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LOCAL BOUNDARY REGULARITY FOR THE NAVIER–STOKES EQUATIONS IN NONENDPOINT BORDERLINE LORENTZ SPACES

Abstract. We prove local regularity up to the flat part of the boundary, for certain classes of distributional solutions that are $L_\infty L^{3,q}$ with q finite. The corresponding result, for the interior case, was proven recently by Wang and Zhang, see also work by Phuc. For local regularity, up to the flat part of the boundary, $q = 3$ was established by G.A Seregin. Our result can be viewed as an extension of this to $L^{3,q}$ with q finite. New scale-invariant bounds, refined pressure decay estimates near the boundary and development of a convenient new ϵ-regularity criterion are central themes in providing this extension.

§1. Introduction

In this paper we are going to prove local regularity, up to the flat part of the boundary, for certain classes of weak solutions to the three dimensional Navier Stokes equations. The main assumption is the velocity field belongs to $L_\infty L^{3,q}$ with q finite.

In the local theory, two cases are distinguished: interior cases and boundary cases. The first result, regarding local interior regularity criteria for the Navier–Stokes in terms of the velocity, was established by Serrin in [6]. Later the following generalisation was proven by Struwe in [17].

Theorem 1.1. *Let v be a divergent free vector field defined in the unit parabolic cylinder $Q = B\times]-1,0[$, where B is the unit ball in $\mathbb{R}^3$ centered at the origin. Suppose that v satisfies the three conditions*

$$v \in L_{2,\infty}(Q) \cap W_2^{1,0}(Q), \tag{1.1}$$

$$\int\limits_Q \big(v.\partial_t\phi - v\otimes v : \nabla\phi + \nabla v : \nabla\phi\big)dz = 0 \tag{1.2}$$

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for all smooth solenodial functions ϕ that are compactly supported in Q denoted $C^{\infty}_{0,0}(Q)$, and

$$v \in L_{s,l}(Q), \quad 3/s + 2/l = 1, \quad s > 3. \tag{1.3}$$

Then v is locally bounded in Q.

Before further comment, we give the required notation. Henceforth, " . " defines the scalar product of vectors and " : " the product of tensors. If X is a Banach space with norm $\|\cdot\|_X$, then $L_s(a,b;X)$, $a < b$, means the usual Banach space of strongly measurable X-valued functions $f(t)$ on $]a,b[$ with finite norm

$$\|f\|_{L_s(a,b;X)} := \left(\int\limits_a^b \|f(t)\|^s_X dt \right)^{\frac{1}{s}} < +\infty$$

for $s \in [1,\infty[$, and with the usual modification if $s = \infty$. With this notation if $Q_T = \Omega \times]0,T[$ then

$$L_{s,l}(Q_T) := L_l(0,T;L_s(\Omega)).$$

We define the following Sobolev space with the mixed norm:

$$W^{1,0}_{m,n}(Q_T) = \{v \in L_{m,n}(Q_T) : \|v\|_{L_{m,n}(Q_T)} + \|\nabla v\|_{L_{m,n}(Q_T)} < \infty\}.$$

For $s = 3$ the following local interior regularity result was proven in [3]. Namely the following:

Theorem 1.2. *Suppose that a pair of functions (u,p) satisfies the Navier–Stokes equations in $Q(1)$ in the sense of distributions. In addition, assume that*

$$u \in L_{2,\infty}(Q) \cap L_2(-1,0;W^1_2(B)), \tag{1.4}$$

and

$$p \in L_{\frac{3}{2}}(Q). \tag{1.5}$$

Suppose further that

$$u \in L_{3,\infty}(Q) \tag{1.6}$$

for some $q \in]3,\infty[$. Then the velocity function u is Holder continuous on $\bar{Q}(1/2)$.

Here we define:

$$\begin{aligned}B(x_0,R)&=\{x\in\mathbb{R}^3:|x-x_0|<R\},\\ B(\theta)&=B(0,\theta),\;\; B=B(1),\\ Q(z_0,R)&=B(x_0,R)\times]t_0-R^2,t_0[,\;\; z_0=(x_0,t_0),\\ Q(\theta)&=Q(0,\theta).\end{aligned}$$

Theorem 1.2 is was proven in [3] by ad absurdum. A suitable rescaling and limiting procedure is performed and gives a nontrivial solution to Navier–Stokes equations in $\mathbb{R}^3\times]-\infty,0[$. Then contradiction is then obtained by proving a Liouville type theorem involving backward uniqueness for a certain class of parabolic operators. Subsequently, a version of Theorem 1.2 was proven up to the flat part of the boundary in [24]. This was also done near the curved part of the boundary in [18]. In the context of Lorentz spaces, the interior result is proven in [27]. Namely the assumptions and statement proved are the same as in Theorem 1.2 except (1.6) is replaced by

$$u\in L_\infty(-1,0;L^{3,q}(B)) \tag{1.7}$$

with q finite. Recently, a version is proven in [19] but with the additional restriction that

$$p\in L_2(-1,0;L_1(B)). \tag{1.8}$$

We will first present a different proof to [27] of Theorem 1.2 under the assumption that $u\in L_\infty(-1,0;L^{3,q}(B))$. The contradiction argument has the same spirit as that used in [3, 19] and [27]. The difference is the development of new estimates of certain scale invariant quantities associated with the pressure and velocity, these may be of independent interest. Our reasons for first presenting a different proof of the known interior result are two fold. Firstly the estimates of scale invariant quantities used in our version, mostly carry over (with some adjustment) to the case involving the flat part of the boundary. Secondly, the interior result is an important prerequisite for proving the boundary case.

Now, we can state our main goal of proving local regularity up to flat part of the boundary for nonendpoint Lorentz spaces.

Theorem 1.3. *Let a pair of functions v and p have the following differentiability properties*:

$$v\in L_{2,\infty}(Q^+(2))\cap W^{1,0}_2(Q^+(2)),\quad p\in L_{\frac32}(Q^+(2)). \tag{1.9}$$

Suppose that v and p satisfies the Navier–Stokes equations

$$\partial_t v + \operatorname{div}\, v \otimes v - \Delta v = -\nabla p, \ \operatorname{div}\, v = 0 \tag{1.10}$$

in $Q^+(2)$ in the sense of distributions along with the boundary condition

$$v(x,t) = 0, \ \ x \in \Gamma(2) \text{ and } -4 < t < 0. \tag{1.11}$$

Assume, in addition, that there exists $3 \leqslant q < \infty$ such that

$$v \in L_\infty(-4,0; L^{3,q}(B^+(2)). \tag{1.12}$$

Then v is Holder continuous in the closure of the set $Q^+(1/2)$.

We explain some notation. Setting $x' = (x_1, x_2) \in \mathbb{R}^2$, we introduce the following definitions:

$$B^+(x_0,R) = \{x \in B(x_0,R) : x = (x', x_3), \ x_3 > x_{03}\},$$
$$B(\theta) = B(0,\theta), \ \ B = B(1) \ \ B^+(\theta) = B^+(0,\theta), \ \ B^+ = B^+(1),$$
$$\Gamma(x_0,R) = \{x \in B(x_0,R) : x_3 = x_{30}\}, \quad \Gamma(\theta) = \Gamma(0,\theta), \quad \Gamma = \Gamma(1),$$
$$Q^+(z_0,R) = B^+(x_0,R) \times]t_0 - R^2, t_0[,$$
$$Q^+(\theta) = Q^+(0,\theta), \ \ Q^+ = Q^+(1).$$

Before commenting further let us define the Lorentz spaces. For a measurable function $f : \Omega \to \mathbb{R}$ define:

$$d_{f,\Omega}(\alpha) := |\{x \in \Omega : |f(x)| > \alpha\}|. \tag{1.13}$$

Given a measurable subset $\Omega \subset \mathbb{R}^n$, the Lorentz space $L^{p,q}(\Omega)$ (with $p \in]0,\infty[$, $q \in]0,\infty]$) is the set of all measurable functions g on Ω such that the quasinorm $\|g\|_{L^{p,q}(\Omega)}$ is finite. Here:

$$\|g\|_{L^{p,q}(\Omega)} := \Big(p \int\limits_0^\infty \alpha^q d_{g,\Omega}(\alpha)^{\frac{q}{p}} \frac{d\alpha}{\alpha}\Big)^{\frac{1}{q}}, \tag{1.14}$$

$$\|g\|_{L^{p,\infty}(\Omega)} := \sup_{\alpha>0} \alpha d_{g,\Omega}(\alpha)^{\frac{1}{p}}. \tag{1.15}$$

It is well known that for $q \in]0,\infty[$, $q_1 \in]0,\infty]$ and $q_2 \in]0,\infty]$ with $q_1 < q_2$ we have the embedding $L^{p,q_1} \hookrightarrow L^{p,q_2}$ and the inclusion is known to be strict. Roughly speaking, the second index of Lorentz spaces gives information regarding nature of logarithmic bumps. For example, using decreasing rearrangements, it can be verified that for any $1 > \beta > 0, q > 3$ we have

$$|x|^{-1} |\log(|x|^{-1})|^{-\beta} \chi_{|x|<1}(x) \in L^{3,q}(\mathbb{R}^3) \quad \text{if and only if} \quad q > \frac{1}{\beta}. \tag{1.16}$$

In this way Theorem 1.3 gives a strengthening of the previous result obtained in [24]. It should be stressed that, at the time of writing, the question of interior regularity is open for the critical norm $L^{3,\infty}(B)$ that contains $|x|^{-1}$. We mention that interior regularity results, that have a smallness condition on $L_\infty(L^{3,\infty})$ norm, have been obtained in [7, 9] and [16], for example.

Now we can explicitly describe the challenges presented by the boundary, in order to motivate the method used. The proof also proceeds by contradiction and uses the proof of a Liouville type theorem via backward uniqueness. The major differences lie in the treatment of the pressure. Unlike the interior case, if we scale and blow up the Navier–Stokes equations at singular boundary points we do not know if we can obtain a limiting pressure for the boundary case that lies in the space $L_\infty(-\infty,0;L^{\frac{3}{2},\frac{q}{2}}(\mathbb{R}^3_+))$. Unfortunately, we cannot even show that there is a reasonable global norm of the limiting pressure which is finite. In our investigation, we were only able to demonstrate that the limiting pressure has the same integrability, on compact space-time subsets of $\mathbb{R}^3_+\times]-\infty,0[$, as for the original pressure. This creates major difficulties with the epsilon regularity used in the the interior regularity result in Lorentz spaces, presented in [19]. This criteria needs the limiting pressure to have more local integrability in time than that assumed for the original pressure in Theorem 1.3. Our investigation is in the same spirit as that of [24]. The main differences are that we have to strengthen Lemmas on the decay of the pressure (Proposition 2.5, 2.6 of that paper), together with the development of a convenient epsilon regularity criteria for interior regularity of suitable weak solutions. Both of these points use the scale invariant estimates used in our version of the proof of the corresponding interior result. The epsilon regularity criteria may be of independent interest, as it provides a strengthening to the statement given in [12] and [15].

