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Z. Kabluchko, A. E. Litvak, D. Zaporozhets

MEAN WIDTH OF REGULAR POLYTOPES AND EXPECTED MAXIMA OF CORRELATED GAUSSIAN VARIABLES

Abstract. An old conjecture states that among all simplices inscribed in the unit sphere, the regular one has the maximal mean width. We restate this conjecture probabilistically and prove its asymptotic version. We also show that the mean width of the regular simplex with $2n$ vertices is remarkably close to the mean width of the regular crosspolytope with the same number of vertices. We establish several formulas conjectured by S. Finch on projection length W of the regular cube, simplex and crosspolytope onto a line with random direction. Finally, we prove distributional limit theorems for W as the dimension of the regular polytope goes to ∞.

§1. Conjecture on the mean width

1.1. Introduction. The mean width of a compact convex body $K \subset \mathbb{R}^n$ is the expected length of a projection of this body onto a line with uniformly chosen, random direction. That is, the mean width equals $\mathbf{E}\,[W_K]$, where

$$W_K = \sup_{t\in K}\langle U, t\rangle - \inf_{t\in K}\langle U, t\rangle,$$

and U is uniformly distributed on the unit sphere $\mathbb{S}^{n-1} \subset \mathbb{R}^n$.

How should $n+1$ points be arranged on the $(n-1)$-dimensional unit sphere so as to maximize the mean width of their convex hull? An old conjecture states (see [15, Section 9.10.2]) that the arrangement must be *regular*.

Key words and phrases: Gumbel distribution, mean width, intrinsic volumes, regular simplex, regular crosspolytope, maxima of Gaussian processes, random projections, extreme value theory.

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The mean width is just a multiple of the first intrinsic volume V_1, namely

$$V_1(K) = \sqrt{\pi}\, \frac{\Gamma(\frac{n+1}{2})}{\Gamma(\frac{n}{2})}\, \mathbf{E}\,[W_K]; \tag{1}$$

see [20, p. 210]. The first intrinsic volume has the advantage of not depending on the dimension of the surrounding space. Hence the conjecture can be formulated as follows:

$$\sup_{x_1,\ldots,x_{n+1}\in\mathbb{S}^{n-1}} V_1(\mathrm{conv}(x_1,\ldots,x_{n+1})) = V_1(T_n), \tag{2}$$

where T_n is a regular simplex with $n+1$ vertices inscribed in the sphere $\mathbb{S}^{n-1}$, and conv denotes the convex hull.

This question is surprisingly hard. Several authors [3,4,14,22] assumed the existence of a proof, but the problem is still open. Besides very natural formulation in Convex Geometry this problem is very important in Information Theory, as it is closely related to the the long-standing simplex code conjecture [9].

1.2. Probabilistic statement. The conjecture can be reformulated in terms of Gaussian processes in the following way. Throughout the paper, $\eta = (\eta_1,\ldots,\eta_n)$ denotes a standard Gaussian vector in $\mathbb{R}^n$. Consider a compact set $K \subset \mathbb{R}^n$. Using the fact that the norm and the direction of η are independent, it is not difficult to derive Sudakov's formula

$$V_1(\mathrm{conv}(K)) = \sqrt{2\pi}\, \mathbf{E} \sup_{x\in K} \langle \eta, x\rangle \tag{3}$$

(see [21] for details and for a generalization to the infinite-dimensional case, or Theorem 3.1 in the present paper for a more general result). This probabilistic interpretation of the first intrinsic volume allows to reformulate the conjecture as follows.

Proposition 1.1. *For every integer $n \geqslant 2$ the following two statements are equivalent:*

(i) *One has*

$$\sup_{x_1,\ldots,x_n\in\mathbb{S}^{n-2}} V_1(\mathrm{conv}(x_1,\ldots,x_n)) = V_1(T_{n-1}), \tag{4}$$

and the equality is attained iff $x_1,\ldots,x_n$ are vertices of a regular simplex.

(ii) *For every centered Gaussian vector* $(\xi_1, \dots, \xi_n)$ *satisfying*

$$\mathbf{E}\,\xi_1^2 = \dots = \mathbf{E}\,\xi_n^2 = 1, \tag{5}$$

one has

$$\mathbf{E}\,\max\{\xi_1, \dots, \xi_n\} \leqslant \sqrt{\frac{n}{n-1}}\,\mathbf{E}\,\max\{\eta_1, \dots, \eta_n\}, \tag{6}$$

and the equality is attained iff $\mathbf{E}\,[\xi_i \xi_j] = -1/(n-1)$ *for all* $i \neq j$.

