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ON THE GROTHENDIECK–SERRE CONJECTURE CONCERNING PRINCIPAL G-BUNDLES OVER SEMI-LOCAL DEDEKIND DOMAINS

Abstract. Let R be a semi-local Dedekind domain and let K be the field of fractions of R. Let G be a reductive semisimple simply connected R-group scheme such that every semisimple normal R-subgroup scheme of G contains a split R-torus $\mathbb{G}_{m,R}$. We prove that the kernel of the map

$$H^1_{\text{ét}}(R,G) \to H^1_{\text{ét}}(K,G)$$

induced by the inclusion of R into K, is trivial. This result partially extends the Nisnevich theorem [10, Thm.4.2].

§1. Introduction

A well-known conjecture due to J.-P. Serre and A. Grothendieck [15, Remarque, p.31], [7, Remarque 3, p. 26–27], and [8, Remarque 1.11.a] asserts that given a regular local ring R and its field of fractions K and given a reductive group scheme G over R the map

$$H^1_{\text{ét}}(R,G) \to H^1_{\text{ét}}(K,G),$$

induced by the inclusion of R into K, has trivial kernel.

The Grothendieck–Serre conjecture holds for semi-local regular rings containing a field. That is proved in [6] and in [11]. The first of these two papers is heavily based on results of [13] and [12]. For the detailed history of the topic see, for instance, [6]. Assuming that R is not equicharacteristic, the conjecture has been established only in the case where G is an R-torus [2] and in the case where G is a reductive group scheme over a discrete valuation ring R [10, Theorem 4.2]. In the present paper, we extend the latter result to the case of an isotropic semisimple simply

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connected reductive group scheme over a semi-local Dedekind domain R; see Theorem 3.4.

§2. Preliminaries

2.1. Parabolic subgroups and elementary subgroups. Let A be a commutative ring. Let G be an isotropic reductive group scheme over A, and let P be a parabolic subgroup of G in the sense of [4]. Since the base $\operatorname{Spec} A$ is affine, the group P has a Levi subgroup L_P [4, Exp. XXVI Cor. 2.3]. There is a unique parabolic subgroup P^- in G which is opposite to P with respect to L_P, that is $P^- \cap P = L_P$, cf. [4, Exp. XXVI Th. 4.3.2]. We denote by U_P and U_{P^-} the unipotent radicals of P and P^- respectively.

Definition 2.1. *The* elementary subgroup $E_P(A)$ corresponding to P *is the subgroup of $G(A)$ generated as an abstract group by $U_P(A)$ and $U_{P^-}(A)$.*

Note that if L'_P is another Levi subgroup of P, then L'_P and L_P are conjugate by an element $u \in U_P(A)$ [4, Exp. XXVI Cor. 1.8], hence $E_P(A)$ does not depend on the choice of a Levi subgroup or of an opposite subgroup P^-, respectively. We suppress the particular choice of L_P or P^- in this context.

Definition 2.2. *A parabolic subgroup P in G is called* strictly proper*, if it intersects properly every normal semisimple subgroup of G.*

We will use the following result that is a combination of [14] and [4, Exp. XXVI, §5].

Lemma 2.3. *Let G be a reductive group scheme over a commutative ring A, and let R be a commutative A-algebra. Assume that A is a semilocal ring. Then the subgroup $E_P(R)$ of $G(R)$ is the same for any minimal parabolic A-subgroup P of G. If, moreover, G contains a strictly proper parabolic A-subgroup, the subgroup $E_P(R)$ is the same for any strictly proper parabolic A-subgroup P.*

Proof. See [16, Theorem 2.1]. □

2.2. Torus actions on reductive groups. Let R be a commutative ring with 1, and let $S = (\mathbb{G}_{\mathrm{m},R})^N = \operatorname{Spec}(R[x_1^{\pm 1}, \dots, x_N^{\pm 1}])$ be a split N-dimensional torus over R. Recall that the character group $X^*(S) =$

$\mathrm{Hom}_R(S, \mathbb{G}_{\mathrm{m},R})$ of S is canonically isomorphic to $\mathbb{Z}^N$. If S acts R-linearly on an R-module V, this module has a natural $\mathbb{Z}^N$-grading

$$V = \bigoplus_{\lambda \in X^*(S)} V_\lambda,$$

where

$$V_\lambda = \{v \in V \mid s \cdot v = \lambda(s)v \text{ for any } s \in S(R)\}.$$

Conversely, any $\mathbb{Z}^N$-graded R-module V can be provided with an S-action by the same rule.

