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H. Mikayelyan

STABILIZATION TECHNIQUE APPLIED TO CURVE SHORTENING FLOW IN THE PLANE

ABSTRACT. The method proposed by T. I. Zelenjak is applied to the mean curvature flow in the plane. A new type of monotonicity formula for star-shaped curves is obtained.

1. Introduction

One of the classical problems combining geometry and PDEs is the mean curvature flow. Gerhard Huisken proved that convex surfaces converge in finite time to points in asymptotically spheric fashion (see [3]). In dimension two this result was proven by M. Gage and R. Hamilton in [1]. In [2] Grayson showed that in the plane any closed embedded curve shrinks to a convex one in finite time and thus also shrinks to a point. This result is not true in higher dimensions where other types of singularities may occur if the initial curve is not convex (see [6]).

A powerful tool in proving many properties of solutions is the monotonicity formula of Gerhard Huisken (see [4]). In the present paper we apply a general method developed by T. Zelenjak in [5] to mean curvature flow in the plane, and derive the monotonicity formula of Huisken. We also derive a new monotonicity formula for star-shaped curves. The presented approach is general and systematic, and we believe can be very useful in generalizations of the mean curvature flow, where no monotonicity formula is known. Our main motivation was the derivation of such a monotonicity formula for the anisotropic mean curvature flow, which still remains a challenge.

2. The formulation of the problem

We consider a closed curve in $\mathbb{R}^2$ moving by its curvature with an anisotropy given by a function g:

$$\partial_t \gamma = g(\nu)\kappa\nu,$$

Key words and phrases: mean curvature flow, monotonicity formula.

where $\gamma : \mathbb{R}_+ \times S^1 \to \mathbb{R}^2$ is the curve parametrization, κ is the curvature and ν is the normal vector.

Note that in this form we fix a certain parametrization which has no tangential component. For a general parametrization we will get

$$\partial_t \gamma \cdot \nu = g(\nu)\kappa. \tag{1}$$

If we take now $\gamma(t,x) = \begin{pmatrix} u_1(t,x) \\ u_2(t,x) \end{pmatrix}$ we get the following

$$\begin{pmatrix} \partial_t u_1 \\ \partial_t u_2 \end{pmatrix} = g(u_1', u_2') \frac{-u_1'' u_2' + u_1' u_2''}{({u_1'}^2 + {u_2'}^2)^2} \begin{pmatrix} -u_2' \\ u_1' \end{pmatrix}$$

where $'$ means the x-derivative.

Assume the first singularity appears at point 0 after finite time T. We rescale the parametrization in the following way

$$\tau = -\log(T-t), \tilde{\gamma}(\tau, x) = (T-t)^{-\frac{1}{2}} \gamma(t,x)$$

and arrive at

$$\begin{pmatrix} \partial_\tau v_1 \\ \partial_\tau v_2 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + g(v_1', v_2') \frac{-v_1'' v_2' + v_1' v_2''}{({v_1'}^2 + {v_2'}^2)^2} \begin{pmatrix} -v_2' \\ v_1' \end{pmatrix}, \tag{2}$$

where $\tilde{\gamma}(\tau, x) = \begin{pmatrix} v_1(\tau,x) \\ v_2(\tau,x) \end{pmatrix}$.

In the paper we will carry out a significant part of the computations for the anisotropic flow, but we are able to do the final part of the computations only for the isotropic case.

Remark 1. Note that in the isotropic case $g \equiv 1$ the stationary solution of (2) is the circle with the radius $\sqrt{2}$.