§2. Local interior regularity in nonendpoint borderline Lorentz spaces

Here is the explicit statement, which we provide a proof of. An alternative proof can be found in [27].

Theorem 2.1. *Suppose that a pair of functions (u,p) satisfies the Navier–Stokes equations in $Q(1)$ in the sense of distributions. In addition, assume* *that*

$$u \in L_{2,\infty}(Q) \cap L_2(-1,0;W^1_2(B)), \tag{2.1}$$

and

$$p \in L_{\frac{3}{2}}(Q). \tag{2.2}$$

Suppose further that

$$u \in L_\infty(-1,0;L^{3,q}(B)) \tag{2.3}$$

for some $q \in]3,\infty[$. *Then the velocity function* u *is Holder continuous on* $\bar{Q}(1/2)$.

We briefly recap the definition of a suitable weak solution to the Navier–Stokes equations given by Lin in [15]. Here it is.

Definition 2.2. *Let* ω *be an open set in* $\mathbb{R}^3$. *We say that a pair* v *and* p *are a suitable weak solution to the Navier–Stokes equations on the set* $\omega\times]-T_1,T[$ *if they satisfy the Navier Stokes equations in the sense of distributions. Moreover they are required to satisfy the following conditions.*

$$v \in L_{2,\infty}(\omega\times]T_1,T[) \cap L_2(-T_1,T;W^1_2(\omega)), \tag{2.4}$$

$$p \in L_{\frac{3}{2}}(\omega\times]-T_1,T[). \tag{2.5}$$

For a.a $t \in]-T_1,T[$ *and for all non negative cut-off functions* $\phi \in C^\infty_0(\mathbb{R}^4)$ *vanishing in a neighbourhood of the parabolic boundary*

$$\partial^{'}Q = \omega \times \{t=-T_1\} \cup \partial\omega \times [-T_1,T],$$

v *and* p *satisfy the local energy inequality*

$$\int\limits_{\omega} \phi(x,t)|v(x,t)|^2dx + 2\int\limits_{\omega\times]-T_1,t[} \phi|\nabla v|^2dxdt^{'}$$
$$\leqslant \int\limits_{\omega\times]-T_1,t[} [|v|^2(\partial_t\phi+\Delta\phi)+v.\nabla\phi(|v|^2+2p)]dxdt^{'}. \tag{2.6}$$

Let us proceed with a Lemma. The analogous Lemma (Lemma 4.1) was stated and proven in [19]. As the proof of this statement is essentially unchanged we omit it.

Lemma 2.3. *Suppose that the pair of functions* (u,p) *satisfy the hypothesis of Theorem* 2.1. *Then* (u,p) *forms a suitable weak solution to Navier–Stokes equations in* $Q(5/6)$ *with a generalized energy inequality and furthermore* $u \in L_4(Q)$. *Moreover the inequality*

$$\|u(\,\cdot\,,t)\|_{L^{3,q}(B(3/4))} \leqslant \|u\|_{L_\infty(-(3/4)^2,0;L^{3,q}(B(3/4)))}, \tag{2.7}$$

holds for all $t \in [-(3/4)^2, 0]$, *and the function*

$$t \to \int\limits_{B(3/4)} u(x,t)w(x)\,dx$$

is continuous on $[-(3/4)^2, 0]$ *for any* $w \in L^{\frac{3}{2}, \frac{q}{q-1}}(B(3/4))$.

Next, we recap the rescaling prodecure for the proof of Theorem 2.1 used by Escauriza, Seregin, Sverak in [3]. Suppose the conditions for (u,p) of Theorem 2.1 hold. Then by the previous Lemma (u,p) form a suitable weak solution to the Navier–Stokes equations in $Q(5/6)$ and for $t \in [-(3/4)^2, 0]$ we have

$$u(\,\cdot\,,t) \in L^{3,q}(B(3/4)). \tag{2.8}$$

The rescaling procedure arises from assuming Theorem 2.1 is false. Thus, u has no representative that is Holder continuous on $\bar{Q}(1/2)$. This implies that there exists a singular point $z_0 \in \bar{Q}(1/2)$ such that there is no parabolic neighbourhood of O_{z_0} of z_0 where u has a Holder continuous representative on $O_{z_0} \cap Q$. By Lemma 3.3 of [22], there exists a universal constant $c_0 > 0$ and a sequence of numbers $R_k \in]0,1[$ such that $R_k \to 0$ as $k \to +\infty$ and

$$A(z_0, R_k; u) = \sup_{t_0 - R_k^2 \leqslant s \leqslant t_0} \frac{1}{R_k} \int\limits_{B(x_0,R_k)} |u(x,s)|^2 dx \geqslant c_0 \tag{2.9}$$

for any $k = 1, 2, \ldots$.

For each $\Omega = \omega \times]a,b[$, where $\omega \Subset \mathbb{R}^3$ and $-\infty < a < b \leqslant 0$, we choose a large $k_0 = k_0(\Omega) \geqslant 1$ such that for all $k \geqslant k_0$ we have for $(x,t) \in \Omega$:

$$x_0 + R_k x \in B(2/3),$$

and

$$t_0 + R_k^2 t \in]-(2/3)^2, 0[.$$

Given such an Ω we perform the Navier–Stokes scaling as follows:

$$u^k(x,t) := R_k u(x_0 + R_k x, t_0 + R_k^2 t), \tag{2.10}$$

$$p^k(x,t) := R_k^2 p(x_0 + R_k x, t_0 + R_k^2 t). \tag{2.11}$$

As done in [19] we may decompose the pressure

$$p = \tilde{p} + h, \tag{2.12}$$

where h is harmonic in B, and $\tilde{p} := R_i R_j[(u_1 u_j)\chi_B]$. It is clear that

$$p^k = \tilde{p^k} + h^k. \tag{2.13}$$

Here,

$$\tilde{p^k}(x,t) := R_k^2\tilde{p}(x_0 + R_k x, t_0 + R_k^2 t), \quad h^k(x,t) := R_k^2\tilde{h}(x_0 + R_k x, t_0 + R_k^2 t)$$

for any $(x,t) \in \Omega$ and $k \geqslant k_0(\Omega)$. Now under the hypothesis of Theorem 2.1, Lemma 2.3 gives that (u,p) is a suitable weak solution to the Navier Stokes equations in $Q(5/6)$. It is not difficult to see that this implies (u^k, p^k) is a suitable weak solution to the Navier Stokes equations in Ω.

Now define the relevant following scale invariant functional $(0 < r < 1)$

$$A(z_0, r; u) := \sup_{t_0 - r^2 \leqslant t \leqslant t_0} r^{-1} \int\limits_{B(x_0,r)} |u(x,t)|^2\, dx, \tag{2.14}$$

$$B(z_0, r; u) := r^{-1} \int\limits_{Q(z_0,r)} |\nabla u(x,t)|^2\, dx\, dt, \tag{2.15}$$

$$C_\infty(z_0, r; u) := r^{-2} \int\limits_{t_0 - r^2}^{t_0} \|u\|^3_{L^{3,\infty}(B(x_0,r))}\, dt, \tag{2.16}$$

$$D_\infty(z_0, r; p) := r^{-2} \int\limits_{t_0 - r^2}^{t_0} \|p\|^{\frac{3}{2}}_{L^{\frac{3}{2},\infty}(B(x_0,r))}\, dt. \tag{2.17}$$

Now let us state a new estimate, which may be of independent interest, that we use heavily in Theorem 2.1 and Theorem 1.3. The proof of this is contained in the Appendix.

Lemma 2.4. *Let (u,p) be a suitable weak solution in $Q(z_0, 1)$. Then for $0 < r < 1$ the following holds (c is some universal constant):*

$$A(z_0, r/2; u) + B(z_0, r/2; u) \leqslant c(C_\infty(z_0, r; u)^{\frac{4}{3}} + C_\infty(z_0, r; u)^{\frac{2}{3}} + D_\infty(z_0, r; p)^{\frac{4}{3}}). \tag{2.18}$$

Now, let us make more explicit the role that this estimate plays by means of a Proposition.

Proposition 2.5. *The rescaled velocity and pressure the following uniform estimates for $k \geqslant k_0(\Omega)$:*

$$\|p^k\|_{L_{\frac{3}{2}}(a,b;L^{\frac{3}{2},\frac{q}{2}}(\omega))} \leqslant C(\Omega)[\|p\|_{L_{\frac{3}{2}}(Q)} + \|u\|^2_{L_\infty(-1,0;L^{3,q}(B))}], \tag{2.19}$$

$$\int\limits_a^b \sup_\omega |h^k(x,t)|^{\frac{3}{2}}dt \leqslant CR_k\big(\|p\|^{\frac{3}{2}}_{L_{\frac{3}{2}}(Q)} + \|u\|^3_{L_\infty(-1,0;L^{3,q}(B))}\big), \qquad (2.20)$$

$$\|u^k(\,\cdot\,,t)\|_{L^{3,q}(\omega)} \leqslant \|u\|_{L_\infty(-1,0;L^{3,q}(B))} \qquad (2.21)$$

and

$$\|u^k\|_{L_\infty(a,b;L_2(\omega)}+\|\nabla u^k\|_{L_2(a,b;L_2(\omega)} \leqslant C(\Omega, \|p\|_{L_{\frac{3}{2}}(Q)}, \|u\|_{L_\infty(-1,0;L^{3,q}(B))}) \qquad (2.22)$$

for all sufficiently large k depending only on Ω.

