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THE COMMUTATORS OF CLASSICAL GROUPS

Abstract. In his seminal paper, half a century ago, Hyman Bass established a commutator formula in the setting of (stable) general linear group which was the key step in defining the K_1 group. Namely, he proved that for an associative ring A with identity,

$$E(A) = [E(A), E(A)] = [\mathrm{GL}(A), \mathrm{GL}(A)],$$

where $\mathrm{GL}(A)$ is the stable general linear group and $E(A)$ is its elementary subgroup. Since then, various commutator formulas have been studied in stable and non-stable settings, and for a range of classical and algebraic like-groups, mostly in relation to subnormal subgroups of these groups. The major classical theorems and methods developed include some of the splendid results of the heroes of classical algebraic K-theory; Bak, Quillen, Milnor, Suslin, Swan and Vaserstein, among others.

One of the dominant techniques in establishing commutator type results is localisation. In this note we describe some recent applications of localisation methods to the study (higher/relative) commutators in the groups of points of algebraic and algebraic-like groups, such as general linear groups, $\mathrm{GL}(n, A)$, unitary groups $\mathrm{GU}(2n, A, \Lambda)$ and Chevalley groups $G(\Phi, A)$. We also state some of the intermediate results as well as some corollaries of these results.

This note provides a general overview of the subject and covers the current activities. It contains complete proofs of several main results to give the reader a self-contained source. We have borrowed the proofs from our previous papers and expositions [38–50, 99, 100, 129–132].

Key words and phrases: general linear groups, unitary groups, Chevalley groups, elementary subgroups, elementary generators, localisation, relative subgroups, conjugation calculus, commutator calculus, Noetherian reduction, the Quillen–Suslin lemma, localisation-completion, commutator formulae, commutator width, nilpotency of K_1, nilpotent filtration.

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Everybody knows there is no fineness or accuracy of suppression;
if you hold down one thing you hold down the adjoining.
Saul Bellow

§1. Introduction

Let A be a ring and I be a two sided ideal of A. In his seminal paper [16], fifty years ago, Bass laid out a theory now known as the classical algebraic K-theory (as opposed to the higher algebraic K-theory introduced by Quillen [89]). He considered the stable general linear group $\mathrm{GL}(A) = \bigcup_{n=1}^{\infty} \mathrm{GL}(n, A)$ and its stable elementary subgroup $E(A) = \bigcup_{n=1}^{\infty} E(n, A)$ and defined the stable $K_1(A)$ as the quotient $\mathrm{GL}(A)/E(A)$ (see §4 for details). Relating the group structure of $\mathrm{GL}(A)$ to the ideal structure of A, he went on to establish an exact sequence naturally relating K_1 to the group K_0, previously defined by Grothendieck and Serre. In order the coset space $K_1(A)$ to be a well-defined group, Bass proved his famous "Whitehead lemma" ([16, Theorem 3.1], see Lemma 6), i.e.,

$$E(A, I) = [E(A), E(A, I)] = [\mathrm{GL}(A), \mathrm{GL}(A, I)].$$

In particular when $I = A$, it follows that $E(A)$ is a normal subgroup of $\mathrm{GL}(A)$.

He further proved that if $n \geqslant \max\{\mathrm{sr}(A), 3\}$, where $\mathrm{sr}(A)$ is the stable range of A, then

$$E(n, A, I) = [\mathrm{GL}(n, A), E(n, A, I)]. \tag{1}$$

Again, when $I = A$, it follows that $E(n, A)$ is a normal subgroup of $\mathrm{GL}(n, A)$.

The next natural question arose was whether $E(n, A)$ is a normal subgroup of $\mathrm{GL}(n, A)$ below the stable range as well. In the non-stable case, there is no "room" available for manoeuvring as in the stable case (see the proof of Whitehead Lemma 6). Thus, one is forced to put some finiteness assumption on the ring. Indeed, Gerasimov [33] produced examples of rings A for which, for any $n \geqslant 2$, $E(n, A)$ is as far from being normal in $\mathrm{GL}(n, R)$, as one can imagine.

A major contribution in this direction came with the work of Suslin [105, 114] who showed that if A is a *module finite ring*, namely, a ring that is finitely generated as module over its centre, and $n \geqslant 3$, then $E(n, A)$ is a normal subgroup of $\mathrm{GL}(n, A)$. That Suslin's normality theorem (and the methods developed to prove it) implies the standard commutator formulae of the type (1) in full force was somewhat later observed independently by Borewicz–Vavilov [23] and Vaserstein [115]. In these work it was established that, for a module finite ring A and a two-sided ideal I of A and $n \geqslant 3$, we have (see §6)

$$[E(n,A), \mathrm{GL}(n,A,I)] = E(n,A,I).$$

The focus then shifted to the relative commutators with two ideals. In his paper, Bass already proved that for a ring A and two sided ideals I, J, and $n \geqslant \max(\mathrm{sr}(A)+1, 3)$,

$$[E(n,A,I), \mathrm{GL}(n,A,J)] = [E(n,A,I), E(n,A,J)]. \tag{2}$$

Mason and Stothers, building on Bass' result improved the formula, with the same assumptions, to

$$[\,\mathrm{GL}(n,A,I), \mathrm{GL}(n,A,J)] = [E(n,A,I), E(n,A,J)].$$

Later, in a series of the papers, the authors with A. Stepanov proved that the commutator formula (2) is valid for any module finite ring A and $n \geqslant 3$ (see Theorem 1A).

Since Suslin's work, five major noticeably different methods have been developed for arbitrary rings to prove such commutator formulae results (and carried out in different classical groups):

- Suslin's direct factorisation method [105, 106, 61] (see also [37]);
- Suslin's factorisation and patching method [114, 59, 15];
- Quillen–Suslin–Vaserstein's localisation and patching method [105, 115, 110, 107];
- Bak's localisation-completion method [10, 38, 14];
- Stepanov–Vavilov–Plotkin's decomposition of unipotents [124, 128, 99, 125].

Suslin's result makes it possible to define the non-stable $K_{1,n} := \mathrm{GL}(n,A)/E(n,A)$, when $n \geqslant 3$, for module finite rings. The study of these non-stable K_1's is known to be very difficult. There are examples due to van der Kallen [56] and Bak [10] which show that non-stable K_1 can be non-abelian and the natural question is how non-abelian it can be?

The breakthrough came with the work of Bak [10], who showed that $K_{1,n}$ is nilpotent by abelian if $n \geqslant 3$ and the ring satisfies some dimension condition (e.g. has a centre with finite Krull dimension). His method consists of some "conjugation calculus" on elementary elements, plus simultaneously applying localisation-patching and completion. This is the method which opened doors to establishing the so called, higher commutator formulas and will be employed in this paper.

Localisation is one of the most powerful ideas in the study of classical groups over rings. It allows to reduce many important problems over various classes of rings subject to commutativity conditions, to similar problems for semi-local rings. Both methods – the Quillen-Suslin and Bak's approach (particularly the latter)– rely on a large body of common calculations, and technical facts, known as *conjugation calculus* and *commutator calculus*. Often times these calculations are even referred to as the *yoga of conjugation*, and the *yoga of commutators*, to stress the overwhelming feeling of technical strain and exertion. We use variations of these methods to prove multiple commutator formulas for general linear group of the following type (see §10):

$$\begin{aligned}&\big[E(n,A,I_0),\mathrm{GL}(n,A,I_1),\mathrm{GL}(n,A,I_2),\dots,\mathrm{GL}(n,A,I_m)\big]\\&=\big[E(n,A,I_0),E(n,A,I_1),E(n,A,I_2),\dots,E(n,A,I_m)\big].\end{aligned}\tag{3}$$

First note that one can produce examples of a commutative ring A and ideals I, J and K such that (see §7)

$$[E(n,A,I),E(n,A,J)]\neq E(n,A,IJ),$$

and (see §11)

$$\big[[E(A,I),E(A,J)],E(A,K)\big]\neq\big[E(A,I),[E(A,J),E(A,K)]\big].$$

So higher commutator formulas of the form (3) is far from trivial. We will observe that using some commutator calculus, and induction, the proof of (3) reduces to prove the base of induction, i.e., to prove

$$\begin{aligned}&\big[[E(n,A,I),\mathrm{GL}(n,A,J)],\mathrm{GL}(n,A,K)\big]\\&=\big[[E(n,A,I),E(n,A,J)],E(n,A,K)\big].\end{aligned}\tag{4}$$

The proof of (4) constitutes the bulk of work and uses a variation of localisation method first developed in [10].

The path to full-scale generalisation of these results from general linear groups to other classical groups was anything but straightforward. For instance, in the unitary case, due to the following circumstances,

- the presence of long and short roots,
- complicated elementary relations,
- non-commutativity,
- non-trivial involution,

• non-trivial form parameter,

these yoga calculations tend to be especially lengthy, and highly involved. In this paper, for a comparison, we only provide one proof in the case of unitary groups (which has not been appeared before). Namely, whereas the proof of Lemma 1A in the setting of general linear groups is only a half of a page, the proof of its counterpart in the unitary setting, Lemma 1B, constitutes more than 4 pages.

The aim of this note is to start with the original Bass' Whitehead lemma and continue to establish the (higher) commutator formulas. We trace the literature on this theme, provide proofs to the main results in the setting of the general linear group and formulate the results in other classical-like groups. We aim to provide a self-contained source from the results scattered in the literature.

§2. The groups, an overview

In this paper we consider algebraic-like or classical-like group functors G. We let $G(A)$ to be the group of points of G over a ring A. Note that groups of types other than A_l only exist over commutative rings. Typically, $G(A)$ is one of the following groups.

A. General linear group $\mathrm{GL}(n, A)$ of degree n over a ring A.

In this context the ring A does not have to be commutative. However, we have to impose *some* commutativity conditions for our results to hold. One of the well behaved classes is the class of quasi-finite rings. Recall, that a ring A is called *module finite* if it is finitely generated as a module over its centre. *Quasi-finite* rings are direct limits of inductive systems of module finite rings (see §3.3). To avoid unnecessary repetitions, in the sequel, speaking of ideals of an associative ring A, we always mean *two-sided* ideals of A.

B. Unitary groups $\mathrm{GU}(2n, A, \Lambda)$ over a form ring (A, Λ).

In this setting A is a [not necessarily commutative] ring with involution $\overline{} : A \to A$ and a form parameter Λ (see §5). As in the case of general linear groups, we usually assume that A is module finite over a commutative ring R. In general, Λ is not an R-module. Thus, R has to be replaced by its subring R_0, generated by all $\xi\overline{\xi}$ with $\xi \in R$.

C. Chevalley groups $G(\Phi, A)$ of type Φ over a commutative ring A.

Chevalley groups are indeed *algebraic*, and the ground rings are *commutative* in this case, which usually makes life easier.

Together with the algebraic-like group $G(A)$ we consider the following subgroups.

- First of all, the elementary group $E(A)$, generated by elementary unipotents.
  - In the linear case, the elementary generators are elementary [linear] transvections $e_{ij}(\xi)$, $1 \leqslant i \neq j \leqslant n$, $\xi \in A$.
  - In the unitary case, the elementary generators are elementary unitary transvections $T_{ij}(\xi)$, $1 \leqslant i \neq j \leqslant -1$, $\xi \in A$. In the even hyperbolic case they come in two modifications. They can be short root type, $i \neq \pm j$, when the parameter ξ can be any element of A. On the other hand, for the long root type $i = -j$ and the parameter ξ must belong to [something defined in terms of] the form parameter Λ.
  - Finally, for Chevalley groups, the elementary generators are the elementary root unipotents $x_\alpha(\xi)$ for a root $\alpha \in \Phi$ and a ring element $\xi \in A$.

Further, let $I \trianglelefteq A$ be an ideal of A. We also consider the following relative subgroups.

- The elementary group $E(I)$ of level I, generated by elementary unipotents of level I.
- The relative elementary group $E(A, I) = E(I)^{E(A)}$ of level I.
- The principal congruence subgroups $G(A, I)$ of level I, the kernel of reduction homomorphism $\rho_I : G(A) \longrightarrow G(A/I)$.
- The full congruence subgroups $C(A, I)$ of level I, the inverse image of the centre of $G(A/I)$ with respect to ρ_I.

We use the usual notation for these groups in the above contexts A–C as shown below.

$G(A)$	$\mathrm{GL}(n, A)$	$\mathrm{GU}(n, A, \Lambda)$	$G(\Phi, A)$
$E(A)$	$E(n, A)$	$\mathrm{EU}(n, A, \Lambda)$	$E(\Phi, A)$
$E(I)$	$E(n, I)$	$\mathrm{FU}(n, I, \Gamma)$	$E(\Phi, I)$
$E(A, I)$	$E(n, A, I)$	$\mathrm{EU}(n, I, \Gamma)$	$E(\Phi, A, I)$
$G(A, I)$	$\mathrm{GL}(n, A, I)$	$\mathrm{GU}(n, I, \Gamma)$	$G(\Phi, A, I)$
$C(A, I)$	$C(n, A, I)$	$\mathrm{CU}(n, I, \Gamma)$	$C(\Phi, A, I)$

There are two more general contexts, where localisation methods have been successfully used, in particular,

D. Isotropic reductive groups $G(A)$,

E. Odd unitary groups $U(V,q)$,

however we don't pursue these groups here (see [84–87, 74, 96]).

§3. Preliminaries

We gather here basic results in group and ring theory, which will be used throughout this note.

3.1. Commutators. Let G be a group. For any $x, y \in G$, ${}^x y = xyx^{-1}$ denotes the left x-conjugate of y. Let $[x,y] = xyx^{-1}y^{-1}$ denote the commutator of x and y. Sometimes the double commutator $[[x,y],z]$ will be denoted simply by $[x,y,z]$ and

$$[[A,B],C] = [A,B,C].$$

Thus we write $[A_1, A_2, A_3, \dots, A_n]$ for $[\dots[[A_1,A_2],A_3],\dots,A_n]$ and call it the *standard form* of the multiple commutator formulas.

The following formulas will be used frequently (sometimes without giving a reference to them),

(C1) $[x,yz] = [x,y]({}^y[x,z])$.

(C1$^+$) An easy induction, using identity (C1), shows that

$$[x, \prod_{i=1}^{k} u_i] = \prod_{i=1}^{k} {}^{\prod_{j=1}^{i-1} u_j}[x, u_i],$$

where by convention $\prod_{j=1}^{0} u_j = 1$.

(C2) $[xy,z] = ({}^x[y,z])[x,z]$.

(C2$^+$) As in (C1$^+$), we have

$$[\prod_{i=1}^{k} u_i, x] = \prod_{i=1}^{k} {}^{\prod_{j=1}^{k-i} u_j}[u_{k-i+1}, x].$$

(C3) (the Hall-Witt identity): ${}^x[[x^{-1},y],z]\ {}^z[[z^{-1},x],y]\ {}^y[[y^{-1},z],x] = 1$;

(C4) $[x, {}^y z] = {}^y[{}^{y^{-1}}x, z]$;

(C5) $[{}^y x, z] = {}^y[x, {}^{y^{-1}} z]$.

(C6) If H and K are subgroups of G, then $[H,K] = [K,H]$.

(C7) If F, H and K are subgroups of G, then

$$\big[[F,H],K\big] \leqslant \big[[F,K],H\big]\big[F,[H,K]\big].$$

In §11.1 we will provide an example that even in the setting of elementary subgroups of a linear group

$$\big[[F,H],K\big] \neq \big[F,[H,K]\big].$$

(C8) $(xy)^2 = x^2y^2\big[y^{-1},x^{-1}\big]\big[[x^{-1},y^{-1}]y^{-1}\big]$.

One can write numerous identities involving commutators. The reader is referred to [51, 52] for more samples of these identities.

3.2. Let A be a ring and I, J and K be two sided ideals. We denote by

$$I \circ J := IJ + JI,$$

the symmetrised product of ideals $I, J \trianglelefteq A$. In the commutative case it coincides with their usual product. In general, the symmetrised product is not associative. Thus, when writing something like $I \circ J \circ K$ we have to specify the order in which products are formed.

3.3. Limit of rings. An R-algebra A is called *module finite* over R, if A is finitely generated as an R-module. An R-algebra A is called *quasi-finite* over R if there is a direct system of module finite R-subalgebras A_i of A such that $\varinjlim A_i = A$.

Suppose A is an R-algebra and I is an index set. By a direct system of subalgebras A_i/R_i, $i \in I$, of A, we shall mean a set of subrings R_i of R and a set of subrings A_i of A such that each A_i is naturally an R_i-algebra and such that given $i, j \in I$, there is a $k \in I$ such that $R_i \leqslant R_k$, $R_j \leqslant R_k$, $A_i \leqslant A_k$, and $A_j \leqslant A_k$.

Proposition 1. *An R-algebra A is quasi-finite over R if and only if it satisfies the following equivalent conditions:*

(1) *There is a direct system of subalgebras A_i/R_i of A such that each A_i is module finite over R_i and such that $\varinjlim R_i = R$ and $\varinjlim A_i = A$.*
(2) *There is a direct system of subalgebras A_i/R_i of A such that each A_i is module finite over R_i and each R_i is finitely generated as a $\mathbb{Z}$-algebra and such that $\varinjlim R_i = R$ and $\varinjlim A_i = A$.*

3.4. Stable rank of rings. Let us recall the linear case first. These results are most conveniently stated in terms of the new type of dimension for rings, introduced by Bass, stable rank. Since later we shall discuss generalisations of this notion, we recall here its definition.

A row $(a_1, \dots, a_n) \in {}^nA$ is called *unimodular* if the elements $a_1, \dots, a_n$ generate A as a *right* ideal, i.e. $a_1A + \dots + a_nA = A$, or, what is the same, there exist $b_1, \dots, b_n \in A$ such that $a_1b_1 + \dots + a_nb_n = 1$.

A row $(a_1, \dots, a_{n+1}) \in {}^{n+1}A$ is called *stable*, if there exist $b_1, \dots, b_n \in A$ such that the *right* ideal generated by $a_1+a_{n+1}b_1, \dots, a_n+a_{n+1}b_n$ coincides with the *right* ideal generated by $a_1, \dots, a_{n+1}$.

One says that the *stable rank* of the ring A equals n and writes $\mathrm{sr}(A) = n$ if every unimodular row of length $n+1$ is stable, but there exists a non-stable unimodular row of length n. If such n does not exist (i.e. there are non-stable unimodular rows of arbitrary length) we say that the stable rank of A is infinite.

It turned out that stable rank, on one hand, most naturally arises in the proof of results pertaining to linear groups and, on the other hand, it can be easily estimated in terms of other known dimensions of a commutative ring A, say of its Krull dimension $\dim(A)$, or its Jacobson dimension $j(A) = \dim(\mathrm{Max}(A))$. Here, $\mathrm{Max}(A)$ is the subspace of all maximal ideals of the topological space $\mathrm{Spec}(A)$, the set of all prime ideals of A, equipped with the Zariski Topology. Then $j(A)$ is the dimension of the topological space $\mathrm{Max}(A)$. Let us state a typical result in this spirit due to Bass.

Theorem 2. *Let A be a ring finitely generated as a module over a commutative ring R. Then* $\mathrm{sr}(A) \leqslant \dim(\mathrm{Max}(R)) + 1$.

The right hand side should be thought of as a condition expressing (a weaker form of) stability for not necessarily unimodular rows. In [28] and [57] it is shown that already $\mathrm{asr}(A) \leqslant \dim(\mathrm{Max}(R)) + 1$, where $\mathrm{asr}(A)$ stands for the *absolute stable rank*.

§4. General linear groups

Let $G = \mathrm{GL}(n, A)$ be the general linear group of degree n over an associative ring A with 1. Recall that $\mathrm{GL}(n, A)$ is the group of all two-sided invertible square matrices of degree n over A, or, in other words, the multiplicative group of the full matrix ring $M(n, A)$. When one thinks of $A \mapsto \mathrm{GL}(n, A)$ as a functor from rings to groups, one writes GL_n. In the sequel for a matrix $g \in G$ we denote by g_{ij} its matrix entry in the position

(i,j), so that $g=(g_{ij})$, $1\leqslant i,j\leqslant n$. The inverse of g will be denoted by $g^{-1}=(g'_{ij})$, $1\leqslant i,j\leqslant n$.

A crucial role is played by the elementary subgroup of $\mathrm{GL}(n,A)$. As usual we denote by e (or sometimes 1) the identity matrix of degree n and by e_{ij} a standard matrix unit, i.e., the matrix that has 1 in the position (i,j) and zeros elsewhere. An elementary matrices $e_{i,j}(\xi)$ is a matrix of the form

$$e_{i,j}(\xi)=e+\xi e_{ij},\qquad \xi\in A,\quad 1\leqslant i\neq j\leqslant n.$$

An elementary matrices $e_{i,j}(\xi)$ only differs from the identity matrix in the position (i,j), $i\neq j$, where it has ξ instead of 0. In other words, multiplication by an elementary matrix on the left/right performs what in an undergraduate linear algebra course would be called a row/column elementary transformation 'of the first kind'.

If there is no danger we simply write $e_{ij}(\xi)$ instead of $e_{i,j}(\xi)$.

The *elementary subgroup* $E(n,A)$ of the general linear group $\mathrm{GL}(n,A)$ is generated by all the elementary matrices. That is,

$$E(n,A)=\langle e_{ij}(\xi),\ \xi\in A,\ 1\leqslant i\neq j\leqslant n\rangle.$$

Both for the development of the theory and for the sake of applications one has to extend these definitions to include relative groups. For a two-sided ideal I of A, one defines the corresponding reduction homomorphism

$$\pi_I:\mathrm{GL}(n,A)\longrightarrow \mathrm{GL}(n,A/I),\qquad (g_{ij})\mapsto (g_{ij}+I).$$

Now the *principal congruence subgroup* $\mathrm{GL}(n,A,I)$ of level I is the kernel of reduction homomorphism π_I, while the *full congruence subgroup* $C(n,A,I)$ of level I is the inverse image of the centre of $\mathrm{GL}(n,A/I)$ with respect to this homomorphism. Clearly both are normal subgroups of $\mathrm{GL}(n,A)$.