Main result. For the solutions of (2), with $g \equiv 1$, which are star-shaped with respect to the origin, we prove the monotonicity formula

$$\frac{d}{d\tau} \int_{S^1} \sqrt{\frac{v_1'^2 + v_2'^2}{v_1^2 + v_2^2}} \Big(f(\psi) + \Big(\frac{\log(v_1^2 + v_2^2)}{2} + \frac{v_1^2 + v_2^2}{4} \Big) \cos\psi \Big) dx$$

$$= - \int_{S^1} |\partial_\tau \gamma \cdot \nu|^2 \sqrt{\frac{v_1'^2 + v_2'^2}{v_1^2 + v_2^2}} \frac{1}{\cos\psi} dx, \tag{3}$$

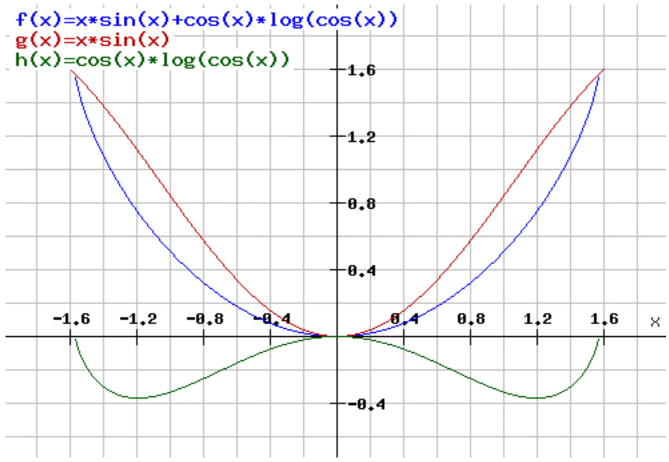

Fig. 1

where ψ is the angle between the outer normal direction $(v_2', -v_1')$ and the position vector (v_1, v_2), and

$$f(\psi) = \psi \sin\psi + \cos\psi \log(\cos\psi) \tag{4}$$

is a positive, even and convex function defined in the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$ (see Figure 1).

Remark 2. One can check numerically that the function

$$f(\psi) + \alpha\cos\psi,$$

is positive and strictly convex in the interval $(-\pi/2, \pi/2)$ for $\alpha < 1$. Since in the limit the curve will converge to a circle with radius $\sqrt{2}$, the coefficient of the $\cos\psi$-term in the left hand side of the formula (3) will be

$$\frac{\log(v_1^2 + v_2^2)}{2} + \frac{v_1^2 + v_2^2}{4} \approx \frac{\log 2}{2} + \frac{1}{2} = 0.8465...$$

This means that the quantity depending on ψ on the left-hand side of (3), at least for large times, is a convex function with minimum in the origin, and thus measures the $W^{1,2}$-deviation of the curve from its final limit, which is the circle.

3. Monotonicity formula by Zelenjak's approach

In this section we adapt the method proposed by T. I. Zelenjak in [5] to the mean curvature flow in the plane.

For the system (2) we want to obtain a monotonicity formula of the form

$$\frac{d}{d\tau}\int_{S^1} F(v_1,v_2,v_1',v_2')dx = -\int_{S^1} |\partial_\tau\gamma\cdot\nu|^2\rho(v_1,v_2,v_1',v_2')dx, \tag{5}$$

where ρ is positive.

Note that in the isotropic case $g\equiv 1$ the well-known Huisken's monotonicity formula (see [4,6]) in this notations will correspond to

$$F(\xi_1,\xi_2,\eta_1,\eta_2)=\rho(\xi_1,\xi_2,\eta_1,\eta_2)=e^{-\frac{|\xi|^2}{4}}|\eta|.$$

Differentiating the left hand side of (5) and integrating by parts we get

$$\partial_\tau v_1\left[\frac{\partial F}{\partial\xi_1}-\frac{\partial^2 F}{\partial\xi_1\partial\eta_1}v_1'-\frac{\partial^2 F}{\partial\xi_2\partial\eta_1}v_2'-\frac{\partial^2 F}{\partial\eta_1^2}v_1''-\frac{\partial^2 F}{\partial\eta_1\partial\eta_2}v_2''\right] \tag{6}$$

$$+\partial_\tau v_2\left[\frac{\partial F}{\partial\xi_2}-\frac{\partial^2 F}{\partial\xi_1\partial\eta_2}v_1'-\frac{\partial^2 F}{\partial\xi_2\partial\eta_2}v_2'-\frac{\partial^2 F}{\partial\eta_1\partial\eta_2}v_1''-\frac{\partial^2 F}{\partial\eta_2^2}v_2''\right]. \tag{7}$$