Let G be a reductive group scheme over R in the sense of [4]. Assume that S acts on G by R-group automorphisms. The associated Lie algebra functor $\mathrm{Lie}(G)$ then acquires a $\mathbb{Z}^N$-grading compatible with the Lie algebra structure,

$$\mathrm{Lie}(G) = \bigoplus_{\lambda \in X^*(S)} \mathrm{Lie}(G)_\lambda.$$

We will use the following version of [4, Exp. XXVI Prop. 6.1].

Lemma 2.4. *Let $L = \mathrm{Cent}_G(S)$ be the subscheme of G fixed by S. Let $\Psi \subseteq X^*(S)$ be an R-subsheaf of sets closed under addition of characters.*

(i) *If $0 \in \Psi$, then there exists a unique smooth connected closed subgroup U_Ψ of G containing L and satisfying*

$$\mathrm{Lie}(U_\Psi) = \bigoplus_{\lambda \in \Psi} \mathrm{Lie}(G)_\lambda. \tag{1}$$

Moreover, if $\Psi = \{0\}$, then $U_\Psi = L$; if $\Psi = -\Psi$, then U_Ψ is reductive; if $\Psi \cup (-\Psi) = X^*(S)$, then U_Ψ and $U_{-\Psi}$ are two opposite parabolic subgroups of G with the common Levi subgroup $U_{\Psi \cap (-\Psi)}$.

(ii) *If $0 \notin \Psi$, then there exists a unique smooth connected unipotent closed subgroup U_Ψ of G normalized by L and satisfying* (1).

Proof. The statement immediately follows by faithfully flat descent from the standard facts about the subgroups of split reductive groups proved in [4, Exp. XXII]; see the proof of [4, Exp. XXVI Prop. 6.1]. □

Definition 2.5. *The sheaf of sets*

$$\Phi = \Phi(S, G) = \{\lambda \in X^*(S) \setminus \{0\} \mid \mathrm{Lie}(G)_\lambda \neq 0\}$$

is called the system of relative roots of G with respect to S.

Choosing a total ordering on the $\mathbb{Q}$-space $\mathbb{Q}\otimes_{\mathbb{Z}}X^*(S)\cong\mathbb{Q}^n$, one defines the subsets of positive and negative relative roots Φ^+ and Φ^-, so that Φ is a disjoint union of Φ^+, Φ^-, and $\{0\}$. By Lemma 2.4 the closed subgroups

$$U_{\Phi^+\cup\{0\}}=P,\qquad U_{\Phi^-\cup\{0\}}=P^-$$

are two opposite parabolic subgroups of G with the common Levi subgroup $\mathrm{Cent}_G(S)$. Thus, if a reductive group G over R admits a non-trivial action of a split torus, then it has a proper parabolic subgroup. The converse is true Zariski-locally, see Lemma 2.6 below.

2.3. Relative roots and subschemes. In order to prove our main result, we need to use the notions of relative roots and relative root subschemes. These notions were initially introduced and studied in [14], and further developed in [17].

Let R be a commutative ring. Let G be a reductive group scheme over R. Let P be a parabolic subgroup scheme of G over R, and let L be a Levi subgroup of P. By [4, Exp. XXII, Prop. 2.8] the root system Φ of $G_{\overline{k(s)}}$, $s\in\operatorname{Spec}R$, is constant locally in the Zariski topology on $\operatorname{Spec}R$. The type of the root system of $L_{\overline{k(s)}}$ is determined by a Dynkin subdiagram of the Dynkin diagram of Φ, which is also constant Zariski-locally on $\operatorname{Spec}R$ by [4, Exp. XXVI, Lemme 1.14 and Prop. 1.15]. In particular, if $\operatorname{Spec}R$ is connected, all these data are constant on $\operatorname{Spec}R$.

Lemma 2.6. [17, Lemma 3.6] *Let G be a reductive group over a connected commutative ring R, P be a parabolic subgroup of G, L be a Levi subgroup of P, and $\bar{L}$ be the image of L under the natural homomorphism $G\to G^{ad}\subseteq\mathrm{Aut}(G)$. Let D be the Dynkin diagram of the root system Φ of $G_{\overline{k(s)}}$ for any $s\in\operatorname{Spec}A$. We identify D with a set of simple roots of Φ such that $P_{\overline{k(s)}}$ is a standard positive parabolic subgroup with respect to D. Let $J\subseteq D$ be the set of simple roots such that $D\setminus J\subseteq D$ is the subdiagram corresponding to $L_{\overline{k(s)}}$. Then there are a unique maximal split subtorus $S\subseteq\mathrm{Cent}(\bar{L})$ and a subgroup $\Gamma\leqslant\mathrm{Aut}(D)$ such that J is invariant under Γ, and for any $s\in\operatorname{Spec}R$ and any split maximal torus $T\subseteq\bar{L}_{\overline{k(s)}}$ the kernel of the natural surjection*