Proof. It is clear (2.21) follows from Lemma 2.3 along with the easily seen property that $u_0 \in L^{3,q}(\Omega)$ and $u_{0,\lambda}(x) = \lambda u_0(\lambda x)$ implies

$$\|u_{0,\lambda}\|_{L^{3,q}(\Omega/\lambda)} = \|u_0\|_{L^{3,q}(\Omega)}.$$

Such spaces are called critical spaces for the Navier–Stokes equations. Clearly, (2.22) follows from Lemma 2.4 having established (2.19)–(2.21). Hence, we focus only on proving (2.19)–(2.20). Note that we use the notation from (2.12)–(2.13). Lemma 2.3 and inverse Navier–Stokes scaling gives us the following for $t \in]a,b[$:

$$\|\tilde{p^k}(\,\cdot\,,t)\|_{L^{\frac{3}{2},\frac{q}{2}}(\omega)} \leqslant \|\tilde{p}(\,\cdot\,,t_0+R_k^2t)\|_{L^{\frac{3}{2},\frac{q}{2}}(B(\frac{2}{3}))} \leqslant C\|u\|^2_{L_\infty(-1,0;L^{3,q}(B))}. \qquad (2.23)$$

Here, we used the well known fact that Calderon Zygmund singular integral operators are bounded linear operators on $L^{p,q}$ with $p \in]1,\infty[$ and $q \in]0,\infty]$. Using (2.23) we see it is sufficient to only prove (2.20). Since h is a harmonic function in B for a.a $t \in]-1,0[$, we have

$$\|h\|_{L_{\frac{3}{2}}(-1,0;L_\infty(B(\frac{3}{4}))} \leqslant C\|h\|_{L_{\frac{3}{2}}(-1,0;L_1(B))} \leqslant C(\|\tilde{p}\|_{L_{1,\frac{3}{2}}(Q)} + \|p\|_{L_{\frac{3}{2}}(Q)}) \qquad (2.24)$$

$$\leqslant C(\|\tilde{p}\|_{L_{\frac{3}{2}}(-1,0;L^{\frac{3}{2},\frac{q}{2}}(B))}+\|p\|_{L_{\frac{3}{2}}(Q)}) \leq c(\|p\|_{L_{\frac{3}{2}}(Q)} + \|u\|^2_{L_\infty(-1,0;L^{3,q}(B))}). \qquad (2.25)$$

In the above, we used what is known as O'Neil's inequality. Namely if $0 < p,q,r \leqslant \infty$, $0 < s_1, s_2 \leqslant \infty$, $\frac{1}{p}+\frac{1}{q}=\frac{1}{r}$ and $\frac{1}{s_1}+\frac{1}{s_2}=\frac{1}{s}$ then:

$$\|fg\|_{L^{r,s}(\Omega)} \leqslant C(p,q,s_1,s_2)\|f\|_{L^{p,s_1}(\Omega)}\|g\|_{L^{q,s_2}(\Omega)}. \qquad (2.26)$$

Using (2.24), one infers

$$\int\limits_a^b \sup_\omega |h^k(x,t)|^{\frac{3}{2}}\,dt \leqslant R_k \int\limits_{-(3/4)^2}^{0} \sup_\omega |h(x_0+R_kx,s)|^{\frac{3}{2}}\,ds$$

$$\leqslant R_k\|h\|^{\frac{3}{2}}_{L_{\frac{3}{2}}(-1,0;L_\infty(B(\frac{3}{4})))} \leqslant CR_k(\|p\|^{\frac{3}{2}}_{L_{\frac{3}{2}}(Q)} + \|u\|^3_{L_\infty(-1,0;L^{3,q}(B))}). \quad (2.27)$$

The remaining part of the proof of Theorem 2.1 is proven in more or less an identical way to the method in [19]. □

§3. Local regularity near the boundary in nonendpoint borderline Lorentz spaces

Define the mixed Sobolev space:

$$W^{2,1}_{m,n}(Q_T) = \{v \in L_{m,n}(Q_T) : \|v\|_{L_{m,n}(Q_T)} + \|\nabla v\|_{L_{m,n}(Q_T)} + \|\nabla^2 v\|_{L_{m,n}(Q_T)} + \|\partial_t v\|_{L_{m,n}(Q_T)} < \infty\}.$$

First we show that the assumptions of Theorem 1.3 immediately give spatial smoothing and a local energy inequality.

Proposition 3.1. *Suppose (v,p) satisfy the hypotheses of Theorem 1.3. Then for $\tau \in]0,1[$ we have*

$$v \in L_4(Q^+(2)) \cap W^{2,1}_{\frac{9}{8},\frac{3}{2}}(Q^+(2\tau)) \cap W^{2,1}_{\frac{4}{3}}(Q^+(2\tau)), \quad (3.1)$$

$$p \in W^{1,0}_{\frac{9}{8},\frac{3}{2}}(Q^+(2\tau)) \cap W^{1,0}_{\frac{4}{3}}(Q^+(2\tau)). \quad (3.2)$$

In addition, the inequality

$$\|v(\,\cdot\,,t)\|_{L^{3,q}(B^+(2\tau)} \leqslant \|v\|_{L_\infty(-4,0;L^{3,q}(B^+(2))} \quad (3.3)$$

holds for all $t \in]-(2\tau)^2,0[$, and the function

$$t \to \int\limits_{B(2\tau)} v(x,t)w(x)\,dx$$

is continuous on $[-(2\tau)^2,0]$ for any $w \in L^{\frac{3}{2},\frac{q}{q-1}}(B^+(2\tau))$.

Moreover, for a.a $t \in]-1,0[$ and for all non negative functions $\phi \in C_0^\infty(\mathbb{R}^4)$, vanishing in a neighbourhood of the parabolic boundary $\partial' Q(2)$

of $Q(2)$, v and p satisfy the inequality

$$\int\limits_{B^+(2)} \phi(x,t)|v(x,t)|^2\,dx + 2 \int\limits_{B^+(2)\times]-2,t[} \phi|\nabla v|^2\,dx\,dt^{'}$$
$$\leqslant \int\limits_{B^+(2)\times]-2,t[} [|v|^2(\partial_t\phi+\Delta\phi)+v.\nabla\phi(|v|^2+2p)]dxdt^{'}. \tag{3.4}$$

Proof. First we remark that (3.4) and (3.3) is a consequence of (3.1)-(3.2), together with arguments used in the previous section on the interior case, so we focus on only proving (3.1)-(3.2). We proceed in a slightly different way to [24]. Indeed we instead use the results regarding local boundary regularity for the Stokes system developed by Seregin in [25]. Under the assumptions of Theorem 1.3 we clearly have that

$$v \in W^{1,0}_{\frac{9}{8},\frac{3}{2}}(Q^+(2)) \cap W^{1,0}_{\frac{4}{3}}(Q^+(2)), \tag{3.5}$$

$$p \in L_{\frac{9}{8},\frac{3}{2}}(Q^+(2)) \cap L_{\frac{4}{3}}(Q^+(2)). \tag{3.6}$$

Clearly from Sobolev embedding we have $v \in L_{6,2}(Q^+(2))$. By a well known characterisation of Lorentz spaces:

$$L_4(B^+(2)) = (L^{3,q}(B^+(2)), L_6(B^+(2)))_{\frac{1}{2},4}. \tag{3.7}$$

The proof of this and notation used in (3.7) can be found in [1], for example. By well known properties of interpolation spaces (see, for example, section 3.5 of [1]):

$$\|v(\,\cdot\,,t)\|_{L_4(B^+(2))} \leqslant C\|v(\,\cdot\,,t)\|^{\frac{1}{2}}_{L_{3,q}(B^+(2))}\|v(\,\cdot\,,t)\|^{\frac{1}{2}}_{L_6(B^+(2))}.$$

Thus,

$$\|v\|_{L_4(Q^+(2))} \leqslant C\|v\|^{\frac{1}{2}}_{L_\infty(-4,0;L^{3,q}(B^+(2))}\|v\|^{\frac{1}{2}}_{W^{1,0}_2(Q^+(2))}. \tag{3.8}$$

By Holder's inequality we obtain

$$\|v.\nabla v\|_{L_{\frac{4}{3}}(Q^+(2))} \leqslant C\|v\|^{\frac{1}{2}}_{L_\infty(-4,0;L^{3,q}(B^+(2))}\|v\|^{\frac{3}{2}}_{W^{1,0}_2(Q^+(2))}. \tag{3.9}$$

It is well known that the assumption $v \in W^{1,0}_2(Q^+(2)) \cap L_{2,\infty}(Q^+(2))$ implies:

$$v.\nabla v \in L_{\frac{9}{8},\frac{3}{2}}(Q^+(2)). \tag{3.10}$$

From here (3.5)–(3.6) combined with (3.9)–(3.10) are enough to obtain the higher regularity. This is seen from the local boundary regularity for the Stokes system, namely Lemma 1.1 of [25]. □

3.1. Pressure estimates. It is first necessary to recap some appropriate definitions and a Lemma that will be necessary for our investigation. The first definition was given by Seregin in [23]. Later on in [25], Seregin gave a more general definition of suitable weak solution near flat part of boundary, but the one stated below is sufficient for our purposes.

Definition 3.2. *A pair of functions v and p is called a suitable weak solution to the Navier Stokes equations in $Q^+(z_0,R)$ near $\Gamma(x_0,R)\times[t_0-R^2,t_0]$ if they satisfy the following conditions. They have the differentiability properties*

$$v\in L_\infty(t_0-R^2,t_0;L_2(B^+(x_0,R))\cap W^{1,0}_2(Q^+(z_0,R))\cap W^{2,1}_{\frac{9}{8},\frac{3}{2}}(Q^+(z_0,R)), \tag{3.11}$$

$$p\in W^{1,0}_{\frac{9}{8},\frac{3}{2}}(Q^+(z_0,R)). \tag{3.12}$$

The pair v and p satisfies the Naiver Stokes equations a.e in $Q^+(z_0,R)$ and the boundary condition

$$v(x,t)=0,\quad x_3=x_{03}\ \textit{and}\quad t_0-R^2<t<t_0. \tag{3.13}$$

For a.a $t\in]t_0-R^2,t_0[$ and for all non negative cut-off functions $\phi\in C^\infty_0(\mathbb{R}^4)$ vanishing in a neighbourhood of the parabolic boundary

$$\partial^{'}Q(z_0,R)=B(x_0,R)\times\{t=t_0-R^2\}\cup\partial B(x_0,R)\times[t_0-R^2,t_0]$$

of the cylinder $Q(z_0,R)$, v and p satisfy the local energy inequality

$$\int\limits_{B^+(x_0,R)}\phi(x,t)|v(x,t)|^2dx+2\int\limits_{B^+(x_0,R)\times]t_0-R^2,t[}\phi|\nabla v|^2\,dx\,dt^{'}$$

$$\leqslant\int\limits_{B^+(x_0,R)\times]t_0-R^2,t[}[|v|^2(\partial_t\phi+\Delta\phi)+v.\nabla\phi(|v|^2+2p)]dxdt^{'}. \tag{3.14}$$