In the right hand side of (5) using (2) we obtain

$$-\rho(\xi,\eta)\frac{-\partial_\tau v_1 v_2'+\partial_\tau v_2 v_1'}{({v_1'}^2+{v_2'}^2)^{\frac12}}\left(\frac{-v_1v_2'+v_2v_1'}{2({v_1'}^2+{v_2'}^2)^{\frac12}}+g(v_1',v_2')\frac{-v_1''v_2'+v_2''v_1'}{({v_1'}^2+{v_2'}^2)^{\frac32}}\right)= \tag{8}$$

$$-\rho\partial_\tau v_1\left[\frac{v_1{v_2'}^2-v_2v_1'v_2'}{2({v_1'}^2+{v_2'}^2)}+g(v_1',v_2')\frac{{v_2'}^2}{({v_1'}^2+{v_2'}^2)^2}v_1''-g(v_1',v_2')\frac{v_1'v_2'}{({v_1'}^2+{v_2'}^2)^2}v_2''\right] \tag{9}$$

$$-\rho\partial_\tau v_2\left[\frac{v_2{v_1'}^2-v_1v_1'v_2'}{2({v_1'}^2+{v_2'}^2)}-g(v_1',v_2')\frac{v_1'v_2'}{({v_1'}^2+{v_2'}^2)^2}v_1''+g(v_1',v_2')\frac{{v_1'}^2}{({v_1'}^2+{v_2'}^2)^2}v_2''\right]. \tag{10}$$

Remark 3. Note that we do not use (2) but its weak form similar to (1), which means that what we obtain will work for any parametrization of the curve.

We now require that the square brackets of (6) and (9) as well as (7) and (10) be equal. Moreover, we require that

$$D^2_\eta F(\xi,\eta) = \rho(\xi,\eta)g(\eta)\begin{pmatrix} \frac{\eta_2^2}{(\eta_1^2+\eta_2^2)^2} & -\frac{\eta_1\eta_2}{(\eta_1^2+\eta_2^2)^2} \\ -\frac{\eta_1\eta_2}{(\eta_1^2+\eta_2^2)^2} & \frac{\eta_1^2}{(\eta_1^2+\eta_2^2)^2} \end{pmatrix}$$
$$= \rho(\xi,\eta)g(\eta)|\eta|^{-1}D^2|\eta| \quad (11)$$

where g is homogeneous of order 0.

Introducing radial coordinates $(|\eta|,\phi)$ for η it is easy to check that for a given ρ one can find an F satisfying (11) if and only if

$$\rho(\xi,\eta) = c(\xi,\eta)|\eta|,$$

where c is homogeneous of order 0 with respect to η, and

$$\int_0^{2\pi} c(\xi,\phi)g(\phi)\cos\phi d\phi = \int_0^{2\pi} c(\xi,\phi)g(\phi)\sin\phi d\phi = 0, \text{ for all } \xi.$$

Moreover, F is homogeneous of order 1 in η variable and we can write $F(\xi,\eta) = f(\xi,\phi)|\eta|$. The formula (11) becomes now

$$D^2F = (\partial_{\phi\phi}f + f)D^2|\eta| = c(\xi,\phi)g(\phi)D^2|\eta|. \quad (12)$$

The solution of $f'' + f = h$ can be calculated by the following formula

$$f(\phi) = c_1\cos\phi + c_2\sin\phi + \int_0^\phi h(\tau)\sin(\phi-\tau)d\tau. \quad (13)$$

What now remains to achieve our aim of (6)=(9) and (7)=(10), is to make sure that

$$\frac{\partial F}{\partial \xi_1} - \frac{\partial^2 F}{\partial\xi_1\partial\eta_1}\eta_1 - \frac{\partial^2 F}{\partial\xi_2\partial\eta_1}\eta_2 = \rho\frac{-\xi_1\eta_2^2+\xi_2\eta_1\eta_2}{2(\eta_1^2+\eta_2^2)} \quad (14)$$

and

$$\frac{\partial F}{\partial \xi_2} - \frac{\partial^2 F}{\partial\xi_1\partial\eta_2}\eta_1 - \frac{\partial^2 F}{\partial\xi_2\partial\eta_2}\eta_2 = \rho\frac{-\xi_2\eta_1^2+\xi_1\eta_1\eta_2}{2(\eta_1^2+\eta_2^2)}. \quad (15)$$