$$X^*(T)\cong\mathbb{Z}\,\Phi\xrightarrow{\ \pi\ }X^*(S_{\overline{k(s)}})\cong\mathbb{Z}\,\Phi(S,G)\tag{2}$$

is generated by all roots $\alpha\in D\setminus J$, and by all differences $\alpha-\sigma(\alpha)$, $\alpha\in J$, $\sigma\in\Gamma$.

In [14], we introduced a system of relative roots Φ_P with respect to a parabolic subgroup P of a reductive group G over a commutative ring R. This system Φ_P was defined independently over each member $\operatorname{Spec} A = \operatorname{Spec} A_i$ of a suitable finite disjoint Zariski covering

$$\operatorname{Spec} R = \coprod_{i=1}^{m} \operatorname{Spec} A_i,$$

such that over each $A = A_i$, $1 \leqslant i \leqslant m$, the root system Φ and the Dynkin diagram D of G is constant. Namely, we considered the formal projection

$$\pi_{J,\Gamma}\colon\ \mathbb{Z}\,\Phi \longrightarrow \mathbb{Z}\,\Phi/\left\langle D\setminus J;\ \alpha-\sigma(\alpha),\ \alpha\in J,\ \sigma\in\Gamma\right\rangle,$$

and set $\Phi_P = \Phi_{J,\Gamma} = \pi_{J,\Gamma}(\Phi)\setminus\{0\}$. The last claim of Lemma 2.6 allows to identify $\Phi_{J,\Gamma}$ and $\Phi(S,G)$ whenever $\operatorname{Spec} R$ is connected.

Definition 2.7. *In the setting of Lemma* 2.6 *we call* $\Phi(S,G)$ *a* system of relative roots with respect to the parabolic subgroup P over R *and denote it by* Φ_P*.*

If A is a field or a local ring, and P is a minimal parabolic subgroup of G, then Φ_P is nothing but the relative root system of G with respect to a maximal split subtorus in the sense of [1] or, respectively, [4, Exp. XXVI §7].

We have also defined in [14] irreducible components of systems of relative roots, the subsets of positive and negative relative roots, simple relative roots, and the height of a root. These definitions are immediate analogs of the ones for usual abstract root systems, so we do not reproduce them here.

Let R be a commutative ring with 1. For any finitely generated projective R-module V, we denote by $W(V)$ the natural affine scheme over R associated with V, see [4, Exp. I, §4.6]. Any morphism of R-schemes $W(V_1)\to W(V_2)$ is determined by an element $f\in \operatorname{Sym}^*(V_1^\vee)\otimes_R V_2$, where Sym^* denotes the symmetric algebra, and $V_1^\vee$ denotes the dual module of V_1. If $f\in\operatorname{Sym}^d(V_1^\vee)\otimes_R V_2$, we say that the corresponding morphism is homogeneous of degree d. By abuse of notation, we also write $f: V_1\to V_2$ and call it *a degree d homogeneous polynomial map from V_1 to V_2*. In this context, one has

$$f(\lambda v)=\lambda^d f(v)$$

for any $v\in V_1$ and $\lambda\in R$.

Lemma 2.8. [17, Lemma 3.9]. *In the setting of Lemma* 2.6, *for any* $\alpha \in \Phi_P = \Phi(S,G)$ *there exists a closed* S*-equivariant embedding of* R*-schemes*

$$X_\alpha \colon W\big(\mathrm{Lie}(G)_\alpha\big) \to G,$$

satisfying the following condition.

