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AN ALTERNATIVE APPROACH TOWARDS THE HIGHER ORDER DENOISING OF IMAGES. ANALYTICAL ASPECTS

Abstract. We investigate theoretical aspects of a variational model for the denoising of images which can be interpreted as a substitute for a higher order approach. In this model, the smoothness term that usually involves the highest derivatives is replaced by a mixed expression for a second unknown function in which only derivatives of lower order occur. Our main results concern existence and uniqueness as well as the regularity properties of the solutions to this variational problem established under various assumptions imposed on the growth rates of the different parts of the energy functional.

Bibliography: 56 titles.

Dedicated to Gregory A. Seregin on his 65^{th} birthday

§1. Introduction

Suppose that we are given a bounded Lipschitz domain $\Omega \subset \mathbb{R}^2$ being in addition convex, e.g. a rectangle, together with a function $f \colon \Omega \to \mathbb{R}$, for which we assume

$$f \in L^\infty(\Omega)\,. \tag{1.1}$$

For a definition of the various Lebesgue and Sobolev spaces used throughout this paper and their elementary properties the reader is referred, e.g., to the textbook of Adams [2].

The function f acts as an observed image and these recorded data might be defective in various manners. As a matter of fact there exists a large variety of different techniques used for the denoising of the observed image based on variational or PDE methods. Without being complete we quote $[1, 4, 5, 9, 13, 20, 21, 23, 24, 36, 46, 47, 54, 55]$, and the references therein.

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The most elementary approach in the variational setting is to discuss the first order quadratic variational problem

$$I[u] := \int_\Omega (u-f)^2 \,\mathrm{d}x + \alpha \int_\Omega |\nabla u|^2 \,\mathrm{d}x \;\to\; \min \quad \text{in } W_2^1(\Omega)\,, \tag{1.2}$$

where the quality of data fitting is measured through the quantity $\int_\Omega (u-f)^2 \,\mathrm{d}x$ and where $\alpha > 0$ denotes a positive parameter being under our disposal. For a detailed analysis of (1.2) and related nonquadratic first order variational approaches we refer e.g. to the monographs [5] and [23].

Since first order variational methods and their associated second order PDEs do not allow to preserve affine functions, there have been many proposals for higher order models and related PDEs as presented in [17, 26, 35, 42, 48, 56]. For instance, a natural extension of (1.2) is the second order denoising of f where the energy I from (1.2) is replaced by the functional J introduced below and where one looks at the problem

$$J[w] := \int_\Omega (w-f)^2 \,\mathrm{d}x + \alpha \int_\Omega |\nabla^2 w|^2 \,\mathrm{d}x \;\to\; \min \quad \text{in } W_2^2(\Omega)\,, \tag{1.3}$$

$\nabla^2 w := (\partial_\alpha \partial_\beta w)_{1\leqslant \alpha,\beta \leqslant 2}$ representing the Hessian matrix of w.

In Section 2 we will briefly sketch the proof of

Theorem 1.1. *Let* (1.1) *hold. Then problem* (1.3) *admits a unique solution, which in addition belongs to the space* $W^4_{p,\mathrm{loc}}(\Omega)$ *for any finite* p.

The Euler–Lagrange equation associated to (1.3) is of fourth order. One major goal of this paper is to analyse an alternative to (1.3) whose Euler–Lagrange equations consist of several lower order PDEs which from a computational point of view are more pleasant to handle. To be precise we will analyse the problem

$$\begin{aligned} &E[u,v] \to \min \quad \text{in } X := W_2^1(\Omega) \times W_2^1(\Omega,\mathbb{R}^2),\\ &E[u,v] := \int_\Omega (u-f)^2 \,\mathrm{d}x + \alpha_1 \int_\Omega |\nabla u - v|^2 \,\mathrm{d}x + \alpha_2 \int_\Omega |\nabla v|^2 \,\mathrm{d}x \end{aligned} \tag{1.4}$$

and nonquadratic variants of it. The three terms in this energy are named data term, coupling term, and smoothness term, and we call models of this type coupling models.

Since 1990, related coupling models have been proposed for a number of image analysis problems such as shape from shading [39], stereo reconstuction [6], direct Lagrangian strain tensor computation from image sequences [38], depth upsampling [27], and focus fusion [37]. The aforementioned articles, however, focus on modelling aspects and do not investigate the analytical theory behind these models. In the context of image denoising, Chambolle and Lions [21] have proposed a coupling model with quadratic data term, L^1 coupling term, and L^1 smoothness term. They related it to an infimal convolution approach for the filtered image $u = u_1 + u_2$ where the first derivatives of u_1 and the second derivatives of u_2 are penalized by corresponding smoothness terms. Setting $v := \nabla u_2$ relates both models. A variant of the Chambolle–Lions model has been studied by Chan et al. [22], and discrete counterparts are investigated by Setzer et al [49]. In 2010, Bredies et al. [14] have introduced the concept of total generalized variation (TGV) for image denoising, using Radon measures and functions of bounded deformation. In the second-order case, this resembles a coupling model with L^1 coupling term and L^1 smoothness term with a symmetrized derivative. For any order k, TGV models can be related to an appropriately defined k-fold infimal convolution. Bredies and Valkonen [16] considered an L^2 data term with TGV penalty of order 2 and established existence of solutions and stability w.r.t. the data. In the 1D setting, exact solutions for piecewise affine images are derived by Papafitsoros and Bredies [45]. Bredies et al. [15] study also properties of solutions when the L^2 data term is replaced by an L^1 term. Burger et al. [18] consider an L^1-like coupling (using the Radon norm of a finite Radon measure) together with an L^p norm penalization in the coupling variable for $p \in (1, \infty)$. They come up with well-posedness results, and they show that the case $p = 2$ is equivalent to a Huber variant of total variation (TV) regularization [47], for which they obtain analytic solutions in 1D. The case $p = \infty$ is studied in another work of Burger et al. [19]. They derive exact solutions for 1D data f being piecewise constant or piecewise affine step functions. Moreover, they show that their functional promotes piecewise affine structures in the reconstructed images, which is comparable to TGV under specific conditions.

This discussion shows that deriving analytical results for coupling models constitutes a very active current research topic. Most results so far have been obtained for specific choices of the data, coupling and smoothness term. The goal of the present paper is to aim at an analytic theory for coupling models that is of fairly general character. While we start with analysing the prototypical quadratic model (1.4), we will also generalize our results to nonquadratic energies. First of all, however, we establish the following results for the problem (1.4):

Theorem 1.2. *Suppose that α_1 and α_2 are positive parameters and let* (1.1) *hold. Then we have:*

i) *The problem* (1.4) *admits a unique solution* $(u, v) \in X$.

ii) *For any finite p we have* $u \in W^2_{p,\mathrm{loc}}(\Omega)$ *and* $v \in W^3_{p,\mathrm{loc}}(\Omega, \mathbb{R}^2)$, *hence* $u \in C^{1,\beta}(\Omega)$ *and* $v \in C^{2,\beta}(\Omega, \mathbb{R}^2)$ *for all* $\beta \in (0, 1)$.

iii) *If* $v = \nabla u$, *then u is an affine function.*

iv) *Suppose that* $u \in W^2_2(\Omega)$. *Then it holds*

$$\int_\Omega |\nabla u - v|^2 \,\mathrm{d}x \leqslant \alpha_1^{-1}\alpha_2 \left[\int_\Omega \left|\nabla^2 u\right|^2 \,\mathrm{d}x - \int_\Omega |\nabla v|^2 \,\mathrm{d}x\right].$$

v) *We have the identities*

$$\int_\Omega v \,\mathrm{d}x = \int_\Omega \nabla u \,\mathrm{d}x\,, \quad \int_\Omega u \,\mathrm{d}x = \int_\Omega f \,\mathrm{d}x \ \textit{and}\ u = f - \alpha_1 \Delta(\operatorname{div} v) \ \textit{on}\ \Omega\,.$$

Remark 1.1. i) Apart from trivial cases the field v is not the gradient of the function u. However, according to the first identity in v) of Theorem 1.2, we have that "v equals ∇u in the mean".

ii) The third equation in Theorem 1.2 v) provides a nice formula for the restored image u in terms of the data f.

iii) Coming back to the comments stated after Theorem 1.1 it is easy to see (compare (3.12) and (3.13)) that the Euler–Lagrange system for problem (1.4) consists of three second order equations involving u and the components of v.

iv) We leave it to the reader to discuss iterated variants of problem (1.4) with results in the spirit of Theorem 1.2.

Our paper is organized as follows: Section 2 contains the proof of Theorem 1.1. In Section 3 we present the proof of Theorem 1.2 proceeding in several steps. Non-quadratic energies are the subject of Sections 4 and 5. We investigate the questions of existence and uniqueness for energies of e.g. power growth allowing different exponents for the data term and the regularizing quantities. Here we also touch the TV-case. Section 6 contains the analysis of the regularity properties of minima for power growth models. In Section 7 we briefly sketch some related variational problems. Discrete model formulations, computational aspects, and numerical experiments will be addressed in another paper.

§2. Proof of Theorem 1.1

Probably the proof of Theorem 1.1 is well known but for the reader's convenience we recall its basic idea formulated in

Lemma 2.1. *There is a positive constant $C = C(\Omega)$ such that*

$$\big(\|w\|_2 := \|w\|_{L^2(\Omega)},\ \textit{etc.}\big)$$

$$\|\nabla w\|_2 \leqslant C\left[\|w\|_2 + \|\nabla^2 w\|_2\right] \tag{2.1}$$

is true for any function $w \in W_2^2(\Omega)$.

Proof of Lemma 2.1. If (2.1) is false, then there exists a sequence $w_k \in W_2^2(\Omega)$ for which

$$\|\nabla w_k\|_2 > k\left[\|w_k\|_2 + \|\nabla^2 w_k\|_2\right] . \tag{2.2}$$

Passing to the normalized sequence $\widetilde{w}_k := w_k/\|\nabla w_k\|_2$, we obtain from (2.2)

$$1 = \|\nabla \widetilde{w}_k\|_2 > k\left[\|\widetilde{w}_k\|_2 + \|\nabla^2 \widetilde{w}_k\|_2\right] , \tag{2.3}$$

and (2.3) in particular implies $\sup_k \|\widetilde{w}_k\|_{W_2^2(\Omega)} < \infty$. Thus, we find $w \in W_2^2(\Omega)$ such that

$$\widetilde{w}_k \rightharpoondown w \text{ in } W_2^2(\Omega) \tag{2.4}$$

at least for a subsequence. By the Rellich–Kondrachov theorem (see, e.g. [2], p. 144, Theorem 6.2) the embedding $W_2^2(\Omega) \hookrightarrow W_2^1(\Omega)$ is compact, so that by (2.4) $\nabla \widetilde{w}_k \to \nabla w$ in $L^2(\Omega, \mathbb{R}^2)$. But then (compare (2.3)) $\|\nabla w\|_2 = 1$, whereas the inequality stated in (2.3) clearly implies the contradiction $w = 0$. □

If we apply Lemma 2.1 to a minimizing sequence of problem (1.3), we obtain boundedness of this sequence in the space $W_2^2(\Omega)$. Thus, there exists a weak limit w of a subsequence, which is clearly J-minimizing. Uniqueness follows from the strict convexity of the functional $J[w]$. Finally, we quote e.g. [43] to see that $w \in W^4_{p,loc}(\Omega)$ for any $p < \infty$.

□

§3. Proof of Theorem 1.2

From now on we assume that the assumptions of Theorem 1.2 hold (w.l.o.g.) for $\alpha_1 = \alpha_2 = 1/2$. We proceed in several steps, an alternative approach towards existence is presented in Section 4.