After differentiating this equations in η_1 and η_2 respectively we obtain

$$-\frac{\partial^3 F}{\partial\xi_1\partial\eta_1^2}\eta_1 - \frac{\partial^3 F}{\partial\xi_2\partial\eta_1^2}\eta_2 = \partial_{\eta_1}\left(\rho\frac{-\xi_1\eta_2^2+\xi_2\eta_1\eta_2}{2(\eta_1^2+\eta_2^2)}\right)$$

and

$$-\frac{\partial^3 F}{\partial\xi_1\partial\eta_2^2}\eta_1-\frac{\partial^3 F}{\partial\xi_2\partial\eta_2^2}\eta_2=\partial_{\eta_2}\left(\rho\frac{-\xi_2\eta_1^2+\xi_1\eta_1\eta_2}{2(\eta_1^2+\eta_2^2)}\right),$$

where we can substitute the value of $D^2_\eta F$ from (11). The two equations turn out to be the same and can be written in terms of c as follows

$$2g(\eta)\langle\eta, D_\xi c\rangle-|\eta|^2\langle\xi, D_\eta c\rangle=-c\cdot\langle\xi,\eta\rangle. \tag{16}$$

Remark 4. If we differentiate (14) with respect to η_2 and add (15) differentiated with respect to η_1 we will get the same as (16).

Taking $c=e^b$ and rewriting (16) in polar coordinates in η variable we arrive at

$$\nabla_{\xi_1,\xi_2,\phi}b\cdot\begin{pmatrix}2g(\phi)\cos\phi\\ 2g(\phi)\sin\phi\\ \xi_1\sin\phi-\xi_2\cos\phi\end{pmatrix}=-\xi_1\cos\phi-\xi_2\sin\phi.$$

The coordinate transformation

$$\begin{array}{l}\tilde{\xi}_1=\xi_1\cos\phi+\xi_2\sin\phi\\ \tilde{\xi}_2=\xi_1\sin\phi-\xi_2\cos\phi\\ \tilde{\phi}=\phi\end{array},$$

brings us to the following first order linear PDE

$$\nabla_{\tilde{\xi}_1\tilde{\xi}_2,\phi}b\cdot\begin{pmatrix}2g(\phi)-\tilde{\xi}_2^2\\ \tilde{\xi}_1\tilde{\xi}_2\\ \tilde{\xi}_2\end{pmatrix}=-\tilde{\xi}_1. \tag{17}$$

In the isotropic case $g\equiv 1$ the solution $b(\tilde{\xi},\phi)=-\frac{|\tilde{\xi}|^2}{4}$ gives us Huisken's famous monotonicity formula.

It remains a challenge to find a solution to (17) in the general anisotropic case which would correspond to the Huisken's one.

4. A new monotonicity formula

Let us observe that another obvious solution to (17) different from Huisken's one is

$$b(\tilde{\xi},\phi)=-\log|\tilde{\xi}_2|. \tag{18}$$

This gives the function

$$\rho(\xi,\eta)=|\eta|c(\xi,\eta)=\frac{|\eta|}{|\xi_1\sin\phi-\xi_2\cos\phi|}=\frac{|\eta|^2}{|\langle\xi,\eta_\nu\rangle|},$$

where η_ν is the vector η rotated by 90 degrees clockwise and thus showing in outer normal direction.

From now on we will consider the isotropic case $g \equiv 1$. Obviously we cannot solve (12) globally because ρ is not integrable, but if we assume that our domain is always star-shaped with respect to the origin and the angle ψ between ξ and η_ν remains between $-\pi/2$ and $\pi/2$, we can solve (12) locally. Now we just solve the equation

$$\partial_{\psi\psi} f + f = \frac{1}{\cos\psi} \tag{19}$$

in the interval $(-\pi/2, \pi/2)$. The general solution is

$$f(\psi) + a(\xi)\cos\psi + b(\xi)\sin\psi, \tag{20}$$

where

$$f(\psi) = \psi\sin\psi + \cos\psi\log(\cos\psi). \tag{21}$$

Let us first take $a = b = 0$.