(∗) *Let* R'/R *be any ring extension such that* $G_{R'}$ *is split with respect to a maximal split* R'*-torus* $T \subseteq L_{R'}$. *Let* e_δ, $\delta \in \Phi$, *be a Chevalley basis of* $\mathrm{Lie}(G_{R'})$, *adapted to* T *and* P, *and* $x_\delta \colon \mathbf{G}_\mathrm{a} \to G_{R'}$, $\delta \in \Phi$, *be the associated system of* 1*-parameter root subgroups* (*e.g.* $x_\delta = \exp_\delta$ *of* [4, Exp. XXII, Th. 1.1]). *Let*

$$\pi : \Phi = \Phi(T, G_{R'}) \to \Phi_P \cup \{0\}$$

be the natural projection. Then for any $u = \sum\limits_{\delta\in\pi^{-1}(\alpha)} a_\delta e_\delta \in \mathrm{Lie}(G_{R'})_\alpha$ *one has*

$$X_\alpha(u) = \Big(\prod_{\delta\in\pi^{-1}(\alpha)} x_\delta(a_\delta)\Big) \cdot \prod_{i\geqslant 2}\Big(\prod_{\theta\in\pi^{-1}(i\alpha)} x_\theta(p^i_\theta(u))\Big), \qquad (3)$$

where every $p^i_\theta : \mathrm{Lie}(G_{R'})_\alpha \to R'$ *is a homogeneous polynomial map of degree* i, *and the products over* δ *and* θ *are taken in any fixed order.*

Definition 2.9. *Closed embeddings* X_α, $\alpha \in \Phi_P$, *satisfying the statement of Lemma* 2.8, *are called* relative root subschemes of G with respect to the parabolic subgroup P.

Relative root subschemes of G with respect to P, actually, depend on the choice of a Levi subgroup L in P, but their essential properties stay the same, so we usually omit L from the notation.

We will use the following properties of relative root subschemes.

Lemma 2.10. [14, Theorem 2, Lemma 6, Lemma 9] *Let* X_α, $\alpha \in \Phi_P$, *be as in Lemma* 2.8. *Set* $V_\alpha = \mathrm{Lie}(G)_\alpha$ *for short. Then*

(i) *There exist degree* i *homogeneous polynomial maps* $q^i_\alpha : V_\alpha \oplus V_\alpha \to V_{i\alpha}$, $i > 1$, *such that for any* R*-algebra* R' *and for any* $v, w \in V_\alpha \otimes_R R'$ *one has*

$$X_\alpha(v)X_\alpha(w) = X_\alpha(v + w) \prod_{i>1} X_{i\alpha}\left(q^i_\alpha(v, w)\right). \qquad (4)$$

(ii) *For any* $g \in L(R)$, *there exist degree* i *homogeneous polynomial maps* $\varphi^i_{g,\alpha} \colon V_\alpha \to V_{i\alpha}$, $i \geqslant 1$, *such that for any* R*-algebra* R' *and for any*

$v \in V_\alpha \otimes_R R'$ *one has*

$$gX_\alpha(v)g^{-1} = \prod_{i \geqslant 1} X_{i\alpha}\left(\varphi^i_{g,\alpha}(v)\right).$$

(iii) (Generalized Chevalley commutator formula) *For any* $\alpha, \beta \in \Phi_P$ *such that* $m\alpha \neq -k\beta$ *for all* $m, k \geqslant 1$, *there exist polynomial maps*

$$N_{\alpha\beta ij} \colon V_\alpha \times V_\beta \to V_{i\alpha+j\beta},\ i, j > 0,$$

homogeneous of degree i *in the first variable and of degree* j *in the second variable, such that for any* R*-algebra* R' *and for any for any* $u \in V_\alpha \otimes_R R'$, $v \in V_\beta \otimes_R R'$ *one has*

$$[X_\alpha(u), X_\beta(v)] = \prod_{i,j>0} X_{i\alpha+j\beta}\big(N_{\alpha\beta ij}(u, v)\big) \tag{5}$$

(iv) *For any subset* $\Psi \subseteq X^*(S) \setminus \{0\}$ *that is closed under addition, the morphism*

$$X_\Psi \colon W\Big(\bigoplus_{\alpha\in\Psi} V_\alpha\Big) \to U_\Psi, \qquad (v_\alpha)_\alpha \mapsto \prod_\alpha X_\alpha(v_\alpha),$$

where the product is taken in any fixed order, is an isomorphism of schemes.

Lemma 2.11. *In the notation of Lemma* 2.6, *let* $\Phi^\pm$ *be the set of positive and negative roots such that* $D \subseteq \Phi^+$. *Set* $\Phi_P^\pm = \pi(\Phi^\pm) \setminus \{0\}$, $P^+ = P$, *and let* P^- *be the opposite parabolic subgroup to* P *such that* $P \cap P^- = L$. *Then for any* R*-algebra* R', *one has*

$$U_{P^\pm}(R') = \left\langle X_\alpha(R' \otimes_R V_\alpha),\ \alpha \in \Phi_P^\pm \right\rangle.$$