Again, let $I\trianglelefteq A$ be a two-sided ideal of A, and let $x=e_{ij}(\xi)$ be an elementary matrix. Somewhat loosely we say that x is of level I, provided $\xi\in I$. One can consider the subgroup generated in $\mathrm{GL}(n,A)$ by all the elementary matrices of level I:

$$E(n,I)=\langle e_{ij}(\xi),\xi\in I,1\leqslant i\neq j\leqslant n\rangle.$$

This group is contained in the absolute elementary subgroup $E(n,A)$ and does not depend on the choice of an ambient ring A with 1. However, in general $E(n,I)$ has little chances to be normal in $\mathrm{GL}(n,A)$. The relative elementary subgroup $E(n,A,I)$ is defined as the normal closure of $E(n,I)$

in $E(n,A)$:

$$E(n,A,I) = \langle e_{ij}(\xi), \xi \in I, 1 \leqslant i \neq j \leqslant n \rangle^{E(n,A)}.$$

We have the following relations among elementary matrices which will be used in the paper. We refer to these relations in the text by (E).

(E1) $e_{i,j}(a)e_{i,j}(b) = e_{i,j}(a+b)$.

(E2) $[e_{i,j}(a), e_{k,l}(b)] = 1$ if $i \neq l, j \neq k$.

(E3) $[e_{i,j}(a), e_{j,k}(b)] = e_{i,k}(ab)$ if $i \neq k$.

Essentially, the following result was first established in the context of Chevalley groups by Michael Stein [97]. The next approximation is the paper by Jacques Tits [112], where it is proven that $E(n,A,I)$ is generated by its intersections with the fundamental SL_2. Nevertheless, the earliest reference, where we could trace this result, was the paper by Leonid Vaserstein and Andrei Suslin [121]. We follow the proof given in [10, Lemma 4.8] (see also in [132, Theorem 11]).

Lemma 3. *Let A be a ring and I be a two-sided ideal of A. Then $E(n,A,I)$ is generated as a group by the elements*

$$z_{ij}(a,\alpha) := {}^{e_{ji}(a)}e_{ij}(\alpha) = e_{ji}(a)e_{ij}(\alpha)e_{ji}(-a),$$

where $i \neq j$, $a \in A$ and $\alpha \in I$.

Proof. By definition, $E(n,A,I)$ is generated by the elements ${}^{e}e_{ij}(\alpha)$, where $i \neq j$, $e \in E(n,A)$, and $\alpha \in I$. If e is the identity matrix, let $l(e) = 0$ and otherwise, let $l(e)$ denote the least number of elementary matrices required to write e as a product of elementary matrices. The proof is by induction on $l(e)$.

We need the following identity in order to reduce the length of e in the induction proof. Let i, j, k be distinct natural numbers and $a, b \in A$ and $\alpha \in I$. Then one can check by straightforward multiplication that

$$\begin{aligned} {}^{e_{ij}(a)e_{ji}(b)}e_{ij}(\alpha) &= e_{kj}(-\alpha(1+ba))e_{ki}(\alpha b)e_{ik}(-ab\alpha b)e_{ij}(ab\alpha) \\ &\times ({}^{e_{jk}(b)}e_{kj}(\alpha))e_{ij}(\alpha)e_{ik}((ab-1)\alpha b)e_{jk}(b\alpha b)({}^{e_{ij}(a)}e_{ji}(-b\alpha b)) \\ &\times ({}^{e_{ki}(1)}e_{ik}(\alpha b))e_{kj}(\alpha ba)e_{ij}(\alpha ba). \quad (5) \end{aligned}$$

We proceed by induction. If $l(e) = 0$, there is nothing to prove. Suppose $l(e) = 1$. Then $e = e_{kl}(a)$ for some $1 \leqslant k \neq l \leqslant n$. If $(k,l) = (j,i)$, there is nothing to prove. If $(k,l) \neq (j,i)$ then by (E), ${}^{e_{kl}(a)}e_{ij}(\alpha)$ is either $e_{ij}(\alpha)$ or $e_{i'j'}(\alpha')e_{ij}(\alpha)$ for an elementary matrix $e_{i'j'}(\alpha')$ such that $\alpha' \in I$.

Suppose $l(e) \geqslant 2$. Write $e = e'e_{mn}(b)e_{kl}(a)$, where $l(e') = l(e) - 2$. If $(k,l) \neq (j,i)$, then applying the paragraph above, one can finish by induction on $l(e)$. Suppose $(k,l) = (l,i)$. If $(m,n) = (i,j)$ then applying (5), one can finish by induction on $l(e)$. Suppose $(m,n) \neq (i,j)$. If $m \neq i$ and $n \neq j$ then by (E)

$$e_{mn}(b)e_{ji}(a) = e_{ji}(a)e_{mn}(b).$$

It is not possible that $(m,n) = (j,i)$, because then it would follow that $e = e'e_{ji}(b+a)$ and thus, that $l(e) \leqslant l(e') + 1$. Since $(m,n) \neq (j,i)$, it follows from (E) that ${}^{e_{mn}(b)}e_{ij}(q)$ is either $e_{ij}(\alpha)$ or $e_{i'j'}(\alpha')e_{ij}(\alpha)$, for an elementary matrix $e_{i'j'}(\alpha')$, where $\alpha' \in I$ and one is done again by induction on $l(e)$. There remain now two cases to check; namely, $(m,n) = (m,j)$ with $m \neq i$ and $(m,n) = (i,n)$ with $n \neq j$. In the first case,

$$\begin{aligned} {}^{e_{mj}(b)e_{ji}(a)}e_{ij}(\alpha) &= {}^{e_{mi}(ba)e_{ji}(a)e_{mj}(b)}e_{ij}(\alpha) \\ &= {}^{e_{mi}(ba)e_{ji}(a)}e_{ij}(\alpha) \\ &= {}^{e_{ji}(a)e_{mi}(ba)}e_{ij}(\alpha) \\ &= {}^{e_{ji}(a)}(e_{mj}(ba\alpha)e_{ij}(\alpha)). \end{aligned}$$

Thus, one can finish by induction on $l(e)$. The second case is checked similarly. □

Using Lemma 3, it is not hard to prove that $E(n,A,I^2) \leqslant E(n,I)$ (see [10, Corollary 4.9] and [112, Proposition 2]). This containment can be slightly generalised to the case of two ideals. This will be established in Lemma 1A which will be used throughout the paper.

The first step in the construction of algebraic K-theory was done by Hyman Bass in [16] almost half century ago. There is a standard embedding

$$\mathrm{GL}(n,A) \longrightarrow \mathrm{GL}(n+1,A), \qquad g \mapsto \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}, \tag{6}$$

called the stabilisation map, which allows us to identify $\mathrm{GL}(n,A)$ with a subgroup in $\mathrm{GL}(n+1,A)$. Now we can consider the stable general linear group

$$\mathrm{GL}(A) = \varinjlim_n \mathrm{GL}(n,A),$$

which is the direct limit (effectively the union) of the $\mathrm{GL}(n,A)$ under the stabilisation embeddings.

Since the stabilisation map sends $E(n,A)$ to $E(n+1,A)$, we can define the stable elementary group $E(A) = \varinjlim E(n,A)$. This subgroup is called the (absolute) *elementary group* of degree n over A.

Applying the stabilisation embeddings to the families $\mathrm{GL}(n,A,I)$ and $E(n,A,I)$ generates stable versions $\mathrm{GL}(A,I)$ and $E(A,I)$, respectively, of these groups. There is no stable version of $C(n,A,I)$, though, as the stability map does not send $C(n,A,I)$ into $C(n+1,A,I)$.

A crucial observation known as the Whitehead lemma, asserts that modulo $E(A)$ the product of two matrices in $\mathrm{GL}(n,A)$ is the same as their direct sum, and in particular, $E(A) = [\mathrm{GL}(A), \mathrm{GL}(A)]$. Such identities in the stable case can be established easily, as there is enough room to arrange the matrices inside $\mathrm{GL}(A)$. For the pedagogical reason we include the proof of the following identity (see Lemma 6)

$$E(A,I) = [E(A), E(A,I)] = [\mathrm{GL}(A), E(A,I)] = [\mathrm{GL}(A), \mathrm{GL}(A,I)].$$

The main theme of this note is to establish the non-stable identities of this type.

First, we need some lemmas.

Lemma 4. *Let A be a ring and I be a two sided ideal of A. Any $n \times n$ upper/lower triangular matrix with 1 on the main diagonal and elements of I as non-zero entries belong to $E(n,I)$.*

Proof. Let x be an upper triangular matrix with 1 on the diagonal and elements of I as non-zero entries, i.e., $x = (a_{ij}) \in M_n(A)$ with $a_{ii} = 1$, $1 \leqslant i \leqslant n$ and $a_{ij} \in I$ for $j > i$. Then the matrix

$$x' = (a'_{ij}) = x e_{12}(-a_{12}) e_{23}(-a_{23}) \dots e_{n-1,n}(-a_{n-1,n}) \tag{7}$$

is still upper triangular with 1 on the main diagonals and 0 on $j - i = 1$. Note that since $a_{ij} \in I$, all the elementary matrices in (7) are in $E(n,I)$.

Now the matrix

$$x'' = (a''_{ij}) = x' e_{13}(-a'_{13}) e_{24}(-a'_{24}) \dots e_{n-2,n}(-a'_{n-2,n}),$$

is again upper triangular with 1 on the main diagonals and 0 on $j - i = 1, 2$. Here also $a'_{ij} \in I$ and so all the elementary matrices are in $E(n,I)$. Continuing in this fashion, by induction, $x^{(n-1)}$ is the identity matrix. Note that all elementary matrices involved are in $E(n,I)$. It follows that $A \in E(n,I)$. The lower triangular case is similar. $\square$

Lemma 5. *Let A be an associative ring and let $I \trianglelefteq A$ be a two-sided ideal of A. Then for any $x, y \in \mathrm{GL}(n, A, I)$ one has*

$$\begin{pmatrix} xyx^{-1}y^{-1} & 0 \\ 0 & 1 \end{pmatrix} \in E(2n, A, I). \tag{8}$$

Proof. Following Bass [16, Lemma 1.7], we first show that

$$\begin{pmatrix} xy & 0 \\ 0 & 1 \end{pmatrix} \equiv \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \pmod{E(2n, A, I)}, \tag{9}$$

and

$$\begin{pmatrix} yx & 0 \\ 0 & 1 \end{pmatrix} \equiv \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \pmod{E(2n, A, I)}, \tag{10}$$

which then immediately implies (8).

Write $y = 1 + q$, where $q \in M_n(I)$. Furthermore, let

$$\alpha = \begin{pmatrix} yx & 0 \\ 0 & 1 \end{pmatrix}, \beta = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}, \tau_1 = \begin{pmatrix} 1 & (yx)^{-1}q \\ 0 & 1 \end{pmatrix},$$
$$\tau_2 = \begin{pmatrix} 1 & -x^{-1}q \\ 0 & 1 \end{pmatrix}, \tau_3 = \begin{pmatrix} 1 & 0 \\ -y^{-1}qx & 1 \end{pmatrix}, \sigma = \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}.$$

By Lemma 4, $\tau_1, \tau_2, \tau_3 \in E(2n, I)$, $\sigma \in E(2n, A)$ and thus by definition $\sigma^{-1}\tau_2\sigma \in E(2n, A, I)$. We get $\tau := \tau_1\sigma^{-1}\tau_2\sigma\tau_3 \in E(2n, A, I)$. Now a simple matrix calculation shows

$$\alpha\tau_1 = \begin{pmatrix} yx & q \\ 0 & 1 \end{pmatrix}, \alpha\tau_1\sigma^{-1} = \begin{pmatrix} yx - qa & q \\ -x & 1 \end{pmatrix} = \begin{pmatrix} x & q \\ -x & 1 \end{pmatrix},$$
$$\alpha\tau_1\sigma^{-1}\tau_2 = \begin{pmatrix} x & -q+q \\ -x & 1+q \end{pmatrix} = \begin{pmatrix} x & 0 \\ -x & y \end{pmatrix},$$
$$\alpha\tau_1\sigma^{-1}\tau_2\sigma = \begin{pmatrix} x & 0 \\ yx - x & y \end{pmatrix} = \begin{pmatrix} x & 0 \\ qx & b \end{pmatrix}.$$

Finally

$$\alpha\tau = \alpha\tau_1\sigma^{-1}\tau_2\sigma\tau_3 = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} = \beta.$$

This shows the Identity (10). Plugging $x = y^{-1}$ into this identity we obtain

$$\begin{pmatrix} y^{-1} & 0 \\ 0 & y \end{pmatrix} \in E(2n, A, I).$$

Thus

$$\begin{pmatrix} xy & 0 \\ 0 & 1 \end{pmatrix} \equiv \begin{pmatrix} xy & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} y^{-1} & 0 \\ 0 & y \end{pmatrix} \equiv \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix},$$

which is Identity (9). □

Lemma 6. *For an associative ring A and an ideal $I \trianglelefteq A$ one has*

$$E(A,I) = [E(A), E(A,I)] = [\mathrm{GL}(A), E(A,I)] = [\mathrm{GL}(A), \mathrm{GL}(A,I)]. \quad (11)$$

Proof. The elements of $E(A,I)$ are generated by $xe_{ij}(\alpha)x^{-1}$, where $e_{ij}(\alpha) \in E(I)$ and $x \in E(A)$. Writing

$$xe_{ij}(\alpha)x^{-1} = [x, e_{ij}(\alpha)]e_{ij}(\alpha) = [x, e_{ij}(\alpha)][e_{ik}(1), e_{kj}(\alpha)] \in [E(A), E(A,I)],$$

it follows that

$$E(A,I) \leqslant [E(A), E(A,I)].$$

Thus we have

$$E(A,I) \leqslant [E(A), E(A,I)] \leqslant [\mathrm{GL}(A), E(A,I)] \leqslant [\mathrm{GL}(A), \mathrm{GL}(A,I)].$$

We show $[\mathrm{GL}(A), \mathrm{GL}(A,I)] \leqslant E(A,I)$. Let $x \in \mathrm{GL}(A)$ and $y \in \mathrm{GL}(A,I)$. Then for a sufficiently large n, $x \in \mathrm{GL}(n,A)$ and $y \in \mathrm{GL}(n,A,I)$. By Lemma 5,

$$\begin{pmatrix} xyx^{-1}y^{-1} & 0 \\ 0 & 1 \end{pmatrix} \in E(2n,A,I) \leqslant E(A,I).$$

This finishes the proof. □

At this point Bass defines

$$K_1(A) = \mathrm{GL}(A)/E(A) = \mathrm{GL}(A)/[\mathrm{GL}(A), \mathrm{GL}(A)]$$

as the abelianisation of $\mathrm{GL}(A)$. Indeed algebraic K-theory was born as Bass observed that the functors K_0 and K_1 together with their relative versions fit into a unified theory with important applications in algebra, algebraic geometry and number theory. In the same manner, the relative K_1-functor of a pair (A,I) is defined as

$$K_1(A,I) = \mathrm{GL}(A,I)/E(A,I).$$

As one of important applications in algebra, Bass [16] relates the normal subgroup structure of $\mathrm{GL}(A)$ to the ideal structure of A. This leap in generality is considered as the starting point of the modern theory of linear groups.

Theorem 7. *Let A be an arbitrary associative ring and $H \leqslant \mathrm{GL}(A)$ be a subgroup normalised by the elementary group $E(A)$. Then there exists a unique ideal $I \trianglelefteq A$ such, that*

$$E(A,I) \leqslant H \leqslant \mathrm{GL}(A,I).$$

Conversely, any subgroup H satisfying these inclusions is (*by Lemma* 6) *normal in* $\mathrm{GL}(A)$.

Quite remarkably this result holds for arbitrary associative rings. Thus, an explicit enumeration of all normal subgroups of $\mathrm{GL}(A)$ amounts to the calculation of $K_1(A, I)$ for all ideals I in A.

The group K_1 answers essentially the question as to how far $\mathrm{GL}(n, A)$ falls short of being spanned by elementary generators. A few years later Milnor [80, 81], building on the work of Steinberg [103, 104] and Moore [82], introduced the group K_2, which measures essentially to which extent all relations among elementary generators follow from the obvious ones.

For any associative ring A, a two-sided ideal $I \trianglelefteq A$ and a fixed n we consider the quotient

$$K_1(n, A, I) = \mathrm{GL}(n, A, I)/E(n, A, I).$$

In general, the elementary subgroup $E(n, R, I)$ does not have to be normal in the congruence subgroup $\mathrm{GL}(n, A, I)$. In particular, $K_1(n, A, I)$ is a pointed set, rather than a group. However, we will see when A is quasi-finite and $n \geqslant 3$, the $K_1(n, A, I)$ is a group. Similarly, we define

$$\mathrm{SK}_1(n, A, I) = \mathrm{SL}(n, A, I)/E(n, A, I),$$

consult [10] for the definition of $\mathrm{SL}(n, A, I)$ for quasi-finite rings.

The stability embedding of the general linear groups (see (6)) sends $E(n, A, I)$ inside $E(n+1, A, I)$. In particular, by the homomorphism theorem it induces stability map

$$\psi_n : K_1(n, A, I) \longrightarrow K_1(n+1, A, I),$$

which is a group homomorphism when both sides are groups. Clearly, ψ_n restricts to a map between $\mathrm{SK}_1(n, A, I)$'s.

The following results, known as the surjective and injective stability for K_1 are due to Bass and to Bass–Vaserstein, respectively.

Lemma 8. *Let A be an associative ring and let $I \trianglelefteq A$ be a two-sided ideal of A. Consider the stability map*

$$\psi_n : K_1(n, A, I) \longrightarrow K_1(n+1, A, I).$$

Then

(1) *If $n \geqslant \mathrm{sr}(A)$, then ψ_n is surjective. In other words*

$$\mathrm{GL}(n+1, A, I) = \mathrm{GL}(n, A, I)E(n+1, A, I).$$

(2) *If* $n \geqslant \mathrm{sr}(A)+1$*, then* ψ_n *is injective. In other words*

$$\mathrm{GL}(n,A,I)\cap E(n+1,A,I)=E(n,A,I).$$

§5. Unitary groups

The notion of Λ-quadratic forms, quadratic modules and generalised unitary groups over a form ring (A,Λ) were introduced by Anthony Bak in his Thesis who studied their K-theory (see [7, 8]).

Although the quadratic setting is much more complicated than the linear one, it is being gradually established that most results concerning the K-theory of general linear groups can be carried over to the K-theory of general quadratic groups.

In this section we *briefly* review the most fundamental notation and results that will be constantly used in the present paper. We refer to [8, 37, 60, 15, 38, 44, 111, 66] for details, proofs, and further references.

5.1. Let R be a commutative ring with 1, and A be an (not necessarily commutative) R-algebra. An involution, denoted by $\overline{}$, is an anti-homomorphism of A of order 2. Namely, for $\alpha,\beta\in A$, one has $\overline{\alpha+\beta}=\overline{\alpha}+\overline{\beta}$, $\overline{\alpha\beta}=\overline{\beta}\,\overline{\alpha}$ and $\overline{\overline{\alpha}}=\alpha$. Fix an element $\lambda\in\mathrm{Cent}(A)$ such that $\lambda\overline{\lambda}=1$. One may define two additive subgroups of A as follows:

$$\Lambda_{\min}=\{\alpha-\lambda\overline{\alpha}\mid\alpha\in A\},\qquad \Lambda_{\max}=\{\alpha\in A\mid\alpha=-\lambda\overline{\alpha}\}.$$

A *form parameter* Λ is an additive subgroup of A such that

(1) $\Lambda_{\min}\leqslant\Lambda\leqslant\Lambda_{\max}$,

(2) $\alpha\Lambda\overline{\alpha}\leqslant\Lambda$ for all $\alpha\in A$.

The pair (A,Λ) is called a *form ring.*

5.2. Let $I\trianglelefteq A$ be a two-sided ideal of A. We assume I to be involution invariant, i.e., such that $\overline{I}=I$. Set

$$\Gamma_{\max}(I)=I\cap\Lambda,\quad \Gamma_{\min}(I)=\{\xi-\lambda\overline{\xi}\mid\xi\in I\}+\langle\xi\alpha\overline{\xi}\mid\xi\in I,\alpha\in\Lambda\rangle.$$

A *relative form parameter* Γ in (A,Λ) of level I is an additive group of I such that

(1) $\Gamma_{\min}(I)\leqslant\Gamma\leqslant\Gamma_{\max}(I)$,

(2) $\alpha\Gamma\overline{\alpha}\leqslant\Gamma$ for all $\alpha\in A$.

The pair (I, Γ) is called a *form ideal*.

In the level calculations we will use sums and products of form ideals. Let (I, Γ) and (J, Δ) be two form ideals. Their sum is artlessly defined as $(I + J, \Gamma + \Delta)$, it is immediate to verify that this is indeed a form ideal.

Guided by analogy, one is tempted to set $(I, \Gamma)(J, \Delta) = (IJ, \Gamma\Delta)$. However, it is considerably harder to correctly define the product of two relative form parameters. The papers [35, 36, 38] introduce the following definition

$$\Gamma\Delta = \Gamma_{\min}(IJ) + {}^{J}\Gamma + {}^{I}\Delta,$$

where

$${}^{J}\Gamma = \langle \xi\Gamma\overline{\xi} \mid \xi \in J \rangle, \qquad {}^{I}\Delta = \langle \xi\Delta\overline{\xi} \mid \xi \in I \rangle.$$

One can verify that this is indeed a relative form parameter of level IJ if $IJ = JI$.

However, in the present paper we do not wish to impose any such commutativity assumptions. Thus, we are forced to consider the symmetrised products

$$I \circ J = IJ + JI, \qquad \Gamma \circ \Delta = \Gamma_{\min}(IJ + JI) + {}^{J}\Gamma + {}^{I}\Delta$$

The notation $\Gamma \circ \Delta$ – as also $\Gamma\Delta$ is slightly misleading, since in fact it depends on I and J, not just on Γ and Δ. Thus, strictly speaking, one should speak of the symmetrised products of *form ideals*

$$(I, \Gamma) \circ (J, \Delta) = (IJ + JI, \Gamma_{\min}(IJ + JI) + {}^{J}\Gamma + {}^{I}\Delta).$$

Clearly, in the above notation one has

$$(I, \Gamma) \circ (J, \Delta) = (I, \Gamma)(J, \Delta) + (J, \Delta)(I, \Gamma).$$

5.3. A *form algebra over a commutative ring* R is a form ring (A, Λ), where A is an R-algebra and the involution leaves R invariant, i.e., $\overline{R} = R$. A form algebra (A, Λ) is called *module finite*, if A is finitely generated as an R-module. A form algebra (A, Λ) is called *quasi-finite*, if there is a direct system of module finite R-subalgebras A_i of A such that $\varinjlim A_i = A$ (see §3.3).

In general Λ is not an R-module. This forces us to replace R by its subring R_0, generated by all $\alpha\overline{\alpha}$ with $\alpha \in R$. Clearly, all elements in R_0 are invariant with respect to the involution, i. e. $\overline{r} = r$, for $r \in R_0$.