As mentioned before $f(\psi)$ is a positive, bounded, convex, even function in the interval $(-\pi/2, \pi/2)$ (see Figure). The corresponding function F is

$$F(\xi,\eta) = \frac{|\eta|}{|\xi|} f(\psi) = \frac{|\eta|}{|\xi|}(\psi\sin\psi + \cos\psi\log(\cos\psi)),$$

where ψ is the angle between position vector $\xi = (v_1, v_2)$ and the outer normal ν showing in the direction $(\eta_2, -\eta_1) = (v_2', -v_1')$.

Now we need to check whether the function F satisfies the equations (14) and (15). The answer is no. We obtain in (14) and (15)

$$\frac{\eta_2}{|\xi|^2} \neq -\frac{\eta_2}{2}$$

and

$$-\frac{\eta_1}{|\xi|^2} \neq \frac{\eta_1}{2}$$

respectively (see Remark 4). This means we have an additional term in the formula (5)

$$\frac{d}{d\tau}\int_{S^1}\sqrt{\frac{v_1'^2+v_2'^2}{v_1^2+v_2^2}}\,f(\psi)dx + \int_{S^1}|\partial_\tau\gamma\cdot\nu|^2\sqrt{\frac{v_1'^2+v_2'^2}{v_1^2+v_2^2}}\,\frac{1}{\cos\psi}dx$$
$$= -\int_{S^1}(v_2'\partial_\tau v_1 - v_1'\partial_\tau v_2)\Big(\frac{1}{2}+\frac{1}{v_1^2+v_2^2}\Big)dx. \tag{22}$$

5. The "repaired" formula

In order to obtain a monotonicity formula without additional terms we need to go back to the general solution of (19). The idea is that by adding a term linear in η to F we do not create problems in (11), so let us find a function $a(r)$ such that the function

$$F(\xi,\eta)=\frac{|\eta|}{|\xi|}f(\psi)+a(|\xi|)|\eta|\cos\psi$$

solves (14) and (15), so we do not have additional terms. Substituting F we obtain

$$\frac{\eta_2}{|\xi|^2}-\eta_2\Big(a'(|\xi|)+\frac{a(|\xi|)}{|\xi|}\Big)=-\frac{\eta_2}{2}$$

and

$$-\frac{\eta_1}{|\xi|^2}+\eta_1\Big(a'(|\xi|)+\frac{a(|\xi|)}{|\xi|}\Big)=\frac{\eta_1}{2}$$

respectively, and now need to solve

$$ra'(r)+a(r)=\frac{r}{2}+\frac{1}{r}. \tag{23}$$

The solution is $a(r)=\frac{r}{4}+\frac{\log r}{r}$ and

$$F(\xi,\eta)=\frac{|\eta|}{|\xi|}f(\psi)+|\eta|\Big(\frac{|\xi|}{4}+\frac{\log|\xi|}{|\xi|}\Big)\cos\psi.$$

Thus we obtain the following monotonicity formula

$$\frac{d}{d\tau}\int_{S^1}\sqrt{\frac{v_1'^2+v_2'^2}{v_1^2+v_2^2}}\Big(f(\psi)+\Big(\frac{\log(v_1^2+v_2^2)}{2}+\frac{v_1^2+v_2^2}{4}\Big)\cos\psi\Big)dx$$
$$=-\int_{S^1}|\partial_\tau\gamma\cdot\nu|^2\sqrt{\frac{v_1'^2+v_2'^2}{v_1^2+v_2^2}}\frac{1}{\cos\psi}dx. \tag{24}$$

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Mathematcal Sciences
The University of Nottingham Ningbo China
Taikang Dong Lu Nr. 199, Ningbo 315100
PR China
E-mail: Hayk.Mikayelyan@nottingham.edu.cn

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