Consequently,

$$E_P(R') = \left\langle X_\alpha(R' \otimes_R V_\alpha),\ \alpha \in \Phi_P \right\rangle.$$

Proof. By the choice of D the parabolic subgroup $P_{\overline{k(s)}}$ is the standard positive parabolic subgroup of $G_{\overline{k(s)}}$ corresponding to a closed set of roots $\Psi \supseteq \Phi^+$. By the choice of $J \subseteq D$, one has

$$\Psi = \Phi^+ \cup \big(\mathbb{Z}(D \setminus J) \cap \Phi^-\big).$$

Then, clearly, $\pi(\Psi) = \Phi_P^+ \cup \{0\}$. Similarly, P^- corresponds to the set $(-\Psi)$ and $\pi(-\Psi) = \Phi_P^- \cup \{0\}$. Then the unipotent radicals $U_{P^\pm}$ correspond to the closed unipotent subsets

$$\pi\big(\Phi^\pm \setminus \mathbb{Z}(D \setminus J)\big) = \Phi_P^\pm \subseteq \Phi_P.$$

Then Lemma 2.10 (iv) finishes the proof. $\square$

§3. Main Theorem

All commutative rings are assumed to be unital. For any commutative ring R and $n \geqslant 3$, we denote by $E_n(R)$ the usual elementary subgroup of $\mathrm{GL}_n(R)$.

Lemma 3.1. *Let R be a commutative ring, let G be a reductive group scheme over R, and let $i : G \to \mathrm{GL}_{n,R}$ be a closed embedding of G as a closed R-subgroup, where $n \geqslant 3$. Assume that G contains a non-central 1-dimensional subtorus $H \cong \mathbb{G}_{\mathrm{m},R}$, and let $P = P^+$ and P^- be the corresponding two opposite parabolic subgroups constructed as in Lemma* 2.4. *Then one has $E_P(R) \leqslant E_n(R)$.*

Proof. Let $Q = Q^+$ and Q^- be the two parabolic R-subgroups of $\mathrm{GL}_{n,R}$ corresponding to $H \leqslant G \leqslant \mathrm{GL}_{n,R}$, and let $M = \mathrm{Cent}_{\mathrm{GL}_{n,R}}(H)$ be their common Levi subgroup. We show that $U_P(R) \leqslant U_Q(R)$. Clearly, this implies the claim of the lemma. By [3, Proposition 2.8.3(3)] this is true if R is a field. In general, take $g \in U_P(R)$. It is enough to show that $g \in U_Q(R_m)$ for any maximal localization R_m of R. Let

$$\rho : R_m \to R_m/mR_m = l$$

be the residue homomorphism. By the above $\rho^*(g) \in U_Q(l)$. Recall that $\Omega_Q = U_{Q^+} M U_{Q^-} \cong U_{Q^+} \times M \times U_{Q^-}$ is an open subscheme of $\mathrm{GL}_{n,R}$ [4, Exp. XXVI, Remarque 4.3.6]. Hence

$$g \in U_{Q^+}(R_m) M(R_m) U_{Q^-}(R_m). \tag{6}$$

Let $L = P \cap P^- = \mathrm{Cent}_G(H)$ be the Levi subgroup of P and P^-. Let $\bar{H} \subseteq G^{ad}$ be the image of H under the natural homomorphism $G \to G^{ad}$. Clearly, $\bar{H} \cong \mathbb{G}_{\mathrm{m},R_m}$ is a split subtorus of the center of the image $\bar{L}$ of L in G^{ad}. Let $S \leqslant \mathrm{Cent}(\bar{L}_{R_m})$ be the split torus constructed in Lemma 2.6 (applied to the connected ring R_m). Then $\bar{H}_{R_m} \leqslant S$. The embeddings X_α, $\alpha \in \Phi(S, G_{R_m})$, are S-equivariant, hence they are $\bar{H}_{R_m}$-equivariant. Since $H \leqslant L$ preserves the subschemes $U_{P^\pm}$, and $\mathrm{Cent}(G) \leqslant L$, this implies that the embeddings X_α are H_{R_m}-equivariant.