It is immediate, that any form parameter Λ is an R_0-module. This simple fact will be used throughout. This is precisely why we have to localise in multiplicative subsets of R_0, rather than in those of R itself (see §12.4).

We now recall the basic notation and facts related to Bak's generalised unitary groups and their elementary subgroups.

5.4. Let, as above, A be an associative ring with 1. For natural m, n we denote by $M(m,n,A)$ the additive group of $m \times n$ matrices with entries in A. In particular $M(m,A) = M(m,m,A)$ is the ring of matrices of degree m over A. For a matrix $x \in M(m,n,A)$ we denote by x_{ij}, $1 \leqslant i \leqslant m$, $1 \leqslant j \leqslant n$, its entry in the position (i,j). Let e be the identity matrix and e_{ij}, $1 \leqslant i,j \leqslant m$, be a standard matrix unit, i.e. the matrix which has 1 in the position (i,j) and zeros elsewhere.

As usual, $\mathrm{GL}(m,A) = M(m,A)^*$ denotes the general linear group of degree m over A. The group $\mathrm{GL}(m,A)$ acts on the free right A-module $V \cong A^m$ of rank m. Fix a base $e_1, \ldots, e_m$ of the module V. We may think of elements $v \in V$ as columns with components in A. In particular, e_i is the column whose i-th coordinate is 1, while all other coordinates are zeros.

In the unitary setting, we are only interested in the case, when $m = 2n$ is even. We usually number the base as follows: $e_1, \ldots, e_n, e_{-n}, \ldots, e_{-1}$. All other occurring geometric objects will be numbered accordingly. Thus, we write

$$v = (v_1, \ldots, v_n, v_{-n}, \ldots, v_{-1})^t,$$

where $v_i \in A$, for vectors in $V \cong A^{2n}$.

The set of indices will be always ordered accordingly, $\Omega = \{1, \ldots, n, -n, \ldots, -1\}$. Clearly, $\Omega = \Omega^+ \sqcup \Omega^-$, where $\Omega^+ = \{1, \ldots, n\}$ and $\Omega^- = \{-n, \ldots, -1\}$. For an element $i \in \Omega$ we denote by $\varepsilon(i)$ the sign of Ω, i.e. $\varepsilon(i) = +1$ if $i \in \Omega^+$, and $\varepsilon(i) = -1$ if $i \in \Omega^-$.

5.5. For a form ring (A, Λ), one considers the *hyperbolic unitary group* $\mathrm{GU}(2n, A, \Lambda)$, see [15, §2]. This group is defined as follows:

One fixes a symmetry $\lambda \in \mathrm{Cent}(A)$, $\lambda\overline{\lambda} = 1$ and supplies the module $V = A^{2n}$ with the following λ-hermitian form $h : V \times V \longrightarrow A$,

$$h(u,v) = \overline{u}_1 v_{-1} + \cdots + \overline{u}_n v_{-n} + \lambda \overline{u}_{-n} v_n + \cdots + \lambda \overline{u}_{-1} v_1.$$

and the following Λ-quadratic form $q : V \longrightarrow A/\Lambda$,

$$q(u) = \overline{u}_1 u_{-1} + \cdots + \overline{u}_n u_{-n} \mod \Lambda.$$

In fact, both forms are engendered by a sesquilinear form f,

$$f(u,v) = \overline{u}_1 v_{-1} + \cdots + \overline{u}_n v_{-n}.$$

Now, $h = f + \lambda\overline{f}$, where $\overline{f}(u,v) = \overline{f(v,u)}$, and $q(v) = f(u,u) \mod \Lambda$.

By definition, the hyperbolic unitary group $\mathrm{GU}(2n, A, \Lambda)$ consists of all elements from $\mathrm{GL}(V) \cong \mathrm{GL}(2n, A)$ preserving the λ-hermitian form h and the Λ-quadratic form q. In other words, $g \in \mathrm{GL}(2n, A)$ belongs to $\mathrm{GU}(2n, A, \Lambda)$ if and only if

$$h(gu, gv) = h(u, v) \quad \text{and} \quad q(gu) = q(u), \qquad \text{for all} \quad u, v \in V.$$

When the form parameter is not maximal or minimal, these groups are not algebraic. However, their internal structure is very similar to that of the usual classical groups. They are also often times called *general quadratic groups*, or *classical-like groups*.

The groups introduced by Bak in his Thesis [7] gather all *even* classical groups under one umbrella. Linear groups, symplectic groups, (even) orthogonal groups, (even) classical unitary groups, are all special cases of his construction. Not only that, Bak's construction allows to introduce a whole new range of *classical like groups*, taking into account hybridisation, defect groups, and other such phenomena in characteristic 2, which before [7] were considered pathological, and required separate analysis outside of the general theory.

To give the idea of how it works, let us illustrate how Bak's construction specialises in the case of hyperbolic groups.

• In the case when involution is trivial, $\lambda = -1$, $\Lambda = \Lambda_{\max} = R$, one gets the split symplectic group $G(2n, R, \Lambda) = \mathrm{Sp}(2n, R)$.

• In the case when involution is trivial, $\lambda = 1$, $\Lambda = \Lambda_{\min} = 0$, one gets the split even orthogonal group $G(2n, R, \Lambda) = O(2n, R)$.

• In the case when involution is non-trivial, $\lambda = -1$, $\Lambda = \Lambda_{\max}$, one gets the classical quasi-split even unitary group $G(2n, R, \Lambda) = U(2n, R)$.

• Let R^o be the ring opposite to R and $R^e = R \oplus R^o$. Define an involution on R^e by $(x, y^o) \mapsto (y, x^o)$ and set $\lambda = (1, 1^o)$. Then there is a unique form parameter $\Lambda = \{(x, -x^o) \mid x \in R\}$. The resulting unitary group

$$G(2n, R^e, \Lambda) = \{(g, g^{-t}) \mid g \in \mathrm{GL}(n, R)\}$$

may be identified with the general linear group $\mathrm{GL}(n, R)$.

Thus, in particular the hyperbolic unitary groups cover Chevalley groups of types A_l, C_l and D_l.

5.6. *Elementary unitary transvections* $T_{ij}(\xi)$ correspond to the pairs $i, j \in \Omega$ such that $i \neq j$. They come in two stocks. Namely, if, moreover,

$i \neq -j$, then for any $\xi \in A$ we set

$$T_{ij}(\xi) = e + \xi e_{ij} - \lambda^{(\varepsilon(j)-\varepsilon(i))/2}\overline{\xi}e_{-j,-i}.$$

These elements are also often called *elementary short root unipotents*. On the other side for $j = -i$ and $\alpha \in \lambda^{-(\varepsilon(i)+1)/2}\Lambda$ we set

$$T_{i,-i}(\alpha) = e + \alpha e_{i,-i}.$$

These elements are also often called *elementary long root elements*.

Note that $\overline{\Lambda} = \overline{\lambda}\Lambda$. In fact, for any element $\alpha \in \Lambda$ one has $\overline{\alpha} = -\overline{\lambda}\alpha$ and thus $\overline{\Lambda}$ coincides with the set of products $\overline{\lambda}\alpha$, $\alpha \in \Lambda$. This means that in the above definition $\alpha \in \overline{\Lambda}$ when $i \in \Omega^+$ and $\alpha \in \Lambda$ when $i \in \Omega^-$.

Subgroups $X_{ij} = \{T_{ij}(\xi) \mid \xi \in A\}$, where $i \neq \pm j$, are called *short root subgroups*. Clearly, $X_{ij} = X_{-j,-i}$. Similarly, subgroups $X_{i,-i} = \{T_{i,-i}(\alpha) \mid \alpha \in \lambda^{-(\varepsilon(i)+1)/2}\Lambda\}$ are called *long root subgroups*.

The *elementary unitary group* $\mathrm{EU}(2n, A, \Lambda)$ is generated by elementary unitary transvections $T_{ij}(\xi)$, $i \neq \pm j$, $\xi \in A$, and $T_{i,-i}(\alpha)$, $\alpha \in \lambda^{-(\varepsilon(i)+1)/2}\Lambda$, see [15, §3].

5.7. Elementary unitary transvections $T_{ij}(\xi)$ satisfy the following *elementary relations*, also known as *Steinberg relations*. These relations will be used throughout this paper.

(R1) $T_{ij}(\xi) = T_{-j,-i}(-\lambda^{(\varepsilon(j)-\varepsilon(i))/2}\overline{\xi})$,

(R2) $T_{ij}(\xi)T_{ij}(\zeta) = T_{ij}(\xi + \zeta)$,

(R3) $[T_{ij}(\xi), T_{hk}(\zeta)] = e$, where $h \neq j, -i$ and $k \neq i, -j$,

(R4) $[T_{ij}(\xi), T_{jh}(\zeta)] = T_{ih}(\xi\zeta)$, where $i, h \neq \pm j$ and $i \neq \pm h$,

(R5) $[T_{ij}(\xi), T_{j,-i}(\zeta)] = T_{i,-i}(\xi\zeta - \lambda^{-\varepsilon(i)}\overline{\zeta\xi})$, where $i \neq \pm j$,

(R6) $[T_{i,-i}(\alpha), T_{-i,j}(\xi)] = T_{ij}(\alpha\xi)T_{-j,j}(-\lambda^{(\varepsilon(j)-\varepsilon(i))/2}\overline{\xi}\alpha\xi)$, where $i \neq \pm j$.

Relation (R1) coordinates two natural parameterisations of the same short root subgroup $X_{ij} = X_{-j,-i}$. Relation (R2) expresses additivity of the natural parameterisations. All other relations are various instances of the Chevalley commutator formula. Namely, (R3) corresponds to the case, where the sum of two roots is not a root, whereas (R4), and (R5) correspond to the case of two short roots, whose sum is a short root, and a long root, respectively. Finally, (R6) is the Chevalley commutator formula for the case of a long root and a short root, whose sum is a root. Observe that any two long roots are either opposite, or orthogonal, so that their sum is never a root.

5.8. There is a standard embedding

$$G(2n,A,\Lambda)\longrightarrow G\big(2(n+1),A,\Lambda\big),\qquad \begin{pmatrix} a & b\\ c & d\end{pmatrix}\mapsto\begin{pmatrix} a & 0 & 0 & b\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ c & 0 & 0 & d\end{pmatrix}$$

called the stabilisation map. In fact some other sources may give a slightly different picture of the right hand side. How the right hand side exactly looks, depends on the ordered basis we choose. With the ordered basis which is used in [8], the standard embedding has the form

$$G(2n,R,\Lambda)\longrightarrow G\big(2(n+1),R,\Lambda\big),\qquad \begin{pmatrix} a & b\\ c & d\end{pmatrix}\mapsto\begin{pmatrix} a & 0 & b & 0\\ 0 & 1 & 0 & 0\\ c & 0 & d & 0\\ 0 & 0 & 0 & 1\end{pmatrix}.$$

Define

$$G(A,\Lambda)=\varinjlim_n G(2n,R,\Lambda)$$

and

$$E(A,\Lambda)=\varinjlim_n E(2n,R,\Lambda).$$

The groups $G(I,\Gamma)$ and $E(I,\Gamma)$ are defined similarly.

One could ask, whether one can carry over Bass' results discussed in §4 to the unitary case? Bak, and in a slightly narrower situation, Vaserstein, established unitary versions of Whitehead's lemma, which in particular implies the following result.

Theorem 9. *Let (A,Λ) be an arbitrary form ring, and (I,Γ) be its form ideal, then*

$$E(I,\Gamma)=[E(A,\Lambda),E(I,\Gamma)]=[G(A,\Lambda),E(I,\Gamma)]=[G(A,\Lambda),G(I,\Gamma)].$$

Now, similarly to the linear case, this allows one to introduce the *unitary K-functor*

$$K_1(I,\Gamma)=G(I,\Gamma)/E(I,\Gamma).$$

A version of unitary K-theory modelled upon the unitary groups has been systematically developed by Bass in [18]. Note that, in some literature, the notation KU is used to denote the unitary K-groups. In other literature, the functor above is called a *quadratic K-functor* and the notation KQ is used. (For a lexicon of notations, see [8, §14]).

As another piece of structure, parallel to the linear situation, let us mention the description of normal subgroups in $G(A,\Lambda)$, that holds over an arbitrary ring.

Theorem 10. *Let (A,Λ) be an arbitrary form ring. If $H \leqslant G(A,\Lambda)$ is a subgroup normalised by $E(A,\Lambda)$, then for a unique form ideal (I,Γ), one has*

$$E(I,\Gamma) \leqslant H \leqslant G(I,\Gamma).$$

Conversely, these inclusions guarantee that H is automatically normal in $G(A,\Lambda)$.

§6. Towards non-stable K-theory

One of the major contributions toward non-stable K-theory of rings is the work of Suslin [105, 114]. He proved that if A is a module finite ring, namely, a ring that is finitely generated as a module over its center, and $n \geqslant 3$, then $E(n,A)$ is a normal subgroup of $\mathrm{GL}(n,A)$. Therefore the non-stable K_1-group, i.e., $\mathrm{GL}(n,A)/E(n,A)$, is well-defined. Later, Borevich and Vavilov [23] and Vaserstein [115], building on Suslin's method, established the *standard commutator formula*:

Theorem 11 (Suslin, Borevich–Vavilov, Vaserstein)**.** *Let A be a module finite ring, I a two-sided ideal of A and $n \geqslant 3$. Then $E(n,A,I)$ is normal in $\mathrm{GL}(n,A)$, i.e.,*

$$[E(n,A,I),\mathrm{GL}(n,A)] = E(n,A,I).$$

Furthermore,

$$[E(n,A),\mathrm{GL}(n,A,I)] = E(n,A,I).$$

One natural question that arises here is whether one has a "finer" mixed commutator formula involving two ideals. In fact this had already been established by Bass for general linear groups of degrees sufficiently larger than the stable rank, when he proved his celebrated classification of subgroups of GL_n normalized by E_n (see [16, Theorem 4.2]):

Theorem 12 (Bass)**.** *Let A be a ring, I, J two-sided ideals of A and $n \geqslant \max(\mathrm{sr}(A)+1,3)$. Then*

$$[E(n,A,I),\mathrm{GL}(n,A,J)] = [E(n,A,I),E(n,A,J)].$$

Later, Mason and Stothers, building on Bass' result, proved ([78, Theorem 3.6, Corollary 3.9], and [76, Theorem 1.3]).

Theorem 13 (Mason-Stothers)**.** *Let A be a ring, I, J two-sided ideals of A and $n \geqslant \max(\mathrm{sr}(A)+1, 3)$. Then*

$$[\,\mathrm{GL}(n,A,I), \mathrm{GL}(n,A,J)] = [E(n,A,I), E(n,A,J)].$$

As the Bass Theorem 12 and the Mason and Stothers Theorem 13 are the starting point of this paper, below we present a new proof of Theorem 13.

Lemma 14. *For any $n \geqslant 1$ one has*

$$[\mathrm{GL}(n,A,I), \mathrm{GL}(n,A,J)] \leqslant [\mathrm{GL}(n,A,I), E(2n,A,J)].$$

Proof. Indeed, if $x \in \mathrm{GL}(n,A,I)$ and $y \in \mathrm{GL}(n,A,J)$. By Whitehead lemma one has

$$y = \begin{pmatrix} y & 0 \\ 0 & e \end{pmatrix} \equiv \begin{pmatrix} e & 0 \\ 0 & y \end{pmatrix} \pmod{E(2n,A,J)}.$$

Since $E(2n,A,J)$ is normal in $\mathrm{GL}(2n,A,J)$, one has

$$y = \begin{pmatrix} y & 0 \\ 0 & e \end{pmatrix} = \begin{pmatrix} e & 0 \\ 0 & y \end{pmatrix} z,$$

for some $z \in E(2n,A,J)$. Since the first factor on the right commutes with $x = \begin{pmatrix} x & 0 \\ 0 & e \end{pmatrix}$ one has $[x,y] = [x,z]$, as claimed. $\square$

Lemma 15. *For any $n \geqslant \max(\mathrm{sr}(A)+1, 3)$ the stability map*

$$\begin{aligned}&[E(n,A,I), E(n,A,J)]/E(n,A,IJ+JI)\\ &\qquad \longrightarrow [E(n+1,A,I), E(n+1,A,J)]/E(n+1,A,IJ+JI)\end{aligned}$$

is an isomorphism.

Proof. Clearly,

$$\begin{aligned}&[E(n,A,I), E(n,A,J)]/E(n,A,IJ+JI)\\ &\qquad \leqslant \mathrm{GL}(n,A,IJ+JI)/E(n,A,IJ+JI) = K_1(n,A,IJ+JI).\end{aligned}$$

By Theorem 3A, $[E(n,A,I), E(n,A,J)]$ is generated by $[E(2,A,I), E(2,A,J)]$ as a normal subgroup of $\mathrm{GL}(n,A)$. Since $K_1(n,A,IJ+JI)$ is central in the quotient $\mathrm{GL}(n,A)/E(n,A,IJ+JI)$, for $n \geqslant \mathrm{sr}(A)$, the stability map is surjective and becomes an isomorphism one step further, when the stability map

$$K_1(n,A,IJ+JI) \longrightarrow K_1(n+1,A,IJ+JI)$$

becomes an isomorphism by Lemma 8. □

Lemma 1A. *Let A be a ring and I, J be two-sided ideals of A. Then*

$$\begin{aligned} E(n,A,IJ+JI) &\leqslant [E(n,I),E(n,J)] \leqslant [E(n,A,I),E(n,A,J)] \\ &\leqslant [E(n,A,I),\mathrm{GL}(n,A,J)] \leqslant [\,\mathrm{GL}(n,A,I),\mathrm{GL}(n,A,J)] \\ &\leqslant \mathrm{GL}(n,A,IJ+JI). \end{aligned}$$

Proof. We first show

$$E(n,A,IJ+JI) \leqslant [E(n,I),E(n,J)]. \tag{12}$$

By Lemma 3, let ${}^{e_{i,j}(a)}e_{j,i}(\beta)$ be a generator of $E(n,A,IJ+JI)$, where $a \in A$ and $\beta \in IJ+JI$. It suffices to show that ${}^{e_{i,j}(a)}e_{j,i}(\alpha\beta) \in [E(n,I),E(n,J)]$, where $\alpha \in I$ and $\beta \in J$. Using (E3), we have

$$\begin{aligned} &{}^{e_{i,j}(a)}e_{j,i}(\alpha\beta) = {}^{e_{i,j}(a)}\big[e_{j,k}(\alpha),e_{k,i}(\beta)\big] \\ &= \big[{}^{e_{i,j}(a)}e_{j,k}(\alpha),{}^{e_{i,j}(a)}e_{k,i}(\beta)\big] \\ &= \big[[e_{i,j}(a),e_{j,k}(\alpha)]e_{j,k}(\alpha),e_{k,i}(\beta)[e_{k,i}(-\beta),e_{i,j}(a)]\big] \\ &= \big[e_{i,k}(a\alpha)e_{j,k}(\alpha),e_{k,i}(\beta)e_{k,j}(-\beta a)\big] \\ &\in [E(n,I),E(n,J)]. \end{aligned}$$

This shows (12). We are left to show that

$$[\,\mathrm{GL}(n,A,I),\mathrm{GL}(n,A,J)] \leqslant \mathrm{GL}(n,A,IJ+JI). \tag{13}$$

Let $x \in \mathrm{GL}(n,A,I)$ and $y \in \mathrm{GL}(n,A,J)$. Then $x = e + x_1$ and $x^{-1} = e + x_2$ for some $x_1, x_2 \in M(n,I)$ such that $x_1 + x_2 + x_1x_2 = 0$. Similarly, $y = e + y_1$ and $y^{-1} = e + y_2$ for some $y_1, y_2 \in M(n,J)$ such that $y_1 + y_2 + y_1y_2 = 0$. Then the following equality holds modulo $IJ + JI$.

$$\begin{aligned} [x,y] &= (e+x_1)(e+y_1)(e+x_2)(e+y_2) \\ &= e + x_1 + x_2 + x_1x_2 + y_1 + y_2 + y_1y_2 = e \end{aligned}$$

which proves (13). □

A stable version of Lemma 1A implies that

$$[E(R,I),E(R,J)]/E(R,I\circ J)$$

lives inside $K_1(R, I \circ J)$.

