the following condition of local boundary regularity proved in [5].

Proposition 2.4. *Let w and π be a suitable weak solution to the Navier–Stokes system in $Q^+(R)$. There exist two universal positive constants ε_0 and c_0 such that if*

$$\frac{1}{R^2}\int\limits_{Q^+(R)}(|w|^3+|\pi-[\pi]_{B^+(R)}|^{\frac{3}{2}})\,dz<\varepsilon_0,$$

then the function $z\mapsto w(z)$ is Hölder continuous in the closure of $Q^+(R/2)$ and

$$\sup_{z\in Q^+(R/2)}|w(z)|\leqslant\frac{c_0}{R}.$$

Let us select a positive number $r=r(M)<1/2$ such that

$$c(M)r^2<\frac{\varepsilon_0}{2}.$$

Then we can pick up $\varepsilon=\varepsilon(M)$ such that

$$c\Big(\frac{1}{r(M)^2}\varepsilon+\frac{1}{r(M)^{\frac{3}{2}}}\,d(\varepsilon,M)\Big)<\frac{\varepsilon_0}{2}.$$

From Proposition 2.4 and from (2.6), it follows that the function $z\mapsto u(z)$ is Hölder continuous in the closure of $Q^+(r(M)/2)$ and

$$|u(0)|\leqslant c(M)=2c_0/r(M).$$

□

The scaled version of Proposition 2.3 is as follows.

Proposition 2.5. *Let u and p be a suitable weak solution to the Navier–Stokes equations in $Q^+(R)$. Given $M > 0$, there exist positive numbers $\varepsilon = \varepsilon(M)$ and $c = c(M)$ such that if two conditions*

$$\frac{1}{R^2} \int\limits_{Q^+(R)} |u|^3 \, dx \, dt < \varepsilon(M)$$

and

$$\frac{1}{R^2} \int\limits_{Q^+(R)} |p - [p]_{B^+(R)}|^{\frac{3}{2}} \, dx \, dt < M$$

hold, then $z = 0$ is a regular point of u and therefore u is Hölder continuous in the closure of a parabolic vicinity of $z = 0$. Moreover,

$$|u(0)| \leqslant \frac{c(M)}{R}.$$

Now, we wish to show the following.

Proposition 2.6. *Let u and p be a suitable weak solution to the Navier–Stokes equations in Q^+. Given $M > 0$, there exist positive numbers $\varepsilon_1 = \varepsilon_1(M)$ and $c_1 = c_1(M)$ such that if two conditions*

$$\int\limits_{Q^+} |u|^3 \, dx \, dt < \varepsilon_1(M)$$

and

$$\int\limits_{Q^+} |p - [p]_{B^+}|^{\frac{3}{2}} \, dx \, dt < M$$

hold, then u is Hölder continuous in the closure of $Q^+(1/2)$. Moreover,

$$\sup_{Q^+(1/2)} |u(z)| \leqslant c_1(M).$$

Proof. For $z_0 = (x_0, t_0) \in \overline{Q}^+(1/2)$, we have

$$\frac{1}{(1/2)^2} \int\limits_{Q^+(z_0,1/2)} |u|^3 \, dz \leqslant 4 \int\limits_{Q^+} |u|^3 \, dz < 4\varepsilon_1(M)$$

and

$$\frac{1}{(1/2)^2} \int\limits_{Q^+(z_0,1/2)} |p - [p]_{B^+(x_0)}|^{\frac{3}{2}} \, dz \leqslant 4c \int\limits_{Q^+} |p - [p]_{B^+}|^{\frac{3}{2}} \, dz \leqslant 4cM.$$

We complete the proof by letting

$$\varepsilon_1(M) = \frac{1}{r}\varepsilon(4cM), \qquad c_1(M) = 2c(4cM). \qquad \Box$$

Now, Theorem 2.2 follows from obvious scaling and shift and from Proposition 2.6.

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