Before stating and proving a certain Lemma, let us introduce some notation. Various mean values of integrable functions are denoted as follows

$$[p]_\Omega=\frac{1}{|\Omega|}\int\limits_\Omega p(x,t)\,dx,$$

$$(v)_\omega=\frac{1}{\omega}\int\limits_\omega v\,dz.$$

Take $q \in [3,\infty]$ and introduce the following scale invariant quantities

$$C_q(z_0,r;v) := \frac{1}{r^2}\int\limits_{t_0-r^2}^{t_0} \|v(\,\cdot\,,t)\|^3_{L^{3,q}(B(x_0,r))}dt \tag{3.15}$$

$$D_q(z_0,r;p) := \frac{1}{r^2}\int\limits_{t_0-r^2}^{t_0} \|p(\,\cdot\,,t)-[p]_{B(x_0,r)}(t)\|^{\frac{3}{2}}_{L^{\frac{3}{2},\frac{q}{2}}(B(x_0,r))}dt \tag{3.16}$$

The following generalises Lemma 2.1 of [24]. Let us Remark that the Lemma and proof is very similar to Lemma 3.1 in [21]. As it is slightly different we outline a proof in the Appendix, for the convenience of the reader. Here it is:

Lemma 3.3. *Let $v \in L_3(Q(z_0,R))$ and $p \in L_{\frac{3}{2}}(Q(z_0,R))$ satisfy the Navier Stokes equations in the sense of distributions. Then, for $0 < r \leqslant \rho \leqslant R$, we have*

$$D_q(z_0,r;p) \leq c\Big[\Big(\frac{r}{\rho}\Big)^{\frac{5}{2}} D_q(z_0,\rho;p) + \Big(\frac{\rho}{r}\Big)^2 C_q(z_0,\rho;v)\Big]. \tag{3.17}$$

This enables us to prove a certain generalisation to Proposition 2.5 in [24]. Namely the following.

Proposition 3.4. *Assume all the criteria of Lemma 3.3 are fulfilled. And let, in addition, for $q \in [3,\infty[$, that*

$$\|v\|_{L_\infty(t_0-R^2,t_0;L^{3,q}(B(x_0,R))} \leqslant L < \infty. \tag{3.18}$$

Then, for any $\gamma \in]0,1[$, there exists a constant $c_1(L,\gamma)$ such that, for $0 < r \leqslant R$, we have

$$D_q(z_0,r;p) \leqslant c_1\left[\Big(\frac{r}{R}\Big)^{\frac{5}{2}\gamma} D_q(z_0,R;p) + 1\right]. \tag{3.19}$$

Proof. Clearly from (3.3) we have

$$D_q(z_0,\tau^{k+1}R;p) \leqslant c\left[\tau^{\frac{5}{2}} D_q(z_0,\tau^k R;p) + \frac{L^3}{\tau^2}\right] \tag{3.20}$$

for any $0 < \tau < 1$. We choose τ such that $c\tau^{\frac{5}{2}(1-\gamma)} \leq 1$. The proof then follows immediately from iterating (3.20). □

Before introducing the relevant statements we are required to introduce further notation. Firstly, quantities already defined but with $'+'$ superscript denote integration over half-balls. For example,

$$A^+(z_0,r;u) := \sup_{t_0-r^2\leqslant t\leqslant t_0} r^{-1} \int\limits_{B^+(x_0,r)} |u(x,t)|^2\,dx.$$

Now define the following:

$$D_1^+(z_0,r;p) := \frac{1}{r^{\frac{3}{2}}} \int\limits_{t_0-r^2}^{t_0} \Big(\int\limits_{B^+(x_0,r)} |\nabla p|^{\frac{9}{8}} dx\Big)^{\frac{4}{3}}\,dt. \tag{3.21}$$

The following Lemma was proven in [23] and is Lemma 7.2 there. We state it without proof.

Lemma 3.5. *Consider a pair of functions v and p that is a suitable weak solution to the Navier Stokes equations in $Q^+(z_0,R)$ near $\Gamma(x_0,R)\times[t_0-R^2,t_0]$. Then for any $0<r\leqslant\rho\leqslant R$, we have*

$$D_1^+(z_0,r;p) \leqslant c\Big\{ \Big(\frac{r}{\rho}\Big)^2 [D_1^+(z_0,\rho;p) + (B^+(z_0,\rho;v))^{\frac{3}{4}}] + \Big(\frac{\rho}{r}\Big)^{\frac{3}{2}} (A^+(z_0,\rho;v))^{\frac{1}{2}} B^+(z_0,\rho;v)\Big\}. \tag{3.22}$$

Next we state a new estimate. The statement and proof in the Appendix is for the interior case, but there is no distinction in the proof for the flat part of the boundary. It is very similar to Lemma 2.4, and is a key part that allows us to work in the context of nonendpoint critical Lorentz spaces.

Lemma 3.6. *Let (u,p) be a suitable weak solution in $Q^+(z_0,1)$ near $\Gamma(x_0,1)\times[t_0-1,t_0]$. Then for $0<r<1$ the following holds (c is some universal constant):*

$$A^+(z_0,r/2;u) + B^+(z_0,r/2;u) \leqslant c\Big(C_\infty^+(z_0,r;u)^{\frac{4}{3}} + C_\infty^+(z_0,r;u)^{\frac{2}{3}} + D_1^+(z_0,r;p)^{\frac{2}{3}} C_\infty^+(z_0,r;u)^{\frac{1}{3}}\Big). \tag{3.23}$$

Now we can prove a generalisation of Proposition 2.6 in [24].

Proposition 3.7. *Let $q\in[3,\infty[$. Assume that (v,p) is a suitable weak solution in $Q^+(z_0,R)$ near $\Gamma(x_0,R)\times[t_0-R^2,t_0]$. Suppose, in addition,*

$$\|v\|_{L_\infty(t_0-R^2,t_0;L^{3,q}(B(z_0,R)))} \leqslant L < \infty. \tag{3.24}$$

Then, for any $\gamma \in]0,1[$, there exists a constant c_2 depending on γ and L only such that, for $0 < r \leqslant R$, we have

$$D_1^+(z_0, r; p) \leqslant c_2 \left[\left(\frac{r}{R}\right)^{2\gamma} D_1^+(z_0, R; p) + 1 \right]. \tag{3.25}$$

Proof. Let $\rho \leqslant \frac{R}{2}$. Then by Lemma 3.6 and (3.24) we have

$$A^+(z_0, \rho; v) + B^+(z_0, \rho; v) \leq c[L^4 + L^2 + (D_1^+(z_0, 2\rho; p))^{\frac{2}{3}} L].$$

By O'Neils inequality for $0 \leqslant s \leqslant R$:

$$\int\limits_{B(x_0,s)} |v(x,t)|^2 \, dx \leqslant C|B(0,s)|^{\frac{1}{3}} \| |v|^2(\,\cdot\,,t)\|_{L^{\frac{3}{2},\infty}(B(x_0,s))}.$$

Thus, it is clear that

$$A^+(z_0, s; v) \leqslant CL^2. \tag{3.26}$$

Thus, using these facts with by Lemma 3.5, we see that for any $0 < r \leqslant \rho \leqslant \frac{R}{2}$:

$$D_1^+(z_0, r; p) \leq c \left(\frac{r}{\rho}\right)^2 [D_1^+(z_0, \rho; p) + ((L^4 + L^2 + (D_1^+(z_0, 2\rho; p))^{\frac{2}{3}} L)^{\frac{3}{4}}]$$

$$+ \left(\frac{\rho}{r}\right)^{\frac{3}{2}} L[L^4 + L^2 + (D_1^+(z_0, 2\rho; p))^{\frac{2}{3}} L]\}.$$

Then by Young's inequality, it is not so difficult to see for $0 < r \leqslant \rho \leqslant \frac{R}{2}$:

$$D_1^+(z_0, r; p) \leqslant c(L) \left[\left(\frac{r}{\rho}\right)^2 (D_1(z_0, 2\rho; p) + 1) + \left(\frac{\rho}{r}\right)^{\frac{17}{2}} \right].$$

Clearly this implies that for $0 < r \leqslant \rho \leqslant R$ that

$$D_1^+(z_0, r; p) \leqslant c(L) \left[\left(\frac{r}{\rho}\right)^2 (D_1(z_0, \rho; p) + 1) + \left(\frac{\rho}{r}\right)^{\frac{17}{2}} \right].$$

The conclusion now follows by identical reasoning as Proposition 3.4. □

3.2. Proof of Theorem 1.3. We let

$$L=\|v\|_{L_\infty(-4,0;L^{3,q}(B^+(2)))}<\infty.$$

Let us describe the rescaling procedure taken from [24]. Assume Theorem 1.3 is false. Let $z_0\in\bar{Q}^+(1/2)$ be a singular point. Then we know that all conclusions of Proposition 3.1 hold. Hence, (v,p) form a suitable weak solution to the Navier Stokes equations in $Q^+(1)$ near $\Gamma(0,1)\times[-1,0]$. Furthermore for every $t\in[-1,0]$

$$\|v(\,\cdot\,,t)\|_{L^{3,q}(B^+(1))}\leqslant L. \tag{3.27}$$

From Theorem 2.1 we can only have z_0 lies on the boundary $\bar{\Gamma}(0,1/2)$. Without loss of generality (translation invariance), we assume that $z_0=0$ and that all conclusions of Proposition 3.1 (together with (3.27)) hold on the slightly smaller domain $Q^+(1/2)$.