Proof of Theorem 13. By Lemma 14 one has

$$\begin{aligned}E(n,A,IJ+JI) &\leqslant [E(2n,A,I),E(2n,A,J)]\\ &\leqslant [\mathrm{GL}(n,A,I),\mathrm{GL}(n,A,J)] \leqslant [\mathrm{GL}(n,A,I),E(2n,A,J)]\\ &\leqslant [\mathrm{GL}(2n,A,I),E(2n,A,J)] = [E(2n,A,I),E(2n,A,J)].\end{aligned}$$

By Lemma 1A one has $[\mathrm{GL}(n,A,I),\mathrm{GL}(n,A,J)] \leqslant \mathrm{GL}(n,R,IJ+JI)$. On the other hand, by Lemma 15

$$[E(2n,A,I),E(2n,A,J)] \cap \mathrm{GL}(n,R,IJ+JI) \leqslant [E(n,A,I),E(n,A,J)],$$

so that

$$[\mathrm{GL}(n,A,I),\mathrm{GL}(n,A,J)] = [E(n,A,I),E(n,A,J)],$$

as claimed. □

There are (counter)examples that the Mason–Stothers Theorem does not hold for an arbitrary module finite ring [10]. However, recently Stepanov and Vavilov [129] proved Bass' Theorem 12 for any commutative ring and $n \geqslant 3$. The authors, using Bak's localisation and patching method, extended the theorem to all module finite rings [49]. Then in [131], using the Hall-Witt identity, a very short proof for this theorem was found. We include this proof here. We refer to Bass' Theorem in this setting as the *generalised commutator formula.*

Theorem 1A (Generalized commutator formula)**.** *Let A be a module finite R-algebra and I,J be two-sided ideals of A. Then*

$$[E(n,A,I),\mathrm{GL}(n,A,J)] = [E(n,A,I),E(n,A,J)].$$

Proof. We first prove

$$[E(n,A,I),\mathrm{GL}(n,A,J)] \leqslant [E(n,A,I),E(n,A,J)]. \tag{14}$$

Writing $E(n,A,I) = [E(n,A),E(n,A,I)]$ by Theorem 11 and then using the three subgroup lemma, i.e., $[[F,H],L] \leqslant [[F,L],H][F,[H,L]]$ for three normal subgroups F,H and L of a group G, we have

$$\begin{aligned}&[E(n,A,I),\mathrm{GL}(n,A,J)] = \big[[E(n,A),E(n,A,I)],\mathrm{GL}(n,A,J)\big]\\ &\leqslant \big[[E(n,A),\mathrm{GL}(n,A,J)],E(n,A,I)\big]\big[E(n,A),[E(n,A,I),\mathrm{GL}(n,A,J)]\big].\end{aligned}$$

But using Theorem 11,

$$\big[[E(n,A),\mathrm{GL}(n,A,J)],E(n,A,I)\big] = [E(n,A,I),E(n,A,J)].$$

On the other hand using Theorem 1A twice, along with Theorem 11 again, we get

$$\begin{aligned}\big[E(n,A),[E(n,A,I),\mathrm{GL}(n,A,J)]\big] &\leqslant [E(n,A),\mathrm{GL}(n,A,IJ+JI)]\\ &\leqslant E(n,A,IJ+JI) \leqslant [E(n,A,I),E(n,A,J)].\end{aligned}$$

The inclusion (14) now follows. The opposite inclusion is obvious. □

In the similar manner one can establish the generalised commutator formula in the setting of unitary groups and Chevalley groups. Again, in these setting the calculations are more challenging. We include the proof of the unitary version of Lemma 1A as an indication of complexity of calculations. Recall from §5 that

$$(I,\Gamma)\circ(J,\Delta)=(IJ+JI,\Gamma_{\min}(IJ+JI)+{}^J\Gamma+{}^I\Delta).$$

Lemma 1B. *Let (I,Γ) and (J,Δ) be two form ideals of a form ring (A,Λ). Then*

$$\begin{aligned}\mathrm{EU}(2n,(I,\Gamma)\circ(J,\Delta)) &\leqslant [\mathrm{FU}(2n,I,\Gamma),\mathrm{FU}(2n,J,\Delta)]\\ &\leqslant [\mathrm{EU}(2n,I,\Gamma),\mathrm{EU}(2n,J,\Delta)] \leqslant [\mathrm{GU}(2n,I,\Gamma),\mathrm{GU}(2n,J,\Delta)]\\ &\leqslant \mathrm{GU}(2n,(I,\Gamma)\circ(J,\Delta)).\end{aligned}$$

Proof. We first show

$$\mathrm{EU}(2n,(I,\Gamma)\circ(J,\Delta)) \leqslant [\mathrm{FU}(2n,I,\Gamma),\mathrm{FU}(2n,J,\Delta)]. \tag{15}$$

It is well known that $\mathrm{EU}(2n,(I,\Gamma)\circ(J,\Delta))$ is generated by $T_{i,j}(\alpha)^{T_{j,i}(\xi)}$ with $\alpha\in I\circ J$, $\xi\in A$ when $i\neq\pm j$ and with $\alpha\in\lambda^{-(\varepsilon(i)+1)/2}\Gamma\circ\Delta$ and $\xi\in\lambda^{(\varepsilon(i)-1)/2}\Lambda$ when $i=-j$. We divide the proof into cases according the length of the elementary element.

Case I. $T_{i,j}(\alpha)$ is a short root, namely $i\neq\pm j$. Then $\alpha\in I\circ J$. It is sufficient show that $T_{i,j}(a_1b_1+a_2b_2)^{T_{j,i}(\xi)}\in[\mathrm{FU}(2n,I,\Gamma),\mathrm{FU}(2n,J,\Delta)]$ for any $a_1,a_2\in I$ and $b_1,b_2\in J$. By (R2), we have

$$T_{i,j}(a_1b_1+a_2b_2)^{T_{j,i}(\xi)}=T_{i,j}(a_1b_1)^{T_{i,j}(\xi)}T_{i,j}(a_2b_2)^{T_{j,i}(\xi)}.$$

We will show the first factor of the right hand side of the above equation, and left the second to the reader. Choose a $k\neq\pm i,\pm j$. Using (R4), the first factor can be rewritten as a commutator

$$\begin{aligned}T_{i,j}(a_1b_1)^{T_{j,i}(\xi)} &= [T_{i,k}(a_1), T_{k,j}(b_1)]^{T_{j,i}(\xi)}\\ &= [T_{i,k}(a_1)^{T_{j,i}(\xi)}, T_{k,j}(b_1)^{T_{j,i}(\xi)}].\\ &= \big[[T_{i,k}(a_1), T_{j,i}(-\xi)]T_{i,k}(a_1), [T_{k,j}(b_1), T_{j,i}(-\xi)]T_{k,j}(b_1)\big].\end{aligned}$$

Again by (R4), we have

$$\begin{aligned}&\big[[T_{i,k}(a_1), T_{j,i}(-\xi)]T_{i,k}(a_1), [T_{k,j}(b_1), T_{j,i}(-\xi)]T_{k,j}(b_1)\big]\\ &\qquad = \big[T_{j,k}(a_1\xi)T_{i,k}(a_1), T_{k,i}(-\xi b_1)T_{k,j}(b_1)\big].\end{aligned}$$

Clearly $a_1\xi, a_1 \in I$ and $-\xi b_1, b_1 \in J$, thus

$$\begin{aligned}T_{i,j}(a_1b_1)^{T_{j,i}(\xi)} &= \big[T_{j,k}(a_1\xi)T_{i,k}(a_1), T_{k,i}(-\xi b_1)T_{k,j}(b_1)\big]\\ &\in [\mathrm{FU}(2n, I, \Gamma), \mathrm{FU}(2n, J, \Delta)]. \qquad (16)\end{aligned}$$

This finishes the proof of Case I.

Case II. $T_{i,j}(\alpha)$ is a long root, namely $i = -j$. Therefore we have $\alpha \in \lambda^{-(\varepsilon(i)+1)/2}\Gamma \circ \Delta$. Without loss of generality, we may assume that $i < 0$. Hence $\alpha \in \Gamma \circ \Delta$. By definition,

$$\Gamma \circ \Delta = \Gamma^J + \Delta^I + \Gamma_{min}(IJ + JI).$$

It suffices to show that

$$T_{i,-i}(\alpha_1 + \alpha_2 + \alpha_3)^{T_{-i,i}(\xi)} \in [\mathrm{FU}(2n, I, \Gamma), \mathrm{FU}(2n, J, \Delta)]$$

with $\alpha_1 \in \Gamma^J$, $\alpha 2 \in \Delta^I$ and $\alpha_2 \in \Gamma_{min}(IJ + JI)$. By (R2),

$$T_{i,-i}(\alpha_1+\alpha_2+\alpha_3)^{T_{-i,i}(\xi)} = T_{i,-i}(\alpha_1)^{T_{-i,i}(\xi)}T_{i,-i}(\alpha_2)^{T_{-i,i}(\xi)}T_{i,-i}(\alpha_3)^{T_{-i,i}(\xi)}.$$

We prove one by one that each of the factors above belongs to

$$[\mathrm{FU}(2n, I, \Gamma), \mathrm{FU}(2n, J, \Delta)].$$

Since $\alpha_1 \in \Gamma^J$, we may rewrite $\alpha_1 = a\gamma\overline{a}$ with $a \in J$ and $\gamma \in \Gamma$. Therefore

$$T_{i,-i}(\alpha_1)^{T_{-i,i}(\xi)} = T_{i,-i}(a\gamma\overline{a})^{T_{-i,i}(\xi)}.$$

Choose a $j \neq i$ and $j < 0$. Equation (R6) implies that

$$\begin{aligned}&T_{i,-i}(a\gamma\overline{a})^{T_{-i,i}(\xi)} = \big(T_{i,-j}(-a\gamma)[T_{i,j}(a), T_{j,-j}(\gamma)]\big)^{T_{-i,i}(\xi)}\\ &= T_{i,-j}(-a\gamma)^{T_{-i,i}(\xi)}[T_{i,j}(a), T_{j,-j}(\gamma)]^{T_{-i,i}(\xi)}\\ &= [T_{i,-j}(-a\gamma), T_{-i,i}(\xi)]T_{i,-j}(-a\gamma)[T_{i,j}(a), T_{j,-j}(\gamma)]^{T_{-i,i}(\xi)}. \qquad (17)\end{aligned}$$

Again by (R6), the first factor

$$[T_{i,-j}(-a\gamma), T_{-i,i}(\xi)] = T_{-j,j}(-\overline{\lambda}\gamma a\xi\overline{\gamma a})T_{-i,-j}(\overline{\lambda}\xi\overline{\gamma a}).$$

Because $\gamma a\xi\overline{\gamma a} \in \Gamma_{min}(I \circ J)$ and $\overline{\lambda}\xi\overline{\gamma a} \in I \circ J$, we have

$$\begin{aligned} T_{-j,j}(-\overline{\lambda}\gamma a\xi\overline{\gamma a})T_{-i,-j}(\overline{\lambda}\xi\overline{\gamma a}) &\in \mathrm{FU}(2n, (I,\Gamma) \circ (J,\Delta)) \\ &\leqslant [\mathrm{FU}(2n, I, \Gamma), \mathrm{FU}(2n, J, \Delta)]. \end{aligned}$$

Furthermore, $\gamma a \in I \circ J$ implies the second factor of (17)

$$T_{i,j}(-\gamma a) \in \mathrm{FU}(2n, (I,\Gamma) \circ (J,\Delta)) \leqslant [\mathrm{FU}(2n, I, \Gamma), \mathrm{FU}(2n, J, \Delta)].$$

As for the last factor of (17),

$$\begin{aligned} [T_{i,j}(a), T_{j,-j}(\gamma)]^{T_{-i,i}(\xi)} &= [T_{i,j}(a)^{T_{-i,i}(\xi)}, T_{j,-j}(\gamma)^{T_{-i,i}(\xi)}] \\ &= \big[[T_{i,j}(a), T_{-i,i}(\xi)]T_{i,j}(a), T_{j,-j}(\gamma)\big]. \end{aligned}$$

Apply (R6) to the first component of the commutator above shows that

$$[T_{i,j}(a), T_{-i,i}(\xi)]T_{i,j}(a) = T_{-j,j}(\lambda\overline{a}\xi a)T_{-i,j}(-\xi a)T_{i,j}(a) \in \mathrm{FU}(2n, J, \Delta).$$

Thus

$$T_{i,-i}(\alpha_1)^{T_{-i,i}(\xi)} \in [\mathrm{FU}(2n, I, \Gamma), \mathrm{FU}(2n, J, \Delta)].$$

A similar argument, which is left to the reader, shows that

$$T_{i,-i}(\alpha_2)^{T_{-i,i}(\xi)} \in [\mathrm{FU}(2n, I, \Gamma), \mathrm{FU}(2n, J, \Delta)].$$

For the third factor of (16), we have

$$T_{i,-i}(\alpha_3)^{T_{-i,i}(\xi)} \in T_{i,-i}(\Gamma_{\min}(IJ + JI))^{T_{-i,i}(\xi)}.$$

By definition,

$$\Gamma_{\min}(IJ + JI) = \{a - \lambda\overline{a} \mid a \in IJ + JI\} + \langle b\gamma\overline{b} \mid b \in IJ + JI, \gamma \in \Lambda\rangle.$$

Hence, we shall show that for any given α_4 and α_5 which belong to the first and second summands of the above equation, respectively, one has

$$T_{i,-i}(\alpha_4)^{T_{-i,i}(\xi)} \text{ and } T_{i,-i}(\alpha_5)^{T_{-i,i}(\xi)} \in [\mathrm{FU}(2n, I, \Gamma), \mathrm{FU}(2n, J, \Delta)].$$

For the first inclusion, take a typical generator $c_1d_1 + d_2c_1 - \lambda\overline{c_1d_1 + d_2c_2}$ of $\{a - \lambda\overline{a} \mid a \in IJ + JI\}$ with $c_1, c_2 \in I$ and $d_1, d_2 \in J$. It suffices to prove that

$$\begin{aligned} &T_{i,-i}(c_1d_1 + d_2c_2 - \lambda\overline{c_1d_1 + d_2c_2})^{T_{-i,i}(\xi)} \\ &\qquad = T_{i,-i}(c_1d_1 - \lambda\overline{c_1d_1})^{T_{-i,i}(\xi)}T_{i,-i}(d_2c_2 - \lambda\overline{d_2c_2})^{T_{-i,i}(\xi)} \end{aligned}$$

belongs to above relative commutator subgroup. We shall prove this inclusion for $T_{i,-i}(c_1d_1 - \lambda\overline{c_1d_1})^{T_{-i,i}(\xi)}$ and the rest follows by the same arguments.

Choose a $j \neq i$ and $j < 0$. Using (R5), we get

$$\begin{aligned} &T_{i,-i}(c_1d_1 - \lambda\overline{c_1d_1})^{T_{-i,i}(\xi)} = [T_{i,j}(c_1), T_{j,-i}(d_1)]^{T_{-i,i}(\xi)} \\ &= [T_{i,j}(c_1)^{T_{-i,i}(\xi)}, T_{j,-i}(d_1)^{T_{-i,i}(\xi)}] \\ &= \big[[T_{i,j}(c_1), T_{-i,i}(\xi)]T_{i,j}(c_1), [T_{j,-i}(d_1), T_{-i,i}(\xi)]T_{j,-i}(d_1)\big]. \end{aligned}$$

By (R6), $[T_{i,j}(c_1), T_{-i,i}(\xi)]$ can be written as a product of elements from $\mathrm{FU}(2n, I, \Gamma)$ and $[T_{j,-i}(d_1), T_{-i,i}(\xi)]$ a product of elements from $\mathrm{FU}(2n, J, \Delta)$. Thus

$$T_{i,-i}(\alpha_4)^{T_{-i,i}(\xi)} \in [\mathrm{FU}(2n, I, \Gamma), \mathrm{FU}(2n, J, \Delta)].$$

Finally, as $\alpha_5 \in \langle b\gamma\overline{b} \mid b \in IJ + JI, \gamma \in \Lambda\rangle$, we reduce our proof by (R2) to the case

$$\alpha_5 = \big(\sum_k a_k\big)\gamma\big(\overline{\sum_k a_k}\big), \qquad \text{with} \quad a_k \in IJ + JI.$$

By induction, it can be further reduced to

$$\alpha_5 \quad = \quad (a_1b_1 + b_2a_2)\gamma\overline{a_1b_1 + b_2a_2}$$

with $a_1, a_2 \in I$ and $b_1, b_2 \in J$. The above equation can be rewritten as

$$\begin{aligned} \alpha_5 &= a_1b_1\gamma\overline{a_1b_1} + b_2a_2\gamma\overline{b_2a_2} + a_1b_1\gamma\overline{a_2b_2} + a_2b_2\gamma\overline{a_1b_1} \\ &= a_1b_1\gamma\overline{a_1b_1} + b_2a_2\gamma\overline{b_2a_2} + (a_1b_1\gamma\overline{a_2b_2} - \lambda\overline{a_1b_1\gamma\overline{a_2b_2}}). \end{aligned}$$

The last summand is of the same form as α_4's, hence it follows immediately by the proof of α_4 that

$$T_{i,-i}(a_1b_1\gamma\overline{a_2b_2} - \lambda\overline{a_1b_1\gamma\overline{a_2b_2}})^{T_{-i,i}(\xi)} \in [\mathrm{FU}(2n, I, \Gamma), \mathrm{FU}(2n, J, \Delta)].$$

Now consider the first two summands. Note that

$$a_1b_1\gamma\overline{a_1b_1} = a_1(b_1\gamma\overline{b}_1)\overline{a}_1$$

and

$$b_2a_2\gamma\overline{b_2a_2} = b_2(a_2\gamma\overline{a}_2)\overline{b}_2.$$

By the definition of relative form parameter, $a_1(b_1\gamma\overline{b}_1)\overline{a}_1$ and $b_2(a_2\gamma\overline{a}_2)\overline{b}_2$ belong to Δ^I and Γ^J respectively. The proofs for α_1 and α_2 show that

$$\begin{aligned}&T_{i,-i}(a_1(b_1\gamma\overline{b}_1)\overline{a}_1)^{T_{-i,i}(xi)}T_{i,-i}(a_2(b_2\gamma\overline{b}_2)\overline{a}_2)^{T_{-i,i}(xi)}\\&\in [\mathrm{FU}(2n,I,\Gamma),\mathrm{FU}(2n,J,\Delta)].\end{aligned}$$

This proves (15). We are left to show that

$$[\mathrm{GU}(2n,I,\Gamma),\mathrm{GU}(2n,J,\Delta)] \leqslant \mathrm{GU}(2n,(I,\Gamma)\circ(J,\Delta)). \tag{18}$$

We first show that (18) holds for the *stable* unitary groups, namely that

$$[\mathrm{GU}(I,\Gamma),\mathrm{GU}(J,\Delta)] \leqslant \mathrm{GU}((I,\Gamma)\circ(J,\Delta)). \tag{19}$$

In the stable level, we have inclusions

$$\mathrm{EU}((I,\Gamma)\circ(J,\Delta)) \leqslant [\mathrm{EU}(I,\Gamma),\mathrm{EU}(J,\Delta)] \leqslant [\mathrm{GU}(I,\Gamma),\mathrm{GU}(J,\Delta)] \tag{20}$$

and

$$[\mathrm{EU}(I,\Gamma),\mathrm{EU}(J,\Delta)] \leqslant \mathrm{GU}((I,\Gamma)\circ(J,\Delta)). \tag{21}$$

Since the subgroup $[\mathrm{GU}(I,\Gamma),\mathrm{GU}(J,\Delta)]$ is normalized by $E(A,\Lambda)$, applying Theorem 10, we can conclude that there exists a unique form ideal (K,Ω) such that

$$\mathrm{EU}(K,\Omega) \leqslant [\mathrm{GU}(I,\Gamma),\mathrm{GU}(J,\Delta)] \leqslant \mathrm{GU}(K,\Omega). \tag{22}$$

By Identity (C7), we get

$$\begin{aligned}&[[\mathrm{GU}(I,\Gamma),\mathrm{GU}(J,\Delta)],\mathrm{EU}(A,\Lambda)]\\&\leqslant [[\mathrm{GU}(I,\Gamma),\mathrm{EU}(A,\Lambda)],\mathrm{GU}(J,\Delta)]\cdot[[\mathrm{GU}(J,\Delta),\mathrm{EU}(A,\Lambda)],\mathrm{GU}(I,\Gamma)].\end{aligned}$$

But the absolute commutator formula implies that

$$\begin{aligned}&[[\mathrm{GU}(I,\Gamma),\mathrm{EU}(A,\Lambda)],\mathrm{GU}(J,\Delta)]\cdot[[\mathrm{GU}(J,\Delta),\mathrm{EU}(A,\Lambda)],\mathrm{GU}(I,\Gamma)]\\&\qquad= [\mathrm{EU}(I,\Gamma),\mathrm{EU}(J,\Delta)].\end{aligned} \tag{23}$$

Thus,

$$[[\mathrm{GU}(I,\Gamma),\mathrm{GU}(J,\Delta)],\mathrm{EU}(A,\Lambda)] \leqslant [\mathrm{EU}(I,\Gamma),\mathrm{EU}(J,\Delta)]. \tag{24}$$

Again by the general commutator formula and (21), we have

$$\begin{aligned}\mathrm{EU}((I,\Gamma)\circ(J,\Delta)) &= [\mathrm{EU}((I,\Gamma)\circ(J,\Delta)),\mathrm{EU}(A,\Lambda)]\\&\leqslant [[\mathrm{EU}(I,\Gamma),\mathrm{EU}(J,\Delta)],\mathrm{EU}(A,\Lambda)]\\&\leqslant [\mathrm{GU}((I,\Gamma)\circ(J,\Delta)),\mathrm{EU}(A,\Lambda)] = \mathrm{EU}((I,\Gamma)\circ(J,\Delta)).\end{aligned} \tag{25}$$

Forming another commutator of (24) with $\mathrm{EU}(A,\Lambda)$ and applying the inequalities obtained in (25) we get

$$\big[[[\mathrm{GU}(I,\Gamma),\mathrm{GU}(J,\Delta)],\mathrm{EU}(A,\Lambda)],\mathrm{EU}(A,\Lambda)\big]=\mathrm{EU}((I,\Gamma)\circ(J,\Delta)).$$

Using inclusions (22), we see that

$$\begin{aligned}
\mathrm{EU}(K,\Omega)&=[[\mathrm{EU}(K,\Omega),\mathrm{EU}(A,\Lambda)],\mathrm{EU}(A,\Lambda)]\\
&\leqslant\big[[[\mathrm{GU}(I,\Gamma),\mathrm{GU}(J,\Delta)],\mathrm{EU}(A,\Lambda)],\mathrm{EU}(A,\Lambda)\big]=\mathrm{EU}((I,\Gamma)\circ(J,\Delta))\\
&\qquad=[[\mathrm{EU}((I,\Gamma)\circ(J,\Delta)),\mathrm{EU}(A,\Lambda)],\mathrm{EU}(A,\Lambda)]\\
&\qquad\leqslant\big[[[\mathrm{GU}(I,\Gamma),\mathrm{GU}(J,\Delta)],\mathrm{EU}(A,\Lambda)],\mathrm{EU}(A,\Lambda)\big]\\
&\qquad\qquad\leqslant[[\mathrm{GU}(K,\Omega),\mathrm{EU}(A,\Lambda)],\mathrm{EU}(A,\Lambda)]=\mathrm{EU}(K,\Omega).
\end{aligned}$$

Thus, we can conclude that $\mathrm{EU}(K,\Omega)=\mathrm{EU}((I,\Gamma)\circ(J,\Delta))$. This implies that $(K,\Omega)=(I,\Gamma)\circ(J,\Delta)$, see the second paragraph of the proof of [37, Theorem 5.4.10]. Substituting this equality in (22), we see that inclusion (19) holds at the stable level, as claimed.

Let φ denote the usual stability embedding $\varphi:\mathrm{GU}(2n,A,\Lambda)\to\mathrm{GU}(A,\Lambda)$. Then

$$\begin{aligned}
\varphi([\,\mathrm{GU}(2n,I,\Gamma),\mathrm{GU}(2n,J,\Delta)])&=[\varphi(\,\mathrm{GU}(2n,I,\Gamma)),\varphi(\,\mathrm{GU}(2n,J,\Delta))]\\
&<[\,\mathrm{GU}(I,\Gamma),\mathrm{GU}(J,\Delta)].
\end{aligned}$$

In particular, the result at the stable level implies that

$$\varphi([\,\mathrm{GU}(2n,I,\Gamma),\mathrm{GU}(2n,J,\Delta)])\leqslant\varphi(\,\mathrm{GU}(2n,A,\Lambda))\cap\mathrm{GU}((I,\Gamma)\circ(J,\Delta)).$$

On the other hand,

$$\varphi(\,\mathrm{GU}(2n,A,\Lambda))\cap\mathrm{GU}((I,\Gamma)\circ(J,\Delta))=\varphi(\,\mathrm{GU}(2n,(I,\Gamma)\circ(J,\Delta))).$$

Since φ is injective, we can conclude that

$$[\mathrm{GU}(2n,I,\Gamma),\mathrm{GU}(2n,J,\Delta)]\leqslant\mathrm{GU}(2n,(I,\Gamma)\circ(J,\Delta)).$$

This finishes the proof. $\square$

We can state the unitary version of generalised commutator formula.