Proof. First of all note that

$$\sup_{x_1,\dots,x_n \in \mathbb{S}^{n-2}} V_1(\mathrm{conv}(x_1, \dots, x_n)) = \sup_{y_1,\dots,y_n \in \mathbb{S}^{n-1}} V_1(\mathrm{conv}(y_1, \dots, y_n)) \tag{7}$$

because there is an $(n-1)$-dimensional affine subspace (and hence, an $(n-2)$-dimensional sphere of radius at most 1) containing $y_1, \dots, y_n$. Therefore, we can restate (i) as follows:

$$\sup_{y_1,\dots,y_n \in \mathbb{S}^{n-1}} V_1(\mathrm{conv}(y_1, \dots, y_n)) = V_1(T_{n-1}), \tag{8}$$

and the equality is attained iff $y_1, \dots, y_n$ are vertices of a regular simplex centered at the origin. Let $\{e_1, \dots, e_n\}$ be a standard orthonormal basis in $\mathbb{R}^n$. As a realization of such a simplex we can take the convex hull of the points

$$v_i := \sqrt{\frac{n}{n-1}}\Big(e_i - \frac{e_1 + \dots + e_n}{n}\Big), \quad i = 1, \dots, n.$$

To see this, note that the $(n-1)$-dimensional regular simplex

$$S_{n-1} := \mathrm{conv}(e_1, \dots, e_n)$$

can be inscribed in an $(n-2)$-dimensional sphere of radius $\sqrt{(n-1)/n}$ centered at $(e_1 + \dots + e_n)/n$. It follows from (3) applied to $K = S_{n-1}$ that

$$V_1(T_{n-1}) = \sqrt{\frac{n}{n-1}}\,V_1(S_{n-1}) = \sqrt{2\pi}\sqrt{\frac{n}{n-1}}\,\mathbf{E}\,\max\{\eta_1, \dots, \eta_n\}. \tag{9}$$

To any points $y_1, \dots, y_n \in \mathbb{S}^{n-1}$ we associate a centered Gaussian vector $(\xi_1, \dots, \xi_n)$ such that $\mathbf{E}\,\xi_1^2 = \dots = \mathbf{E}\,\xi_n^2 = 1$ via

$$\xi_1 := \langle \eta, y_1 \rangle, \quad \dots, \quad \xi_n := \langle \eta, y_n \rangle.$$

If we agree to identify two Gaussian vectors if they have the same distribution and two tuples $(y_1, \dots, y_n)$ and $(y'_1, \dots, y'_n)$ if $\langle y_i, y_j \rangle = \langle y'_i, y'_j \rangle$ for all $i,j \in \{1,\dots,n\}$, then this correspondence becomes one-to-one because $\mathrm{Cov}(\xi_i,\xi_j) = \langle y_i, y_j\rangle$. It follows from (3) that

$$\sqrt{2\pi}\,\mathbf{E}\,\max\{\xi_1,\dots,\xi_n\} = V_1(\mathrm{conv}(y_1,\dots,y_n)).$$

The Gaussian vector corresponding to the points $v_1,\dots,v_n$ satisfies

$$\mathbf{E}\,[\xi_i\xi_j] = \langle v_i, v_j\rangle = -1/(n-1), \quad i \neq j.$$

Taken together, the above considerations show the equivalence of (i) and (ii). □

1.3. Asymptotic version of the conjecture. We now show that (2) holds *asymptotically.*

Theorem 1.2. *For some absolute constant $C > 0$ and all $n \in \mathbb{N}$,*

$$V_1(T_n) \leqslant \sup_{x_1,\dots,x_{n+1}\in\mathbb{S}^{n-1}} V_1\big(\mathrm{conv}(x_1,\dots,x_{n+1})\big) \leqslant V_1(T_n)\Big(1+C\frac{\log\log n}{\log n}\Big).$$

Proof. The first inequality is trivial because we can take $x_1,\dots,x_{n+1}$ to be the vertices of T_n. Replacing n by $n-1$ and using (7) we can restate that second inequality as follows: For all $n \geqslant 2$,

$$\sup_{x_1,\dots,x_n\in\mathbb{S}^{n-1}} V_1\big(\mathrm{conv}(x_1,\dots,x_n)\big) \leqslant V_1(T_{n-1})\Big(1 + C\frac{\log\log n}{\log n}\Big)$$

Fix $x_1,\dots,x_n \in \mathbb{S}^{n-1}$. For $k = 1,\dots,n$ define Gaussian random variables $\xi_k := \langle x_k, \eta\rangle$ and note that ξ_k has zero mean and unit variance. It is known (see, e.g., [8, p. 138]) that

$$\mathbf{E}\,\max\{\xi_1,\dots,\xi_n\} \leqslant \sqrt{2\,\log n}. \tag{10}$$

We provide a proof for the sake of completeness. For $t > 0$ one has

$$\exp\big(t\,\mathbf{E}\,\max\{\xi_1,\dots,\xi_n\}\big) \leqslant \mathbf{E}\,\exp\big(t\max\{\xi_1,\dots,\xi_n\}\big)$$
$$= \mathbf{E}\,\max\{\mathrm{e}^{t\xi_1},\dots,\mathrm{e}^{t\xi_n}\} \leqslant \sum_{k=1}^{n}\mathbf{E}\,\mathrm{e}^{t\xi_k} = n\,\mathrm{e}^{t^2/2}.$$

Letting $t = \sqrt{2\,\log n}$ yields (10).