Step 1. *Uniqueness*

We claim that there exists at most one E-minimizer $(u, v) \in X$ for the functional E defined in (1.4). In fact, suppose that $(u_1, v_1), (u_2, v_2) \in X$ are E-minimizing. Then $u_1 \neq u_2$ on a set with positive measure yields

$$\int_\Omega \left(\frac{1}{2}u_1 + \frac{1}{2}u_2 - f\right)^2 \mathrm{d}x < \frac{1}{2}\int_\Omega (u_1 - f)^2 \mathrm{d}x + \frac{1}{2}\int_\Omega (u_2 - f)^2 \mathrm{d}x$$

leading to the contradiction

$$E\left(\frac{1}{2}u_1 + \frac{1}{2}u_2, \frac{1}{2}v_1 + \frac{1}{2}v_2\right) < \frac{1}{2}E(u_1, v_1) + \frac{1}{2}E(u_2, v_2) = \inf_X E\,.$$

For the same reason we obtain $v_1 = v_2$ a.e. since otherwise

$$\int_\Omega \left|\frac{1}{2}(v_1 + v_2) - \nabla u\right|^2 \mathrm{d}x < \frac{1}{2}\int_\Omega |v_1 - \nabla u|^2 \mathrm{d}x + \frac{1}{2}\int_\Omega |v_2 - \nabla u|^2 \mathrm{d}x\,,$$

$$u := u_1 = u_2\,.$$

Step 2. *Construction of an appropriate E-minimizing sequence*

Fix an E-minimizing sequence (u_n, v_n) from the space X and consider for $\varepsilon > 0$ the perturbed energy $E_\varepsilon(u, v) := E(u, v) + \varepsilon \int_\Omega |v|^2 \mathrm{d}x$, $(u, v) \in X$. For any $n \in \mathbb{N}$ we can find a sufficiently small number $\varepsilon(n)$ such that

$$E_{\varepsilon(n)}(u_n, v_n) \leqslant E(u_n, v_n) + \frac{1}{n}\,. \tag{3.1}$$

Let us denote by $(\overline{u}_n, \overline{v}_n) \in X$ the unique solution of the problem $E_{\varepsilon(n)} \to \min$ in X. Note that for the perturbed problem the existence of a minimizer is immediate. We claim that $(\overline{u}_n, \overline{v}_n)$ is an E-minimizing sequence: due to the $E_{\varepsilon(n)}$–minimality of $(\overline{u}_n, \overline{v}_n)$, inequality (3.1) implies for any $n \in \mathbb{N}$

$$E(\overline{u}_n, \overline{v}_n) \leqslant E_{\varepsilon(n)}(\overline{u}_n, \overline{v}_n) \leqslant E_{\varepsilon(n)}(u_n, v_n) \leqslant E(u_n, v_n) + \frac{1}{n}\,,$$

and since

$$\lim_{n\to\infty} E(u_n, v_n) = \inf_X E$$

it follows

$$\lim_{n\to\infty} E\left(\overline{u}_n, \overline{v}_n\right) = \inf_X E\,.$$

Step 3. *Compactness of the minimizing sequence* $(\overline{u}_n, \overline{v}_n)$.

From $E_{\varepsilon_n}\left(\overline{u}_n, \overline{v}_n\right) \leqslant E_{\varepsilon_n}(0,0)$ it follows

$$\sup_n \int\limits_\Omega |\overline{u}_n|^2 \, \mathrm{d}x < \infty\,, \tag{3.2}$$

$$\sup_n \int\limits_\Omega |\nabla \overline{v}_n|^2 \, \mathrm{d}x < \infty\,, \tag{3.3}$$

$$\sup_n \int\limits_\Omega \left|\nabla \overline{u}_n - \overline{v}_n\right|^2 \, \mathrm{d}x < \infty\,. \tag{3.4}$$

Moreover, we have

$$E_{\varepsilon_n}\left(\overline{u}_n, \overline{v}_n\right) \leqslant E_{\varepsilon_n}\left(\overline{u}_n + t\varphi, \overline{v}_n\right)$$

for e.g. any $\varphi \in W_2^1(\Omega)$ with compact support in Ω and all $t \in \mathbb{R}$, hence

$$0 = \frac{d}{\mathrm{d}t}_{|t=0} E\left(\overline{u}_n + t\varphi, \overline{v}_n\right) = \int\limits_\Omega 2\left(\overline{u}_n - f\right)\varphi \, \mathrm{d}x + \int\limits_\Omega \left(\nabla \overline{u}_n - \overline{v}_n\right)\cdot \nabla\varphi \, \mathrm{d}x\,,$$

and in conclusion

$$\int\limits_\Omega \nabla \overline{u}_n \cdot \nabla\varphi \, \mathrm{d}x = -\int\limits_\Omega \operatorname{div} \overline{v}_n \varphi \, \mathrm{d}x - 2\int\limits_\Omega \left(\overline{u}_n - f\right)\varphi \, \mathrm{d}x\,. \tag{3.5}$$

We apply (3.5) with the choice $\varphi := \eta^2 \overline{u}_n$ with arbitrary $\eta \in C_0^\infty(\Omega)$, $0 \leqslant \eta \leqslant 1$. Using (3.2) and (3.3) on the r.h.s. of (3.5), an elementary calculation yields

$$\int\limits_\Omega \eta^2 \left|\nabla \overline{u}_n\right|^2 \, \mathrm{d}x \leqslant \mathrm{const}(\eta) < \infty$$

with $\mathrm{const}(\eta)$ being independent of n. Thus,

$$\sup_n \int\limits_{\Omega^*} \left|\nabla \overline{u}_n\right|^2 \, \mathrm{d}x \leqslant c(\Omega^*) < \infty \tag{3.6}$$

for any subdomain Ω^* with $\overline{\Omega^*} \subset \Omega$. If we put together (3.4) and (3.6) it follows

$$\sup_n \int_{\Omega^*} |\overline{v}_n|^2 \, dx \leqslant c(\Omega^*) < \infty \tag{3.7}$$

for any Ω^* as above.

We now claim that (3.3) and (3.7) imply the key estimate (to be established in Step 5)

$$\sup_n \int_{\Omega} |\overline{v}_n|^2 \, dx < \infty \,, \tag{3.8}$$

which means that

$$\sup_n \|\overline{v}_n\|_{W_2^1(\Omega,\mathbb{R}^2)} < \infty \tag{3.9}$$

and therefore (see (3.2), (3.4) and (3.8))

$$\sup_n \|\overline{u}_n\|_{W_2^1(\Omega)} < \infty \,. \tag{3.10}$$

From (3.9) and (3.10) we deduce the existence of $(u, v) \in X$ such that

$$\overline{u}_n \rightharpoondown u \text{ in } W_2^1(\Omega)\,, \quad \overline{v}_n \rightharpoondown v \text{ in } W_2^1(\Omega, \mathbb{R}^2)\,,$$

at least for a subsequence and by lower semicontinuity this yields

$$E(u, v) \leqslant \liminf_{n\to\infty} E(\overline{u}_n, \overline{v}_n)\,,$$

and since by Step 2 $(\overline{u}_n, \overline{v}_n)$ is an E-minimizing sequence, the E-minimality of (u, v) follows.

Step 4. *Regularity of u and v.*

From the minimality of (u, v) we get

$$\begin{aligned} 0 &= \frac{d}{dt}\bigg|_{t=0} E\,(u + t\varphi, v + t\Psi) \\ &= 2\int_\Omega (u - f)\varphi \, dx + \int_\Omega \nabla v : \nabla\Psi \, dx + \int_\Omega (\nabla u - v)\cdot(\nabla\varphi - \Psi)\, dx \end{aligned} \tag{3.11}$$

for any $\varphi \in C_0^\infty(\Omega)$, $\Psi \in C_0^\infty(\Omega, \mathbb{R}^2)$. By (3.11) we clearly have the equations

$$\Delta v = v - \nabla u \quad \text{weakly on } \Omega\,, \tag{3.12}$$

$$\Delta u = \operatorname{div} v + 2(u - f) \quad \text{weakly on } \Omega\,, \tag{3.13}$$

and we can argue as follows (compare, e.g., [43]):

i) The r.h.s. of (3.12) is of class $L^2(\Omega, \mathbb{R}^2)$, and by standard potential theory we obtain $v \in W^2_{2,\mathrm{loc}}(\Omega; \mathbb{R}^2)$. Hence, by Sobolev's embedding theorem we get

$$\nabla v \in L^p_{\mathrm{loc}}\left(\Omega; \mathbb{R}^{2x2}\right) \quad \text{for all } p < \infty\,. \tag{3.14}$$

ii) Again by Sobolev's embedding theorem, $u \in W^1_2(\Omega)$ implies

$$u \in L^p(\Omega) \quad \forall\, p < \infty\,. \tag{3.15}$$

iii) By (3.14) and (3.15) we see that the r.h.s. of (3.13) is of class $L^p_{\mathrm{loc}}(\Omega)$ for all finite p, hence applying potential theory once more we obtain

$$u \in W^2_{p,\mathrm{loc}}(\Omega) \quad \text{for all } p < \infty\,. \tag{3.16}$$

iv) Finally, (3.14) and (3.16) yield that the r.h.s. of (3.12) is of class $W^1_{p,\mathrm{loc}}(\Omega, \mathbb{R}^2)$ for all $p < \infty$ which gives (potential theory)

$$v \in W^3_{p,\mathrm{loc}}(\Omega, \mathbb{R}^2) \quad \text{for all } p < \infty\,.$$

Together with (3.16) the regularity of the minimizer is established.

Remark 3.1. If f has a certain degree of smoothness, then the statement $(u, v) \in W^2_{p,\mathrm{loc}}(\Omega) \times W^3_{p,\mathrm{loc}}(\Omega, \mathbb{R}^2)$ can be improved.

Step 5. *Proof of estimate* (3.8)

We fix $n \in \mathbb{N}$ and write $\overline{v}$ in place of $\overline{v}_n$. Let $B := B_R(x_0)$ denote a disk such that $2B := B_{2R}(x_0)$ has compact closure in Ω. Finally we consider $\eta \in C^\infty(\mathbb{R}^2), 0 \leqslant \eta \leqslant 1$, $\eta \equiv 0$ on B, $\eta \equiv 1$ outside of $2B$. Then $\eta\overline{v}$ is in $W^1_2(\Omega, \mathbb{R}^2)$ vanishing on B, and from [43], Theorem 3.6.5, it follows (see also Lemma 4.1)

$$\|\eta\overline{v}\|_{L^2(\Omega)} \leqslant c(\Omega)\, \|\nabla(\eta\overline{v})\|_{L^2(\Omega)}\;. \tag{3.17}$$

We have

$$\int_\Omega |\nabla(\eta\overline{v})|^2 \,\mathrm{d}x \leqslant c\left[\int_{2B} |\overline{v}|^2 \,\mathrm{d}x + \int_\Omega |\nabla\overline{v}|^2 \,\mathrm{d}x\right].$$

Hence, $\|\nabla(\eta\overline{v})\|_{L^2(\Omega)} \leqslant \text{const}$ for a constant independent of $\overline{v}$ on account of (3.3) and (3.7). Therefore, (3.17) implies

$$\int_\Omega \eta^2\, |\overline{v}|^2 \,\mathrm{d}x \leqslant \text{const}\,,$$

where the constant is independent of $\overline{v}$, and by the choice of η together with (3.7) our claim (3.8) follows.

Step 6. *Proof of* iii)–v) *of Theorem* 1.2

ad iii). Suppose that we have $v = \nabla u$, which in particular implies $u \in W_2^2(\Omega)$. In (3.11) we observe that actually any $\varphi \in W_2^1(\Omega)$ and all $\Psi \in W_2^1(\Omega, \mathbb{R}^2)$ are admissible. The choices $\Psi := \nabla u$ and $\varphi := 0$ then yield $\nabla^2 u = 0$, hence $u(x) = \xi \cdot x + a$ for some $\xi \in \mathbb{R}^2$, $a \in \mathbb{R}$.

ad iv). If u is in the space $W_2^2(\Omega)$, then the inequality $E(u,v) \leqslant E(u,\nabla u)$ turns into the desired estimate.

ad v). We observe that for any function (or field) w it holds

$$\int_\Omega |w - w_\Omega|^2 \, \mathrm{d}x = \int_\Omega |w|^2 \, \mathrm{d}x - \mathcal{L}^2(\Omega)\, |w_\Omega|^2 \,, \tag{3.18}$$

where $w_\Omega := \fint_\Omega w \, \mathrm{d}x$. So, if for example $\int_\Omega u \, \mathrm{d}x \neq \int_\Omega f \, \mathrm{d}x$, then (3.18) would imply

$$\int_\Omega \left(u - f - (u-f)_\Omega\right)^2 \, \mathrm{d}x < \int_\Omega (u-f)^2 \, \mathrm{d}x \,,$$

hence we obtain the contradiction $E\left(u - (u-f)_\Omega \,, v\right) < E(u,v)$.

For the same reason we get $\int_\Omega v \, \mathrm{d}x = \int_\Omega \nabla u \, \mathrm{d}x$, since otherwise we could replace v by $v - (v - \nabla u)_\Omega$ and decrease energy.