Theorem 1B. *Let $n\geqslant 3$, R be a commutative ring, (A,Λ) be a form ring such that A is a module finite R-algebra. Further, let (I,Γ) and (J,Δ) be two form ideals of the form ring (A,Λ). Then*

$$[\,\mathrm{EU}(2n,I,\Gamma),\mathrm{GU}(2n,J,\Delta)]=[\,\mathrm{EU}(2n,I,\Gamma),\mathrm{EU}(2n,J,\Delta)].$$

Actually, in the *commutative* case the principal congruence subgroup in the left hand side of the equalities can be replaced by the full congruence subgroup. In other words, when R is commutative, one has

$$[E(n,R,I),C(n,R,J)] = [E(n,R,I),E(n,R,J)].$$

Similarly, when A is commutative, one has

$$[\,\mathrm{EU}(2n,I,\Gamma),\mathrm{CU}(2n,J,\Delta)] = [\,\mathrm{EU}(2n,I,\Gamma),\mathrm{EU}(2n,J,\Delta)].$$

On the other hand, it is easy to construct non-commutative counter-examples to these stronger assertions, see [76].

Finally, for Chevalley groups the corresponding result was first officially stated by You Hong [138, Theorem 1], see also [46, Lemmas 17,19].

Lemma 1C. *Let* $\mathrm{rk}(\Phi) \geqslant 2$. *In the cases* $\Phi = \mathrm{B}_2, \mathrm{G}_2$ *assume that* R *does not have residue fields* $\mathbb{F}_2$ *of* 2 *elements and in the case* $\Phi = \mathrm{B}_2$ *assume additionally that any* $c \in R$ *is contained in the ideal* $c^2R + 2cR$.

Then for any two ideals I *and* J *of the ring* R *one has the following inclusion*

$$\begin{aligned} E(\Phi,R,IJ) \leqslant [E(\Phi,R,I),E(\Phi,R,J)] &\leqslant [E(\Phi,R,I),G(\Phi,R,J)] \\ &\leqslant [G(\Phi,R,I),C(\Phi,R,J)] \leqslant G(\Phi,R,IJ). \end{aligned}$$

For groups of rank 2, these additional assumptions are indeed necessary. It is classically known that when the ground ring R has residue fields of 2 elements, the groups of types B_2 and G_2 are not perfect. Thus, the left-most inclusion fails even at the absolute level, when $I = J = R$.

The second assumption for B_2 is not visible at the absolute level. But without that assumption the upper and lower levels of the relative commutator subgroup $[E(\Phi,R,I),E(\Phi,R,J)]$ do not coincide, so that the left-most inclusion in the above lemma should be replaced by

$$E(\Phi,R,IJ,I^2J + 2IJ + IJ^2) \leqslant [E(\Phi,R,I),E(\Phi,R,J)].$$

Here, $E(\Phi,R,I,J)$ is the elementary subgroup corresponding to an admissible pair (I,J) in the sense of Abe, where I is an ideal of R, expressing the short root level (= upper level), whereas a Jordan ideal J, expressing the long root level (= lower level), plays the role of a form parameter. Not to complicate things any further, in the sequel we *always* impose these additional restrictions on R, when $\Phi = \mathrm{B}_2, \mathrm{G}_2$. These two cases, especially that of the group $\mathrm{Sp}(4,R)$, require separate analysis anyway, [143, 144].

Since Chevalley groups of types other than A_l are only defined over commutative rings, we can state the next result with the full congruence subgroup right from the outset. It is (essentially) [46, Theorem 3], with slightly weaker assumptions for Chevalley groups of rank 2.

Theorem 1C. *Let Φ be a reduced irreducible root system, $rk(\Phi) \geqslant 2$. Further, let R be a commutative ring, and $I, J \trianglelefteq R$ be two ideals of R. In the cases $\Phi = \mathrm{B}_2, \mathrm{G}_2$ assume that R does not have residue fields $\mathbb{F}_2$ of 2 elements and in the case $\Phi = \mathrm{B}_2$ assume additionally that any $c \in R$ is contained in the ideal $c^2R + 2cR$. Then*

$$[E(\Phi, R, I), C(\Phi, R, J)] = [E(\Phi, R, I), E(\Phi, R, J)].$$

Actually, relative standard commutator formulas can be proven by localisation, as in [49, 45, 46], and this is precisely the proof on which most generalisations are based. Otherwise, they can be reduced to the *absolute* standard commutator formulas by level calculations, as in [138, 131, 45, 46]. Of course, the usual proofs of the absolute commutator formulas themselves in this generality involve some forms of localisation, at least in the non-commutative case.

Before proceeding to higher generalisations, we dwell a bit more on the structure and generation of the relative commutator subgroups $[E(R, I), E(R, J)]$ that appear in these theorems. These results are essentially elementary, sheer abstract or algebraic group theory, and do not use localisation. But they are useful and amusing, and serve to motivate, prove or amplify our main theorems.

§7. Relative commutator subgroups are not elementary

In view of Theorem 1A, it is natural to ask, whether the commutators of relative elementary subgroups are themselves elementary of the corresponding level, in other words, whether

$$[E(A, I), E(A, J)] = E(A, I \circ J) \tag{26}$$

holds?

This is known to be the case in many important classical situations, for instance, at the absolute level, where $I = A$ or $J = A$. In fact, this equality holds under much weaker assumptions. Specifically, it is easily verified when the ideals I and J are comaximal, $I + J = A$. We will reproduce the proof of this fact in the setting of general linear group from [131]. The proof in the setting of unitary groups and Chevalley groups can be now found in [45, Theorem 3], and [46, Theorem 3], respectively.

Theorem 2A. *Let A be a quasi-finite ring, $n \geqslant 3$. Then for any two comaximal ideals $I, J \trianglelefteq A$, $I + J = A$, one has*

$$[E(n, A, I), E(n, A, J)] = E(n, A, I \circ J).$$

Proof. First observe that an application of (E1) shows that for any ideals I and J of A, we have

$$E(n, A, I)E(n, A, J) = E(n, A, I + J). \tag{27}$$

Since I and J are comaximal, from (27) it follows $E(n, A, I)E(n, A, J) = E(n, A)$.

Now

$$E(n, A, I) = [E(n, A, I), E(n, A)] = [E(n, A, I), E(n, A, I)E(n, A, J)].$$

Thus using Lemma 1A we can write

$$\begin{aligned} E(n, A, I) &\leqslant [E(n, A, I), E(n, A, I)][E(n, A, I), E(n, A, J)] \\ &\leqslant [E(n, A, I), E(n, A, I)]\, \mathrm{GL}(n, A, IJ + JI). \end{aligned}$$

Commuting this inclusion with $E(n, A, J)$, we see that

$$\begin{aligned} &[E(n, A, I), E(n, A, J)] \\ &\leqslant \big[[E(n, A, I), E(n, A, I)], E(n, A, J)\big]\big[\, \mathrm{GL}(n, A, IJ + JI), E(n, A, J)\big]. \end{aligned}$$

Applied to the second factor, the standard commutator formula, Theorem 11, shows that

$$\begin{aligned} &[\mathrm{GL}(n, A, IJ + JI), E(n, A, J)] \\ &\qquad \leqslant [\mathrm{GL}(n, A, IJ + JI), E(n, A)] = E(n, A, IJ + JI). \end{aligned}$$

On the other hand, applying Lemma 1A to the first factor, and then invoking the standard commutator formula again, we have

$$\begin{aligned} \big[[E(n, A, I), E(n, A, J)], E(n, A, I)\big] &\leqslant [\mathrm{GL}(n, A, IJ + JI), E(n, A, I)] \\ &\leqslant [\mathrm{GL}(n, A, IJ + JI), E(n, A)] = E(n, A, IJ + JI). \end{aligned}$$

Thus we have

$$[E(n, A, I), E(n, A, J)] \leqslant E(n, A, IJ + JI).$$

Combining this with Lemma 1A, the proof is complete. $\square$

Theorem 2B. *Let $n \geqslant 3$, and (A, Λ) be an arbitrary form ring for which absolute standard commutator formulae are satisfied. Then for any two comaximal form ideals (I, Γ) and (J, Δ) of the form ring (A, Λ), $I + J = A$, one has the following equality*

$$[\mathrm{EU}(2n, I, \Gamma), \mathrm{EU}(2n, J, \Delta)] = \mathrm{EU}(2n, IJ + JI, {}^J\Gamma + {}^I\Delta + \Gamma_{\min}(IJ + JI)).$$

Theorem 2C. *Let Φ be a reduced irreducible root system, $rk(\Phi) \geqslant 2$. Further, let A be a commutative ring, and $I, J \trianglelefteq A$ be two ideals of A. In the cases $\Phi = \mathrm{B}_2, \mathrm{G}_2$ assume that A does not have residue fields $\mathbb{F}_2$ of 2 elements. Then for any two comaximal ideals $I, J \trianglelefteq A$, $I + J = A$, one has the following equality*

$$[E(\Phi, A, I), E(\Phi, A, J)] = E(\Phi, A, IJ).$$

Observe, that unlike Theorem 1C, in Theorem 2C the extra assumption on R for type B_2 turned out to be redundant (due to more accurate level calculations in terms of admissible pairs).

7.1. Despite Theorem 2A, the relative commutator subgroup $[E(A, I), E(A, J)]$ cannot be always elementary of the form (26). We reproduce from [78, 76] one such example based on the calculation of relative K_1-functors for Dedekind rings of arithmetic type by Hyman Bass, John Milnor and Jean-Pierre Serre [17]. We do not make any attempt to recall the explicit formula for

$$\mathrm{SK}_1(n, A, I) = \mathrm{SL}(n, A, I)/E(n, A, I)$$

in the general case. Instead, we cite the explicit answer for the *first* non-trivial case of Gaussian integers $A = \mathbb{Z}[i]$. Consider the prime ideal $\mathfrak{p} = (1 + i)A$. Then for any $n \geqslant 3$ and any ideal $I \trianglelefteq A$ one has

$$\mathrm{SK}_1(n, A, I) = \mathrm{SK}_1(n, A, \mathfrak{p}^s), \qquad s = \mathrm{ord}_{\mathfrak{p}}(I).$$

On the other hand,

$$|\,\mathrm{SK}_1(n, A, \mathfrak{p}^s)| = \begin{cases} 1, & s \leqslant 3, \\ 2, & s = 4, 5, \\ 4, & s \geqslant 6. \end{cases}$$

Now a straightforward calculation shows that

$$\begin{aligned} E(n, \mathbb{Z}[i], \mathfrak{p}^6) &< [E(n, \mathbb{Z}[i], \mathfrak{p}^3), E(n, \mathbb{Z}[i], \mathfrak{p}^3)] \\ &= [\mathrm{SL}(n, \mathbb{Z}[i], \mathfrak{p}^3), \mathrm{SL}(n, \mathbb{Z}[i], \mathfrak{p}^3)] < \mathrm{SL}(n, \mathbb{Z}[i], \mathfrak{p}^6), \end{aligned}$$

where *both* inclusions are strict. In fact, both indices are equal to 2.

This, and many further examples of arithmetic and algebra-geometric nature show that in general the relative commutator subgroup $[E(n,A,I), E(n,A,J)]$ is *strictly larger* than the relative elementary subgroup $E(n,A,I\circ J)$.

In particular, it follows that in general

$$[E(n,A,I),E(n,A,J)]\neq[E(n,A,K),E(n,A,L)]$$

for two pairs of ideals (I,J) and (K,L), such that $I\circ J=K\circ L$. In fact, this already follows from the previous example, for pairs (I,J) and $(K,L)=(I\circ J,A)$, but it is easy to construct many further examples, much fancier than that.

Summarising the above, we can conclude that in general the double relative commutator subgroups do not reduce to relative elementary subgroups, and reveal some new layers of the internal structure of $\mathrm{K}_1(A,I)$.

Amazingly, all higher multiple commutator subgroups reduce to *double* commutator subgroups. In other words, forming successive commutators of relative elementary subgroups *never* results in anything new inside $K_1(A,K)$, apart from the groups

$$[E(A,I),E(A,J)]/E(A,K)\leqslant K_1(A,K),$$

for some other ideals I and J, such that $I\circ J=K$. We will discuss this in §11.

§8. Generators of relative commutator subgroups

Here, we describe generators of relative commutator subgroups $[E(A,I), E(A,J)]$ as normal subgroups of $E(A)$. These results are elementary algebraic group theory, but they are an essential complement to Theorem 1A, an important tool in the proof of multiple commutator formula, and the starting point for results on relative commutator width.

By Lemma 1A the relative commutator subgroup $[E(A,I),E(A,J)]$ contains the elementary subgroup $E(A,I\circ J)$. In particular, it contains the generators of that group. However, we know that in general $[E(A,I), E(A,J)]$ may be strictly larger, than $E(A,I\circ J)$ (see §7). Thus, we have to produce the missing generators. As in the case of the relative elementary subgroups $E(A,I)$ themselves, these generators will sit in the fundamental SL_2's and are in fact commutators of *some* elementary generators of $E(\Phi,A,I)$ and $E(\Phi,A,J)$.

Lemma 2A. *Let A be a ring and I, J be two-sided ideals of A. Then*

$$[E(n,A,I),E(n,A,J)]$$

is generated as a group by the elements of the form

$$\left.\begin{aligned} &{}^{c}\big[e_{j,i}(\alpha),{}^{e_{i,j}(a)}e_{j,i}(\beta)\big],\\ &{}^{c}[e_{j,i}(\alpha),e_{i,j}(\beta)],\\ &{}^{c}e_{i,j}(\alpha\beta),\\ &{}^{c}e_{i,j}(\beta\alpha),\end{aligned}\right\} \tag{28}$$

where $1 \leqslant i \neq j \leqslant n$, $\alpha \in I$, $\beta \in J$, $a \in A$ and $c \in E(n,A)$.

Proof. A typical generator of $[E(n,A,I),E(n,A,J)]$ is of the form $[e,f]$, where $e \in E(n,A,I)$ and $f \in E(n,A,J)$. Thanks to Lemma 3, we may assume that e and f are products of elements of the form

$$e_i =^{e_{p',q'}(a)} e_{q',p'}(\alpha) \quad \text{and} \quad f_j =^{e_{p,q}(b)} e_{q,p}(\beta),$$

where $a, b \in A$, $\alpha \in I$ and $\beta \in J$, respectively. Applying (C1$^+$) and then (C2$^+$), it follows that $[E(n,A,I),E(n,A,J)]$ is generated by elements of the form

$${}^{c}[{}^{e_{i',j'}(a)}e_{j',i'}(\alpha),{}^{e_{i,j}(b)}e_{j,i}(\beta)],$$

where $c \in E(n,A)$. Furthermore,

$${}^{c}[{}^{e_{i',j'}(a)}e_{j',i'}(\alpha),{}^{e_{i,j}(b)}e_{j,i}(\beta)] =^{ce_{i',j'}(a)} [e_{j',i'}(\alpha),{}^{e_{i',j'}(-a)e_{i,j}(b)}e_{j,i}(\beta)].$$

The normality of $E(n,A,J)$ implies that ${}^{e_{i',j'}(-a)e_{i,j}(b)}e_{j,i}(\beta) \in E(n,A,J)$, which is a product of ${}^{e_{p,q}(a)}e_{q,p}(\beta)$, $a \in A$ and $\beta \in J$ by Lemma 3. Again by (C1$^+$), one reduces the proof to the case of showing that

$$[e_{i',j'}(\alpha),{}^{e_{i,j}(a)}e_{j,i}(\beta)]$$

is a product of the generators listed in (28). We need to consider the following cases:

- If $i' = j, j' = i$: Then there is nothing to prove.
- if $i' = j, j' \neq i$:

$$\begin{aligned}[][e_{j,j'}(\alpha),{}^{e_{i,j}(a)}e_{j,i}(\beta)] &= {}^{e_{i,j}(a)}[{}^{e_{i,j}(-a)}e_{j,j'}(\alpha),e_{j,i}(\beta)]\\ &= {}^{e_{i,j}(a)}[[e_{i,j}(-a),e_{j,j'}(\alpha)]e_{j,j'}(\alpha),e_{j,i}(\beta)]\\ &= {}^{e_{i,j}(a)}[e_{i,j'}(-a\alpha)e_{j,j'}(\alpha),e_{j,i}(\beta)].\end{aligned}$$

Applying now (C2),

$$\begin{aligned}[e_{i,j'}(-a\alpha)e_{j,j'}(\alpha),e_{j,i}(\beta)] &= ({}^{e_{i,j'}(-a\alpha)}[e_{j,j'}(\alpha),e_{j,i}(\beta)])[e_{i,j'}(-a\alpha),e_{j,i}(\beta)]\\ &= [e_{i,j'}(-a\alpha),e_{j,i}(\beta)]\\ &= [e_{j,i}(\beta),e_{i,j'}(-a\alpha)]^{-1}\\ &= e_{j,j'}(-\beta a\alpha)^{-1}\\ &= e_{j,j'}(\beta a\alpha).\end{aligned}$$

Thus

$$[e_{j,j'}(\alpha),{}^{e_{i,j}(a)}e_{j,i}(\beta)] = {}^{e_{i,j}(a)}e_{j,j'}(\beta a\alpha)$$

which satisfies the lemma.

- if $i'\neq j, j'=i$: The argument is similar to the previous case.
- if $i'\neq j, j'\neq i$: We consider four cases:
  - if $i'=i, j'=j$:
    $$[e_{i,j}(\alpha),{}^{e_{i,j}(a)}e_{j,i}(\beta)] = {}^{e_{i,j}(a)}[e_{i,j}(\alpha),e_{j,i}(\beta)].$$
  - if $i'=i, j'\neq j$:
    $$\begin{aligned}[e_{i,j'}(\alpha),{}^{e_{i,j}(a)}e_{j,i}(\beta)] &= {}^{e_{i,j}(a)}[e_{i,j'}(\alpha),e_{j,i}(\beta)]\\ &= {}^{e_{i,j}(a)}e_{j,j'}(-\beta\alpha).\end{aligned}$$
  - if $i'\neq i, j'=j$:
    $$\begin{aligned}[e_{i',j}(\alpha),{}^{e_{i,j}(a)}e_{j,i}(\beta)] &= {}^{e_{i,j}(a)}[e_{i',j}(\alpha),e_{j,i}(\beta)]\\ &= {}^{e_{i,j}(a)}e_{i,i'}(\alpha\beta).\end{aligned}$$
  - if $i'\neq i, j'\neq j$:
    $$[e_{i',j'}(\alpha),{}^{e_{i,j}(a)}e_{j,i}(\beta)] = 1.$$

This finishes the proof. □

Theorem 3A. *Let A be a quasi-finite R-algebra with 1, let $n\geqslant 3$, and let I, J be two-sided ideals of A. Then the mixed commutator subgroup $[E(n,A,I),E(n,A,J)]$ is generated as a group by the elements of the form*

$$\begin{aligned}&[e_{ji}(\alpha),{}^{e_{ij}(a)}e_{ji}(\beta)],\\ &[e_{ji}(\alpha),e_{ij}(\beta)],\\ &z_{ij}(\alpha\beta,a),\\ &z_{ij}(\beta\alpha,a),\end{aligned}\tag{29}$$

where $1 \leqslant i \neq j \leqslant n$, $\alpha \in I$, $\beta \in J$, $a \in A$.

Proof. By Lemma 2A, the current generating set (29) generates $[E(n,A,I),E(n,A,J)]$ as a normal subgroup. Therefore, it suffices to show that any conjugates of the generators (29) is a product of these generators. Let g be a generator listed in (29), and $c \in E(n,A)$. Lemma 1A shows that $g \in \mathrm{GL}(n,A,I\circ J)$. Now applying the general commutator formula (see Theorem 11), one obtains

$$[c,g] \in [\mathrm{GL}(n,A,I\circ J),E(n,A)] = E(n,A,I\circ J).$$

Therefore by Lemma 3, $[c,g]$ is a product of $z_{ij}(\alpha\beta,a)$ and $z_{ij}(\beta\alpha,a)$ with $\alpha \in I$, $\beta \in J$, $a \in A$. It follows immediately that cgc^{-1} is a product of the generators listed in (29). This completes the proof. □

A closer look at the generating set in Theorem 3A reveals an interesting fact that all the generators are taken from $[E(n,I),E(n,A,J)]$. This implies the following corollary.

Corollary 1A. *Let A be a module finite ring and I and J two sided ideals of A. Then*

$$[E(n,I),E(n,A,J)] = [E(n,A,I),E(n,J)] = [E(n,A,I),E(n,A,J)].$$

Corollary 16. *Let A be an quasi-finite algebra with identity, $n \geqslant 3$, and let I, J be two-sided ideals of A. Then the absolute mixed commutator subgroup $[E(n,I),E(n,J)]$ is a normal subgroup of $E(n,A)$.*

Proof. Let g be a typical element in $[E(n,I),E(n,J)]$ and let $c \in E(n,A)$. As in the proof of Theorem 3A, we have

$$[c,g] \in E(n,A,I\circ J) \leqslant [E(n,I),E(n,J)].$$

It follows immediately that $cgc^{-1} \in [E(n,I),E(n,J)]$. Thus $[E(n,I),E(n,J)]$ is a normal subgroup of $E(n,A)$. □

A similar result for unitary groups is [47, Theorem 9], which is more technical. To somewhat shorten the next statement, we describe conditions on the generators in the form $T_{ji}(\xi) \in \mathrm{EU}(2n,I,\Gamma)$. Recall that (as in [15, 38, 44]) that this means that $\xi \in I$, for $i \neq \pm j$, and $\xi \in \Gamma$, for $i = -j$.