As a consequence of Lemma 3.3, proven in [24], there exists a decreasing sequence $R_k<\frac{1}{2}$ tending to zero, together with a universal constant ϵ_3, such that for $k=1,2\ldots$

$$\frac{1}{R_k^2}\int\limits_{Q^+(R_k)}|v|^3dz\geqslant\epsilon_3. \tag{3.28}$$

Extending functions (v,p) outside $Q^+(1/2)$ to zero, for

$$(y,s)\in\mathbb{R}^3_+\times]-\infty,0[$$

define the rescaled functions

$$u^k(y,s)=R_kv(R_ky,R_k^2s),\quad p^k(y,s)=R_k^2p(R_ky,R_k^2s).$$

Next we claim the following properties in the limit:

Proposition 3.8. *There exists a subsequence, still denoted by,* (u^k,p^k), *and a pair of functions* (u_∞,p_∞) *with* $\operatorname{div}u_\infty=0$ *in* $\mathbb{R}^3_+\times]-\infty,0[$, *such that*

$$u^k\overset{*}{\rightharpoonup}u_\infty\quad in\quad L_\infty(-\infty,0;L^{3,q}(\mathbb{R}^3_+)). \tag{3.29}$$

Moreover for any $a>0$

$$|u_\infty|^2,\,\nabla u_\infty\in L_2(Q^+(a)). \tag{3.30}$$

Additionally,

$$(u_\infty,p_\infty)\in W^{2,1}_{\frac{4}{3}}(Q^+(a))\cap W^{2,1}_{\frac{9}{8},\frac{3}{2}}(Q^+(a))\times W^{1,0}_{\frac{4}{3}}(Q^+(a))\cap W^{1,0}_{\frac{9}{8},\frac{3}{2}}(Q^+(a)), \tag{3.31}$$

and (u_∞, p_∞) forms a suitable weak solution to the Navier–Stokes equations in $Q^+(a)$ near $\Gamma(a)\times[-a^2,0]$. One also has that

$$u^k \to u_\infty \quad in \quad C([-a^2,0]; L_s(B^+(a)), \;\; u_\infty(x,0)=0 \tag{3.32}$$

for any $s\in]1,3[$ and for a.a $x\in\mathbb{R}^3_+$. Furthermore, u_∞ satisfies the lower bound

$$\int\limits_{Q^+} |u_\infty|^3 dz \geqslant \epsilon_3. \tag{3.33}$$

Proof. Fix $a>0$ and let $k(a)$ be such that

$$R_k a < \frac{1}{8} \tag{3.34}$$

for all $k \geqslant k(a)$. It is clear that (3.29) follows from identical reasons as discussed in the interior case, and in addition

$$\|u_\infty\|_{L_\infty(-\infty,0;L^{3,q}(\mathbb{R}^3_+))} \leqslant \sup_k \|u_k\|_{L_\infty(-\infty,0;L^{3,q}(\mathbb{R}^3_+))} = L < \infty. \tag{3.35}$$

By inverse scaling and the same reasons discussed in Proposition 3.7 we have

$$\begin{aligned} B^+(0,2a;u^k) + D_1^+(0,2a,p^k) &= B^+(0,2aR_k;v) + D_1^+(0,2aR_k,p) \\ &\leqslant c_0(L)[1 + D_1^+(0,4aR_k,p)]. \end{aligned} \tag{3.36}$$

By Proposition 3.7

$$D_1^+(0,4aR_k,p) \leqslant c_1(L)[8aR_k D_1^+(0,1/2,p)+1] \leqslant c_1(L)[D_1^+(0,1/2,p)+1].$$

Thus,

$$B^+(0,2a;u^k) + D_1^+(0,2a,q^k) \leqslant c_2(L)[D_1^+(0,1/2,p)+1]. \tag{3.37}$$

Since,

$$L_4(B^+(a)) = (L_{3,q}(B^+(a)), L_6(B^+(a)))_{\frac{1}{2},4},$$

we have the interpolative inequality

$$\begin{aligned} \|u^k(\cdot,t)\|_{L_4(B^+(2a))} &\leq C\|u^k(\cdot,t)\|^{\frac{1}{2}}_{L^{3,q}(B^+(2a))}\|u^k(\cdot,t)\|^{\frac{1}{2}}_{L_6(B^+(2a))} \\ &\leqslant CL^{\frac{1}{2}}\|u^k(\cdot,t)\|^{\frac{1}{2}}_{L_6(B^+(2a))}. \end{aligned}$$

From here, it follows that

$$\|u^k\|_{L_4(Q^+(2a))} \leqslant c_0(L,a)(D_1^+(0,1/2,p)+1)^{\frac{1}{4}}. \tag{3.38}$$

Observe that

$$\|u^k.\nabla u^k\|_{L_{\frac{4}{3}}(Q^+(2a))} \leqslant \|u^k\|_{L_4(Q^+(2a))}\|\nabla u^k\|_{L_2(Q^+(2a))}$$

$$\leqslant c_1(L,a)(D_1^+(0,1/2,p)+1)^{\frac{3}{4}}. \tag{3.39}$$

Using multiplicative inequalities it is not so hard to see

$$\|u^k.\nabla u^k\|_{L_{\frac{9}{8},\frac{3}{2}}(Q^+(2a))} \leqslant C_2(a,L)(B^+(0,2a,u^k)^{\frac{2}{3}}$$

$$\leqslant C_3(a,L)(D_1^+(0,1/2,p)+1)^{\frac{2}{3}}. \tag{3.40}$$

One can then use identical arguments to those used in [24] to show that

$$\|u^k\|_{W^{2,1}_{\frac{4}{3}}(Q^+(a))} + \|\nabla p^k\|_{L_{\frac{4}{3}}(Q^+(a))} \leqslant c_4(a,L,D_1^+(0,1/2,p)), \tag{3.41}$$

$$\|u^k\|_{W^{2,1}_{\frac{9}{8},\frac{3}{2}}(Q^+(a))} + \|\nabla p^k\|_{L_{\frac{9}{8},\frac{3}{2}}(Q^+(a))} \leqslant c_5(a,L,D_1^+(0,1/2,p)). \tag{3.42}$$

Having obtained this, (3.29)-(3.31) and (3.33) follow in the same way as presented in [24]. Let us focus on (3.32). By (3.41) and compactness of the embedding

$$W^{2,1}_{\frac{4}{3}}(Q^+(a)) \hookrightarrow C([-a^2,0];L_{\frac{4}{3}}(B^+(a)),$$

we have, taking further subsequences and using further cantor diagonalisation if necessary, that for all $a>0$

$$u^k \to u_\infty \quad \text{in} \quad C([-a^2,0];L_{\frac{4}{3}}(B^+(a)). \tag{3.43}$$

For $s\in]\frac{4}{3},3[$ there exists $\theta(s)\in]0,1[$ such that $L_s=(L_{\frac{4}{3}},L^{3,q})_{\theta(s),s}$. Using this and fact that u^k and u are uniformly bounded in

$$L_\infty(-a^2,0;L^{3,q}(B^+(a))$$

it is simple to infer, by interpolative inequalities, that

$$u^k \to u \text{ in } \mathrm{C([-a^2,0];L_s(B^+(a))}.$$

For any $y\in B^+(a)$, using O'Neil's inequality and inverse scailing gives us

$$\int\limits_{B^+(y,1)} |u_\infty(x,0)|dx \leqslant \int\limits_{B^+(y,1)} |u_\infty(x,0)-u^k(x,0)|\,dx + \int\limits_{B^+(y,1)} |u^k(x,0)|\,dx$$

$$\leqslant \|u_\infty-u^k\|_{C([-(a+1)^2,0];L_1(B^+(a+1))} + |B_1(0)|^{\frac{2}{3}}\|v(\,\cdot\,,0)\|_{L^{3,q}(B^+(R_ky,R_k))}.$$

Then using Proposition 3.1 and the dominated convergence theorem, applied to the distribution function of $v(\,\cdot\,,0)$, we see that

$$\int\limits_{B^+(1)} |u_\infty(x,0)|\,dx = 0. \qquad \square$$

It is necessary to introduce the following notation. Let $i_3 = (0,0,1)$. The following is a generalisation of a result in [24] (Lemma 4.1 there) . Here it is.

Lemma 3.9. *There exists a positive constant* $c_5(L, D_1^+(0,1/2,p))$ *with the following property. Fix* $h>0$ *and* $T>0$ *arbitrarily, then*

$$D_q(e_0, 2h; p_\infty) \leqslant c_5 \tag{3.44}$$

for any $e_0 = (y_0, s_0) \in (\mathbb{R}^3_+ + 3hi_3)\times]-T,0[$.

Proof. Let a be sufficiently large such that $Q(e_0,2h) \in Q^+(a)$. From Proposition 3.8 and the Poincare inequality we have that there exists $C_a(t) \in L_{\frac{3}{2}}(-a^2,0)$ such that

$$p^k - [p^k]_{B^+(a)}(t) \rightharpoonup p_\infty + C_a(t) \text{ in } \mathrm{L}_{\frac{3}{2}}(-\mathrm{a}^2, 0; \mathrm{L}^{\frac{3}{2},\frac{9}{2}}(\mathrm{B}^+(\mathrm{a})).$$

It easily follows from this that

$$p^k - [p^k]_{B(y_0,2h)}(t) \rightharpoonup p_\infty - [p_\infty]_{B(y_0,2h)}(t) \text{ in } \mathrm{L}_{\frac{3}{2}}(-\mathrm{a}^2, 0; \mathrm{L}^{\frac{3}{2},\frac{9}{2}}(\mathrm{B}^+(\mathrm{a})).$$

Thus it is clear that

$$\limsup_{k\to\infty} D_q(e_0, 2h; p^k) \geqslant D_q(e_0, 2h; p_\infty).$$

So, it is sufficient to prove the following bound

$$D_q(e_0, 2h; p^k) \leqslant c_5(L, D_1^+(0,1/2,p)) \tag{3.45}$$

for all k sufficiently large such that

$$x_0^k = y_0 R_k \in B^+(1/8), \quad t_0^k = s_0 R_k^2 > -(1/8)^2. \tag{3.46}$$

By inverse scaling, we have

$$D_q(e_0, 2h; p^k) = D_q(z_0^k, 2hR_k; p), \quad z_0^k = (x_0^k, t_0^k). \tag{3.47}$$

Notice that $d_k = x_{03}^k = \mathrm{dist}(\mathrm{x}_0^\mathrm{k}, \Gamma) \geqslant 3\mathrm{hR_k}$. Furthermore, $Q(z_0^k, 2hR_k) \subset Q(z_0^k, d_k) \subset Q^+(1/4)$. Thus from Proposition 3.4 we see that

$$D_q(z_0^k, 2hR_k; p) \leqslant c(L)\left[\left(\frac{2hR_k}{d_k}\right)^{\frac{5}{4}} D_q(z_0^k, d_k; p) + 1\right]$$
$$\leqslant c(L)\left[D_q(z_0^k, d_k; p) + 1\right]. \quad (3.48)$$

Let $\bar{x}_0^k = (x_{01}^k, x_{02}^k, 0)$ and also $\bar{z}_0^k = (\bar{x}_0^k, t_0^k)$. It is clear that $Q(z_0^k, d_k) \subset Q^+(\bar{z}_0^k, 2d_k)$ and

$$D_q(z_0^k, d_k; p) \leqslant cD_q^+(\bar{z}_0^k, 2d_k; p).$$

Note that we have the following Poincare inequality:

$$D_q^+(\bar{z}_0^k, 2d_k; p) \leqslant D_1^+(\bar{z}_0^k, 2d_k; p).$$

Using this and (3.48) we infer

$$D_q(z_0^k, 2hR_k; p) \leqslant c(L)[D_1^+(\bar{z}_0^k, 2d_k; p) + 1]. \quad (3.49)$$

Clearly $2d_k < \frac{1}{4}$ so we have

$$Q^+(\bar{z}_0^k, 2d_k) \subset Q^+(\bar{z}_0^k, 1/4) \subset Q^+(1/2).$$

Then one can apply Proposition 3.7 to infer that

$$D_1^+(\bar{z}_0^k, 2d_k; p) \leqslant c(L)[8d_k D_1^+(\bar{z}_0^k, 2d_k; p) + 1] \leqslant c(L)[D_1^+(0, 1/2; p) + 1]. \quad (3.50)$$

Thus putting everything together gives

$$D_q(e_0, 2h; q^k) \leqslant c(L)[D_1^+(0, 1/2; p) + 1]. \qquad \square$$

We now state a new regularity criteria, which is convenient for proving Theorem 1.3. It is a generalisation of the basic ϵ-regularity criteria found in [15] and [12]. The proof is contained in the Appendix. Here is the statement.