Finally we use (3.12) to get

$$\Delta(\operatorname{div} v) = \operatorname{div} v - \Delta u \,,$$

thus by (3.13)

$$\Delta(\operatorname{div} v) = 2(u - f) \,.$$

This completes the proof of Theorem 1.2. □

§4. Non-quadratic energies

As before we consider a bounded Lipschitz region $\Omega \subset \mathbb{R}^2$ being also convex. For exponents p, q, $s \in (1,\infty)$ and positive parameters α_1, α_2 we replace the energy E defined in (1.4) through the quantity

$$F[u,v] := \int_\Omega |u - f|^s \, \mathrm{d}x + \alpha_1 \int_\Omega |\nabla u - v|^q \, \mathrm{d}x + \alpha_2 \int_\Omega |\nabla v|^p \, \mathrm{d}x \tag{4.1}$$

with given function $f \in L^\infty(\Omega)$. The energy F is defined on the space

$$Y := W^1_q(\Omega) \times W^1_p(\Omega, \mathbb{R}^2)$$

and finite on the subclass

$$\widetilde{Y} := \left\{ (u,v) \in Y : \, u \in L^s(\Omega), \, v \in L^q(\Omega, \mathbb{R}^2) \right\} ,$$

which might coincide with Y depending on the values of p, q and s. We have:

Theorem 4.1. *Consider arbitrary exponents* $1 < p,\, q,\, s < \infty$ *with the restriction*

$$q \leqslant \frac{2p}{2-p}, \quad \text{if } p < 2\,.$$

Then the problem $F \to \min$ *in* $Y = W^1_q(\Omega) \times W^1_p(\Omega, \mathbb{R}^2)$ *with* F *being defined in* (4.1) *admits a unique solution* $(u_0, v_0) \in Y$. *It holds*

$$\int\limits_\Omega |u_0 - f|^{s-2}(u_0 - f)\, \mathrm{d}x = 0 \quad \text{as well as} \quad \int\limits_\Omega |\nabla u_0 - v_0|^{q-2}(\nabla u_0 - v_0)\, \mathrm{d}x = 0.$$

Remark 4.1. As a matter of fact, the above existence result is valid for more general functionals of the form

$$Y \ni (u,v) \mapsto \int\limits_\Omega h_1(u - f)\, \mathrm{d}x + \int\limits_\Omega h_2(\nabla u - v)\, \mathrm{d}x + \int\limits_\Omega h_3(\nabla v)\, \mathrm{d}x$$

with strictly convex densities

$$h_1 : \mathbb{R} \to [0,\infty)\,, \quad h_2 : \mathbb{R}^2 \to [0,\infty)\,, \quad h_3 : \mathbb{R}^{2\times 2} \to [0,\infty)$$

for which

$$c_1\left(|t|^s - 1\right) \leqslant h_1(t) \leqslant c_2\left(|t|^s + 1\right)\,, \qquad t \in \mathbb{R}\,,$$

$$c_3\left(|\xi|^q - 1\right) \leqslant h_2(\xi) \leqslant c_4\left(|\xi|^q + 1\right)\,, \qquad \xi \in \mathbb{R}^2\,,$$

$$c_5\left(|M|^p - 1\right) \leqslant h_3(M) \leqslant c_6\left(|M|^p + 1\right)\,, \qquad M \in \mathbb{R}^{2\times 2}\,,$$

with positive constants c_i, $i = 1, \ldots, 6$. Typical examples are densities of the form $h_i(Z) := \Phi_i(|Z|)$ with Φ_i: $[0,\infty) \to [0,\infty)$ strictly increasing, strictly convex and of appropriate growth.

Remark 4.2. We note that Theorem 4.1 extends to the limit case $p = 1$, which means that then Y has to be replaced by the space $W^1_q(\Omega) \times BV(\Omega, \mathbb{R}^2)$. We refer to Section 5 for a discussion of the “linear case”.

Proof of Theorem 4.1. Assume that (u_1, v_1), $(u_2, v_2) \in Y$ are F-minimizing. As in Section 3, Step 1, $u_1 \neq u_2$ on a set with positive measure would imply

$$E\left(\frac{1}{2}u_1 + \frac{1}{2}u_2, \frac{1}{2}v_1 + \frac{1}{2}v_2\right) < \frac{1}{2}E(u_1, v_1) + \frac{1}{2}E(u_2, v_2)\,,$$

which follows from the strict convexity of $u \mapsto \int_\Omega |u - f|^s \,\mathrm{d}x$. But then $u_1 = u_2$ and $v_1 = v_2$ is a consequence of the strict convexity of $v \mapsto \int_\Omega |v - \nabla u|^q \,\mathrm{d}x$.

Next we consider a F-minimizing sequence $(u_k, v_k) \in Y$. From formula (4.1) it is immediate that

$$\sup_k \int_\Omega |u_k|^s \,\mathrm{d}x < \infty\,, \tag{4.2}$$

$$\sup_k \int_\Omega |\nabla u_k - v_k|^q \,\mathrm{d}x < \infty\,, \tag{4.3}$$

$$\sup_k \int_\Omega |\nabla v_k|^p \,\mathrm{d}x < \infty\,. \tag{4.4}$$

Note that (4.3) in particular implies

$$\sup_k \int_\Omega |\nabla u_k - v_k| \,\mathrm{d}x < \infty\,. \tag{4.5}$$

Let us assume for technical simplicity that $\Omega = B_R(0)$. The adjustments of the following calculations for general convex domains are left to the reader. For $t \in [0, R]$ we have

$$\int_{B_t(0)} \nabla u_k \,\mathrm{d}x = \int_{\partial B_t(0)} u_k(y)\frac{y}{t}\, d\mathcal{H}^1(y)\,,$$

hence

$$\int_0^R \left| \int_{B_t(0)} \nabla u_k \,\mathrm{d}x \right| \mathrm{d}t \leqslant \int_0^R \int_{\partial B_t(0)} |u_k|\, d\mathcal{H}^1 \,\mathrm{d}t = \int_{B_R(0)} |u_k| \,\mathrm{d}x \leqslant c_7 < \infty$$

on account of (4.2). Let $f_k\colon [0,R] \to [0,\infty)$, $f_k(t) := \left| \int_{B_t(0)} \nabla u_k \,\mathrm{d}x \right|$. Fatou's lemma combined with the previous estimate gives

$$\int_0^R \liminf_{k\to\infty} f_k(t)\,\mathrm{d}t \leqslant c_7,$$

thus $\liminf_{k\to\infty} f_k(t) < \infty$ for $\mathcal{L}^1$-almost all $t \in [0,R]$.

Let us fix such a radius $t \in (0,R)$. Then a subsequence $\widetilde{f}_k$ exists with the property $\widetilde{f}_k(t) \leqslant c_8$ for a suitable positive constant c_8. If $\widetilde{u}_k, \widetilde{v}_k$ denote the corresponding subsequences of u_k and v_k respectively, it is shown that for any $k \in \mathbb{N}$

$$\left| \int_{B_t(0)} \nabla \widetilde{u}_k \,\mathrm{d}x \right| \leqslant c_8\,. \tag{4.6}$$

Combining (4.5) with (4.6) we find

$$\left| \int_{B_t(0)} \widetilde{v}_k \,\mathrm{d}x \right| \leqslant c_9\,, \tag{4.7}$$

and from (4.4) and (4.7) together with Poincarés inequality it follows

$$\sup_k \|\widetilde{v}_k\|_{W^1_p(B_t(0))} < \infty \tag{4.8}$$

for the particular radius t.

In order to proceed and to improve (4.8) to a global bound we recall the following lemma which we already used for obtaining (3.17).

Lemma 4.1. *Let G denote an open and convex set in $\mathbb{R}^n$ with $d :=$ diam$(G) < \infty$. Consider a measurable subset S of G such that $\mathcal{L}^n(S) > 0$. Then for any function $w \in W^1_p(G)$, $1 \leqslant p < \infty$, it holds*

$$\|w - w_S\|_{L^p(G)} \leqslant \left(\frac{\mathcal{L}^n(B_1)}{\mathcal{L}^n(S)}\right)^{1-\frac{1}{n}} \mathrm{d}^n \,\|\nabla w\|_{L^p(G)}\,, \tag{4.9}$$

w_S denoting the mean value of w on S.

Proof. See [34], inequality (7.45). □

In Lemma 4.1 we let $G = B_R(0)$, $S = B_t(0)$, $w = \widetilde{v}_k \in W^1_p(G, \mathbb{R}^2)$ and obtain from (4.9) in combination with (4.4)

$$\sup_k \int\limits_{B_R(0)} \left| \widetilde{v}_k - \left(\fint\limits_S \widetilde{v}_k \, \mathrm{d}y \right) \right|^p \mathrm{d}x < \infty \, ,$$

hence by (4.7) (returning to the old notation)

$$\sup_k \|\widetilde{v}_k\|_{L^p(\Omega)} < \infty \, ,$$

thus (recall (4.4))

$$\sup_k \|\widetilde{v}_k\|_{W^1_p(\Omega)} < \infty \, . \tag{4.10}$$

In case $p \geqslant 2$ (4.10) implies (by Sobolev's theorem)

$$\sup_k \|\widetilde{v}_k\|_{L^q(\Omega)} < \infty \tag{4.11}$$

independent of the choice of q. Thus, by (4.11) and (4.3) we get

$$\sup_k \int\limits_\Omega |\nabla \widetilde{u}_k|^q \, \mathrm{d}x < \infty \, . \tag{4.12}$$

In case $p < 2$ (4.12) follows in the same manner as a consequence of our assumption $q \leqslant \frac{2p}{2-p}$ and Sobolev's embedding theorem.

Finally we claim

$$\sup_k \int\limits_\Omega |\widetilde{u}_k|^q \, \mathrm{d}x < \infty \, , \tag{4.13}$$

which in case $s \geqslant q$ directly follows from (4.2). In case $s < q$ we use (4.2) to get

$$\sup_k \left| \fint\limits_\Omega \widetilde{u}_k \, \mathrm{d}x \right| < \infty \, . \tag{4.14}$$

Since (4.12) in combination with Poincaré s inequality implies

$$\sup_k \int\limits_\Omega \left| \widetilde{u}_k - \fint\limits_\Omega \widetilde{u}_k \, \mathrm{d}y \right|^q \mathrm{d}x < \infty \, , \tag{4.15}$$

we see that (4.13) is a consequence of (4.14) and (4.15).

With (4.12) and (4.13) it is shown that

$$\sup_k \|\widetilde{u}_k\|_{W^1_q(\Omega)} < \infty\,, \tag{4.16}$$

and from (4.10) and (4.16) the claim of Theorem 4.1 follows in a standard manner. □

We finish this section by presenting an alternative argument leading to the compactness of F-minimizing sequences.

Let $m(\Omega) := \int_\Omega \rho\,\mathrm{d}x$ with $0 \neq \rho \in C^1_0(\Omega)$, $0 \leqslant \rho \leqslant 1$, and define

$$M(w) := \frac{1}{m(\Omega)} \int_\Omega w\rho\,\mathrm{d}x$$

for functions or fields w on Ω (mean value of w with weight ρ). It holds

Lemma 4.2. *There is a constant $c = c(p, \Omega, \rho)$ depending on $p \in [1, \infty)$, the domain Ω and the function ρ such that*

$$\|w - M(w)\|_{L^p(\Omega)} \leqslant c\,\|\nabla w\|_{L^p(\Omega)} \tag{4.17}$$

holds for any $w \in W^1_p(\Omega)$.

Proof of Lemma 4.2. If the statement is false, we can find a sequence $w_k \in W^1_p(\Omega)$ such that

$$\|w_k - M(w_k)\|_{L^p(\Omega)} > k\,\|\nabla w_k\|_{L^p(\Omega)}\ . \tag{4.18}$$

Let

$$\widetilde{w}_k := \frac{w_k - M\,(w_k)}{\|w_k - M\,(w_k)\|_{L^p(\Omega)}}\,.$$

We get from (4.18)

$$1 = \|\widetilde{w}_k\|_{L^p(\Omega)} > k\,\|\nabla \widetilde{w}_k\|_{L^p(\Omega)}\ ,$$

which in particular implies

$$\sup_k \|\widetilde{w}_k\|_{W^1_p(\Omega)} < \infty \tag{4.19}$$

and

$$\nabla \widetilde{w}_k \to 0 \quad \text{in } L^p(\Omega)\,. \tag{4.20}$$

Passing to a subsequence we deduce from (4.19) the existence of $\widetilde{w} \in L^p(\Omega)$ such that $\widetilde{w}_k \to \widetilde{w}$ in $L^p(\Omega)$. Thus,

$$\|\widetilde{w}\|_{L^p(\Omega)} = 1\,, \quad M(\widetilde{w}) = 0\,. \tag{4.21}$$

However, (4.20) yields $\nabla\widetilde{w} = 0$, hence $\widetilde{w}$ must be constant contradicting (4.21). $\square$

Remark 4.3. Inequality (4.17) extends to domains in $\mathbb{R}^n, n \geqslant 2$.