By definition of $P = P^+$ and P^-, there is an isomorphism $X^*(H) \cong \mathbb{Z}$ such that

$$\mathrm{Lie}(U_P) = \bigoplus_{n>0} \mathrm{Lie}(G)_n \quad \text{and} \quad \mathrm{Lie}(U_{P^-}) = \bigoplus_{n<0} \mathrm{Lie}(G)_n.$$

Since the embeddings X_α, $\alpha \in \Phi_P$, are H_{R_m}-equivariant, for any R_m-algebra R', any $s \in H(R')$, and any $u \in R' \otimes_{R_m} V_\alpha = R' \otimes_{R_m} \mathrm{Lie}(G)_\alpha$ we have

$$sX_\alpha(u)s^{-1} = X_\alpha(s(u)).$$

By Lemma 2.11 for any $\alpha \in \Phi_P^+$ we have $u \in \mathrm{Lie}\,(U_P)(R')$, hence $s(u) = s^n u$ for some $n = n(u) > 0$. Similarly, $\alpha \in \Phi_P^-$ we have $u \in \mathrm{Lie}\,(U_{P^-})(R')$, hence $s(u) = s^{-n}u$ for some $n = n(u) > 0$.

Applying this result to the ring of Laurent polynomials $R' = R_m[Z^\pm]$ and $s = Z \in H(R')$, we conclude that

$$\begin{aligned} &sU_P(R_m)s^{-1} \subseteq U_P(R_m[Z], ZR_m[Z]);\\ &sU_{P^-}(R_m)s^{-1} \subseteq U_{P^-}(R_m[Z^{-1}], Z^{-1}R_m[Z^{-1}]);\\ &s|_{L(R_m)} = \mathrm{id}. \end{aligned} \tag{7}$$

In particular, one has

$$\begin{aligned} sgs^{-1} \in U_P(R_m[Z], ZR_m[Z]) &\leqslant G(R_m[Z], ZR_m[Z])\\ &\leqslant \mathrm{GL}_n(R_m[Z], ZR_m[Z]). \end{aligned} \tag{8}$$

On the other hand, the analogs of (7) hold for GL_n, $U_{Q^\pm}$, and M in place of G, $U_{P^\pm}$, and L. Therefore by (6) we have

$$sgs^{-1} \in U_{Q^+}(R_m[Z], ZR_m[Z]) \cdot M(R_m) \cdot U_{Q^-}(R_m[Z^{-1}], Z^{-1}R_m[Z^{-1}]).$$

Since one has

$$\begin{aligned} &U_{Q^+}(R_m[Z], ZR_m[Z]) \cdot M(R_m) \cdot U_{Q^-}(R_m[Z^{-1}], Z^{-1}R_m[Z^{-1}])\\ &\qquad \cap \mathrm{GL}_n(R_m[Z], ZR_m[Z]) = U_{Q^+}(R_m[Z], ZR_m[Z]), \end{aligned}$$

we conclude that $sgs^{-1} \in U_{Q^+}(R_m[Z]), ZR_m[Z])$ and thus $g \in U_{Q^+}(R_m)$, as required. ☐

Lemma 3.2. *Let G be an isotropic reductive group scheme over a connected Noetherian commutative ring B, provided with a closed B-embedding $G \to \mathrm{GL}_{n,B}$, $n \geqslant 3$, which is a B-group scheme homomorphism. Assume that G contains a non-central 1-dimensional split subtorus $\mathbb{G}_{\mathrm{m},B}$, and let $P = P^+$ and P^- be the corresponding pair of opposite parabolic subgroups that exist by Lemma 2.4. Assume moreover that B is a subring of a commutative ring A, and let $h \in B$ be a non-nilpotent element. Denote by $F_h : G(A) \to G(A_h)$ the localization homomorphism.*

If $Ah + B = A$, i.e. the natural map $B \to A/Ah$ is surjective, then for any $x \in E_P(A_h)$ there exist $y \in G(A)$ and $z \in E_P(B_h)$ such that $x = F_h(y)z$.

Proof. Since the ring B is connected, by Lemmas 2.6 and 2.8 the group G over B with a parabolic subgroup P is provided with a split B-torus $S \leqslant G^{ad}$, the corresponding system of relative roots $\Phi(S,G) = \Phi_P$ and relative root subschemes $X_\alpha(V_\alpha)$, where $\alpha \in \Phi_P$ and each V_α is a finitely generated projective B-module. By Lemma 2.11 one has

$$E_P(R) = \langle X_\alpha(R \otimes_B V_\alpha),\ \alpha \in \Phi_P \rangle$$

for any B-algebra R.

One has $x = \prod_{i=1}^{m} X_{\beta_i}(c_i)$, $c_i \in A_h \otimes_B V_{\beta_i}$, $\beta_i \in \Phi_P$. We need to show that $x \in F_h(G(A))E_P(B_h)$. Clearly, it is enough to show that

$$E_P(B_h)X_\beta(c) \subseteq F_h(G(A))E_P(B_h) \tag{9}$$

for any $\beta \in \Phi_P$ and $c \in A_h \otimes_B V_\beta$. We can assume that β is a positive relative root without loss of generality. We prove the inclusion (9) by descending induction on the height of β. Let $e_1, \ldots, e_k$ be a set of generators of the B-module V_β.