Theorem 3.10. *Let (u, p) be a suitable weak solution in $Q_1(0)$. Then there exists a universal constants ϵ_0 and c_{0k} (with $k = 1, 2 \ldots$) with the following property. Assume*

$$C_\infty(0, 1; u) + D_\infty(0, 1; p) < \epsilon_0. \quad (3.51)$$

then for any natural number k, $\nabla^{k-1}u$ is Holder continuous in $\bar{Q}(1/2)$ and the following bound is valid:

$$\max_{\bar{Q}(1/2)} |\nabla^{k-1}u(z)| < c_{ok}. \quad (3.52)$$

Now we proceed to the proof of Theorem 1.3. Fix $h \in]0,1[$ arbitrarily and consider an arbitrary point

$$z_0 \in (\mathbb{R}^3_+ + 3hi_3)\times]-100,0[.$$

Following [24], we perform the same pressure decomposition as in the interior case as follows. In the ball $B(x_0,2h)$ decompose the pressure

$$p_\infty = p^1_\infty + p^2_\infty$$

such that

$$p^1_\infty := R_iR_j((u_\infty)_i(u_\infty)_j\chi_{B(x_0,2h)}). \tag{3.53}$$

It is clear that

$$\Delta p^2_\infty(\,\cdot\,,t) = 0 \quad in \quad B(x_0,2h).$$

Thus for the same reasons as previously stated we have

$$\|p^1_\infty(\,\cdot\,,t)\|_{L^{\frac{3}{2},\frac{q}{2}}(B(x_0,h))} \leqslant c\|u_\infty(\,\cdot\,,t)\|^2_{L^{3,q}(B(x_0,h))}. \tag{3.54}$$

It is not so difficult to show that for the harmonic part $p^2_\infty(\,\cdot\,,t)$ we have the following:

$$\sup_{x\in B(x_0,h)}|\nabla p^2_\infty(x,t)| \leqslant \frac{c}{h}\Big(\frac{1}{h^3}\int\limits_{B(x_0,2h)}|p^2_\infty(\,\cdot\,,t)-[p^2_\infty]_{B(x_0,2h)}(t)|\,dx\Big).$$

Clearly, by O'Neil's inequality:

$$\sup_{x\in B(x_0,h)}|\nabla p^2_\infty(x,t)| \leqslant \frac{c}{h}\left(\frac{1}{h^2}\|p^2_\infty(\,\cdot\,,t)-[p^2_\infty]_{B(x_0,2h)}(t)\|_{L^{\frac{3}{2},\frac{q}{2}}(B(x_0,2h))}\right). \tag{3.55}$$

For any $0<\rho<1$ we can use (3.54)–(3.55) along with the Poincare inequality to infer

$$D_q(z_0,h\rho;p_\infty)\leqslant c[D_q(z_0,h\rho;p^1_\infty)+D_q(z_0,h\rho;p^2_\infty)]$$
$$\leqslant \frac{c}{(h\rho)^2}\int\limits_{t_0-4h^2}^{t_0}\|p^1_\infty(\cdot,t)\|^{\frac32}_{L^{\frac32,\frac q2}(B(x_0,2h))}\,dt$$
$$+\,c(h\rho)^{\frac52}\int\limits_{t_0-(h\rho)^2}^{t_0}\sup_{x\in B(x_0,h)}|\nabla p^2_\infty(x,t)|^{\frac32}\,dt$$
$$\leqslant c\left[\frac{1}{(h\rho)^2}\int\limits_{t_0-4h^2}^{t_0}\|u_\infty(\cdot,t)\|^3_{L^{3,q}(B(x_0,h))}\,dt{+}\rho^{\frac52}D_q(z_0,2h,q)\right].$$

By Lemma 3.9 ($c_5=c_5(L,D_1^+(0,1/2,p))$) and the well known embeddings for Lorentz spaces we obtain:

$$C_\infty(z_0,h\rho;u_\infty)+D_\infty(z_0,h\rho;p_\infty)$$
$$\leqslant c\left[\frac{1}{(h\rho)^2}\int\limits_{t_0-4h^2}^{t_0}\|u_\infty(\cdot,t)\|^3_{L^{3,q}(B(x_0,h))}dt+\rho^{\frac52}c_5\right]$$

for any $z_0\in(\mathbb{R}^3_++3hi_3)\times]-100,0[$. Next, fix $\rho(L,D_1^+(0,\frac12;p))\in]0,1[$ such that
$$c\rho^{\frac52}c_5<\frac{\epsilon_0}{2}.$$
Since $u_\infty\in L_\infty(-\infty,0;L^{3,q}(\mathbb{R}^3_+))$ and $q\in]3,\infty[$ we have that for a.a t
$$\lim_{R\to\infty}\|u_\infty(\cdot,t)\|_{L^{3,q}(\mathbb{R}^3_+\setminus B(R))}=0.$$
Thus, by the dominated convergence theorem, we may find $R_1>100$ such that
$$\frac{c}{(h\rho)^2}\int\limits_{-200}^{0}\|u_\infty(\cdot,t)\|^3_{L^{3,q}(\mathbb{R}^3_+\setminus B(\frac{R_1}{4}))}dt<\frac{\epsilon_0}{2}.$$
Using this, it can be inferred that for any $z_0\in(\mathbb{R}^3_++3hi_3)\times]-100,0[$ such that $|x_0|>\frac{R_1}{2}$ we have
$$C_\infty(z_0,h\rho;u_\infty)+D_\infty(z_0,h\rho;p_\infty)<\epsilon_0.$$

Now Theorem 3.10 is applicable. Moreover we obtain that any $z_0 \in (\mathbb{R}^3_+ + 6hi_3) \setminus B(R_1) \times]-50, 0[$ and $k = 1, 2 \dots$

$$|\nabla^k u_\infty(z_0)| \leqslant \frac{c_{0k}}{(h\rho)^{k+1}}.$$

By the interior result of Theorem 2.1, we obtain boundedness of $\nabla^k u_\infty$ on the set $(\mathbb{R}^3_+ + 6hi_3) \cap B(R_1) \times]-50, 0[$. Define the vorticity $\omega_\infty = \nabla \wedge u_\infty$. Then on the set $(\mathbb{R}^3_+ + 6hi_3) \times]-50, 0[$, we have that there exists $M > 0$ such that

$$|\omega_\infty| \leqslant M,$$

$$|\partial_t \omega_\infty - \Delta \omega_\infty| \leqslant M(|\omega_\infty| + |\nabla \omega_\infty|).$$

Furthermore,

$$\omega_\infty(\,\cdot\,, 0) = 0 \ \text{ in } \ \mathbb{R}^3_+.$$

By applying a backward uniqueness theorem (Theorem 5 in [3]), one deduces that

$$\omega_\infty = 0 \quad \text{in} \quad (\mathbb{R}^3_+ + 6hi_3) \times [-50, 0].$$

Since h is arbitrary we infer that

$$\omega_\infty = 0 \ \text{ in } \ \mathbb{R}^3_+ \times [-50, 0].$$

Hence for a.a $t \in [-50, 0]$, u_∞ is a harmonic function, which satisfies the boundary condition $u_\infty(x, t) = 0$ if $x_3 = 0$. But for a.a $t \in [-50, 0]$, $L^{3,q}$ of u_∞ over $\mathbb{R}^3_+$ is finite. By a Liouville Theorem we get for the same t that $u_\infty(\,\cdot\,, t) = 0$ in $\mathbb{R}^3_+$. This contradicts (3.33).

§4. Appendix: ϵ-regularity in weak Lebesgue spaces

First, we begin with stating a well known algebraic Lemma, whose proof is omitted but found in [4].

Lemma 4.1. *Let $I(s)$ be a bounded non negative function for $s \in [R_1, R_2]$. Assume that for every $s, \rho \in [R_1, R_2]$ and $s < \rho$ we have*

$$I(s) \leqslant [A(\rho - s)^{-\alpha} + B(\rho - s)^{-\beta} + C] + \theta I(\rho)$$

with $A, B, C \geqslant 0$, $\alpha > \beta > 0$ and $\theta \in [0, 1[$. Then there holds

$$I(R_1) \leqslant c(\alpha, \theta)[A(R_2 - R_1)^{-\alpha} + B(R_2 - R_1)^{-\beta} + C].$$

Proof of Lemma 2.4. Without loss of generally, consider z_0 to be the origin. Let $0 < \frac{r}{2} \leqslant s < \rho \leqslant r < 1$. Let $\eta_1 \in C_0^\infty(B(\rho))$ such that $0 \leqslant \eta_1 \leqslant 1$ in $\mathbb{R}^3$ and $\eta_1 = 1$ on $B(s)$. Furthermore for $|\alpha| \leq 2$:

$$|\nabla^\alpha \eta_1| \leqslant \frac{C}{(\rho - s)^\alpha}.$$

Let $\eta_2 \in C_0^\infty(-\rho^2, \rho^2)$ such that $0 \leqslant \eta_2 \leqslant 1$ in $\mathbb{R}$ and $\eta_1 = 1$ on $[-s^2, s^2]$. Furthermore :

$$|\eta_1^{'}| \leqslant \frac{C}{(\rho^2 - s^2)} \leqslant \frac{C}{r(\rho - s)} \leqslant \frac{C}{(\rho - s)^2}.$$