On the other hand, it is well-known in the theory of extreme values, see [17, Theorem 1.5.3, p. 14] and [19], that

$$\mathbf{E}\,\max\{\eta_1,\dots,\eta_n\} = \sqrt{2\,\log n} - O\Big(\frac{\log\log n}{\sqrt{2\,\log n}}\Big), \quad n \to \infty. \tag{11}$$

Using (3) and (10), we obtain

$$V_1\big(\operatorname{conv}(x_1,\dots,x_n)\big) = \sqrt{2\pi}\,\mathbf{E}\,\max\{\xi_1,\dots,\xi_n\} \leqslant \sqrt{4\pi\log n}.$$

Combining this with (9) and (11) gives

$$\begin{aligned} V_1\big(\operatorname{conv}(x_1,\dots,x_n)\big) &\leqslant V_1(T_{n-1})\cdot\sqrt{\frac{n-1}{n}}\,\Big(1-O\Big(\frac{\log\log n}{\log n}\Big)\Big)^{-1} \\ &= V_1(T_{n-1})\cdot\Big(1+O\Big(\frac{\log\log n}{\log n}\Big)\Big), \end{aligned}$$

as $n\to\infty$. This proves the claim. □

§2. Regular simplex and regular crosspolytope

In this section we compare the mean width of the regular simplex T_{2n-1} to the mean width of the regular n-dimensional crosspolytope defined by

$$C_n = \operatorname{conv}(\pm e_1,\dots,\pm e_n).$$

Note that both T_{2n-1} and C_n (which can be considered as a degenerate simplex) have $2n$ vertices and can be inscribed in $\mathbb{S}^{2n-2}$. We will show that conjecture (2) is true in this special case, that is $V_1(C_n)\leqslant V_1(T_{2n-1})$. Moreover, we will prove a lower bound which shows that the mean width of T_{2n-1} is remarkably close to the mean width of C_n.

2.1. Mean width and extreme values. It follows from Sudakov's formula (3), see also (9), that

$$V_1(T_{n-1}) = \sqrt{2\pi}\sqrt{\frac{n}{n-1}}\,\mathbf{E}\,\max\{\eta_1,\dots,\eta_n\} = \sqrt{\frac{n}{n-1}}\,V_1(S_{n-1}), \tag{12}$$

$$V_1(C_n) = \sqrt{2\pi}\,\mathbf{E}\,\max\{\pm\eta_1,\dots,\pm\eta_n\} = \sqrt{2\pi}\,\mathbf{E}\,\max\{|\eta_1|,\dots,|\eta_n|\}, \tag{13}$$

where we recall that $S_{n-1}=\operatorname{conv}(e_1,\dots,e_n)$. It is well-known in the theory of extreme values [17, Theorem 1.5.3 on p. 14] that

$$\lim_{n\to\infty}\mathbf{P}\Big[\max\{\eta_1,\dots,\eta_n\}\leqslant u_n+\frac{x}{\sqrt{2\,\log n}}\Big] = \mathrm{e}^{-\mathrm{e}^{-x}}, \tag{14}$$

$$\lim_{n\to\infty}\mathbf{P}\Big[\max\{|\eta_1|,\dots,|\eta_n|\}\leqslant u_{2n}+\frac{x}{\sqrt{2\,\log n}}\Big] = \mathrm{e}^{-\mathrm{e}^{-x}}, \tag{15}$$

where u_n is any sequence satisfying $\sqrt{2\pi} u_n e^{u_n^2/2} \sim n$, for example[1]

$$u_n = \sqrt{2\log n} - \frac{\frac{1}{2}\log\log n + \log(2\sqrt{\pi})}{\sqrt{2\log n}}. \tag{16}$$

Note that (15) (together with (14)) expresses the fact that the minimum and the maximum of $\eta_1, \dots, \eta_n$ become asymptotically independent; see [17, Theorem 1.8.3, p. 28]. Taking the expectation (which is justified by [19]) and noting that the expectation of the Gumbel distribution on the right-hand side of (14) and (15) is the Euler constant γ, we obtain the large n asymptotics

$$V_1(T_{n-1}) = \sqrt{2\pi}\Big(u_n + \frac{\gamma + o(1)}{\sqrt{2\log n}}\Big), \quad n \to \infty, \tag{17}$$

$$V_1(C_n) = \sqrt{2\pi}\Big(u_{2n} + \frac{\gamma + o(1)}{\sqrt{2\log n}}\Big), \quad n \to \infty. \tag{18}$$

These formulas are known; see [2], [12, p. 5], [11, p. 8].