Take any $z \in E_P(B_h)$. By Lemma 3.1 we have $E_P(R) \leqslant E_n(R)$ for any B-algebra R. Take $R = A[Z]$, the ring of polynomials over A. For any $N \geqslant 1$ and $1 \leqslant i \leqslant k$ one has

$$zX_\beta(h^N Ze_i)z^{-1} \in zE_n(A_h[Z], ZA_h[Z])z^{-1} \cap G(A_h[Z]).$$

Since $z \in E_P(B_h) \leqslant E_n(A_h)$, by [18, Lemma 3.3] there exists $N_i \geqslant 1$ and $g_i(Z) \in E_n(A[Z], ZA[Z])$ such that $F_h(g_i(Z)) = zX_\beta(h^{N_i}Ze_i)z^{-1}$. By [9, Lemma 3.5.4] there is $K_i \geqslant 1$ such that $g_i(h^{K_i}Z) \in G(A[Z])$. Summing up, we conclude that there is $N \geqslant 1$ such that

$$zX_\beta(h^N Ze_i)z^{-1} \in F_h\big(G(A[Z])\big) \tag{10}$$

for any $1 \leqslant i \leqslant k$.

On the other hand, note that $Ah + B = A$ implies $Ah^n + B = A$ for any $n \geqslant 1$. Let $M \geqslant 0$ be such that $h^M c \in A \otimes_B V_\beta$. Then one can find $a \in A \otimes_B V_\beta$ and $b \in V_\beta$ such that

$$c = ah^N + h^{-M}b.$$

Write $a = \sum_{i=1}^{k} a_i e_i$, where $a_i \in A$. By the multiplication formula for relative root elements (4) we have

$$X_\beta(c) = X_\beta(ah^N)X_\beta(h^{-M}b)\prod_{j\geqslant 2} X_{j\beta}\left(u_j\right),$$

where $u_j = q^j_\beta(h^N a, h^{-M} b) \in A_h \otimes_B V_{j\beta}$, and, similarly,

$$X_\beta(ah^N) = \prod_{i=1}^{k} X_\beta(a_i h^N e_i) \prod_{j\geqslant 2} X_{j\beta}(v_j),$$

where $v_j \in A \otimes_B V_{j\beta}$. By the choice of N in (10), one has

$$z\left(\prod_{i=1}^{k} X_\beta(a_i h^N e_i)\right) z^{-1} \in F_h(G(A)). \tag{11}$$

It remains to note that, since the height of the relative roots $j\beta$, $j \geqslant 2$, is larger than that of β, the inductive hypothesis version of the inclusion (9) can be applied to all elements $X_{j\beta}(v_j)$ and $X_{j\beta}(u_j)$, $j \geqslant 2$. Since, moreover, z and $X_\beta(h^{-M}b)$ belong to $E_P(B_h)$, we see that

$$z\left(\prod_{j\geqslant 2} X_{j\beta}(v_j)\right) X_\beta(h^{-M}b) \left(\prod_{j\geqslant 2} X_{j\beta}(u_j)\right) z^{-1} \in F_h(G(A))E_P(B_h).$$

Combining this result with (11), we conclude that

$$zX_\beta(c)z^{-1} \in F_h(G(A))\, E_P(B_h),$$

which proves (9). □

Lemma 3.3. *Let R be a Henselian discrete valuation ring. Let K be the field of fractions of R. Let G be a semisimple simply connected R-group scheme such that every semisimple normal R-subgroup scheme of G contains a split R-torus $\mathbb{G}_{m,R}$. Then G contains a strictly proper parabolic R-subgroup P, and*

$$G(K) = G(R)E_P(K).$$

Proof. Since G is a semisimple simply connected R-group scheme, by [4, Exp. XXIV 5.3, Prop. 5.10] there exist finite étale ring extensions R'_i/R, $1 \leqslant i \leqslant n$, and absolutely almost simple simply connected R'_i-group schemes G'_i such that

$$G \cong G_1 \times_{\operatorname{Spec} R} G_2 \times_{\operatorname{Spec} R} \dots \times_{\operatorname{Spec} R} G_n,$$

where $G_i = \mathrm{R}_{R'_i/R}(G'_i)$ are minimal semisimple normal subgroups of G. Clearly,

$$G(K) \cong \prod_{i=1}^{n} G'_i(K \otimes_R R_i) \quad \text{and} \quad G(R) \cong \prod_{i=1}^{n} G'_i(R_i). \tag{12}$$