Let $\phi(x,t) := \eta_1(t)\eta_2(x)$. Hence:

$$|\nabla \phi| \leqslant \frac{C}{\rho - s}, \tag{4.1}$$

$$|\nabla^2 \phi| \leqslant \frac{C}{(\rho - s)^2}, \tag{4.2}$$

$$|\phi_t| \leqslant \frac{C}{(\rho - s)^2}. \tag{4.3}$$

From the local energy inequality we have that for a.a $t \in]-1, 0[$:

$$\int\limits_B \phi(x,t)|u(x,t)|^2\,dx + 2 \int\limits_{B\times]-1,t[} \phi|\nabla u|^2\,dx\,ds$$
$$\leqslant \int\limits_{B\times]-1,t[} (|u|^2(\Delta\phi + \partial_t\phi) + u.\nabla\phi(|u|^2 + 2p)\,dx\,ds. \tag{4.4}$$

Let $I_1(s) := sA(0,s;u)$, $I_2(s) := sB(0,s;u)$ and $I(s) = I_1(s) + I_2(s)$.

$$I(s) \leqslant (1) + (2) + (3). \tag{4.5}$$

Where,

$$(1) := \int\limits_{B\times]-1,t[} (|u|^2(\Delta\phi + \partial_t\phi)\,dx\,ds, \tag{4.6}$$

$$(2) := \int\limits_{Q(\rho)} u.\nabla\phi|u|^2\,dx\,ds, \tag{4.7}$$

$$(3) := \int\limits_{Q(\rho)} 2u.\nabla\phi p\,dx\,ds. \tag{4.8}$$

Let us treat the simplest integral (1) first. By O'Neil's inequality in space and then the Holder inequality in time:

$$(1) \leqslant \int_{-\rho^2}^{0} \|u\|^2_{L^{3,\infty}(B(\rho))} \|\Delta\phi + \partial_t\phi\|_{L_{3,1}(B(\rho))}\, ds$$

$$\leqslant c\frac{\rho}{(\rho-s)^2}\int_{-\rho^2}^{0} \|u\|^2_{L^{3,\infty}(B(\rho))} ds \leqslant \frac{\rho^{\frac{5}{3}}}{(\rho-s)^2}\Big(\int_{-\rho^2}^{0} \|u\|^3_{L^{3,\infty}(B(\rho))} ds\Big)^{\frac{2}{3}}.$$

Now, we treat (3). Again by O'Neil's inequality obtain:

$$(3) \leqslant 2\int_{-\rho^2}^{0} \|u.\nabla\phi\|_{L^{3,1}(B(\rho))} \|p\|_{L_{\frac{3}{2},\infty}(B(\rho))}\, ds. \tag{4.9}$$

By a well known interpolation characterisation of Lorentz spaces:

$$L^{3,1}(B(\rho) = (L_2(B(\rho)), L_6(B(\rho))_{\frac{1}{2},1}.$$

Thus, by well known properties of interpolation spaces and Gagliardo Nirenberg inequality:

$$\begin{aligned}\|u.\nabla\phi\|_{L^{3,1}(B(\rho))} &\leqslant \|u.\nabla\phi\|^{\frac{1}{2}}_{L_2(B(\rho))}\|u.\nabla\phi\|^{\frac{1}{2}}_{L_6(B(\rho))}\\ &\leqslant C\|u.\nabla\phi\|^{\frac{1}{2}}_{L_2(B(\rho))}\|\nabla(u.\nabla\phi)\|^{\frac{1}{2}}_{L_2(B(\rho))}\\ &\leqslant \frac{C\|u\|_{L_2(B(\rho))}}{(\rho-s)^{\frac{3}{2}}} + \frac{C\|u\|^{\frac{1}{2}}_{L_2(B(\rho))}\|\nabla u\|^{\frac{1}{2}}_{L_2(B(\rho))}}{\rho-s}.\end{aligned}$$

Using this and applying the Holder inequality to (4.9) we get

$$(3) \leqslant \frac{C}{(\rho-s)^{\frac{3}{2}}}\Big(\int_{-\rho^2}^{0} \|u\|^3_{L_2(B(\rho))} ds\Big)^{\frac{1}{3}}$$

$$+\frac{C}{\rho-s}\Big(\int_{-\rho^2}^{0} \|u\|^{\frac{3}{2}}_{L_2(B(\rho))}\|\nabla u\|^{\frac{3}{2}}_{L_2(B(\rho))} ds\Big)^{\frac{1}{3}} \times \Big(\int_{-\rho^2}^{0} \|p\|^{\frac{3}{2}}_{L_{\frac{3}{2},\infty}(B(\rho))} ds\Big)^{\frac{2}{3}}.$$

From here, it is not difficult to obtain

$$(3) \leqslant \Big(\frac{Cr^{\frac{2}{3}} I_1(\rho)^{\frac{1}{2}}}{(\rho-s)^{\frac{3}{2}}} + \frac{Cr^{\frac{1}{6}}}{\rho-s} I_2(\rho)^{\frac{1}{4}} I_1(\rho)^{\frac{1}{4}}\Big) \times \Big(\int\limits_{-r^2}^{0} \|p\|^{\frac{3}{2}}_{L^{\frac{3}{2},\infty}(B(r))} ds\Big)^{\frac{2}{3}}.$$

Identical reasoning gives

$$(2) \leqslant \Big(\frac{Cr^{\frac{2}{3}} I_1(\rho)^{\frac{1}{2}}}{(\rho-s)^{\frac{3}{2}}} + \frac{Cr^{\frac{1}{6}}}{\rho-s} I_2(\rho)^{\frac{1}{4}} I_1(\rho)^{\frac{1}{4}}\Big) \times \Big(\int\limits_{-r^2}^{0} \|u\|^{3}_{L^{3,\infty}(B(r))} ds\Big)^{\frac{2}{3}}.$$

Thus, we see by Young's inequality that

$$I(s) \leqslant \frac{r^{\frac{5}{3}}}{(\rho-s)^2}\Big(\int\limits_{-r^2}^{0} \|u\|^{3}_{L^{3,\infty}(B(r))} ds\Big)^{\frac{2}{3}} + \frac{1}{2} I(\rho)$$
$$+\Big(\frac{Cr^{\frac{4}{3}}}{(\rho-s)^3} + \frac{Cr^{\frac{1}{3}}}{(\rho-s)^2}\Big)\Bigg[\Big(\int\limits_{-r^2}^{0} \|u\|^{3}_{L^{3,\infty}(B(r))} ds\Big)^{\frac{4}{3}} + \Big(\int\limits_{-r^2}^{0} \|p\|^{\frac{3}{2}}_{L^{\frac{3}{2},\infty}(B(r))} ds\Big)^{\frac{4}{3}}\Bigg]. \tag{4.10}$$

By Lemma 4.1, we obtain

$$I(r/2) \leq r^{-\frac{1}{3}}\Big(\int\limits_{-r^2}^{0} \|u\|^{3}_{L^{3,\infty}(B(r))} ds\Big)^{\frac{2}{3}}$$
$$+ Cr^{-\frac{5}{3}}\Bigg[\Big(\int\limits_{-r^2}^{0} \|u\|^{3}_{L^{3,\infty}(B(r))} ds\Big)^{\frac{4}{3}} + \Big(\int\limits_{-r^2}^{0} \|p\|^{\frac{3}{2}}_{L^{\frac{3}{2},\infty}(B(r))} ds\Big)^{\frac{4}{3}}\Bigg].$$

From here the conclusion is immediate. □

Remark 4.2. After appropriate relabellings, an analogous estimate holds when (u, p) is a suitable weak solution near the flat part of the boundary.

Next, we define

$$D_1(z_0, r; p) := \frac{1}{r^{\frac{3}{2}}} \int\limits_{t_0-r^2}^{t_0} \Big(\int\limits_{B(x_0,r)} |\nabla p|^{\frac{9}{8}} dx\Big)^{\frac{4}{3}} dt. \tag{4.11}$$

Proposition 4.3. *Let* (u,p) *be a suitable weak solution in* $Q(z_0,1)$ *Then for* $0<r<1$ *the following holds* (c *is some universal constant*):

$$A(z_0,r/2;u)+B(z_0,r/2;u)\leqslant c(C_\infty(z_0,r;u)^{\frac{4}{3}}+C_\infty^+(z_0,r;u)^{\frac{2}{3}} + D_1(z_0,r;p)^{\frac{2}{3}}C_\infty(z_0,r;u)^{\frac{1}{3}}). \quad (4.12)$$

Proof. The set up is the same as Lemma 2.4. The main difference is estimation of

$$(3):=\int\limits_{Q(\rho)} 2u.\nabla\phi p\,dx\,ds.$$

Using the solenodial condition we can write:

$$(3):=\int\limits_{Q(\rho)} 2u.\nabla\phi(p-[p]_{B(r)})\,dx\,ds.$$

We note the Poincare inequality

$$\|p(\,\cdot\,,t)-[p]_{B(r)}(t)\|_{L^{\frac{3}{2},1}(B(r))}\leqslant Cr^{\frac{1}{3}}\|\nabla p(\,\cdot\,,t)\|_{L_{\frac{9}{8}}(B(r))}.$$

Thus, by using O'Neil in space and Holder in time:

$$|(3)|\leqslant\frac{Cr^{\frac{1}{3}}}{\rho-s}\Big(\int\limits_{-r^2}^{0}\|\nabla p\|^{\frac{3}{2}}_{L_{\frac{9}{8}}(B(r))}ds\Big)^{\frac{2}{3}}\Big(\int\limits_{-r^2}^{0}\|u\|^3_{L_{3,\infty}(B(r))}\,ds\Big)^{\frac{1}{3}}.$$

The remainder of the proof is the same as Lemma 2.4. □

Proof of Lemma 3.3. Without loss of generality take $z_0=0$. For a.a $t\in]-\rho^2,0[$, the pressure p meets the equation in sense of distributions in $B(\rho)$:

$$\Delta p(\,\cdot\,,t)=-\mathrm{div}\,\mathrm{div}\,\mathrm{v}(\,\cdot\,,\mathrm{t})\otimes\mathrm{v}(\,\cdot\,,\mathrm{t}).$$

