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CHOW RING OF GENERIC MAXIMAL ORTHOGONAL GRASSMANNIANS

Abstract. We compute the Chow ring of the maximal orthogonal Grassmannian corresponding to a versal torsor, and in particular show that it has no torsion as an abelian group.

§1. Introduction

To each nonsingular quadratic form over a field with trivial discriminant one can associate a projective variety of totally isotropic spaces of dimension half the rank of the form (rounded down), called the *maximal orthogonal Grassmannian*. For general facts about quadratic forms and orthogonal Grassmannians we refer to [2]. In the split situation, that is when this variety has a rational point, its Chow ring was computed essentially by Borel (and formally, over any field, by Demazure). In non-split situation the answer is not known, but some particular results have been obtained. Vishik in [5] computed the image of the restriction map to the algebraic closure in terms of the so called *J-invariant* of a quadratic form. So the main difficulty is to compute the kernel of the restriction map, which consists of the torsion elements. In the present paper we compute the Chow ring in the case of *generic* quadratic forms, in the sense that the Chow ring of the respective torsor is $\mathbb{Z}$, and show that there is no torsion in this situation.

It is convenient not to restrict oneself to the case when the base is a field, but to work over an arbitrary smooth base. Some results in this direction were obtained by Edidin and Graham in [1].

§2. Chow groups of torsors

We fix some field F; all varieties are supposed to be varieties over F.

Recall that an algebraic group S is called *special* if any S-torsor is locally trivial in Zariski topology. Typical examples are GL_n, SL_n and Sp_{2n}.

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Lemma 1. *Let S be a special split reductive group, E be an S-torsor over some smooth base E/S. Then we have*

$$\mathrm{CH}^*(E) \simeq \mathrm{CH}^*(E/S) \otimes_{\mathrm{CH}^*(BS)} \mathbb{Z}\,.$$

Proof. See [3, Lemma 7.1]. □

Recall that a sequence of graded rings

$$A^* \to B^* \to C^*$$

is called *coexact* if the kernel of the map $B^* \to C^*$ coincides with the ideal generated by images of $A^{>0}$.

Lemma 2. *Let G be a reductive groups, S be a special split reductive subgroup of G, E be a G-torsor over some smooth base X. Then the sequence*

$$\mathrm{CH}^*(BS) \otimes_{\mathrm{CH}^*(BG)} \mathrm{CH}^*(X) \to \mathrm{CH}^*(E/S) \to \mathrm{CH}^*(E) \otimes_{\mathrm{CH}^*(X)} \mathbb{Z}$$

is coexact (*and the last map is surjective*).

Proof. First we need to describe the first map. Since E is an S-torsor over E/S, we have a canonical map $\mathrm{CH}^*(BS) \to \mathrm{CH}^*(E/S)$, and the map $\mathrm{CH}^*(X) \to \mathrm{CH}^*(E/S)$ is just the pull-back map. We have to show that these maps agree on $\mathrm{CH}^*(BG)$. This amounts to the fact that

$$\begin{array}{ccc} (E \times G)/S & \longrightarrow & E \\ \downarrow & & \downarrow \\ E/S & \longrightarrow & X \end{array}$$

is a pull-back square.

The surjectivity of the last map is obvious from Lemma 1.

Assume that we have an element $\alpha \in \mathrm{CH}^*(E/S)$ that maps to 0 in $\mathrm{CH}^*(E) \otimes_{\mathrm{CH}^*(X)} \mathbb{Z}$. This means that the image of α in $\mathrm{CH}^*(E)$ can be written as $\sum a_i b_i$ with $a_i \in \mathrm{CH}^{>0}(X)$, $b_i \in \mathrm{CH}^*(E)$. Choose any preimages $\beta_i \in \mathrm{CH}^*(E/S)$ of b_i and adjust α to $\alpha - \sum a_i \beta_i$. Then the image of α becomes 0 in $\mathrm{CH}^*(E)$, and by Lemma 1 α lies in the ideal generated by $\mathrm{CH}^{>0}(BS)$. □

Lemma 3. *In the notation of Lemma 2 assume that $\mathrm{CH}^*(X) = \mathbb{Z}$ (for example X is the spectrum of a field), $\mathrm{CH}^*(E) = \mathbb{Z}$ (that is the torsor is* versal), *and $\mathrm{CH}^*(BS) \otimes_{\mathrm{CH}^*(BG)} \mathbb{Z}$ is a free $\mathbb{Z}$-module of the same rank as $\mathrm{CH}^*(G/S)$. Then $\mathrm{CH}^*(E/S)$ is isomorphic to $\mathrm{CH}^*(BS) \otimes_{\mathrm{CH}^*(BG)} \mathbb{Z}$.*