Lemma 2B. *Let (A,Λ) be a form ring and (I,Γ), (J,Δ) be two form ideals of (A,Λ). Then as a normal subgroup of* $\mathrm{EU}(2n,R,\Lambda)$, $n \geqslant 3$, *the mixed commutator subgroup* $[\mathrm{EU}(2n,I,\Gamma),\mathrm{EU}(2n,J,\Delta)]$ *is generated by the elements of the form*

- $[T_{ji}(\xi), {}^{T_{ij}(\eta)}T_{ji}(\zeta)]$,
- $[T_{ji}(\xi), T_{ij}(\zeta)]$,
- $T_{ij}(\xi\zeta)$ *and* $T_{ij}(\zeta\xi)$,

where $T_{ji}(\xi) \in \mathrm{EU}(2n, I, \Gamma)$, $T_{ji}(\zeta) \in \mathrm{EU}(2n, J, \Delta)$, $T_{ij}(\eta) \in \mathrm{EU}(2n, A, \Lambda)$, *and* $T_{ij}(\theta) \in \mathrm{EU}\,(2n, (I, \Gamma) \circ (J, \Delta))$.

The proof for Chevalley groups is similar, with some additional complications in the rank 2 case. The following result is of [48, Theorem 2].

Lemma 2C. *Let* $\mathrm{rk}(\Phi) \geqslant 2$ *and let* I, J *be two ideals of a commutative ring* R. *In the cases* $\Phi = \mathrm{B}_2, \mathrm{G}_2$ *assume that* R *does not have residue fields* $\mathbb{F}_2$ *of 2 elements and in the case* $\Phi = \mathrm{B}_2$ *assume additionally that any* $c \in R$ *is contained in the ideal* $c^2R + 2cR$.

Then as a normal subgroup of the elementary Chevalley group $E(\Phi, R)$ *the mixed commutator subgroup* $[E(\Phi, R, I), E(\Phi, R, J)]$ *is generated by the elements of the form*

- $[x_\alpha(\xi), {}^{x_{-\alpha}(\eta)}x_\alpha(\zeta)]$,
- $[x_\alpha(\xi), x_{-\alpha}(\zeta)]$,
- $x_\alpha(\xi\zeta)$,

where $\alpha \in \Phi$, $\xi \in I$, $\zeta \in J$, $\eta \in R$.

Actually, the proof of this result in [48] replaces most of explicit fiddling with the Chevalley commutator formula and commutator identities, by a reference to some obvious properties of parabolic subgroups, which makes it *considerably* less computational, than the proofs of Lemma 1A and Lemma 1B in [50, 46].

We sketch the proof of Lemma 2C as well. First of all, observe that these elements indeed belong to the relative commutator subgroups $[E(R, I), E(R, J)]$ by Lemma 1C. Next, recall that the elementary generators of the elementary groups $E(R, I)$ themselves are classically known, and look as follows:

- $z_{ji}(\xi, \eta) = e_{ij}(\eta)e_{ji}(\xi)e_{ij}(-\eta)$, for GL_n, (see Lemma 3).
- $Z_{ji}(\xi, \eta) = T_{ij}(\eta)T_{ji}(\xi)T_{ij}(-\eta)$, for unitary groups, (see [15]).
- $z_\alpha(\xi, \eta) = x_{-\alpha}(\eta)x_\alpha(\xi)x_{-\alpha}(-\eta)$, for Chevalley groups, (see [97, 112, 117, 3]).

Observe, that these generators are precisely the second factors of the first type of generators in the above Lemma 2C, and we use this shorthand notation in the sequel. The usual commutator identities imply that as a

normal subgroup
$$[E(\Phi,R,I),E(\Phi,R,J)]$$
is generated by the commutators of the form $[z_\alpha(\xi,\eta),z_\beta(\zeta,\theta)]$. Since we are working up to elementary conjugation, we can replace these generators by
$$[x_\alpha(\xi),{}^{x_{-\alpha}(-\eta)}z_\beta(\zeta,\theta)].$$
Since the groups $E(\Phi,R,J)$ are normal in $E(\Phi,R)$, the conjugates ${}^{x_{-\alpha}(-\eta)}z_\beta(\zeta,\theta)$ can be again expressed as products of elementary generators. Once more applying commutator identities, we see that as a normal subgroup $[E(\Phi,R,I),E(\Phi,R,J)]$ is generated by the commutators $[x_\alpha(\xi),z_\beta(\zeta,\theta)]$. At this point, we are left with three options:

- $\alpha=\beta$, and we get the first type of generators,
- $\alpha=-\beta$, and we get the second type of generators, up to conjugation,
- $\alpha\neq\pm\beta$. If α and β are strictly orthogonal, then $[x_\alpha(\xi),z_\beta(\zeta,\theta)]=e$. Thus, we can assume that α and β generate an irreducible root system of rank 2, and fiddle with the Chevalley commutator formula therein. Alternatively, we can choose an order such that β is fundamental, whereas α is positive. Then $[x_\alpha(\xi),z_\beta(\zeta,\theta)]$ sits inside the unipotent radical U_β of the minimal (=rank 1) standard parabolic subgroup P_β. On the other hand, by Lemma 1C it sits inside $G(\Phi,R,IJ)$. Clearly, $U_\beta\cap G(\Phi,R,IJ)\leqslant E(\Phi,IJ)$. Thus, in this last case $[x_\alpha(\xi),z_\beta(\zeta,\theta)]$ is a product of generators of the third type.

Theorem 3B. *Let $n\geqslant 3$, R be a commutative ring, (A,Λ) be a form ring such that A is a quasi-finite R-algebra. Further, let (I,Γ) and (J,Δ) be two form ideals of the form ring (A,Λ).*

Then the mixed commutator subgroup $[\mathrm{EU}(2n,I,\Gamma),\mathrm{EU}(2n,J,\Delta)]$ is generated as a group by the elements of the form

- $[T_{ji}(\xi),Z_{ji}(\zeta,\eta)]$,
- $[T_{ji}(\xi),T_{ij}(\zeta)]$,
- $Z_{ij}(\theta,\eta)$,

where $T_{ji}(\xi)\in\mathrm{EU}(2n,I,\Gamma)$, while $T_{ij}(\zeta),Z_{ji}(\zeta,\eta)\in\mathrm{EU}(2n,J,\Delta)$, and $Z_{ij}(\theta,\eta)\in\mathrm{EU}\,(2n,(I,\Gamma)\circ(J,\Delta))$.

Theorem 3C. *Let $\mathrm{rk}(\Phi)\geqslant 2$ and let I, J be two ideals of a commutative ring A. In the cases $\Phi=\mathrm{B}_2,\mathrm{G}_2$ assume that A does not have residue* *fields $\mathbb{F}_2$ of 2 elements and in the case $\Phi = \mathrm{B}_2$ assume additionally that any $c \in A$ is contained in the ideal $c^2A + 2cA$.*

Then the mixed commutator subgroup $[E(\Phi, A, I), E(\Phi, A, J)]$ is generated as a group by the elements of the form

- $[x_\alpha(\xi), z_\alpha(\zeta, \eta)]$,
- $[z_\alpha(\xi), z_{-\alpha}(\zeta)]$,
- $z_\alpha(\xi\zeta, \eta)$,

where $\alpha \in \Phi$, $\xi \in I$, $\zeta \in J$, $\eta, \in A$.

Let us sketch the proof of Theorem 3C. From this proof, it will be clear, why a similar slick argument does not prove Theorem 3A and Theorem 3B for arbitrary associative rings or arbitrary form rings.

The set described in this theorem contains the set described in Lemma 2C, which already generates $[E(\Phi, A, I), E(\Phi, A, J)]$ as a normal subgroup of $E(\Phi, A)$. Therefore, it suffices to show that elementary conjugates of the above generators are themselves products of such generators. Let g be one of these generators and let $h \in E(\Phi, A)$. By Lemma 2C, one has $g \in G(\Phi, A, IJ)$. Now the [absolute] standard commutator formula implies that

$$[h, g] \in [G(\Phi, A, IJ), E(\Phi, A)] = E(\Phi, A, IJ).$$

Being an element $E(\Phi, A, IJ)$, the commutator $[h, g]$ is a product of some elementary generators $z_\alpha(\xi\zeta, \eta)$, where $\alpha \in \Phi$, $\xi \in I$, $\zeta \in J$, $\eta \in A$. Thus, any conjugate $hgh^{-1} = [h, g]g$ is a product of some generators of the third type and the generator g itself.

In fact, mostly this argument relied on *elementary* calculations, such as the one needed to prove Lemma 2C and Theorem 3C. But at one instance we had to invoke a special case of Theorem 1C, the [absolute] standard commutator formula. This last result is not elementary, and certainly it does not hold over arbitrary associative rings. There are explicit counterexamples to the standard commutator formula in this generality, the first of them by Victor Gerasimov [33].

It seems incongruous that [what appears to be] a pure group theoretic result should depend on commutativity conditions. This poses the following problem.

Problem 1. *Find elementary proofs of Theorems* 3A *and* 3B *that work over arbitrary associative rings/form rings.*

By juggling with commutator identities, we succeeded in proving a slightly weaker version of Theorem 3A, with a somewhat larger set of generators, all of them still sitting inside fundamental GL_2's. However, a straightforward calculation, based on induction on the length of the conjugating element, is so long and appalling, that it strongly discouraged us from any attempt to prove the technically much fancier Theorem 3B for arbitrary form rings along these lines.

A closer look at the generators in Theorems 3A–3C shows that all of them in fact belong already to $[E(\Phi, I), E(\Phi, A, J)]$. By symmetry, we may switch the role of factors. In particular, this means that Theorems 3A–3C imply the following curious corollaries.

Corollary 1B. *Let $n \geqslant 3$, R be a commutative ring, (A, Λ) be a form ring such that A is a quasi-finite R-algebra. Further, let (I, Γ) and (J, Δ) be two form ideals of the form ring (A, Λ). Then one has*

$$[\mathrm{FU}(2n, I, \Gamma), \mathrm{EU}(n, J, \Delta)] = [\mathrm{EU}(2n, I, \Gamma), \mathrm{FU}(n, J, \Delta)] \\ = [\mathrm{EU}(2n, I, \Gamma), \mathrm{EU}(n, J, \Delta)].$$

Corollary 1C. *Let $\mathrm{rk}(\Phi) \geqslant 2$ and let I, J be two ideals of a commutative ring A. In the cases $\Phi = \mathrm{B}_2, \mathrm{G}_2$ assume that A does not have residue fields $\mathbb{F}_2$ of 2 elements and in the case $\Phi = \mathrm{B}_2$ assume additionally that any $c \in A$ is contained in the ideal $c^2A + 2cA$. Then one has*

$$[E(\Phi, I), E(\Phi, A, J)] = [E(\Phi, A, I), E(\Phi, J)] = [E(\Phi, A, I), E(\Phi, A, J)].$$

§9. Higher commutators

Once we understand double commutators, it is natural to consider higher commutators of relative elementary subgroups and congruence subgroups. Let G be a group and $H_1, \dots, H_m \leqslant G$ be its subgroups. There are many ways to form a higher commutator of these groups, depending on where we put the brackets. Thus, for three subgroups $F, H, K \leqslant G$ one can form two triple commutators $[[F, H], K]$ and $[F, [H, K]]$. For four subgroups $F, H, K, L \leqslant G$ one can form 5 such commutators

$$[[[F, H], K], L], \qquad [[F, [H, K]], L], \qquad [[F, H], [K, L]], \\ [F, [H, [K, L]]], \qquad [F, [[H, K], L]]. \tag{30}$$

To be exact, there are as many as the Catalan number

$$c_m = \frac{1}{(m+1)}\binom{2m}{m}$$

ways to arrange the brackets involving $m+1$ subgroups in a meaningful way.

Usually, we write $[H_0, H_1, \ldots, H_m]$ for the *left-normed* commutator, defined inductively by

$$[H_0, \ldots, H_{m-1}, H_m] = [[H_0, \ldots, H_{m-1}], H_m].$$

To stress that we consider any commutator of these subgroups, with an arbitrary placement of brackets, we write $[\![H_0, H_2, \ldots, H_m]\!]$. Thus, for instance, $[\![F, H, K, L]\!]$ refers to any of the five arrangements in (30).

Actually, a specific arrangement of brackets usually does not play major role in our results – and in fact any role whatsoever over commutative rings! – apart from one important attribute. Namely, what will matter a lot is the position of the outermost pairs of inner brackets. Namely, every higher commutator subgroup $[\![H_0, H_2, \ldots, H_m]\!]$ can be uniquely written as

$$[\![H_0, H_2, \ldots, H_m]\!] = \big[[\![H_0, \ldots, H_h]\!], [\![H_{h+1}, \ldots, H_m]\!]\big],$$

for some $h = 1, \ldots, m-1$. This h will be called the *cut point* of our multiple commutator. Thus, among the quadruple commutators $[\![F, H, K, L]\!]$, two arrangements, $[[[F, H], K], L]$ and $[[F, [H, K]], L]$, cut at 3; one, $[[F, H], [K, L]]$, cuts at 2; and the remaining two, $[F, [H, [K, L]]]$ $[F, [[H, K], L]]$, cut at 1.

Now, let I_i, $i = 0, 1, \ldots, m$, be ideals of the ring A. Our ultimate objective is to compute the commutator subgroups of congruence subgroups

$$[\![G(A, I_0), G(A, I_1), \ldots, G(A, I_m)]\!],$$

but that is a *highly* strenuous enterprise. So far, we have done it only for the case $G = \mathrm{GL}_n$, provided that m is strictly lager than the Bass–Serre dimension of A.

In §10 we embark on the [somewhat easier] calculation of higher commutators of relative elementary subgroups

$$[\![E(A, I_0), E(A, I_1), \ldots, E(A, I_m)]\!].$$

Even this turns out to be a rather non-trivial task. In fact, we do not see any other way to do that, but to prove a higher analogue of the standard commutator formula, viz.

$$[\![E(A,I_0),G(A,I_1),\dots,G(A,I_m)]\!] = [\![E(A,I_0),E(A,I_1),\dots,E(A,I_m)]\!].$$

This multiple commutator formula will be discussed in §10 and §11. Unlike the *general* multiple commutator formula in which we are ultimately interested, and which only works for finite-dimensional rings, this weaker formula holds over arbitrary quasi-finite/commutative rings.

Amazingly, the resulting *multiple* commutator subgroups will always coincide with some *double* relative commutator subgroups, depending not on the ideals I_i themselves, but only on two symmetrised products of these ideals. Since the symmetrised product of ideals is not associative, some traces of the initial arrangment will still be visible in these symmetrised products. However, for commutative rings the symmetrised product becomes the usual product of ideals, which is associative, so that the result will not depend on the arrangement itself either, but only on its cut point. We discuss these results in §11.

§10. Multiple commutator formula

The following theorem is the main result of the paper [50]. Initially, it was conceived as *part* of the answer to a problem proposed in [129, 131]. As a matter of fact, it turned out to be of significant independent interest. The proof of the following result in [50] is based on a further enhancement of relative localisation which we outline in §13.

Theorem 17. *Let A be a quasi-finite R-algebra with identity and I_i, $i = 0,\dots,m$, be two-sided ideals of A. Then*

$$\begin{aligned}&\big[E(n,A,I_0),\mathrm{GL}(n,A,I_1),\mathrm{GL}(n,A,I_2),\dots,\mathrm{GL}(n,A,I_m)\big]\\ &\qquad = \big[E(n,A,I_0),E(n,A,I_1),E(n,A,I_2),\dots,E(n,A,I_m)\big]. \qquad (31)\end{aligned}$$

Proof. We prove the statement by induction. For $m = 1$ this is the generalised commutator formula Theorem 1A

$$[E(n,A,I_0),\mathrm{GL}(n,A,I_1)] = [E(n,A,I_0),E(n,A,I_1)].$$

For $m = 2$, this will be proved in Theorem 6A which will be the first step of induction. Suppose the statement is valid for $m-1$ (i.e., there are m ideals in the commutator formula). To prove (31), using Theorem 6A, we have

$$\Big[[[E(n,A,I_0),\mathrm{GL}(n,A,I_1)],\mathrm{GL}(n,A,I_2)\big],\mathrm{GL}(n,A,I_3),\dots,\mathrm{GL}(n,A,I_m)\Big]$$
$$=\Big[[[E(n,A,I_0),E(n,A,I_1)],E(n,A,I_2)\big],\mathrm{GL}(n,A,I_3),\dots,\mathrm{GL}(n,A,I_m)\Big].$$

By Lemma 1A, $[E(n,A,I_0),E(n,A,I_1)]\leqslant \mathrm{GL}(n,A,I_0I_1+I_1I_0)$. Thus

$$\Big[[[E(n,A,I_0),E(n,A,I_1)],E(n,A,I_2)\big],\mathrm{GL}(n,A,I_3),\dots,\mathrm{GL}(n,A,I_m)\Big]$$
$$\leqslant\Big[\big[\,\mathrm{GL}(n,A,I_0I_1+I_1I_0),E(n,A,I_2)\big],\mathrm{GL}(n,A,I_3),\dots,\mathrm{GL}(n,A,I_m)\Big].$$

Since there are m ideals involved in the commutator subgroups in the right hand side, by induction we get

$$\Big[\big[\,\mathrm{GL}(n,A,I_0I_1+I_1I_0),E(n,A,I_2)\big],\mathrm{GL}(n,A,I_3),\dots,\mathrm{GL}(n,A,I_m)\Big]$$
$$=\Big[\big[E(n,A,I_0I_1+I_1I_0),E(n,A,I_2)\big],E(n,A,I_3),\dots,E(n,A,I_m)\Big].$$

Finally again by Lemma 1A,

$$E(n,A,I_0I_1+I_1I_0)\leqslant[E(n,A,I_0),E(n,A,I_1)].$$

Replacing this in the above equation we obtain that the left hand side of (31) is contained in the right hand side. The opposite inclusion is obvious. This completes the proof. □

Theorem 4A. *Let $n\geqslant 3$, let A be a quasi-finite ring with 1 and let $I_i\trianglelefteq A$, $i=0,\dots,m$, be ideals of A. Then one has*

$$[\![E(n,A,I_0),\mathrm{GL}(n,A,I_2),\dots,\mathrm{GL}(n,A,I_m)]\!]$$
$$=[\![E(n,A,I_0),E(n,A,I_2),\dots,E(n,A,I_m)]\!].$$

In this theorem the arrangement of brackets on the left hand side may be arbitrary. But it is essential that the placement of brackets on the right hand side coincides with that on the left hand side. Without this assumption the equality may fail dramatically, even if all factors are elementary, as we shall see in §11.1. Of course, the same observation applies to the theorems below.

For unitary groups, similar result is established in [47], by essentially the same method. However, as one could expect, the necessary calculations are tangibly more complicated and require a completely different level of technical strain.

Theorem 4B. *Let $n \geqslant 3$ and let (A, Λ) be a form ring such that A is a quasi-finite R-algebra over a commutative ring R. Further, let (I_i, Γ_i), $i = 0, \ldots, m$, be form ideals of (A, Λ). Then*

$$\begin{aligned}&[\![\mathrm{EU}(2n, I_0, \Gamma_0), \mathrm{GU}(2n, I_1, \Gamma_1), \ldots, \mathrm{GU}(2n, I_m, \Gamma_m)]\!]\\ &\qquad = [\![\mathrm{EU}(2n, I_0, \Gamma_0), \mathrm{EU}(2n, I_1, \Gamma_1), \ldots, \mathrm{EU}(2n, I_m, \Gamma_m)]\!].\end{aligned}$$

Finally, let us pass to Chevalley groups. We believe that at this point we possess two independent proofs of the following result. One of them, by the authors, is conventional, and involves a further elaboration of the relative commutator calculus in the style of [46]. Another one, by A. Stepanov, is somewhat shorter, and employs his method of universal localisation [98]. But the definitive expositions are still missing.

Theorem 4C. *Let $\mathrm{rk}(\Phi) \geqslant 2$ and let $I_i \trianglelefteq A$, $i = 0, \ldots, m$, be ideals of a commutative ring A. In the cases $\Phi = \mathrm{B}_2, \mathrm{G}_2$ assume that A does not have residue fields $\mathbb{F}_2$ of 2 elements and in the case $\Phi = \mathrm{B}_2$ assume additionally that any $c \in A$ is contained in the ideal $c^2A + 2cA$.*

Then one has

$$\begin{aligned}&[\![E(\Phi, A, I_0), G(\Phi, A, I_)), \ldots, G(\Phi, A, I_m)]\!]\\ &\qquad = [\![E(\Phi, A, I_0), E(\Phi, A, I_1), \ldots, E(\Phi, A, I_m)]\!].\end{aligned}$$

These theorems are broad generalisations of the double commutator formulas. Let us explain, why they do not reduce to the double formula. Consider *three* ideals I, J, K of A and form the commutator $[[E(A, I), G(A, J)], G(A, K)]$. The double commutator formula implies that

$$[[E(A, I), G(A, J)], G(A, K)] = [[E(A, I), E(A, J)], G(A, K)].$$

But as we know, the relative commutator subgroup $[E(A, I), E(A, J)]$ may be strictly larger, than $E(A, I \circ J)$ (see §7.1), so it is not at all clear, why the equality

$$[[E(A, I), E(A, J)], G(A, K)] = [[E(A, I), E(A, J)], E(A, K)]$$

should hold.

This is indeed the key new leap in the proof of Theorem 17, and the commutator calculus developed in [49, 45, 46] is not powerful enough here. This step requires a new layer of the relative commutator calculus, which we discuss in §13.

§11. Multiple $\rightsquigarrow$ double

11.1. In connection with Theorems 6 and 7 it is natural to ask, whether the equality

$$[[E(A,I),E(A,J)],E(A,K)] = [E(A,I),[E(A,J),E(A,K)]] \qquad (32)$$

holds for any three ideals I, J and K of A. If this were the case, one could drop the requirement that the arrangement of brackets on the left hand side and the right hand side of these theorems should coincide.

However, in general this equality fails, as can be shown by easy examples. Let us retreat to the case of GL_n. In fact, setting here $K = A$ we see that

$$\begin{aligned} E(n,A,I\circ J) &= [E(n,A,I\circ J),E(n,A)] \\ &\leqslant [[E(n,A,I),E(n,A,J)],E(n,A)] \leqslant [\mathrm{GL}(n,A,I\circ J),E(n,A)] \\ &= [E(n,A,I\circ J),E(n,A)] = E(n,A,I\circ J). \end{aligned}$$

This shows that in this case one has

$$[[E(A,I),E(A,J)],E(A,K)] = E(n,A,I\circ J).$$

On the other hand, for $K = A$, we have

$$[E(A,I),[E(A,J),E(A,K)]] = [E(A,I),[E(A,J)].$$

Thus, in this case if the associativity of commutators (32) holds, we obtain

$$[E(n,A,I),E(n,A,J)] = E(n,A,I\circ J).$$

However, as we know from the example provided in §7.1, this equality does not hold, in general.