Since each G_i contains $\mathbb{G}_{\mathrm{m},R}$, one readily sees that each G'_i is isotropic, i.e. contains $\mathbb{G}_{\mathrm{m},R'_i}$ (see e.g. the proof of [13, Theorem 11.1]). By Lemma 2.4 G'_i has a proper parabolic R'_i-subgroup P'_i. Then $P_i = \mathrm{R}_{R'_i/R}(P'_i)$ is a proper parabolic R-subgroup of G_i, and

$$P = P_1 \times_{\operatorname{Spec} R} P_2 \times_{\operatorname{Spec} R} \ldots \times_{\operatorname{Spec} R} P_n$$

is a strictly proper parabolic R-subgroup of G. We have

$$E_P(K) = \prod_{i=1}^{n} E_{P_i}(K) \cong \prod_{i=1}^{n} E_{P'_i}(K \otimes_R R'_i). \tag{13}$$

Fix an i, $1 \leqslant i \leqslant n$, and abbreviate $A = R'_i$, $H = G'_i$, $P' = P'_j$. Since the map $R \to A$ is finite étale, the ring A is a product of a finite number of Henselian discrete valuation rings A_j, $1 \leqslant j \leqslant m$, and $K \otimes_R A$ is the product of their respective fraction fields L_j, $1 \leqslant j \leqslant m$. By [5, Lemme 4.5] one has

$$H(K \otimes_R A) = \prod_{j=1}^{m} H(L_j) = \prod_{j=1}^{m} H(A_j) E_{P'}(L_j) = H(A) E_{P'}(K \otimes_R A).$$

Combining this result with (12) and (13), we deduce that

$$G(K) \cong \prod_{i=1}^{n} G'_i(R'_i) E_{P'_i}(K \otimes_R R'_i) \cong \prod_{i=1}^{n} G_i(R) E_{P_i}(K) = G(R) E_P(K),$$

as required. □

Theorem 3.4. *Let R be a semi-local Dedekind domain. Let K be the field of fractions of R. Let G be a reductive semisimple simply connected R-group scheme such that every semisimple normal R-subgroup scheme of G contains a split R-torus $\mathbb{G}_{m,R}$. Then the map*

$$H^1_{\acute{e}t}(R, G) \to H^1_{\acute{e}t}(K, G)$$

of pointed sets induced by the inclusion of R into K has trivial kernel.

Proof. We prove the theorem by induction on the number of maximal ideals in R. If R is local then the theorem holds by [10]. Let $n > 1$ be an integer and suppose the theorem holds for all Dedekind domains containing strictly less than n maximal ideals. Prove that the theorem holds for a Dedekind domain R with exactly n maximal ideals. Let $\mathfrak{m} \subset R$ be a maximal ideal and let $f \in \mathfrak{m}$ be its generator. Let R' be the Henselization of R at the maximal ideal $\mathfrak{m}$ and let R_f be the localization of R at f. Let L' be the fraction field of R'.

Let $\mathcal{E}$ be a principal G-bundle over R which is trivial over the field K. By the inductive hypothesis $\mathcal{E}$ is trivial over R_f and over R'. Thus we may assume that $\mathcal{E}$ is obtained by patching over $\operatorname{Spec} L'$ of two trivial principal G-bundles $G_f := G \times_{\operatorname{Spec} R} \operatorname{Spec} R_f$ and $G' := G \times_{\operatorname{Spec} R} \operatorname{Spec} R'$ using an element $x \in G(L')$.

By Lemma 3.3 G contains a strictly proper parabolic R-subgroup P, and one has

$$G(L') = G(R').E_P(L')$$

So, $x = x''.x'$ for some $x'' \in G(R')$ and $x' \in E_P(L')$. Replacing the patching element $x = x''.x'$ with $x' \in E_P(L')$ we do not change the isomorphism class of the principal G-bundle $\mathcal{E}$ over R. Moreover, by Lemma 3.2 one can present x' in the form $x' = y.z$ with $y \in G(R')$ and $z \in E_P(R_f)$. The latter yields the triviality of the principal G-bundle $\mathcal{E}$ over R, since $\operatorname{Spec} R_f$ and $\operatorname{Spec} R'$ form a covering of $\operatorname{Spec} R$ for the Nisnevich topology. □

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