Proof. Consider the base change $E \to X$; under this map E becomes a trivial G-torsor over E, so we have a pull-back map

$$\mathrm{CH}^*(E/S) \to \mathrm{CH}^*(E \times G/S).$$

The latter is isomorphic to $\mathrm{CH}^*(E) \otimes \mathrm{CH}^*(G/S)$ by [4, Lemma 6.1], that is to $\mathrm{CH}^*(G/S)$ by our assumption. After tensoring with $\mathbb{Q}$ this map becomes surjective. Indeed, $\mathrm{CH}^*(G) \otimes \mathbb{Q} = \mathbb{Q}$, and by Lemma 1 the map

$$\mathrm{CH}^*(BS) \otimes \mathbb{Q} \to \mathrm{CH}^*(G/S) \otimes \mathbb{Q}$$

is surjective. Now we have a commutative diagram

$$\begin{array}{ccc} \mathrm{CH}^*(BS) \otimes \mathbb{Q} & \longrightarrow & \mathrm{CH}^*(E/S) \otimes \mathbb{Q} \\ & \searrow & \downarrow \\ & & \mathrm{CH}^*(G/S) \otimes \mathbb{Q}, \end{array}$$

so the vertical map is surjective as well.

Composing with the map from Lemma 2 we obtain a map of two free $\mathbb{Z}$-modules of the same rank

$$\mathrm{CH}^*(BS) \otimes_{\mathrm{CH}^*(BG)} \mathbb{Z} \to \mathrm{CH}^*(G/S)$$

that becomes surjective after tensoring with $\mathbb{Q}$. This means that this map is injective, and a fortiori the left map from Lemma 2 is injective, but it is also surjective by the assumption $\mathrm{CH}^*(E) = \mathbb{Z}$. □

§3. Maximal orthogonal Grassmannian

Now we apply the results of the previous section to the case $G = \mathrm{SO}_{2n+1}$, $S = \mathrm{GL}_n$ embedded as a Levi subgroup of the maximal parabolic subgroup of type P_n in G. Note that over fields we don't lose generality considering only odd dimensional forms, because there is an isomorphism

$$\mathrm{SO}_{2n+2}/P_{n+1} \simeq \mathrm{SO}_{2n+1}/P_n,$$

and in any nonsingular even dimensional form one can find a nonsingular subform of codimension 1.

The Chow ring of $B\,\mathrm{SO}_{2n+1}$ was computed in [4], the answer is

$$\mathbb{Z}[c_2, c_3, \dots, c_{2n+1}]/(2c_i \text{ for odd } i),$$

where c_i's are the Chern classes of the natural representation. The weights of the natural representations of SO_{2n+1} are 0 and $\pm x_i$, where x_i stand for the weights of the natural representation of GL_n. So c_k goes to

$\sigma_k(x_1^2,\dots,x_n^2)$ in $\mathrm{CH}^*(BT)$ for even k and to 0 for odd k, where T is the maximal torus and σ_i's are the elementary symmetric functions. Of course the image actually lies in $\mathrm{CH}^*(\mathrm{BGL}_n)$ considered as a subring in $\mathrm{CH}^*(BT)$. We have

$$\sum_k(-1)^k\sigma_k(x_1^2,\dots,x_n^2)t^{2k}=\prod_i(1-tx_i^2)=\prod_i(1-tx_i)\prod_i(1+tx_i)$$
$$=\sum_{l,m}(-1)^l\sigma_l(x_1,\dots,x_n)\sigma_m(x_1,\dots,x_n)t^{l+m}.$$

So the left-hand term in the sequence of Lemma 2 is

$$\mathrm{CH}^*(X)[\xi_1,\dots,\xi_n]/(\xi_k^2+2\xi_{2k}+2\sum_{l=1}^{k-1}(-1)^{l+k}\xi_l\xi_{2k-l}-c_{2k}(V)),\qquad(*)$$

where V stands for the natural vector bundle of rank $2n+1$ over X determined by E and ξ_i's are the images of the Chern classes of the natural representation of GL_n.

In particular when $\mathrm{CH}^*(X)=\mathbb{Z}$ we see that $\mathrm{CH}^*(BS)\otimes_{\mathrm{CH}^*(BG)}\mathbb{Z}$ is a free $\mathbb{Z}$-module of rank 2^n (with a basis spanned on $\xi_{i_1}\dots\xi_{i_k}$ with pair-wise distinct $i_1,\dots,i_k$), and we can apply Lemma 3 to get the main result:

Theorem 1. *Let E be SO_{2n+1}-torsor over X such that $\mathrm{CH}^*(E)=\mathbb{Z}$ and $\mathrm{CH}^*(X)=\mathbb{Z}$ (for example, a versal torsor over a field). Then $\mathrm{CH}^*(E/\,\mathrm{GL}_n)$ is isomorphic to $(*)$, in particular, it is a free $\mathbb{Z}$-module of rank 2^n.*

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