Decompose the pressure so that

$$p=p_1+p_2,$$

where

$$p_1(\,\cdot\,,t):=R_iR_j(\chi_{B(\rho)}v_iv_j(\,\cdot\,,t)). \quad (4.13)$$

Here R_i is Riesz operator and we adopt the summation convention. It is not difficult to notice that in $B(\rho)$:

$$\Delta p_2(\,\cdot\,,t)=0. \quad (4.14)$$

For p_1 we have

$$\|p_1(\,\cdot\,,t)\|^{\frac{3}{2}}_{L^{\frac{3}{2},\frac{q}{2}}(B(\rho))} \leqslant c\|v\otimes v(\,\cdot\,,t)\|^{\frac{3}{2}}_{L^{\frac{3}{2},\frac{q}{2}}(B(\rho))} \leqslant c\|v(\,\cdot\,,t)\|^{3}_{L^{3,q}(B(\rho))}. \tag{4.15}$$

Let $0<r\leqslant \frac{\rho}{2}$. Using O'Neil's inequality it is not difficult to see

$$|[p_1]_{B(r)}|(t) \leqslant \frac{c\|p_1(\,\cdot\,,t)\|_{L^{\frac{3}{2},\frac{q}{2}}(B(r))}}{r^2}.$$

One can then easily obtain

$$D_q(0,r;p_1)\leqslant\frac{c}{r^2}\Big(\int\limits_{-r^2}^{0}\|p_1(\,\cdot,t)\|^{\frac{3}{2}}_{L^{\frac{3}{2},\frac{q}{2}}(B(r))}dt\Big)\leqslant\frac{c}{r^2}\Big(\int\limits_{-\rho^2}^{0}\|p_1(\,\cdot,t)\|^{\frac{3}{2}}_{L^{\frac{3}{2},\frac{q}{2}}(B(\rho))}dt\Big)$$

Using (4.15), it is not difficult to see

$$D_q(0,r,p)\leqslant c\left[\left(\frac{\rho}{r}\right)^2 C_q(0,\rho,v)+D_q(0,r,p_2)\right]. \tag{4.16}$$

Since p_2 is a harmonic function, we see that

$$\begin{aligned}\sup_{x\in B(r)}|p_2(x,t)-[p_2]_{B(r)}(t)| &\leqslant cr\sup_{x\in B(\frac{\rho}{2})}|\nabla p_2(x,t)|\\ &\leqslant \frac{cr}{\rho^4}\int\limits_{B(\rho)}|p_2(x,t)-[p_2]_{B(\rho)}(t)|\,dx\\ &\leqslant \frac{cr}{\rho^3}\|p_2(\,\cdot\,,t)-[p_2]_{B(\rho)}(t)\|_{L_{\frac{3}{2},\frac{q}{2}}(B(\rho))}.\end{aligned}$$

The last line follows from O'Neils inequality and fact that

$$\|\chi_\Omega\|_{L_{p,q}(\Omega)}\leqslant C_{p,q}|\Omega|^{\frac{1}{p}}.$$

The remaining parts of the proof use this last fact, but are otherwise identical to that in [24]. The remaining details are an exercise for the reader. □

Before proving the main statement we introduce some notation:

$$C(z_0,r;u):=r^{-2}\int\limits_{Q(z_0,r)}|u|^3\,dz, \tag{4.17}$$

$$D(z_0,r;p):=r^{-2}\int\limits_{Q(z_0,r)}|p|^{\frac{3}{2}}\,dz. \tag{4.18}$$

The following version of ϵ-regularity criteria of Caffarelli–Kohn–Nirenberg will be important to us in the sequel:

Lemma 4.4. *Let* (u,p) *be a suitable weak solution in* $Q(R)$*. Then there exists a universal constants* ϵ_0 *and* c_{0k} *(with* $k = 1, 2 \dots$*) with the following property. Assume*

$$C(0,R;u) + D(0,R;p) < \epsilon_0. \tag{4.19}$$

then for any natural number k*,* $\nabla^{k-1}u$ *is Holder continuous in* $\bar{Q}(R/2)$ *and the following bound is valid*:

$$\max_{\bar{Q}(R/2)} |\nabla^{k-1}u(z)| < c_{ok}R^{-k}. \tag{4.20}$$

A short proof can be found in [15] for example, a detailed one in [12]. Now we state our main result, here it is:

Theorem 4.5. *Let* (u,p) *be a suitable weak solution in* Q*. Then there exists a universal constants* ϵ_0 *and* c_{0k} *(with* $k = 1, 2 \dots$*) with the following property. Assume*

$$C_\infty(0,1;u) + D_\infty(0,1;p) < \epsilon_0. \tag{4.21}$$

then for any natural number k*,* $\nabla^{k-1}u$ *is Holder continuous in* $\bar{Q}(1/8)$ *and the following bound is valid*:

$$\max_{\bar{Q}(1/8)} |\nabla^{k-1}u(z)| < c_{ok}. \tag{4.22}$$

Proof. From Lemma 2.4 and (4.21) it follows that

$$A(0,1/2;u) + B(0,1/2;u) \leqslant c\left(\epsilon_0 + \epsilon_0^2\right)^{\frac{2}{3}}. \tag{4.23}$$

Here c will always denote some universal constant. By interpolation and the Sobolev embedding theorem, one can show that

$$C(0,1/2,u) \leqslant c[A(0,1/2;u)^{\frac{3}{4}}B(0,1/2;u)^{\frac{3}{4}} + A(0,1/2;u)^{\frac{3}{2}}].$$

Thus, by (4.23) we have

$$C(0,1/2,u) \leqslant c\left(\epsilon_0 + \epsilon_0^2\right). \tag{4.24}$$

For similar reasons it is not so difficult to see that

$$\|\mathrm{div}(u\otimes u)\|_{L_{\frac{9}{8},\frac{3}{2}}(Q(\frac{1}{2}))} \leqslant c[A(0,1/2;u) + A(0,1/2;u)^{\frac{1}{3}}B(0,1/2;u)^{\frac{2}{3}}].$$

Thus,

$$\|\,\mathrm{div}(u\otimes u)\|_{L_{\frac{9}{8},\frac{3}{2}}(Q(\frac{1}{2}))} \leqslant \left(\epsilon_0 + \epsilon_0^2\right)^{\frac{2}{3}}. \tag{4.25}$$

By Holder's inequality it is obvious that

$$\|u\|_{W^{1,0}_{\frac{9}{8},\frac{3}{2}}(Q(\frac{1}{2}))} \leqslant c(A(u,0;1/2)^{\frac{1}{2}} + B(u,0;1/2)^{\frac{1}{2}})$$

$$\leqslant c\left(\epsilon_0 + \epsilon_0^2\right)^{\frac{1}{3}} \tag{4.26}$$

Using O'Neil's inequality, we have

$$\int\limits_{B(1/2)} |p(x,t)|^{\frac{9}{8}} dx \leqslant c\||p(\,\cdot\,,t)|^{\frac{9}{8}}\|_{L^{\frac{8}{3},\infty}(B(\frac{1}{2}))} = c\|p(\,\cdot\,,t)\|^{\frac{9}{8}}_{L^{3,\infty}(B(\frac{1}{2}))}.$$

Hence,

$$\|p\|_{L_{\frac{9}{8},\frac{3}{2}}(Q(\frac{1}{2}))} \leqslant c\epsilon_0^{\frac{2}{3}}. \tag{4.27}$$

Local interior regularity theory for Stokes equation gives

$$\begin{aligned}
&\|\partial_t u\|_{L_{\frac{9}{8},\frac{3}{2}}(Q(\frac{1}{4}))} + \|\nabla^2 u\|_{L_{\frac{9}{8},\frac{3}{2}}(Q(\frac{1}{4}))} + \|\nabla p\|_{L_{\frac{9}{8},\frac{3}{2}}(Q(\frac{1}{4}))}\\
&\leqslant c[\|\operatorname{div}((u\otimes u)\|_{L_{\frac{9}{8},\frac{3}{2}}(Q(\frac{1}{2}))} + \|u\|_{L_{\frac{9}{8},\frac{3}{2}}(Q(\frac{1}{2}))}\\
&+ \|\nabla u\|_{L_{\frac{9}{8},\frac{3}{2}}(Q(\frac{1}{2}))} + \|p\|_{L_{\frac{9}{8},\frac{3}{2}}(Q(\frac{1}{2}))}].
\end{aligned} \tag{4.28}$$

Using this together with (4.25)–(4.27) obtain that

$$\|\nabla p\|_{L_{\frac{9}{8},\frac{3}{2}}(Q(\frac{1}{4}))} \leqslant c[\left(\epsilon_0 + \epsilon_0^2\right)^{\frac{1}{3}} + \left(\epsilon_0 + \epsilon_0^2\right)^{\frac{2}{3}}].$$

Thus, by the Poincare inequality:

$$\|p - [p]_{,\frac{1}{4}}\|_{L_{\frac{3}{2}}(Q(\frac{1}{4}))} \leqslant c[\left(\epsilon_0 + \epsilon_0^2\right)^{\frac{1}{3}} + \left(\epsilon_0 + \epsilon_0^2\right)^{\frac{2}{3}}].$$

But by O'Neil $|[p(\,\cdot\,,t)]_{,\frac{1}{4}}| \leqslant c\|p(\,\cdot\,,t)\|_{L^{\frac{3}{2},\infty}(B)}$. Therefore, we conclude

$$\|p\|_{L_{\frac{3}{2}}(Q(\frac{1}{4}))} \leqslant c[\left(\epsilon_0 + \epsilon_0^2\right)^{\frac{1}{3}} + \left(\epsilon_0 + \epsilon_0^2\right)^{\frac{2}{3}}]. \tag{4.29}$$

This along with (4.24) gives

$$C(u,0;1/4) + D(u,0;1/4) \leqslant c[\left(\epsilon_0 + \epsilon_0^2\right)^{\frac{1}{2}} + \left(\epsilon_0 + \epsilon_0^2\right)].$$

Choosing ϵ_0 sufficiently small, gives the conclusion by Lemma 4.4. □

Remark 4.6. With certain adjustments to the proof, namely local boundary regularity for the Stokes system and the boundary analogue of Lemma 4.4 (see [23] and [25]), we have the following near the flat part of the boundary.

Let (u,p) be a suitable weak solution to the Navier–Stokes equations in $Q^+(1)$ near $\Gamma(0,1)\times]-1,0[$. Then there exists universal constants ϵ_0 and c_0 such that if

$$C^+_\infty(0,1;u) + D^+_\infty(0,1;p) < \epsilon_0,$$

then u is Holder continuous in $\bar{Q}^+(1/2)$ and the following is valid:

$$\sup_{\bar{Q}^+(1/2)} |u(z)| \leqslant c_0.$$

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