Application of Lemma 4.2. As in the proof of Theorem 4.1 we choose an F-minimizing sequence $(u_k, v_k) \in Y$. In order to justify (4.10) (for the whole sequence v_k) we see that on account of (4.17) it is enough to show the validity of

$$\sup_k |M\,(v_k)| < \infty\,. \tag{4.22}$$

Recalling (4.5) our claim (4.22) follows from

$$\sup_k |M\,(\nabla u_k)| < \infty\,. \tag{4.23}$$

But for (4.23) we just observe

$$M(\nabla u_k) = \frac{1}{m(\Omega)} \int_\Omega \nabla u_k \rho \, \mathrm{d}x = -\frac{1}{m(\Omega)} \int_\Omega \nabla \rho u_k \, \mathrm{d}x\,,$$

thus (4.23) is a consequence of (4.2), and we end up with (4.10) and (4.16) for the sequence (u_k, v_k). $\square$

Remark 4.4. In Section 7 we will apply a more refined variant of Lemma 4.2.

§5. Linear growth models

The techniques used during the proof of Theorem 4.1 can be adjusted to the linear setting, more precisely:

Theorem 5.1. *Let Ω denote a bounded Lipschitz domain in $\mathbb{R}^2$ being in addition convex. Suppose we are given $f \in L^\infty(\Omega)$, parameters α_1, $\alpha_2 > 0$ and exponents $s \in (1, \infty)$, $q \in (1, 2]$. Then the problem*

$$\int_\Omega |u - f|^s \, \mathrm{d}x + \alpha_1 \int_\Omega |\nabla u - v|^q \, \mathrm{d}x + \alpha_2 \int_\Omega |\nabla v| \to \min \ \ in\ W^1_q(\Omega) \times BV(\Omega, \mathbb{R}^2)$$

admits a unique solution.

Remark 5.1. For a definition of the space $BV(\Omega)$ consisting of L^1-functions having finite total variation we refer the reader to, e.g., [3] or [33].

Remark 5.2. The quantity $\int\limits_\Omega |\nabla v|$ (= total variation of the matrix-valued measure ∇v) can be replaced by ,e.g.,

$$\int\limits_\Omega \left(\varepsilon + |\nabla v|^2\right)^{1/2} \quad \text{or} \quad \int\limits_\Omega \Phi_\mu\left(|\nabla v|\right) \text{ with } \mu > 1$$

and

$$\Phi_\mu(t) := \int\limits_0^t \int\limits_0^s (1+r)^{-\mu}\,\mathrm{d}r\,\mathrm{d}s$$

$$= \begin{cases} \frac{1}{\mu-1} t + \frac{1}{\mu-1}\frac{1}{\mu-2}(t+1)^{-\mu+2} - \frac{1}{\mu-1}\frac{1}{\mu-2}\,, & \mu \neq 2\,, \\ t - \ln(1+t)\,, & \mu = 2\,. \end{cases}$$

More generally, we can work with $\int\limits_\Omega \Phi\left(|\nabla v|\right)$, where $\Phi : [0,\infty) \to [0,\infty)$ is increasing and convex satisfying $c_1(t-1) \leqslant \Phi(t) \leqslant c_2(t+1)$, $t \geqslant 0$, with positive constants c_1, c_2.

Simultaneously we can discuss $\int\limits_\Omega \Psi\left(|\nabla u - v|\right)\mathrm{d}x$ in place of $\int\limits_\Omega |\nabla u - v|^q \mathrm{d}x$, provided that $\Psi : [0,\infty) \to [0,\infty)$ is strictly increasing and strictly convex with the property $c_3(t^q - 1) \leqslant \Psi(t) \leqslant c_4(t^q + 1)$, $t \geqslant 0$, with constants $c_3, c_4 > 0$. Clearly the same comments concern the term $\int\limits_\Omega |u - f|^s\,\mathrm{d}x$.

Proof of Theorem 5.1. We can follow exactly the lines of the proof of Theorem 4.1 provided we have a BV-variant of Lemma 4.1, i.e. inequality (4.9) holds in case $w \in BV(G)$ with $\|\nabla w\|_{L^p(G)}$ replaced by the total variation $\int\limits_G |\nabla w|$.

If we apply [33], Theorem 1.17, p.14, to approximate $w \in BV(G)$ through a sequence $w_j \in C^\infty(G) \cap W_1^1(G)$, which means

$$\lim_{j\to\infty} \int\limits_G |w_j - w|\,\mathrm{d}x = 0\,, \quad \lim_{j\to\infty} \int\limits_G |\nabla w_j|\,\mathrm{d}x = \int\limits_G |\nabla w|\,,$$

and quote Lemma 4.1 with $p = 1$ and for the functions w_j, then the BV-version of Lemma 4.1 follows. $\square$

Let us finally look at "linear coupling".

Theorem 5.2. *Consider Ω, f, α_1, $\alpha_2 > 0$ as in Theorem 5.1 and let $s \in (1, \infty)$.*

i) *The problem*

$$\int_\Omega |u - f|^s \, \mathrm{d}x + \alpha_1 \int_\Omega |v - \nabla u| + \alpha_2 \int_\Omega |\nabla v| \to \min$$

admits at least one solution $(u, v) \in BV(\Omega) \times BV(\Omega, \mathbb{R}^2)$ with u being unique.

ii) *If $\int_\Omega |v - \nabla u|$ is replaced by $\int_\Omega \Phi(|v - \nabla u|)$ with $\Phi : [0, \infty) \to [0, \infty)$ strictly increasing and strictly convex satisfying in addition the estimate*

$$a(t-1) \leqslant \Phi(t) \leqslant A(t+1) , \quad t \geqslant 0 ,$$

with constants $a, A > 0$, then we obtain a unique minimizer (u, v) in the class $BV(\Omega) \times BV(\Omega, \mathbb{R}^2)$.

Remark 5.3. As stated in Remark 5.2, the quantities $\int_\Omega |u - f|^s \, \mathrm{d}x$ and $\int_\Omega |\nabla v|$ can be modified in the usual way.

§6. Regularity results for non-quadratic energies

In this section we replace the energy F from (4.1) by its non-degenerate variant

$$\begin{aligned} K[u, v] := \int_\Omega |u - f|^s \, \mathrm{d}x + \alpha_1 \int_\Omega \left(\nu_1 + |\nabla u - v|^2 \right)^{q/2} \, \mathrm{d}x \\ + \alpha_2 \int_\Omega \left(\nu_2 + |\nabla v|^2 \right)^{p/2} \, \mathrm{d}x \end{aligned} \tag{6.1}$$

with exponents s, q, $p \in (1, \infty)$ and parameters α_1, α_2, ν_1, $\nu_2 > 0$.

Theorem 6.1. *(subquadratic case) Assume that $1 < q < p \leqslant 2$ or $1 < q \leqslant p < 2$. Let $(u, v) \in Y := W^1_q(\Omega) \times W^1_p(\Omega, \mathbb{R}^2)$ denote the unique K minimizer in the space Y. Then it holds:*

i) *The field v is continuously differentiable in Ω, the first partial derivatives satisfying a local Hölder-condition. Moreover, v is in the space $W^2_{2,\mathrm{loc}}(\Omega, \mathbb{R}^2)$.*

ii) *If we impose the restriction* $s \leqslant \frac{q}{2-q}$, *then we have the corresponding results for the function* u.

Remark 6.1. Probably Theorem 6.1 extends to the degenerate situation for which $\nu_1 = 0$ or $\nu_2 = 0$. The reader is referred to the papers of, e.g., Tolksdorf [53] and Di Benedetto [25] on degenerate elliptic systems and equations.

Remark 6.2. As usual we can replace the p-part (q-part) of the functional K by any non-degenerate p-elliptic (q-elliptic) density. For the quantity $\int\limits_\Omega |u-f|^s \,\mathrm{d}x$ we have even more flexibility.

Remark 6.3. The reader being interested in a version of Theorem 6.1 for $p > 2$ should consult the paper [8] for the necessary adjustments.

Remark 6.4. If $1 < \mu < 2$ and if in (6.1) the quantity $\int\limits_\Omega (\nu_2 + |\nabla v|^2)^{p/2} \,\mathrm{d}x$ is replaced by $\int\limits_\Omega \Phi_\mu(|\nabla v|)$ with Φ_μ from Remark 5.2, we expect regularity results in the spirit of Theorem 6.1, provided the coupling term is also of linear growth and $s \leqslant 2$. We refer to [10] and [11].

Proof of Theorem 6.1. We first will investigate the regularity of v. The basic idea goes back to the work of Frehse and Seregin outlined in paper [29], in which they study a model for plasticity with logarithmic hardening. An application of this technique in the context of the denoising of images has been presented in [9].

For notational simplicity we let $\alpha_1 = \alpha_2 = \nu_1 = \nu_2 = 1$ and abbreviate $G\colon \mathbb{R}^2 \to \mathbb{R}$, $G(\xi) := \big(1 + |\xi|^2\big)^{q/2}$, $H\colon \mathbb{R}^{2\times 2} \to \mathbb{R}$, $H(M) := \big(1 + |M|^2\big)^{p/2}$.

From $K[u,v] \leqslant K[u,w]$ for all $w \in W^1_p(\Omega,\mathbb{R}^2)$ it follows that v is the minimizer of the functional

$$\widetilde{K} : W^1_p(\Omega,\mathbb{R}^2) \ni w \mapsto \int\limits_\Omega G(w - \nabla u)\,\mathrm{d}x + \int\limits_\Omega H(\nabla w)\,\mathrm{d}x\,,$$

(u,v) denoting the unique K-minimizer in the space Y, whose existence is guaranteed by Theorem 4.1 and Remark 4.1.

The next calculations have to be justified by introducing the quadratic regularization $(0 < \delta < 1)$

$$\widetilde{K}_\delta : W^1_2(\Omega,\mathbb{R}^2) \ni w \mapsto \widetilde{K}(w) + \frac{\delta}{2}\int\limits_\Omega |\nabla w|^2\,\mathrm{d}x\,,$$

which means that in the following we actually work with the unique $\widetilde{K}_\delta$-minimizers v_δ being sufficiently regular, e.g., of class $W^2_{2,\mathrm{loc}}(\Omega, \mathbb{R}^2)$. Moreover, the sequence $\{v_\delta\}$ has nice convergence properties, which enables us to transfer uniform estimates obtained for the sequence $\{v_\delta\}$ to the limit function v. The details of this routine approximation procedure are outlined in [9, 29] and can also be found in various places in the monographs [32] and [7].