11.2. To motivate the next theorem, let us calculate these triple commutators. Combining Lemma 1A and Theorem 1A, we see that

$$\begin{aligned} [E(n,A,I\circ J),E(n,A,K)] &\leqslant \big[[E(n,A,I),E(n,A,J)],E(n,A,K)\big] \\ &\leqslant [\mathrm{GL}(n,A,I\circ J),E(n,A,K)] = [E(n,A,I\circ J),E(n,A,K)]. \end{aligned}$$

In other words,

$$\big[[E(n,A,I),E(n,A,J)],E(n,A,K)\big] = [E(n,A,I\circ J),E(n,A,K)].$$

Similarly, one can verify that

$$\big[E(n,A,I),[E(n,A,J),E(n,A,K)]\big] = [E(n,A,I),E(n,A,J\circ K)].$$

Plugging in the above calculation Theorem 4A instead of Theorem 1A, we get the following amazing corollary. It asserts that multiple commutators of relative elementary subgroups can always be expressed as *double* commutators of such subgroups, corresponding to some symmetrised product ideals. The following is observed in [42].

Theorem 5A. *Let A be a quasi-finite ring with 1 and let $I_i \trianglelefteq A$, $i = 0, \dots, m$, be ideals of A. Consider an arbitrary configuration of brackets $[\![\dots]\!]$ and assume that the outermost pairs of brackets between positions h and $h+1$. Then one has*

$$[\![E(n,A,I_0), E(n,A,I_1), \dots, E(n,A,I_m)]\!] \\ = [E(n,A,I_0 \circ \dots \circ I_h), E(n,A,I_{h+1} \circ \dots \circ I_m)],$$

where the bracketing of symmetrised products on the right hand side coincides with the bracketing of the commutators on the left hand side.

Proof. Alternated application of Lemma 1A and Theorem 1A shows that

$$\begin{aligned}
&\big[[\![E(n,A,I_0), E(n,A,I_1), \dots, E(n,A,I_k)]\!],\\
&[\![E(n,A,I_{k+1}), \dots, E(n,A,I_m)]\!]\big]\\
&\leqslant \big[\,\mathrm{GL}(n,A,I_0 \circ \dots \circ I_k), [\![E(n,A,I_{k+1}), \dots, E(n,A,I_m)]\!]\big]\\
&= \big[E(n,A,I_0 \circ \dots \circ I_k), [\![E(n,A,I_{k+1}), \dots, E(n,A,I_m)]\!]\big]\\
&\leqslant [E(n,A,I_0 \circ \dots \circ I_k), \mathrm{GL}(n,A,I_{k+1} \circ \dots \circ I_m)]\\
&= [E(n,A,I_0 \circ \dots \circ I_k), E(n,A,I_{k+1} \circ \dots \circ I_m)]\\
&\leqslant \big[[\![E(n,A,I_0), E(n,A,I_1), \dots, E(n,A,I_k)]\!],\\
&[\![E(n,A,I_{k+1}), \dots, E(n,A,I_m)]\!]\big],
\end{aligned}$$

as claimed. $\square$

For the unitary case it is [47, Theorem 7].

Theorem 5B. *Let (A,Λ) be a quasi-finite ring with 1 and let (I_i,Γ_i), $i = 0, \dots, m$, be form ideals of the form ring (A,Λ). Consider an arbitrary configuration of brackets $[\![\dots]\!]$ and assume that the outermost pairs of brackets between positions h and $h+1$. Then one has*

$$[\![\mathrm{EU}(2n,I_0,\Gamma_0), \mathrm{EU}(2n,I_1,\Gamma_1), \dots, \mathrm{EU}(2n,I_m,\Gamma_m)]\!] \\ = [\mathrm{EU}(2n,(I_0,\Gamma_0) \circ \dots \circ (I_h,\Gamma_h)), \mathrm{EU}(2n,(I_{h+1},\Gamma_{h+1}) \circ \dots \circ (I_m,\Gamma_m))].$$

Of course, similar result also holds in the context of Chevalley groups, once we have Theorem 6C.

Theorem 6C. *Let A be a commutative ring with 1 and let $I_i \trianglelefteq A$, $i = 0,\dots,m$, be ideals of A. Consider an arbitrary configuration of brackets $[\![\dots]\!]$ and assume that the outermost pairs of brackets between positions h and $h+1$. Then one has*

$$[\![E(\Phi,A,I_0),E(\Phi,A,I_1),\dots,E(\Phi,A,I_m)]\!]$$
$$= [E(\Phi,A,I_0\dots I_h),E(\Phi,A,I_{h+1}\dots I_m)].$$

§12. Localisation

12.1. In this paper we only use central localisation. Namely, for an R-algebra A, we consider the localisation with respect to a multiplicative closed subset of R.

First, we fix some notation. Let R be a commutative ring with 1, S be a multiplicative closed subset in R and A be an R-algebra. Then $S^{-1}R$ and $S^{-1}A$ are the corresponding localisation. We mostly use localisation with respect to the two following types of multiplicative systems.

• *Principal localisation*: S coincides with $\langle s\rangle = \{1, s, s^2, \dots\}$, for some non-nilpotent $s \in R$, in this case we usually write $\langle s\rangle^{-1}R = R_s$ and $\langle s\rangle^{-1}A = A_s$.

• *Localisation at a maximal ideal*: $S = R \setminus \mathfrak{m}$, for some maximal ideal $\mathfrak{m} \in \operatorname{Max}(R)$ in R, in this case we usually write $(R\setminus\mathfrak{m})^{-1}R = R_\mathfrak{m}$ and $(A\setminus\mathfrak{m})^{-1}A = A_\mathfrak{m}$.

We denote by $F_S : A \longrightarrow S^{-1}A$ the canonical ring homomorphism called the *localisation homomorphism*. For the two special cases above, we write $F_s : A \longrightarrow A_s$ and $F_\mathfrak{m} : A \longrightarrow A_\mathfrak{m}$, respectively.

When we write an element as a fraction, like a/s or $\dfrac{a}{s}$ we *always* think of it as an element of some localisation $S^{-1}A$, where $s \in S$. If s were actually invertible in R, we would have written as^{-1} instead.

Ideologically, all proofs using localisations are based on the interplay of the three following observations:

• Functors of points $A \rightsquigarrow G(A)$ are compatible with localisation,

$$g \in G(A) \quad\Longleftrightarrow\quad F_\mathfrak{m}(g) \in G(A_\mathfrak{m}), \quad \text{for all } \mathfrak{m} \in \operatorname{Max}(A).$$

• Elementary subfunctors $A \rightsquigarrow E(A)$ are compatible with factorisation, for any $I \trianglelefteq A$ the reduction homomorphism $\rho_I : E(A) \longrightarrow E(A/I)$ is surjective.

• On a [semi-]local ring A the values of semi-simple groups and their elementary subfunctors coincide, $G(A) = E(A)$.

The following property of the functors G and E will be crucial for what follows: they are continuous functors, i.e., they *commute with direct limits*. In other words, if $A = \varinjlim A_i$, where $\{A_i\}_{i\in I}$ is an inductive system of rings, then

$$G(\varinjlim A_i) = \varinjlim G(A_i), \qquad E(\varinjlim A_i) = \varinjlim E(A_i).$$

We use this property in the two following situations.

• *Noetherian reduction*: let A_i be the inductive system of all finitely generated subrings of A with respect to inclusion. Then

$$G(A) = \varinjlim G(A_i), \qquad E(A) = \varinjlim E(A_i).$$

This allows to reduce most of the proofs to the case of Noetherian rings.

• *Reduction to principal localisations*: let S be a multiplicative closed set in R and let A_s, $s \in S$, be the corresponding inductive system with respect to the principal localisation homomorphisms: $F_t : A_s \longrightarrow A_{st}$. Then

$$G(S^{-1}A) = \varinjlim G(A_s), \qquad E(S^{-1}A) = \varinjlim E(A_s).$$

This reduces localisation in any multiplicative system to the principal localisation.

12.2. Injectivity of localisation homomorphism. Most localisation proofs rely on the injectivity of localisation homomorphism F_S. As observed in §12.1, we can only consider *principal* localisation homomorphisms F_s. Of course, F_s is injective when s is regular. Thus, localisation proofs are particularly easy for integral domains. A large part of what follows are various devices to fight with the presence of zero-divisors.

When s is a zero-divisor, F_s is not injective on the group $G(A)$ itself. But its restrictions to appropriate congruence subgroups often are. Here is an important typical case, i.e., Noetherian ring.

Lemma 18. *Let A be a module finite R-algebra, where R is a commutative Noetherian ring. Then for any $s \in R$, there exists a positive integer l such that the homomorphism $F_s : \mathrm{GL}(n, A, s^l A) \longrightarrow \mathrm{GL}(n, A_s)$ is injective.*

Proof. The homomorphism $F_s : \mathrm{GL}(n, A, s^l A) \longrightarrow \mathrm{GL}(n, A_s)$ is injective whenever $F_s : s^k A \longrightarrow A_s$ is injective. Let $\mathfrak{a}_i = \mathrm{Ann}_R(s^i)$ be the annihilator of s^i in A. Since R is Noetherian, and A is finite over R, A is Noetherian and so there exists k such that $\mathfrak{a}_k = \mathfrak{a}_{k+1} = \cdots$. If $s^k a$ vanishes in A_s, then $s^i s^k a = 0$ for some i. But since $\mathfrak{a}_{k+i} = \mathfrak{a}_k$, already $s^k a = 0$ and thus $s^k A$ injects in A_s. □

Another important trick to override the presence of zero-divisors consists in throwing in polynomial variables. Namely, instead of the ring R itself we consider the polynomial ring $R[t]$ in the variable t. In that ring t is not a zero-divisor, so that the localisation homomorphism F_t is injective. We can use that, and then specialise t to any $s \in R$.

Actually, throwing in polynomial variables has more than one use. The elementary subfunctors $R \rightsquigarrow E(R)$ are compatible with localisation, i.e.,

$$g \in E(R) \quad \Longrightarrow \quad F_{\mathfrak{m}}(g) \in E(R_{\mathfrak{m}}), \quad \text{for all } \mathfrak{m} \in \mathrm{Max}(R),$$

but the converse implication does not hold, for otherwise $E(R)$ would coincide with the [semi-simple part of] $G(R)$ for all commutative rings.

The following remarkable observation was due to Daniel Quillen at the level of K_0, and was first applied by Andrei Suslin at the level of K_1, in the context of solving Serre's conjecture, and its higher analogues [105]. See [63] for a description of Quillen–Suslin's idea in its historical development. We refer to the following result as Quillen–Suslin's lemma.

Theorem 19. *Let $g \in G(R[t], tR[t])$. Then $g \in E(R[t])$ if and only if $F_{\mathfrak{m}}(g) \in E(R_{\mathfrak{m}}[t])$, for all $\mathfrak{m} \in \mathrm{Max}(R)$.*

12.3. Let (A, Λ) be a form algebra over a commutative ring R with 1, and let S be a multiplicative subset of R_0 (see §5.3). For any R_0-module M one can consider its localisation $S^{-1}M$ and the corresponding localisation homomorphism $F_S : M \longrightarrow S^{-1}M$. By definition of the ring R_0 both A and Λ are R_0-modules, and thus can be localised in S.

12.4. Localisation of form rings. In the setting of form rings, we need to adjust the ground field of the localisation. For a form ring (A, Λ), where A is an R-algebras, the form Λ is not necessarily an R-module (see §5.3). This forces us to replace R by its subring R_0, generated by all $\alpha\overline{\alpha}$ with $\alpha \in R$. Clearly, all elements in R_0 are invariant with respect to the involution, i. e. $\overline{r} = r$, for $r \in R_0$. Furthermore, Λ is an R_0-module.

As in the setting of general linear group (§12.1), we mostly use localisation in the unitary setting with respect to the following two types of multiplicative closed subsets of R_0.

• *Principal localisation*: for any $s \in R_0$ with $\overline{s} = s$, the multiplicative closed subset generated by s is defined as $\langle s\rangle = \{1, s, s^2, \dots\}$. The localisation of the form algebra (A, Λ) with respect to multiplicative system $\langle s\rangle$ is usually denoted by (A_s, Λ_s), where as usual $A_s = \langle s\rangle^{-1}A$ and $\Lambda_s = \langle s\rangle^{-1}\Lambda$ are the usual principal localisations of the ring A and the form parameter Λ. Notice that, for each $\alpha \in A_s$, there exists an integer n and an element $a \in A$ such that $\alpha = \dfrac{a}{s^n}$, and for each $\xi \in \Lambda_s$, there exists an integer m and an element $\zeta \in \Lambda$ such that $\xi = \dfrac{\zeta}{s^m}$.

• *Maximal localisation*: consider a maximal ideal $\mathfrak{m} \in \operatorname{Max}(R_0)$ of R_0 and the multiplicative closed set $S_\mathfrak{m} = R_0 \backslash \mathfrak{m}$. We denote the localisation of the form algebra (A, Λ) with respect to $S_\mathfrak{m}$ by $(A_\mathfrak{m}, \Lambda_\mathfrak{m})$, where $A_\mathfrak{m} = S_\mathfrak{m}^{-1}A$ and $\Lambda_\mathfrak{m} = S_\mathfrak{m}^{-1}\Lambda$ are the usual maximal localisations of the ring A and the form parameter, respectively.

In these cases the corresponding localisation homomorphisms will be denoted by F_s and by $F_\mathfrak{m}$, respectively.

The following fact is verified by a straightforward computation.

Lemma 20. *For any $s \in R_0$ and for any $\mathfrak{m} \in \operatorname{Max}(R_0)$ the pairs (A_s, Λ_s) and $(A_\mathfrak{m}, \Lambda_\mathfrak{m})$ are form rings.*

§13. Triple Commutators/Base of induction

We prove Theorem 17 by induction on m. The case of $m = 2$ is precisely the relative commutator formula, Theorem 1A. However, the base of induction for Theorem 17 is $m = 2$, and it is the most demanding part of the induction step. In fact, the proof of the following special case constitutes bulk of the paper [50].

Theorem 6A. *Let $n \geqslant 3$, and let A be a quasi-finite ring. Further, let I, J and K be three two-sided ideals of A. Then*

$$\begin{aligned}&\big[[E(n,A,I), \operatorname{GL}(n,A,J)], \operatorname{GL}(n,A,K)\big]\\ &= \big[[E(n,A,I), E(n,A,J)], E(n,A,K)\big].\end{aligned}$$

As we have just observed, the standard commutator formula, Theorem 1A, implies that

$$\begin{aligned}&\big[[E(n,A,I),\mathrm{GL}(n,A,J)],\mathrm{GL}(n,A,K)\big]\\&=\big[[E(n,A,I),E(n,A,J)],\mathrm{GL}(n,A,K)\big].\end{aligned}$$

Thus, to prove Theorem 6A it remains to establish the following equality

$$\begin{aligned}&\big[[E(n,A,I),E(n,A,J)],\mathrm{GL}(n,A,K)\big]\\&=\big[[E(n,A,I),E(n,A,J)],E(n,A,K)\big].\end{aligned}\tag{33}$$

However, this last equality does not follow from the standard commutator formula. To establish this, we shall use the general "yoga of commutators" which is developed in [49] based on the work of Bak on localisation and patching in general linear groups (see [10, 40] and [44, §13]). In order to make use of this method, one needs to overcome two problems: firstly to devise an appropriate conjugation calculus to approach the identity (33) and secondly to perform the actual calculations. Both of these problems are equally challenging as the nature of the conjugation calculus depends on the problem in hand. In fact the term yoga of commutators is chosen to stress the overwhelming feeling of technical strain and exertion.

In this section we prove Theorem 6A, following [50]. We need the following elementary conjugation calculus, which are Lemmas 7, 8 and 11 from [49], respectively. Note that in Equations 34, 35 and 36 the calculations take place in the group $E(n,A_t)$.

Recall, that for an additive subgroup $\mathfrak{A}$ of R we denote by $E^L(n,\mathfrak{A})$ the subset (not a subgroup!) of $\mathrm{GL}(n,R)$ consisting of products of $\leqslant L$ elementary generators $t_{ij}(\xi)$, $1\leqslant i\neq j\leqslant n$, $\xi\in\mathfrak{A}$.

Lemma 21 (cf. [49]). *Let A be a module finite R-algebra, I,J two-sided ideals of A, $a,b,c\in A$ and $t\in R$. If m,l are given, there is an integer p such that*

$${}^{E^1(n,\frac{c}{t^m})}E(n,t^pA,t^p\langle a\rangle)\leqslant E(n,t^lA,t^l\langle a\rangle),\tag{34}$$

there is an integer p such that

$$\begin{aligned}&{}^{E^1(n,\frac{c}{t^m})}[E(n,t^pA,t^p\langle a\rangle),E(n,t^pA,t^p\langle b\rangle)]\\&\leqslant[E(n,t^lA,t^l\langle a\rangle),E(n,t^lA,t^l\langle b\rangle)],\end{aligned}\tag{35}$$

and there is an integer p such that

$$\big[E(n,t^pA,t^pI),E^1(n,\frac{J}{t^m})\big]\leqslant[E(n,t^lA,t^lI),E(n,t^lA,t^lJ)].\tag{36}$$

By Lemma 21, one easily obtains the following result. The proof is left to the reader.

Lemma 22. *Let A be a module finite R-algebra, I, J two-sided ideals of A and $t \in R$. If m, l, L are given, there is an integer p such that*

$$\big[E(n, t^p A, t^p I), {}^{E^L\left(n, \frac{A}{t^m}\right)} E^1\big(n, \frac{J}{t^m}\big)\big] \leqslant [E(n, t^l A, t^l I), E(n, t^l A, t^l J)]. \tag{37}$$

Denote by $E^L(n, \frac{A}{t^m}, \frac{K}{t^m})$ the product of $\leqslant L$ elements of the form

$$ {}^{E^1\left(n, \frac{A}{t^m}\right)} E^1\big(n, \frac{K}{t^m}\big).$$

In the following two Lemmas, as in Lemma 21, all the calculations take place in the fraction ring A_t. All the subgroups of $\mathrm{GL}(n, A_t)$ used in the Lemmas, such as the ones denoted by $E(n, A, I)$ or $\mathrm{GL}(n, A, J)$, are in fact the homomorphic images of these subgroups in $\mathrm{GL}(n, A)$ under the natural homomorphism $A \to A_t$. Since lemmas such as Lemma 1A and the generalised commutator formula (Theorem 1A) hold for these subgroups in $\mathrm{GL}(n, A)$, they also hold for their corresponding homomorphic images in $\mathrm{GL}(n, A_t)$.

Lemma 23. *Let A be an R-algebra, I, J two-sided ideals of A, and $t \in R$. For any given $e \in \mathrm{GL}(n, A_t, J_t)$ and an integer l, there is an integer p such that for any $g \in \mathrm{GL}(n, A, t^p I)$*

$$[e, g] \in \mathrm{GL}\,(n, A, t^l(IJ + JI)).$$

Proof. Note that all the entries of $g-1$ and $g^{-1}-1$ are in $t^p I$ (to emphasize our convention, they are in the image of $t^p I$ under the homomorphism $\theta : A \to A_t$) and all the entries of $e - 1$ and $e^{-1} - 1$ are in J_t. Choose $k \in \mathbb{N}$ such that one can write all the entries of $e - 1$ and $e^{-1} - 1$ in the form j/t^k, $j \in J$. Let

$$\begin{aligned} g &= 1 + \varepsilon \quad &\text{and} \quad g^{-1} &= 1 + \varepsilon' \\ e &= 1 + \delta \quad &\text{and} \quad e^{-1} &= 1 + \delta'. \end{aligned}$$

A straightforward computation shows that

$$\begin{aligned} \varepsilon + \varepsilon' + \varepsilon\varepsilon' &= \varepsilon + \varepsilon' + \varepsilon'\varepsilon = 0 \\ \delta + \delta' + \delta\delta' &= \delta + \delta' + \delta'\delta = 0. \end{aligned}$$

By the equalities above, one has

$$[e,g]=[1+\delta,1+\varepsilon]=1+\delta'\varepsilon'+\varepsilon\delta'+\varepsilon\delta'\varepsilon'+\delta\delta'\varepsilon'+\delta\varepsilon\delta'+\delta\varepsilon\delta'\varepsilon'.$$

So the entries of $[e,g]-1$ belong to $t^{p-2k}(IJ+JI)$. We finish the proof by choosing $p\geqslant l+2k$. □

Lemma 24. *Let A be a module finite R-algebra, I,J,K two-sided ideals of A and $t\in R$. For any given $e_2\in E(n,A_t,K_t)$ and an integer l, there is a sufficiently large integer p, such that*

$$[e_1,e_2]\in\big[[E(n,A,t^lI),E(n,A,t^lJ)],E(n,A,t^lK)\big],\tag{38}$$

where $e_1\in[E(n,t^pI),E(n,A,J)]$.

Proof. For any given $e_2\in E(n,A_t,K_t)$, one may find some positive integers m and L such that

$$e_2\in E^L(n,\frac{A}{t^m},\frac{K}{t^m}).$$

Applying the identity (C1$^+$) and repeated application of (34) in Lemma 21, we reduce the problem to show that

$$\begin{aligned}&\big[[E(n,t^pI),E(n,A,J)],{}^{c}e_{i',j'}(\frac{\gamma}{t^m})\big]\\&\leqslant\big[[E(n,A,t^lI),E(n,A,t^lJ)],E(n,A,t^lK)\big],\end{aligned}$$

where $c\in E^1(n,\frac{A}{t^m})$ and $\gamma\in K$. We further decompose

$$e_{i',j'}(\frac{\gamma}{t^m})=\big[e_{i',k}(t^{p'}),e_{k,j'}(\frac{\gamma}{t^{m+p'}})\big]$$

for some integer p'. Then

$$\big[e_1,{}^{c}e_{i',j'}(\frac{\gamma}{t^m})\big]=\big[e_1,[{}^{c}e_{i',k}(t^{p'}),{}^{c}e_{k,j'}(\frac{\gamma}{t^{m+p'}})]\big].$$

We use a variant of the Hall–Witt identity (see (C3))

$$[x,[y^{-1},z]]={}^{y^{-1}x}[[x^{-1},y],z]\ {}^{y^{-1}z}[[z^{-1},x],y],$$

to obtain

$$\begin{aligned}&\big[e_1,\big[{}^{c}e_{i',k}(t^{p'}),{}^{c}e_{k,j'}(\frac{\gamma}{t^{m+p'}})\big]\big]\\&\qquad={}^{y^{-1}x}\Big[\big[e_1^{-1},{}^{c}e_{i',k}(-t^{p'})\big],{}^{c}e_{k,j'}(\frac{\gamma}{t^{m+p'}})\Big]\\&\qquad\qquad\times{}^{y^{-1}z}\Big[\big[{}^{c}e_{k,j'}(\frac{-\gamma}{t^{m+p'}}),e_1\big],{}^{c}e_{i',k}(-t^{p'})\Big],\end{aligned}\tag{39}$$

where $x = e_1$, $y = {}^c e_{i',k}(-t^{p'})$, $z = {}^c e_{k,j'}(\frac{\gamma}{t^{m+p'}})$ and as before $c \in E^1(n, \frac{A}{t^m}) \leqslant E^1(n, \frac{A}{t^{m+p'}})$. We will look at each of the two factors of (39) separately.