Dropping the index δ, the $\widetilde{K}$-minimizing property of $v(= v_\delta)$ yields

$$0 = \int_\Omega DH(\nabla v) : \nabla \Psi \,\mathrm{d}x + \int_\Omega DG(v - \nabla u) \cdot \Psi \,\mathrm{d}x \tag{6.2}$$

for $\Psi \in C_0^\infty(\Omega, \mathbb{R}^2)$. If we replace Ψ by $\partial_i \Psi$, $i = 1$, 2 and then choose $\Psi = \eta^2(\partial_i v - \xi_i)$ for some $\xi_i \in \mathbb{R}^2$, we obtain from (6.2)

$$\begin{aligned} &\int_\Omega D^2 H(\nabla v)\left(\partial_i \nabla v, \nabla\left[\eta^2\left(\partial_i v - \xi_i\right)\right]\right) \mathrm{d}x \\ &\qquad = \int_\Omega DG(v - \nabla u) \cdot \partial_i\left(\eta^2\left[\partial_i v - \xi_i\right]\right) \mathrm{d}x\,. \end{aligned} \tag{6.3}$$

Here η is a function from $C_0^1(\Omega)$ such that $0 \leqslant \eta \leqslant 1$, $\eta = 1$ on $B_r(x_0)$, $\operatorname{spt}\eta \subset B_{2r}(x_0)$, $|\nabla \eta| \leqslant c/r$ for a given disk $B_{2r}(x_0) \Subset \Omega$ with $r \leqslant 1$. In equation (6.3) and in what follows we will always take the sum w.r.t. indices repeated twice. Let us introduce the quantity

$$\Theta := \left(D^2 H(\nabla v)(\partial_i \nabla v, \partial_i \nabla v)\right)^{\frac{1}{2}}\,. \tag{6.4}$$

From (6.3) we deduce $\left(T_r(x_0) := B_{2r}(x_0) - \overline{B_r(x_0)}\right)$

$$\begin{aligned} \int_{B_{2r}(x_0)} \eta^2 \Theta^2 \,\mathrm{d}x \leqslant c &\left[\frac{1}{r} \int_{T_r(x_0)} \left|D^2 H(\nabla v)\right| \left|\nabla^2 v\right| |\nabla v - \xi| \,\mathrm{d}x \right. \\ &\left. + \int_{B_{2r}(x_0)} |DG(v - \nabla u)| \left|\partial_i\left(\eta^2\left[\partial_i v - \xi_i\right]\right)\right| \mathrm{d}x \right]. \end{aligned} \tag{6.5}$$

It holds

$$\begin{aligned}\left|D^2H(\nabla v)\right|\left|\nabla^2 v\right| &\leqslant c\left(1+|\nabla v|^2\right)^{\frac{p-2}{2}}|\nabla^2 v| \\ &= c\left[\left(1+|\nabla v|^2\right)^{\frac{p-2}{2}}|\nabla^2 v|^2\right]^{\frac{1}{2}}\left(1+|\nabla v|^2\right)^{\frac{p-2}{4}} \leqslant c\Theta\,,\end{aligned}$$

where we have used the definition of Θ (c.f.(6.4)) as well as the fact that $p \leqslant 2$. Therefore (6.5) yields

$$\begin{aligned}\int\limits_{B_{2r}(x_0)} \eta^2\Theta^2\,\mathrm{d}x &\leqslant c\left[\frac{1}{r}\int\limits_{T_r(x_0)}\Theta\,|\nabla v-\xi|\,\mathrm{d}x + S\right], \\ S &:= \int\limits_{B_{2r}(x_0)} |DG(v-\nabla u)|\left|\partial_i\left(\eta^2\left[\partial_i v-\xi_i\right]\right)\right|\,\mathrm{d}x\,.\end{aligned} \tag{6.6}$$

Let us look at the first term on the r.h.s. of (6.6):

$$\frac{1}{r}\int\limits_{T_r(x_0)}\Theta\,|\nabla u-\xi|\,\mathrm{d}x \leqslant \frac{1}{r}\left[\int\limits_{T_r(x_0)}\Theta^2\,\mathrm{d}x\right]^{\frac{1}{2}}\left[\int\limits_{T_r(x_0)}|\nabla v-\xi|^2\,\mathrm{d}x\right]^{\frac{1}{2}}$$

by Hölder's inequality and if we define $\xi := \fint_{T_r(x_0)}\nabla v\,\mathrm{d}x$, we get after an application of the Sobolev–Poincaré inequality

$$\frac{1}{r}\int\limits_{T_r(x_0)}\Theta\,|\nabla u-\xi|\,\mathrm{d}x \leqslant \frac{c}{r}\left[\int\limits_{T_r(x_0)}\Theta^2\,\mathrm{d}x\right]^{\frac{1}{2}}\int\limits_{T_{2r}(x_0)}\left|\nabla^2 v\right|\,\mathrm{d}x\,.$$

Observe that (see (6.4))

$$\left|\nabla^2 v\right| \leqslant c\Theta\varphi\,,\quad \varphi := \left(1+|\nabla v|^2\right)^{\frac{2-p}{4}}\,, \tag{6.7}$$

hence the previous estimates in combination with (6.6) show

$$\int\limits_{B_{2r}(x_0)}\eta^2\Theta^2\,\mathrm{d}x \leqslant \frac{c}{r}\left[\int\limits_{T_r(x_0)}\Theta^2\,\mathrm{d}x\right]^{\frac{1}{2}}\int\limits_{T_r(x_0)}\Theta\varphi\,\mathrm{d}x + cS\,. \tag{6.8}$$

With the exception of the term S, inequality (6.8) corresponds to the starting inequality (4.22) in [29] (after using the lower bound $\int\limits_{B_r(x_0)} \Theta^2 \, dx$ on the l.h.s. of (6.8)).

So let us have a closer look at S defined in (6.6). From the definition of G it follows

$$|DG\,(v - \nabla u)| \leqslant c\left[1 + |v|^{q-1} + |\nabla u|^{q-1}\right] , \tag{6.9}$$

and according to (6.9) we see

$$\begin{aligned} S \quad &\leqslant \quad c\Bigg[\int\limits_{B_{2r}(x_0)} \left|\partial_i\left(\eta^2\left[\partial_i v - \xi_i\right]\right)\right| \mathrm{d}x + \int\limits_{B_{2r}(x_0)} |v|^{q-1}\left|\partial_i\left(\eta^2\left[\partial_i v - \xi_i\right]\right)\right| \mathrm{d}x \\ &\qquad + \int\limits_{B_{2r}(x_0)} |\nabla u|^{q-1}\left|\partial_i\left(\eta^2\left[\partial_i v - \xi_i\right]\right)\right| \mathrm{d}x\Bigg] =: c\left[T_1 + T_2 + T_3\right] . \end{aligned} \tag{6.10}$$

Clearly

$$\begin{aligned} T_1 &\leqslant c\left[\int\limits_{B_{2r}(x_0)} \eta|\nabla\eta||\nabla v - \xi| \, \mathrm{d}x + \int\limits_{B_{2r}(x_0)} \eta^2\left|\nabla^2 v\right| \mathrm{d}x\right] \\ &\leqslant c\left[\frac{1}{r}\int\limits_{T_r(x_0)} |\nabla v - \xi| \, \mathrm{d}x + \int\limits_{B_{2r}(x_0)} \eta^2\left|\nabla^2 v\right| \mathrm{d}x\right] \\ &\leqslant c\left[\int\limits_{T_r(x_0)} \left|\nabla^2 v\right| \mathrm{d}x + \int\limits_{B_{2r}(x_0)} \eta^2\left|\nabla^2 v\right| \mathrm{d}x\right] \end{aligned}$$

by Poincaré s inequality. In the first integral on the r.h.s. we use (6.7), to the second one we apply Young's inequality with the result $(0 < \varepsilon < 1)$

$$\int\limits_{B_{2r}(x_0)} \eta^2|\nabla^2 v| \, \mathrm{d}x \leqslant \varepsilon \int\limits_{B_{2r}(x_0)} \eta^2\Theta^2 \, \mathrm{d}x + c(\varepsilon)\int\limits_{B_{2r}(x_0)} \varphi^2 \, \mathrm{d}x \, .$$

Therefore we get

$$T_1 \leqslant \varepsilon \int\limits_{B_{2r}(x_0)} \eta^2\Theta^2 \, \mathrm{d}x + c(\varepsilon)\int\limits_{B_{2r}(x_0)} \varphi^2 \, \mathrm{d}x + c\int\limits_{T_r(x_0)} \Theta\varphi \, \mathrm{d}x \, . \tag{6.11}$$

Inserting (6.11) into (6.10) and returning to (6.8) it follows after choosing ε small enough:

$$\begin{aligned}\int\limits_{B_{2r}(x_0)} \eta^2\Theta^2 \,\mathrm{d}x \leqslant \frac{c}{r}\left[\int\limits_{T_r(x_0)} \Theta^2 \,\mathrm{d}x + r^2\right]^{\frac{1}{2}} \int\limits_{T_r(x_0)} \Theta\varphi \,\mathrm{d}x \\ + c\int\limits_{B_{2r}(x_0)} \varphi^2 \,\mathrm{d}x + c\,[T_2+T_3]\ .\end{aligned} \tag{6.12}$$

As it will be shown later it holds $(1+|\nabla v|^2)^{p/4} \in W^1_{2,\mathrm{loc}}(\Omega)$ (uniformly w.r.t. the hidden approximation parameter), thus

$$\nabla v \in L^t_{\mathrm{loc}}\left(\Omega, \mathbb{R}^{2\times 2}\right) \quad \text{for all } t<\infty \tag{6.13}$$

(again uniformly in δ). We use this information during the estimate of T_2:

$$\begin{aligned}T_2 &\leqslant c\left[\int\limits_{B_{2r}(x_0)} |v|^{q-1}\eta^2\left|\nabla^2 v\right| \,\mathrm{d}x + \int\limits_{B_{2r}(x_0)} |v|^{q-1}|\nabla\eta|\,|\nabla v-\xi| \,\mathrm{d}x\right] \\ &=: c\,[U_1+U_2]\ .\end{aligned} \tag{6.14}$$

Recalling the definition of ξ, it follows from (6.13)

$$|\xi| \leqslant c(\varepsilon) r^{-\varepsilon}$$

for any number $\varepsilon > 0$. This gives:

$$U_2 \leqslant c\frac{1}{r}\int\limits_{T_r(x_0)} |v|^{q-1}|\nabla v| \,\mathrm{d}x + c(\varepsilon)\frac{1}{r^{1+\varepsilon}}\int\limits_{T_r(x_0)} |v|^{q-1} \,\mathrm{d}x\,.$$

Combining (6.13) with Sobolev's embedding theorem, we obtain

$$v \in L^\infty_{\mathrm{loc}}\left(\Omega, \mathbb{R}^2\right)$$

(uniformly) and another application of (6.13) together with Hölder's inequality implies the following statement by selecting $\varepsilon > 0$ sufficiently small:

if we choose a number $\beta \in (0,1)$ and if from now on we work on disks $B_{2r}(x_0)$ compactly contained in a subregion $\Omega' \Subset \Omega$, then it holds

$$U_2 \leqslant c(\beta, \Omega') r^\beta\,. \tag{6.15}$$

At the same time (recall (6.7) and use Young's inequality)

$$\begin{aligned} U_1 &\leqslant c \int\limits_{B_{2r}(x_0)} |v|^{q-1}\eta^2\Theta\varphi \,\mathrm{d}x \\ &\leqslant \varepsilon \int\limits_{B_{2r}(x_0)} \eta^2\Theta^2 \,\mathrm{d}x + c(\varepsilon) \int\limits_{B_{2r}(x_0)} |v|^{2q-2}\varphi^2 \,\mathrm{d}x \\ &\leqslant \varepsilon \int\limits_{B_{2r}(x_0)} \eta^2\Theta^2 \,\mathrm{d}x + c(\varepsilon, \Omega', \beta) r^{2\beta} \end{aligned} \tag{6.16}$$

for any $\beta \in (0,1)$ again by (6.13). We insert (6.15) and (6.16) into (6.14), return to (6.12), choose ε small enough and apply (6.13) to the term $\int\limits_{B_{2r}(x_0)} \varphi^2 \,\mathrm{d}x$ on the r.h.s. of (6.12) with the result

$$\begin{aligned} \int\limits_{B_{2r}(x_0)} \eta^2\Theta^2 \,\mathrm{d}x \leqslant c\frac{1}{r}\left[\int\limits_{T_r(x_0)} \Theta^2 \,\mathrm{d}x + r^2\right]^{\frac{1}{2}} \int\limits_{T_r(x_0)} \Theta\varphi \,\mathrm{d}x \\ + c\,(\beta, \Omega')\, r^\beta + c\,T_3,\ \beta \in (0,1)\,. \end{aligned} \tag{6.17}$$

The estimate for T_3 is similar to the discussion of T_2 and uses the assumption $q < 2$: we split

$$\begin{aligned} T_3 &\leqslant c\left[\int\limits_{B_{2r}(x_0)} |\nabla u|^{q-1}\left|\nabla^2 v\right| \eta^2 \,\mathrm{d}x + \frac{1}{r}\int\limits_{T_r(x_0)} |\nabla u|^{q-1}\left|\nabla v - \xi\right| \,\mathrm{d}x\right] \\ &=: c\left[V_1 + \frac{1}{r}V_2\right] \end{aligned}$$

and choose $s \in (2, q/(q-1))$. Let $g := |\nabla u|^{q-1}|\nabla v|$. Then we have (using the previous estimate for $|\xi|$)

$$\begin{aligned}\frac{1}{r}V_2 &\leqslant c(\varepsilon)\left[\frac{1}{r}\int\limits_{T_r(x_0)} g\,\mathrm{d}x + \frac{1}{r^{\varepsilon+1}}\int\limits_{T_r(x_0)} |\nabla u|^{q-1}\,\mathrm{d}x\right]\\ &\leqslant c(\varepsilon)\left[\left(\int\limits_{T_r(x_0)} g^2\,\mathrm{d}x\right)^{\frac{1}{2}} + \frac{1}{r^{\varepsilon}}\left(\int\limits_{T_r(x_0)} |\nabla u|^{2q-2}\,\mathrm{d}x\right)^{\frac{1}{2}}\right]\end{aligned}$$

by Hölder's inequality. For the same reason and by the choice of s

$$\|g\|_{L^2(T_r(x_0))} \leqslant c\, r^{\overline{\beta}}\, \|g\|_{L^s(T_r(x_0))}$$

with suitable positive exponent $\overline{\beta}$. At the same time

$$\int\limits_{T_r(x_0)} g^s\,\mathrm{d}x = \int\limits_{T_r(x_0)} |\nabla u|^{s(q-1)}|\nabla v|^s\,\mathrm{d}x\,,$$

and if we recall $\nabla u \in L^q(\Omega,\mathbb{R}^2)$, $s(q-1) < q$ and (6.13) it follows

$$\left[\int\limits_{T_r(x_0)} g^2\,\mathrm{d}x\right]^{\frac{1}{2}} \leqslant c r^{\overline{\beta}}\,. \tag{6.18}$$

Looking at the remaining term $\frac{1}{r^{\varepsilon}}\left(\int\limits_{T_r(x_0)} |\nabla u|^{2q-2}\,\mathrm{d}x\right)^{1/2}$ and observing $2q-2<q$, we see that for ε small enough this item can also be bounded through a positive power of r, hence by (6.18)

$$\frac{1}{r}V_2 \leqslant c(\Omega')r^{\overline{\beta}} \tag{6.19}$$

after decreasing $\overline{\beta}$ (if necessary). For V_1 we have

$$\begin{aligned}V_1 &\leqslant c\int\limits_{B_{2r}(x_0)} \eta^2\Theta\varphi|\nabla u|^{q-1}\,\mathrm{d}x\\ &\leqslant \varepsilon\int\limits_{B_{2r}(x_0)} \eta^2\Theta^2\,\mathrm{d}x + c(\varepsilon)\int\limits_{B_{2r}(x_0)} \varphi^2|\nabla u|^{2q-2}\,\mathrm{d}x\,.\end{aligned}$$

For sufficiently small ε the first term on the r.h.s. can be absorbed into the l.h.s. of (6.17), to the second integral we apply (6.13) and recall $2q-2<q$ to gain a positive radius power. Combining these estimates with (6.18) and going back to (6.17) it is shown:

there exists $\overline{\beta}\in(0,1)$ such that for any $\Omega'\Subset\Omega$ and any disk $B_{2r}(x_0)\subset\Omega'$, $r\leqslant 1$, it holds (uniformly in δ)

$$\int\limits_{B_r(x_0)}\Theta^2\,\mathrm{d}x\leqslant\frac{c}{r}\left[\int\limits_{T_r(x_0)}\Theta^2\,\mathrm{d}x+r^2\right]^{\frac{1}{2}}\int\limits_{T_r(x_0)}\Theta\varphi\,\mathrm{d}x+c(\Omega')r^{\overline{\beta}}\,. \tag{6.20}$$

Before proceeding we want to verify (6.13) (for the approximation again dropping the index δ). The idea is to bound Θ in $L^2_{\mathrm{loc}}(\Omega)$ uniformly, i.e. through quantities being under our control during the approximation.

To this purpose we return to (6.3) and choose $\xi=0$. Note that the following calculations are valid for any exponents $1<q\leqslant p\leqslant 2$. We get

$$\begin{aligned}\int\limits_{\Omega}D^2H\left(\nabla v\right)\left(\partial_i\nabla v,\partial_i\nabla v\right)\eta^2\,\mathrm{d}x&\\ =-2\int\limits_{\Omega}D^2H(\nabla v)\left(\partial_i\nabla v,\nabla\eta\otimes\partial_i v\right)\eta\,\mathrm{d}x&\\ +\int\limits_{\Omega}DG(v-\nabla u)\cdot\partial_i\left(\eta^2\partial_i v\right)\,\mathrm{d}x&\end{aligned} \tag{6.21}$$

with η as before. Using the Cauchy–Schwarz inequality in the form

$$D^2H(X)(Y,Z)\leqslant D^2H(X)(Y,Y)^{1/2}D^2H(X)(Z,Z)^{\frac{1}{2}}$$

and applying Young's inequality we deduce from (6.21)

$$\begin{aligned}\int\limits_{B_{2r}(x_0)}\Theta^2\eta^2\,\mathrm{d}x\leqslant c\Bigg[&\frac{1}{r^2}\int\limits_{T_r(x_0)}\left|D^2H(\nabla v)\right||\nabla v|^2\,\mathrm{d}x\\ &+\int_{B_{2r}(x_0)}|DG(v-\nabla u)|\left|\partial_i\left(\eta^2\partial_i v\right)\right|\,\mathrm{d}x\Bigg]\,.\end{aligned} \tag{6.22}$$

By the definition of H it is immediate that

$$\int_{T_r(x_0)} \left|D^2 H(\nabla v)\right| |\nabla v|^2 \, \mathrm{d}x \leqslant c \left[\int_\Omega |\nabla v|^p \, \mathrm{d}x + 1\right] \tag{6.23}$$

(the r.h.s. of (6.23) being uniformly bounded for the approximation v_δ!). We have by the definition of G

$$\begin{aligned} &\int_{B_{2r}(x_0)} |DG(v - \nabla u)| \left|\partial_i \left(\eta^2 \partial_i v\right)\right| \mathrm{d}x \\ &\quad\leqslant c \left[\int_{B_{2r}(x_0)} \left|\partial_i \left(\eta^2 \partial_i v\right)\right| \mathrm{d}x + \int_{B_{2r}(x_0)} |v|^{q-1} \left|\partial_i \left(\eta^2 \partial_i v\right)\right| \mathrm{d}x \right. \\ &\qquad\qquad \left. + \int_{B_{2r}(x_0)} |\nabla u|^{q-1} \left|\partial_i \left(\eta^2 \partial_i v\right)\right| \mathrm{d}x \right] \\ &\qquad\qquad\qquad =: c\left[W_1 + W_2 + W_3\right] , \end{aligned}$$

where

$$W_1 \leqslant c \left[\int_{B_{2r}(x_0)} |\nabla \eta| \, |\nabla v| \, \mathrm{d}x + \int_{B_{2r}(x_0)} \left|\nabla^2 v\right| \eta^2 \, \mathrm{d}x \right] .$$

The first integral is controlled, for the second one we use (6.7) and Young's inequality to get

$$\int_{B_{2r}(x_0)} \left|\nabla^2 v\right| \eta^2 \, \mathrm{d}x \leqslant \varepsilon \int_{B_{2r}(x_0)} \Theta^2 \eta^2 \, \mathrm{d}x + c(\varepsilon) \int_{B_{2r}(x_0)} \eta^2 \varphi^2 \, \mathrm{d}x .$$

The ε-term is absorbed in the l.h.s. of (6.22), for the $c(\varepsilon)$term we observe

$$\int_{B_{2r}(x_0)} \eta^2 \varphi^2 \, \mathrm{d}x \leqslant c \left[\int_\Omega |\nabla v|^{2-p} \, \mathrm{d}x + 1\right]$$

by the definition of φ. Therefore (6.22) together with (6.23) yields

$$\int_{B_{2r}(x_0)} \eta^2 \Theta^2 \, \mathrm{d}x \leqslant c(\Omega') + c\left[W_2 + W_3\right] . \tag{6.24}$$

We have

$$
\begin{aligned}
W_2 &\leqslant c\left[\int\limits_{B_{2r}(x_0)} |v|^{q-1}|\nabla v||\nabla\eta|\eta\,\mathrm{d}x + \int\limits_{B_{2r}(x_0)} \eta^2\left|\nabla^2 v\right||v|^{q-1}\,\mathrm{d}x\right]\\
&\leqslant c(\Omega')\left[\int\limits_{\Omega}|\nabla v|^p\,\mathrm{d}x\right]^{\frac{1}{p}}\left[\int\limits_{\Omega}|v|^{(q-1)\frac{p}{p-1}}\,\mathrm{d}x\right]^{1-\frac{1}{p}}\\
&\qquad + c\int\limits_{B_{2r}(x_0)} \eta^2\Theta\varphi|v|^{q-1}\,\mathrm{d}x\,.
\end{aligned}
$$

From $q \leqslant p$ it follows $(q-1)\frac{p}{p-1} \leqslant p$, hence we can bound the integral involving $|v|^{(q-1)\frac{p}{p-1}}$. Splitting the integral involving Θ in the usual manner leads to the quantity $\int\limits_{\Omega} \varphi^2|v|^{2q-2}\,\mathrm{d}x$ with integrand being of order $|\nabla v|^{2-p}|v|^{2q-2}$. But we have (w.l.o.g. $p < 2$)

$$
\int\limits_{\Omega}|\nabla v|^{2-p}|v|^{2q-2}\,\mathrm{d}x \leqslant c\left[\int\limits_{\Omega}|\nabla v|^p\,\mathrm{d}x + \int\limits_{\Omega}|v|^{\frac{p}{2p-2}(2q-2)}\,\mathrm{d}x\right]
$$

with exponent $\frac{p}{2p-2}(2q-2) \leqslant p$, hence (6.24) can be replaced by

$$
\int\limits_{B_{2r}(x_0)} \eta^2\Theta^2\,\mathrm{d}x \leqslant c(\Omega') + cW_3\,. \tag{6.25}
$$

In the same manner as for W_1 and W_2 we get

$$
\begin{aligned}
W_3 &\leqslant c\left[\int\limits_{B_{2r}(x_0)} \eta|\nabla\eta||\nabla u|^{q-1}|\nabla v|\,\mathrm{d}x + \int\limits_{B_{2r}(x_0)} \eta^2|\nabla u|^{q-1}\left|\nabla^2 v\right|\,\mathrm{d}x\right]\\
&\leqslant c(\Omega')\left[\int\limits_{\Omega}|\nabla v|^p\,\mathrm{d}x + \int\limits_{\Omega}|\nabla u|^{\frac{p}{p-1}(q-1)}\,\mathrm{d}x + c\int\limits_{B_{2r}(x_0)}\eta^2|\nabla u|^{q-1}|\nabla^2 v|\,\mathrm{d}x\right],
\end{aligned}
$$

the first two terms being bounded on account of $\frac{p}{p-1}(q-1) \leqslant q$. Finally it holds

$$\int\limits_{B_{2r}(x_0)} \eta^2 |\nabla u|^{q-1} |\nabla^2 v| \,\mathrm{d}x$$
$$\leqslant \varepsilon \int\limits_{B_{2r}(x_0)} \eta^2 \Theta^2 \,\mathrm{d}x + c(\varepsilon) \int\limits_{B_{2r}(x_0)} \eta^2 \varphi^2 \, |\nabla u|^{(q-1)2} \,\mathrm{d}x$$

with

$$\begin{aligned}\varphi^2 \, |\nabla u|^{(q-1)2} &\leqslant c \left(|\nabla v|^{2-p} + 1 \right) |\nabla u|^{(q-1)2} \\ &\leqslant c \left[|\nabla u|^{(q-1)2} + |\nabla v|^p + |\nabla u|^{2(q-1)\frac{p}{2(p-1)}} \right] .\end{aligned}$$

Since the functions on the r.h.s. are integrable, inequality (6.25) clearly implies

$$\int\limits_{\Omega'} \Theta^2 \,\mathrm{d}x \leqslant c(\Omega') \tag{6.26}$$

uniformly for the approximation sequence. We have

$$\Theta^2 = D^2 H(\nabla v) \left(\partial_i \nabla v, \partial_i \nabla v \right) \geqslant c \left(1 + |\nabla v|^2 \right)^{\frac{p-2}{2}} \left| \nabla^2 v \right|^2 \geqslant c \, |\nabla \overline{\varphi}|^2$$

for the function $\overline{\varphi} := \left(1 + |\nabla v|^2\right)^{p/4}$. Clearly $\overline{\varphi} \in L^2(\Omega)$ (uniformly for the approximation) and the previous estimate together with (6.26) shows $|\nabla \overline{\varphi}| \in L^2_{\mathrm{loc}}(\Omega)$ (uniformly). But then $\overline{\varphi} \in L^t_{\mathrm{loc}}(\Omega)$ for any finite t by Sobolev's theorem and (6.13) follows.