By (34) in Lemma 21, for any given p'', one may find a sufficiently large p' such that

$$y = {}^c e_{i',k}(-t^{p'}) \in E(n, t^{p''}A, t^{p''}A) \leqslant E(n, A). \tag{40}$$

Then

$$\begin{aligned}\big[e_1^{-1}, {}^c e_{i',k}(-t^{p'})\big] &\in [[E(n, t^p I), E(n, A, J)], E(n, A)] \\ &\leqslant [\,\mathrm{GL}(n, A, t^p(IJ + JI)), E(n, A)] \\ &\leqslant E(n, A, t^p(IJ + JI)).\end{aligned}$$

Set $p_1 = p$. Thanks to Lemma 1A,

$$\begin{aligned}E(n, A, t^{p_1}(IJ + JI)) &\leqslant \big[E(n, t^{\lfloor \frac{p_1}{2} \rfloor} A), E(n, t^{\lfloor \frac{p_1}{2} \rfloor}(IJ + JI))\big] \\ &\leqslant E(n, t^{\lfloor \frac{p_1}{2} \rfloor} A, t^{\lfloor \frac{p_1}{2} \rfloor}(IJ + JI)).\end{aligned} \tag{41}$$

Hence we obtain that

$$\begin{aligned}&{}^{y^{-1}x}\Big[\big[e_1^{-1}, {}^c e_{i',k}(-t^{p'})\big], {}^c e_{k,j'}\big(\frac{\gamma}{t^{m+p'}}\big)\Big] \\ &\qquad \in {}^{y^{-1}x}\Big[E(n, t^{\lfloor \frac{p_1}{2} \rfloor} A, t^{\lfloor \frac{p_1}{2} \rfloor}(IJ + JI)), {}^c e_{k,j'}\big(\frac{\gamma}{t^{m+p'}}\big)\Big],\end{aligned}$$

where $x \in [E(n, t^{p_1} I), E(n, A, J)]$, $y \in E(n, t^{p''}A, t^{p''}A)$. By Lemma 22, for any given integer l' we may find a sufficiently large p_1 such that

$$\begin{aligned}&{}^{y^{-1}x}\Big[E(n, t^{\lfloor \frac{p_1}{2} \rfloor} A, t^{\lfloor \frac{p_1}{2} \rfloor}(IJ + JI)), {}^c e_{k,j'}\big(\frac{\gamma}{t^{m+p'}}\big)\Big] \\ &\in {}^{y^{-1}x}\big[E(n, t^{2l'}A, t^{2l'}(IJ + JI)), E(n, t^{2l'}A, t^{2l'}K)\big] \\ &\leqslant {}^{y^{-1}x}\big[[E(n, t^{l'}A, t^{l'}I), E(n, t^{l'}A, t^{l'}J)], E(n, t^{2l'}A, t^{2l'}K)\big] \\ &\leqslant {}^{y^{-1}x}\big[[E(n, t^{l'}A, t^{l'}I), E(n, t^{l'}A, t^{l'}J)], E(n, t^{l'}A, t^{l'}K)\big] \\ &= \big[[{}^{y^{-1}x}E(n, t^{l'}A, t^{l'}I), {}^{y^{-1}x}E(n, t^{l'}A, t^{l'}J)], {}^{y^{-1}x}E(n, t^{l'}A, t^{l'}K)\big],\end{aligned}$$

where by definition $y^{-1}x \in E(n, \frac{A}{t^0}, \frac{A}{t^0})$. By (34) in Lemma 21, for any given integer l, we may find a sufficiently large l' such that

$$\begin{aligned}&{}^{y^{-1}x}\big[[E(n, t^{l'}A, t^{l'}I), E(n, t^{l'}A, t^{l'}J)], E(n, t^{l'}A, t^{l'}K)\big] \\ &\qquad \leqslant \big[[E(n, t^l A, t^l I), E(n, t^l A, t^l J)], E(n, t^l A, t^l K)\big].\end{aligned}$$

This shows that for any given l, one may find a sufficiently large p_1 such that the first factor of (39)

$$ {}^{y^{-1}x}\Big[\big[e_1^{-1}, {}^{c}e_{i',k}(-t^{p'})\big], {}^{c}e_{k,j'}\big(\frac{\gamma}{t^{m+p'}}\big)\Big] \\ \in \big[[E(n,t^lA,t^lI),E(n,t^lA,t^lJ)],E(n,t^lA,t^lK)\big]. $$

Next we consider the second factor of (39),

$$ {}^{y^{-1}z}\Big[\big[{}^{c}e_{k,j'}\big(\frac{-\gamma}{t^{m+p'}}\big), e_1\big], {}^{c}e_{i',k}(-t^{p'})\Big]. $$

Set $p_2 = p$. Note that

$$ e_1 \in [E(n,t^{p_2}I),E(n,A,J)] \leqslant \mathrm{GL}\,(n,A,t^{p_2}(IJ+JI)) $$

and

$$ {}^{c}e_{k,j'}\big(\frac{\gamma}{t^{m+p'}}\big) \in {}^{E^1(n,\frac{A}{t^{m+p'}})}E^1\big(n,\frac{K}{t^{m+p'}}\big), $$

where p' is given by (40) from the first part of the proof. We may apply Lemma 23 to find a sufficiently large p_2 such that

$$ \big[{}^{c}e_{k,j'}\big(\frac{-\gamma}{t^{m+p'}}\big), e_1\big] \in \mathrm{GL}\,\big(n,A,t^{p''}(K(IJ+JI)+(IJ+JI)K)\big) \tag{42} $$

for any given p''. Using the commutator formula together with (40), one gets

$$ {}^{y^{-1}z}\Big[\big[{}^{c}e_{k,j'}\big(\frac{-\gamma}{t^{m+p'}}\big), e_1\big], {}^{c}e_{i',k}(-t^{p'})\Big] \\ \in {}^{y^{-1}z}E\big(n,A,t^{p''}(K(IJ+JI)+(IJ+JI)K)\big). $$

Applying Lemma 1A twice, one gets

$$ E\big(n,A,t^{p''}(K(IJ+JI)+(IJ+JI)K)\big) \\ \leqslant \Big[E\big(n,t^{\lfloor\frac{2p''}{3}\rfloor}((IJ+JI)+(IJ+JI))\big),E\big(n,t^{\lfloor\frac{p''}{3}\rfloor}K\big)\Big] \\ \leqslant \Big[\big[E(n,t^{\lfloor\frac{p''}{3}\rfloor}I),E(n,t^{\lfloor\frac{p''}{3}\rfloor}J)\big],E(n,t^{\lfloor\frac{p''}{3}\rfloor}K)\Big]. $$

Hence, we have

$$
\begin{aligned}
{}^{y^{-1}z}\Big[\big[{}^{c}e_{k,j'}\big(\frac{-\gamma}{t^{m+p'}}\big), e_1\big], {}^{c}e_{i',k}(-t^{p'})\Big]
&\leqslant {}^{y^{-1}z}\Big[\big[E(n, t^{\lfloor \frac{p''}{3}\rfloor} I), E(n, t^{\lfloor \frac{p''}{3}\rfloor} J)\big], E(n, t^{\lfloor \frac{p''}{3}\rfloor} K)\Big] \\
&= \Big[\big[{}^{y^{-1}z}E(n, t^{\lfloor \frac{p''}{3}\rfloor} I), {}^{y^{-1}z}E(n, t^{\lfloor \frac{p''}{3}\rfloor} J)\big], {}^{y^{-1}z}E(n, t^{\lfloor \frac{p''}{3}\rfloor} K)\Big].
\end{aligned}
$$

Now applying (34) in Lemma 21 to every component of the commutator above, we may find a sufficiently large p'' such that for any given l,

$$
\begin{aligned}
&\Big[\big[{}^{y^{-1}z}E(n, t^{\lfloor \frac{p''}{3}\rfloor} I), {}^{y^{-1}z}E(n, t^{\lfloor \frac{p''}{3}\rfloor} J)\big], {}^{y^{-1}z}E(n, t^{\lfloor \frac{p''}{3}\rfloor} K)\Big] \\
&\qquad\qquad \leqslant \big[[E(n, t^l A, t^l I), E(n, t^l A, t^l J)], E(n, t^l A, t^l K)\big].
\end{aligned}
$$

Choose p_2 in (42) according to this p'' and then consider p to be the larger of p_1 and p_2. This finishes the Lemma. $\square$

The proof of this result, as also the proofs of similar results for other groups, are mostly prestidigitation and tightrope walking, and similar in spirit to the relative commutator calculus in [49]. However, this piece of commutator calculus operates at a different level of technical sophistication. For instance, now we have to plug in not just the elementary generators, or their conjugates, as in [49, 45, 46], but also the other two types of generators constructed in Theorem 3.

Proof of Theorem 7A. The functors E_n and GL_n commute with direct limits. By Proposition 1 and §12.1, one reduces the proof to the case where A is finite over R and R is Noetherian.

First by the generalized commutator formula (Theorem 1A), we have

$$[E(n, A, I), \mathrm{GL}(n, A, J)] = [E(n, A, I), E(n, A, J)]. \tag{43}$$

Thus it suffices to prove the following equation

$$
\begin{aligned}
&\big[[E(n, A, I), E(n, A, J)], \mathrm{GL}(n, A, K)\big] \\
&= \big[[E(n, A, I), E(n, A, J)], E(n, A, K)\big].
\end{aligned}
$$

By Lemma 2A, $[E(n,A,I),E(n,A,J)]$ is generated by the conjugates in $E(n,A)$ of the following four types of elements

$$\begin{aligned}
e &= \big[e_{j,i}(\alpha), {}^{e_{i,j}(a)}e_{j,i}(\beta)\big],\\
e &= [e_{j,i}(\alpha), e_{i,j}(\beta)],\\
e &= e_{i,j}(\alpha\beta),\\
e &= e_{i,j}(\beta\alpha),
\end{aligned} \tag{44}$$

where $i \neq j$, $\alpha \in I$, $\beta \in J$ and $a \in A$. We claim that for any $g \in \mathrm{GL}(n,A,K)$,

$$[e,g] \in \big[[E(n,A,I),E(n,A,J)],E(n,A,K)\big]. \tag{45}$$

Let $g \in \mathrm{GL}(n,A,K)$. For any maximal ideal $\mathfrak{m}$ of R, the ring $A_\mathfrak{m}$ contains $K_\mathfrak{m}$ as an ideal (K being an ideal of A). Consider the natural homomorphism $\theta_\mathfrak{m} : A \to A_\mathfrak{m}$ which induces a homomorphism (call it $\theta_\mathfrak{m}$ again) on the level of general linear groups, $\theta_\mathfrak{m} : \mathrm{GL}(n,A) \to \mathrm{GL}(n,A_\mathfrak{m})$. Therefore, $\theta_\mathfrak{m}(g) \in \mathrm{GL}(n,A_\mathfrak{m},K_\mathfrak{m})$. Since $A_\mathfrak{m}$ is module finite over the local ring $R_\mathfrak{m}$, $A_\mathfrak{m}$ is semilocal [19, III(2.5), (2.11)], therefore its stable rank is 1. It follows that $\mathrm{GL}(n,A_\mathfrak{m},K_\mathfrak{m}) = E(n,A_\mathfrak{m},K_\mathfrak{m})\,\mathrm{GL}(1,A_\mathfrak{m},K_\mathfrak{m})$ (see [37, Th. 4.2.5]). So $\theta_\mathfrak{m}(g)$ can be decomposed as $\theta_\mathfrak{m}(g) = \varepsilon h$, where $\varepsilon \in E(n,A_\mathfrak{m},K_\mathfrak{m})$ and h is a diagonal matrix all of whose diagonal coefficients are 1, except possibly the k-th diagonal coefficient, and k can be chosen arbitrarily. By (§12.1), there is a $t_\mathfrak{m} \in R\backslash\mathfrak{m}$ such that

$$\theta_{t_\mathfrak{m}}(g) = \varepsilon h, \tag{46}$$

where $\varepsilon \in E(n,A_{t_\mathfrak{m}},K_{t_\mathfrak{m}})$, and h is a diagonal matrix with only one nontrivial diagonal entry which lies in $A_{t_\mathfrak{m}}$.

For any maximal ideal $\mathfrak{m} \lhd R$, choose $t_\mathfrak{m} \in R\backslash\mathfrak{m}$ as above and an arbitrary positive integer $p_\mathfrak{m}$. (We will later choose $p_\mathfrak{m}$ according to Lemma 24.) Since the collection of all $\{t_\mathfrak{m}^{p_\mathfrak{m}} \mid \mathfrak{m} \in \max(R)\}$ is not contained in any maximal ideal, we may find a finite number of $t_{\mathfrak{m}_s}^{p_s} \in R\backslash\mathfrak{m}_s$ and $x_s \in R$, $s = 1,\dots,k$, such that

$$\sum_{s=1}^{k} t_{\mathfrak{m}_s}^{p_s} x_s = 1.$$

In order to prove (45), first we consider the generators of the first kind in (44), namely $e = \big[e_{j,i}(\alpha), {}^{e_{i,j}(a)}e_{j,i}(\beta)\big]$. Consider

$$\begin{aligned} e = \big[e_{j,i}(\alpha), {}^{e_{i,j}(a)}e_{j,i}(\beta)\big] &= \big[e_{j,i}\big(\big(\sum_{s=1}^{k} t_{\mathfrak{m}_s}^{p_s} x_s\big)\alpha\big), {}^{e_{i,j}(a)}e_{j,i}(\beta)\big] \\ &= \big[\prod_{s=1}^{k} e_{j,i}(t_{\mathfrak{m}_s}^{p_s} x_s\alpha), {}^{e_{i,j}(a)}e_{j,i}(\beta)\big]. \end{aligned}$$

By (C2$^+$) identity, $e = \big[\prod_{s=1}^{k} e_{j,i}(t_{\mathfrak{m}_s}^{p_s} x_s\alpha), {}^{e_{i,j}(a)}e_{j,i}(\beta)\big]$ can be written as a product of the following form:

$$\begin{aligned} e = \big({}^{e_k}\big[e_{j,i}(t_{\mathfrak{m}_k}^{p_k} x_k\alpha), {}^{e_{i,j}(a)}e_{j,i}(\beta)\big]\big)\big({}^{e_{k-1}}\big[e_{j,i}(t_{\mathfrak{m}_{k-1}}^{p_{k-1}} x_{k-1}\alpha), {}^{e_{i,j}(a)}e_{j,i}(\beta)\big]\big) \\ \times \cdots \times \big({}^{e_1}\big[e_{j,i}(t_{\mathfrak{m}_1}^{p_1} x_1\alpha), {}^{e_{i,j}(a)}e_{j,i}(\beta)\big]\big), \quad (47) \end{aligned}$$

where $e_1, e_2, \ldots, e_k \in E(n, A)$. Note that from (C2$^+$) it is clear that all e_s, $s = 1, \ldots, k$, are products of elementary matrices of the form $e_{j,i}(A)$. Thus $e_s = e_{j,i}(a_s)$, where $a_s \in A$ and $s = 1, \ldots, k$, which clearly commutes with $e_{j,i}(x)$ for any $x \in A$. So the commutator (47) is equal to

$$\begin{aligned} e = \big(\big[e_{j,i}(t_{\mathfrak{m}_k}^{p_k} x_k\alpha), {}^{e_k e_{i,j}(a)}e_{j,i}(\beta)\big]\big)\big(\big[e_{j,i}(t_{\mathfrak{m}_{k-1}}^{p_{k-1}} x_{k-1}\alpha), {}^{e_{k-1} e_{i,j}(a)}e_{j,i}(\beta)\big]\big) \\ \times \cdots \times \big(\big[e_{j,i}(t_{\mathfrak{m}_1}^{p_1} x_1\alpha), {}^{e_1 e_{i,j}(a)}e_{j,i}(\beta)\big]\big). \quad (48) \end{aligned}$$

Using (C2$^+$) and in view of (48) we obtain that $[e, g]$ is a product of the conjugates in $E(n, A)$ of

$$w_s = \Big[\big[e_{j,i}(t_{\mathfrak{m}_s}^{p_s} x_s\alpha), {}^{e_{j,i}(a_s)e_{i,j}(a)}e_{j,i}(\beta)\big], g\Big],$$

where $a_s \in A$ and $s = 1, \ldots, k$.

For each $s = 1, \ldots, k$, consider $\theta_{t_{\mathfrak{m}_s}}(w_s)$ which we still write as w_s but keep in mind that this image is in $\mathrm{GL}(n, A_{t_{\mathfrak{m}_s}})$.

Note that all $\big[e_{j,i}(t_{\mathfrak{m}_s}^{p_s} x_s\alpha), {}^{e_{j,i}(a_s)e_{i,j}(a)}e_{j,i}(\beta)\big]$, $s = 1, \ldots, k$, differ from the identity matrix only in the ith, jth rows and the ith, jth columns. Since $n > 2$, we can choose h in the decomposition (46) so that it commutes with

$$\big[e_{j,i}(t_{\mathfrak{m}_s}^{p_s} x_s\alpha), {}^{e_{j,i}(a_s)e_{i,j}(a)}e_{j,i}(\beta)\big].$$

This allows us to reduce $\theta_{t_{\mathfrak{m}_s}}(w_s)$ to

$$\Big[\big[e_{j,i}(t_{\mathfrak{m}_s}^{p_s} x_s\alpha), {}^{e_{j,i}(a_s)e_{i,j}(a)}e_{j,i}(\beta)\big], \varepsilon\Big],$$

where $\varepsilon \in E(n, A_{t_{\mathfrak{m}_s}}, K_{t_{\mathfrak{m}_s}})$. By Lemma 24, for any given l_s, there is a sufficiently large p_s, $s = 1, \dots k$, such that

$$\Big[\big[e_{j,i}(t_{\mathfrak{m}_s}^{p_s} x_s \alpha), {}^{e_{j,i}(a_s) e_{i,j}(a)} e_{j,i}(\beta)\big], \varepsilon\Big] \in \big[[E(n, A, t^{l_s} I), E(n, A, t^{l_s} J)], E(n, A, t^{l_s} K)\big].$$

Let us choose l_s to be large enough so that by Lemma 18 the restriction of

$$\theta_{t_{\mathfrak{m}_s}} : \mathrm{GL}(n, A, t_{\mathfrak{m}_s}^{l_s} A) \to \mathrm{GL}(n, A_{t_{\mathfrak{m}_s}})$$

be injective. Then it is easy to see that for any s, we have

$$\Big[\big[e_{j,i}(t_{\mathfrak{m}_s}^{p_s} x_s \alpha), {}^{e_{j,i}(a_s) e_{i,j}(a)} e_{j,i}(\beta)\big], g\Big] \in \big[[E(n, A, I), E(n, A, J)], E(n, A, K)\big].$$

Since relative elementary subgroups E_n are normal in $\mathrm{GL}(n, A)$ (Theorem 11), it follows that $[e, g] \in \big[[E(n, A, I), E(n, A, J)], E(n, A, K)\big]$.

When the generator is of the second kind, $e = [e_{i,j}(\alpha), e_{j,i}(\beta)]$, a similar argument goes through, which is left to the reader.

Now consider the generators of the 3rd and 4th kind, namely, the conjugates of the following two types of elements,

$$e = e_{i,j}(\alpha\beta), \quad \text{or } e = e_{i,j}(\beta\alpha).$$

By the normality of $E(n, A, IJ+JI)$, the conjugates of e are in $E(n, A, IJ+JI)$. Then

$$[e, g] \in [E(n, A, IJ + JI), \mathrm{GL}(n, A, K)].$$

By the generalized commutator formula (Theorem 1A), one obtains

$$[E(n, A, IJ + JI), \mathrm{GL}(n, A, K)] = [E(n, A, IJ + JI), E(n, A, K)].$$

Now applying Lemma 1A, we finally get

$$[E(n, A, IJ + JI), E(n, A, K)] \leqslant \big[[E(n, A, I), E(n, A, J)], E(n, A, K)\big].$$

Therefore $[e, g] \in \big[[E(n, A, I), E(n, A, J)], E(n, A, K)\big]$. This proves our claim. Thus we established (45) for all type of generators e of (44).

To finish the proof, let

$$e \in [E(n, A, I), \mathrm{GL}(n, A, J)] = [E(n, A, I), E(n, A, J)],$$

and $g \in \mathrm{GL}(n, A, K)$. Then by Theorem 3A,

$$e = e_1 \times e_2 \times \cdots \times e_k$$

with e_i takes any of the forms in (29). Thanks to (C2$^+$) identity and the normality of relative elementary subgroups E_n, it suffices to show that

$$[e_i, g] \in \big[[E(n, A, I), E(n, A, J)], E(n, A, K)\big], \quad i = 1, \dots, k.$$

But this is exactly what has been shown above. This completes the proof. □

Similar result for unitary groups is [47, Theorem 7].

Theorem 6B. *Let $n \geqslant 3$, R be a commutative ring, (A, Λ) be a form ring such that A is a quasi-finite R-algebra. Further, let (I, Γ), (J, Δ) and (K, Ω) be three form ideals of a form ring (A, Λ). Then*

$$\begin{aligned}\big[[\,\mathrm{EU}(2n, I, \Gamma), \mathrm{GU}(2n, J, \Delta)], \mathrm{GU}(2n, K, \Omega)\big]\\ = \big[[\,\mathrm{EU}(2n, I, \Gamma), \mathrm{EU}(2n, J, \Delta)], \mathrm{EU}(2n, K, \Omega)\big].\end{aligned}$$

The proof of Theorem 6B is even more toilsome, than that of Theorem 6A. In fact, just the proof of the unitary analogue of the above triple commutator lemma, [47, Lemma 13], consists of some six solid pages of calculations.

After Theorem 6B is established, Theorem 4A, 4B follows by 2–3 pages of artless formal juggling with level calculations and commutator identities, the details of calculations can be found in [49, 47]. For Chevalley groups they are still unpublished.

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