Let us return to (6.20). In order to apply Lemma 4.1 of [29] (with $H := \Theta$, $h := \varphi$) we first let $\beta := \overline{\beta}/2$ and choose an exponent $\alpha \in (0, \beta)$. Then (6.20) implies

$$\int\limits_{B_r(x_0)} \Theta^2 \,\mathrm{d}x \leqslant c \frac{1}{r} \left[r^{2\alpha} + \int\limits_{T_r(x_0)} \Theta^2 \,\mathrm{d}x \right] \cdot \int\limits_{T_r(x_0)} \Theta \varphi \,\mathrm{d}x + c r^{2\beta} \,, \tag{6.20'}$$

and it is easy to check that with (6.20') the estimate (A3.6) in [29] has to be replaced by

$$\int_{B_r(x_0)} \Theta^2 \,\mathrm{d}x \leqslant c \left[\sqrt{\log_2 \frac{2R}{r}} \int_{T_r(x_0)} \Theta^2 \,\mathrm{d}x + r^\alpha \sqrt{\log_2 \frac{2R}{r}} \right] + cr^\beta ,$$

where R is some fixed radius, $r \leqslant R$, and $B_{2R}(x_0) \subset \Omega' \Subset \Omega$.

But since we assume $\alpha < \beta$, the above inequality clearly implies (A3.6) from [29] and as outlined there we obtain

$$\int_{B_r(x_0)} \Theta^2 \,\mathrm{d}x \leqslant K(t) \left|\ln r\right|^{-t} \tag{6.27}$$

for any exponent $t > 1$, any disk $B_{2r}(x_0) \subset \Omega' \Subset \Omega$ with a local constant depending on Ω' and t.

Let $\sigma := DH(\nabla v)$. We have

$$\begin{aligned} \partial_i \sigma : \partial_i \sigma &= D^2 H(\nabla v) \left(\partial_i \nabla v, \partial_i \sigma\right) \\ &\leqslant \left(D^2 H\left(\nabla v\right) \left(\partial_i \nabla v, \partial_i \nabla v\right)\right)^{\frac{1}{2}} \left(D^2 H\left(\nabla v\right) \left(\partial_i \sigma, \partial_i \sigma\right)\right)^{\frac{1}{2}} \leqslant \Theta \left|\nabla \sigma\right| , \end{aligned}$$

where we have used the boundedness of $\left|D^2 H(\nabla v)\right|$ due to $p \leqslant 2$. Hence (6.27) holds with Θ replaced by $|\nabla\sigma|$ and a lemma of Frehse [28] implies the continuity of σ with modulus of continuity being uniform for the approximation.

Next we observe that $DH\colon \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$ is a C^1-diffeomorphism, thus the continuity of ∇v and thereby $v \in C^1(\Omega, \mathbb{R}^2)$ is a consequence of $\nabla v = (DH)^{-1}(\sigma)$.

Having established the local boundedness of ∇v, we return to (6.20) and use the hole-filling technique to prove $\int_{B_r(x_0)} \Theta^2 \,\mathrm{d}x \leqslant \operatorname{const} r^\gamma$ locally, hence $\nabla\sigma$ satisfies a Morrey condition which means that σ and thereby ∇v is locally Hölder-continuous. Note that the local boundedness of ∇v together with, e.g., (6.26) immediately gives $v \in W^2_{2,\mathrm{loc}}(\Omega, \mathbb{R}^2)$.

Let us now look at the function u: from $K[u, v] \leqslant K[u + t\eta, v]$, $\eta \in C^1_0(\Omega)$, we deduce

$$\int_\Omega DG\left(\nabla u - v\right) \cdot \nabla\eta \,\mathrm{d}x = \int_\Omega g\eta \,\mathrm{d}x$$

with suitable function g. Let

$$A\left(x, \xi\right) := DG\left(\xi - v(x)\right) .$$

Since the x-dependence of A is sufficiently nice we can quote, e.g., [41], Chapter 4, Section 3, or [25] to get $u \in C^1(\Omega)$ with locally Hölder-continuous first derivatives.

However, in both references the solution u of the problem

$$\int_\Omega A(x, \nabla u) \cdot \nabla\eta \, \mathrm{d}x = \int_\Omega \eta g \, \mathrm{d}x \,, \quad \eta \in C_0^1(\Omega) \,, \tag{6.28}$$

is assumed to be locally bounded. Therefore we prefer to sketch the regularity of u closely following the lines of the proof of part i) of Theorem 6.1 and to make clear how the assumption $s \leqslant q/(2-q)$ comes into play.

As a matter of fact the following calculations have to be carried out for a quadratic regularization in which the functional

$$W_2^1(\Omega) \ni w \mapsto \frac{\delta}{2} \int_\Omega |\nabla w|^2 \, \mathrm{d}x + K[w, v]$$

is considered with unique minimizer u_δ. We claim

$$\nabla u \in L^t_{\mathrm{loc}}\left(\Omega, \mathbb{R}^2\right) \quad \text{for all } t < \infty \tag{6.29}$$

(uniformly w.r.t. δ).

Clearly (6.29) implies the local boundedness of u as required in the references [41] and [25]. (6.29) corresponds to (6.13) and analogous to the previous arguments it will be deduced from

$$\left(1 + |\nabla u|^2\right)^{q/4} \in W^1_{2,\mathrm{loc}}(\Omega) \,. \tag{6.30}$$

For proving (6.30), we replace η in (6.28) by $\partial_i(\eta^2 \partial_i u)$ with $\eta \in C_0^1(B_{2r}(x_0))$, $0 \leqslant \eta \leqslant 1$, $\eta = 1$ on $B_r(x_0)$, $|\nabla\eta| \leqslant c/r$ for a disk $B_{2r}(x_0) \Subset \Omega$, $r \leqslant 1$. It follows

$$\int_{B_{2r}(x_0)} \partial_i \left\{DG(\nabla u - v)\right\} \cdot \nabla\left(\eta^2 \partial_i u\right) \, \mathrm{d}x = -\int_{B_{2r}(x_0)} g \partial_i \left(\eta^2 \partial_i u\right) \, \mathrm{d}x \,. \tag{6.31}$$

Observing

$$\begin{aligned} \partial_i \left\{DG(\nabla u - v)\right\} &= D^2G(\nabla u - v)(\partial_i \nabla u - \partial_i v, \,\cdot\,) \\ &= D^2G(\nabla u - v)(\partial_i \nabla u, \cdot) - D^2G(\nabla u - v)(\partial_i v, \cdot) \,, \end{aligned}$$

(6.31) can be restated in the form

$$\begin{aligned}
&\int_{B_{2r}(x_0)} D^2G\,(\nabla u - v)\,(\partial_i \nabla u, \partial_i \nabla u)\,\eta^2 \,\mathrm{d}x \\
&= \quad -2\int_{B_{2r}(x_0)} D^2G\,(\nabla u - v)\,(\partial_i \nabla u \eta, \nabla \eta \partial_i u)\,\mathrm{d}x \\
&\quad + \int_{B_{2r}(x_0)} D^2G\,(\nabla u - v)\,(\partial_i v, \partial_i \nabla u)\,\eta^2 \,\mathrm{d}x \\
&\quad + \int_{B_{2r}(x_0)} 2D^2G\,(\nabla u - v)\,(\partial_i v, \nabla \eta\, \eta\, \partial_i u)\,\mathrm{d}x \\
&\quad - \int_{B_{2r}(x_0)} g\,\partial_i \left(\eta^2 \partial_i u\right)\,\mathrm{d}x =: T_1 + T_2 + T_3 + T_4\,. \qquad (6.32)
\end{aligned}$$

Consider a subdomain $\Omega' \Subset \Omega$ and recall $v \in C^1\left(\overline{\Omega'}, \mathbb{R}^2\right)$. Suppose further that $B_{2r}(x_0) \subset \Omega'$. In what follows we denote by "c" local constants, i.e. constants depending additionally on Ω' but being independent of the hidden approximation parameter δ. It holds on account of $q \leqslant 2$ (with varying value of c in each line)

$$|T_2| \leqslant c \int_{B_{2r}(x_0)} \eta^2 \left|\nabla^2 u\right| \mathrm{d}x\,, \qquad (6.33)$$

$$|T_3| \leqslant c \int_{B_{2r}(x_0)} \eta\,|\nabla \eta|\,|\nabla u|\,\mathrm{d}x\,. \qquad (6.34)$$

In T_1 we apply the Cauchy–Schwarz inequality to the bilinear form $D^2G\,(\nabla u - v)$ and then use Young's inequality to get from (6.32)–(6.34)

$$\begin{aligned}
&\int_{B_{2r}(x_0)} \eta^2 D^2G\,(\nabla u - v)\,(\partial_i \nabla u, \partial_i \nabla u)\,\mathrm{d}x \\
&\qquad \leqslant c\left[\frac{1}{r} + \int_{B_{2r}(x_0)} \eta^2 \left|\nabla^2 u\right| \mathrm{d}x + |T_4|\right]. \qquad (6.35)
\end{aligned}$$

On the left-hand side of (6.35) we observe $|\nabla u - v| \leqslant c + |\nabla u|$, hence

$$D^2G(\nabla u - v)(\partial_i \nabla u, \partial_i \nabla u) \geqslant c\left(1 + |\nabla u - v|^2\right)^{\frac{q-2}{2}} \left|\nabla^2 u\right|^2 \geqslant c\left(1 + |\nabla u|^2\right)^{\frac{q-2}{2}} \left|\nabla^2 u\right|^2 .$$

Estimating

$$\int\limits_{B_{2r}(x_0)} \eta^2 \left|\nabla^2 u\right| \,\mathrm{d}x \leqslant \varepsilon \int\limits_{B_{2r}(x_0)} \eta^2 \left(1 + |\nabla u|^2\right)^{\frac{q-2}{2}} \left|\nabla^2 u\right|^2 \,\mathrm{d}x + c(\varepsilon) \int\limits_{B_{2r}(x_0)} \eta^2 \left(1 + |\nabla u|^2\right)^{\frac{2-q}{2}} \,\mathrm{d}x$$

we see that after suitable choice of ε it follows from (6.35)

$$\int\limits_{B_{2r}(x_0)} \eta^2 \left(1 + |\nabla u|^2\right)^{\frac{q-2}{2}} \left|\nabla^2 u\right|^2 \,\mathrm{d}x \leqslant c\left[\frac{1}{r} + |T_4|\right] . \tag{6.36}$$

Let us remark that in (6.36) the constant c also depends on $\|\nabla u\|_{L^q(\Omega)}$. From the definition of g we infer $|g| \leqslant c\,|u - f|^{s-1}$, thus (recall $f \in L^\infty(\Omega)$)

$$|T_4| \leqslant c \int\limits_{B_{2r}(x_0)} |u - f|^{s-1} \left|\partial_i\left(\eta^2 \partial_i u\right)\right| \,\mathrm{d}x \leqslant c\left[\int\limits_{B_{2r}(x_0)} \left|\partial_i\left(\eta^2 \partial_i u\right)\right| \,\mathrm{d}x + \int\limits_{B_{2r}(x_0)} |u|^{s-1} \left|\partial_i\left(\eta^2 \partial_i u\right)\right| \,\mathrm{d}x\right] ,$$

and the first term in $[\ldots]$ is easily handled. Thus, it remains to discuss

$$T_5 := \int\limits_{B_{2r}(x_0)} |u|^{s-1}\, \eta\, |\nabla \eta|\, |\nabla u| \,\mathrm{d}x ,$$

$$T_6 := \int\limits_{B_{2r}(x_0)} |u|^{s-1}\, \eta^2 \left|\nabla^2 u\right| \,\mathrm{d}x .$$

We have

$$
\begin{aligned}
T_5 &\leqslant c(r)\left[\int\limits_{B_{2r}(x_0)} |\nabla u|^q \,\mathrm{d}x + \int\limits_{B_{2r}(x_0)} |u|^{q\frac{s-1}{q-1}} \,\mathrm{d}x\right] \\
&\leqslant c(r)\left[1 + \int\limits_{B_{2r}(x_0)} |u|^{q\frac{s-1}{q-1}} \,\mathrm{d}x\right] .
\end{aligned}
$$

Recalling $\int\limits_{\Omega} |u|^s \,\mathrm{d}x \leqslant c$ it follows

$$
\int\limits_{B_{2r}(x_0)} |u|^{q\frac{s-1}{q-1}} \,\mathrm{d}x \leqslant c \tag{6.37}
$$

in the case $q \geqslant s$ since then $\frac{q}{q-1} \leqslant \frac{s}{s-1}$. Now let $q < s$. Recalling $q \in (1,2)$ we obtain (6.37) provided $q\frac{s-1}{q-1} \leqslant \frac{2q}{2-q}$. But this inequality follows from our hypothesis $s \leqslant \frac{q}{2-q}$.

Finally we estimate

$$
\begin{aligned}
T_6 &\leqslant \varepsilon \int\limits_{B_{2r}(x_0)} \eta^2 \left(1 + |\nabla u|^2\right)^{\frac{q-2}{2}} \left|\nabla^2 u\right|^2 \,\mathrm{d}x \\
&\quad + c(\varepsilon) \int\limits_{B_{2r}(x_0)} \eta^2 \left(1 + |\nabla u|^2\right)^{\frac{2-q}{2}} |u|^{2s-2} \,\mathrm{d}x\,,
\end{aligned}
$$

the first item on the right-hand side being absorbed in the left-hand side of (6.36). For the second term we observe

$$
\begin{aligned}
&\int\limits_{B_{2r}(x_0)} |\nabla u|^{2-q} |u|^{2s-2} \,\mathrm{d}x \\
&\qquad\qquad \leqslant c\left[\int\limits_{B_{2r}(x_0)} |\nabla u|^q \,\mathrm{d}x + \int\limits_{B_{2r}(x_0)} |u|^{(2s-2)\frac{q}{2q-2}} \,\mathrm{d}x\right] \leqslant c
\end{aligned}
$$

on account of (6.37).

Altogether (6.36) yields

$$\int\limits_{B_r(x_0)} \left(1+|\nabla u|^2\right)^{\frac{q-2}{2}} \left|\nabla^2 u\right|^2 \, \mathrm{d}x \leqslant c(r)$$

(uniformly in δ), and (6.30) follows. This proves (6.29).

With this information we return to (6.31) replacing $\partial_i u$ through $\partial_i u - a_i$, where $a_i := \fint_{T_r(x_0)} \partial_i u \, \mathrm{d}x$. Then with the arguments used during the proof of i) of Theorem 6.1 we find $u \in C^{1,\alpha}(\Omega)$ and thereby $u \in W^2_{2,\mathrm{loc}}(\Omega)$. This completes the proof of Theorem 6.1. $\square$

§7. Related variational problems

As in the previous sections let $\Omega \subset \mathbb{R}^2$ denote a bounded Lipschitz domain being in addition convex.

i) *Vector case.*

As a matter of fact all our results remain valid, if the functions u, f: $\Omega \to \mathbb{R}$ are replaced by vector fields u, f: $\Omega \to \mathbb{R}^m$ for some $m > 1$ and if matrix-valued mappings v: $\Omega \to \mathbb{R}^{m\times m}$ are considered.

ii) *Models involving the symmetric gradient of v.*

Roughly speaking it turns out that all our statements on existence, uniqueness and regularity of minimizers remain unchanged, if the Jacobian matrix $\nabla v = (\partial_i v^j)_{1\leqslant i,j\leqslant 2}$ is replaced by its symmetric part

$$\varepsilon(v) := \frac{1}{2}\left(\nabla v + (\nabla v)^T\right) = \frac{1}{2}\left(\partial_i v^j + \partial_j v^i\right)_{1\leqslant i,j\leqslant 2} .$$

Let us look for example at the functional F from (4.1) but now with $\varepsilon(v)$ in place of ∇v. Recall that in (4.1) $p > 1$ is required. Then Korn's inequality (compare [44]) implies

$$W^1_p(\Omega,\mathbb{R}^2) = \left\{w \in L^p(\Omega,\mathbb{R}^2) : \varepsilon(w) \in L^p\left(\Omega,\mathbb{R}^{2\times 2}\right)\right\} ,$$

where on the r.h.s. $\varepsilon(w)$ is defined in the distributional sense. More precisely it holds

$$\|w\|_{W^1_p(\Omega)} \leqslant c\left[\|w\|_{L^p(\Omega)} + \|\varepsilon(w)\|_{L^p(\Omega)}\right] , \tag{7.1}$$

which means that for a minimizing sequence (u_k, v_k) we have to bound $\|v_k\|_{L^p(\Omega)}$ (at least for a subsequence) in order to deduce from (7.1) the boundedness of $\|v_k\|_{W^1_p(\Omega)}$.

Recall that during the proof of Theorem 4.1 we derived $\sup_k \|v_k\|_{L^p(\Omega)} < \infty$ from Lemma 4.1 and now we need a version of Lemma 4.1 with the symmetric gradient on the r.h.s. Unfortunately this result is rather technical and origins in the paper [44], the details even covering the linear case have been presented in [31] and [30].

Fix a function $\rho \in C_0^\infty(\Omega)$ such that

$$0 \leqslant \rho \leqslant 1 \quad \text{and} \quad m(\Omega) := \int_\Omega \rho(x)\,\mathrm{d}x > 0\,. \tag{7.2}$$

For $w : \Omega \to \mathbb{R}^2$ we let

$$\omega_{ij} := \omega_{ij}(w)\,, \quad \omega_{ij} := \frac{1}{2}\left(\partial_j w^i - \partial_i w^j\right)\,, \quad i,\, j = 1,\, 2\,, \tag{7.3}$$

and define

$$\mathcal{R}_w^i(x) := \frac{1}{m(\Omega)}\left[\int_\Omega w^i(y)\rho(y)\,\mathrm{d}y + \sum_{j=1}^2 \int_\Omega \omega_{ij}(y)\rho(y)\,(x_j - y_j)\,\mathrm{d}y\right], \quad i = 1,\, 2\,. \tag{7.4}$$

Lemma 7.1. *Let $1 \leqslant p < \infty$ and consider $w \in W_p^1(\Omega, \mathbb{R}^2)$. Then the field $\mathcal{R}_w$ defined in (7.4) is a rigid motion, i.e. $\varepsilon(\mathcal{R}_w) = 0$ and it holds* ($c = c\,(\Omega, \rho, p) \in (0, \infty)$)

$$\|w - \mathcal{R}_w\|_{L^p(\Omega)} \leqslant c\,\|\varepsilon(w)\|_{L^p(\Omega)}\,. \tag{7.5}$$

Remark 7.1. Clearly Lemma 7.1 holds in any dimension n, i.e. for fields $w\colon \mathbb{R}^n \supset \Omega \to \mathbb{R}^n$.

Theorem 7.1. *The statements of Theorem* 4.1 *remain valid, if in the functional F from* (4.1) *the Jacobian matrix ∇v of v is replaced by $\varepsilon(v)$.*

Proof of Theorem 7.1. We adopt the notation from the proof of Theorem 4.1 observing that (4.4) has to be replaced by

$$\sup_k \int_\Omega |\varepsilon(v_k)|^p\,\mathrm{d}x < \infty\,. \tag{4.4'}$$

Let us drop the index k for the moment. We have by (7.5)

$$\|v\|_{L^p(\Omega)} \leqslant c\left[\|\varepsilon(v)\|_{L^p(\Omega)} + \|\mathcal{R}_v\|_{L^p(\Omega)}\right]. \tag{7.6}$$

Moreover, it holds after an integration by parts

$$\begin{aligned}\mathcal{R}_v^i(x) &= \frac{1}{m(\Omega)}\Bigg[\int_\Omega v^i(y)\rho(y)\,\mathrm{d}y \\ &\quad + \sum_{j=1}^{2}\frac{1}{2}\int_\Omega \Big((v^j(y)\frac{\partial}{\partial y_i}\left\{\rho(y)\,(x_j-y_j)\right\} \\ &\qquad\qquad -v^i(y)\frac{\partial}{\partial y_j}\left\{\rho(y)\,(x_j-y_j)\right\}\Big)\,\mathrm{d}y\Bigg]. \end{aligned} \tag{7.7}$$

The reader should note that the r.h.s. of (7.7) makes sense even for fields $v \in L^1(\Omega,\mathbb{R}^2)$, in particular we can define $\mathcal{R}_{\nabla u}$ via the r.h.s. of (7.7) with v being replaced by $\nabla u \in L^q(\Omega,\mathbb{R}^2)$.

Next observe that

$$\|\mathcal{R}_v\|_{L^p(\Omega)} \leqslant \|\mathcal{R}_{\nabla u}\|_{L^p(\Omega)} + \|\mathcal{R}_v - \mathcal{R}_{\nabla u}\|_{L^p(\Omega)} \tag{7.8}$$

and

$$\begin{aligned}\mathcal{R}_{\nabla u}^i(x) &= \frac{1}{m(\Omega)}\Bigg[-\int_\Omega u(y)\frac{\partial}{\partial y_i}\rho(y)\,\mathrm{d}y \\ &\quad + \sum_{j=1}^{2}\frac{1}{2}\int_\Omega \Big(-u(y)\frac{\partial^2}{\partial y_j\partial y_i}\left\{\rho(y)(x_j-y_j)\right\} \\ &\qquad\qquad +u(y)\frac{\partial^2}{\partial y_i\partial y_j}\left\{\rho(y)\,(x_j-y_j)\right\}\,\mathrm{d}y\Big)\Bigg] \\ &= -\frac{1}{m(\Omega)}\int_\Omega u(y)\frac{\partial}{\partial y_i}\rho(y)\,\mathrm{d}y\,,\end{aligned}$$

hence a bound for $\mathcal{R}_{\nabla u}$ follows from (4.2), more precisely we get from the above equation the inequality

$$|\mathcal{R}_{\nabla u_k}(x)| \leqslant c\int_\Omega |u_k|\,\mathrm{d}y\,,\quad x\in\Omega\,, \tag{7.9}$$

and (4.2) even yields

$$\sup_k \|\mathcal{R}_{\nabla u_k}\|_{L^\infty(\Omega)} < \infty\,.$$

From formula (7.7) we directly infer

$$\left|\mathcal{R}^i_{v_k}(x) - \mathcal{R}^i_{\nabla u_k}(x)\right| \leqslant c \int\limits_\Omega |v_k - \nabla u_k| \, \mathrm{d}y \,, \quad x \in \Omega \,,$$

thus by (4.5)

$$\sup_k \|\mathcal{R}_{v_k} - \mathcal{R}_{\nabla u_k}\|_{L^\infty(\Omega)} < \infty \,. \tag{7.10}$$

From (7.8) - (7.10) it follows

$$\sup_k \|\mathcal{R}_{v_k}\|_{L^p(\Omega)} < \infty \,,$$

thus on account of (7.6) we arrive at

$$\sup_k \|v_k\|_{L^p(\Omega)} < \infty \,. \tag{7.11}$$

Inserting (7.11) and (4.4') into (7.1) we arrive at (4.10) now being valid for the whole sequence (v_k).

The rest of the proof of Theorem 7.1 follows exactly the lines of the proof of Theorem 4.1. □

Let us now turn to the linear case studied in Section 5 again replacing ∇v by $\varepsilon(v)$. The correct class is the space $BD(\Omega)$ introduced in the papers [51, 52] and [50] consisting of fields $v \in L^1(\Omega, \mathbb{R}^2)$ whose distributional symmetric gradient $\varepsilon(v)$ is a matrix valued Radon measure. A short survey of the properties of such fields v with bounded deformation is presented in Appendix A.3 of [32].

Theorem 7.2. *If in Theorem* 5.1 *and* 5.2 *the quantity* $\int_\Omega |\nabla v|$ *is replaced by* $\int_\Omega |\varepsilon(v)|$*, then the conclusions of these theorems remain valid provided the functionals are considered on the spaces* $W^1_q(\Omega) \times BD(\Omega)$ *and* $BV(\Omega) \times BD(\Omega)$*, respectively.*

For the proof of Theorem 7.2 we need a BD-Version of Lemma 7.1, which has been established in [31] and [30].

Lemma 7.2. *With the notation introduced in* (7.2) *and* (7.3) *define* $\mathcal{R}^i_v$ *according to* (7.7)*. Then for* $v \in BD(\Omega)$ *it holds*

$$\|v - \mathcal{R}_v\|_{L^1(\Omega)} \leqslant c(\Omega, \rho) \int\limits_\Omega |\varepsilon(v)| \,. \tag{7.12}$$

With (7.12) we can adjust the arguments used during the proof of Theorem 7.1 to the $BD(\Omega)$-case, the details are left to the reader.

Noting that in [32] and also [29] actually the symmetric gradient is considered, it is easy to transfer the calculations from Section 6 with the result.

Theorem 7.3. *The statements of Theorem* 6.1 *hold if* ∇v *in the functional* K *from* (6.1) *is replaced by* $\varepsilon(v)$.

For regularity results in case $p > 2$ and $\varepsilon(v)$ in place of ∇v the reader should consult the paper [12].

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