2.2. Comparing $V_1(T_{2n-1})$ and $V_1(C_n)$. We are going to show that distance between $V_1(T_{2n-1})$ and $V_1(C_n)$ is in fact much smaller than the bound $o(1/\sqrt{2\log n})$ implied by (17) and (18). First we state the corresponding probabilistic result.

Theorem 2.1. *If $\eta_1, \dots, \eta_{2n}$ are independent standard Gaussian variables, then*

$$\mathbf{E}\max\{\eta_1, \dots, \eta_{2n}\} \leqslant \mathbf{E}\max\{|\eta_1|, \dots, |\eta_n|\} \leqslant \sqrt{\frac{2n}{2n-1}}\,\mathbf{E}\max\{\eta_1, \dots, \eta_{2n}\}.$$

The left-hand side inequality immediately follows from Slepian's lemma [17, Corollary 4.2.3, p. 84] because the random vector $(\eta_1, \dots, \eta_{2n})$ is uncorrelated, whereas the off-diagonal correlations of $(\eta_1, -\eta_1, \dots, \eta_n, -\eta_n)$ are non-positive. The proof of the second estimate will be given in Section 4. Theorem 2.1 together with (12) and (13) implies the following

Corollary 2.2. *For every $n \in \mathbb{N}$,*

$$\sqrt{\frac{2n-1}{2n}}\,V_1(T_{2n-1}) \leqslant V_1(C_n) \leqslant V_1(T_{2n-1}).$$

We now provide a bound which is asymptotically sharper. Its proof will be given in Section 5.

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[1] $a_n \sim b_n$ means $a_n/b_n \to 1$ as $n \to \infty$.

Theorem 2.3. *Let $\eta_1, \eta_2, \ldots,$ be independent standard Gaussian variables. Then, as $n \to \infty$, one has*

$$\mathbf{E}\,\max\{|\eta_1|, \ldots, |\eta_n|\} = \Big(1 + \frac{1+o(1)}{8\,n\,\log n}\Big)\,\mathbf{E}\,\max\{\eta_1, \ldots, \eta_{2n}\}.$$

Combining Theorem 2.3 with (12) and (13) yields the following

Corollary 2.4. *As $n \to \infty$,*

$$V_1(C_n) = V_1(T_{2n-1})\Big(1 - \frac{1+o(1)}{4\,n}\Big), \quad V_1(C_n) = V_1(S_{2n-1})\Big(1 + \frac{1+o(1)}{8\,n\,\log n}\Big).$$

It is possible to obtain further asymptotic terms in (17) and (18), (see, e.g., [17, Eq. (2.4.8), p. 39]) but it seems that none of these expansions can correctly capture the very small difference between the expectations in Theorems 2.1 and 2.3.

§3. Higher moments and limiting distribution of the random width

3.1. Sudakov's formula for higher moments. Given a convex compact set $K \subset \mathbb{R}^n$ we denote by W_K the length of the projection of K onto a uniformly chosen direction, that is

$$W_K = \sup_{t\in K}\langle U, t\rangle - \inf_{t\in K}\langle U, t\rangle, \tag{19}$$

where U has uniform distribution on the sphere $\mathbb{S}^{n-1}$. In this section we study the higher moments of the random variable W_K.

Recall that $\eta = (\eta_1, \ldots, \eta_n)$ denotes a random vector having standard normal distribution on $\mathbb{R}^n$. The *isonormal Gaussian process* is defined by

$$\Xi(t) = \langle \eta, t\rangle, \quad t \in \mathbb{R}^n.$$

It is characterized by

$$\mathbf{E}\left[\,\Xi(t)\right] = 0, \quad \mathbf{E}\left[\,\Xi(t)\,\Xi(s)\right] = \langle t, s\rangle, \quad t, s \in \mathbb{R}^n. \tag{20}$$

For a compact set $K \subset \mathbb{R}^n$ define the range of Ξ over K to be

$$\operatorname*{Range}_{t\in K}\ \Xi(t) = \sup_{t\in K}\ \Xi(t) - \inf_{t\in K}\ \Xi(t).$$

The next theorem generalizes Sudakov's formula (3) to higher moments.

Theorem 3.1. *If the set $K \subset \mathbb{R}^n$ is convex and compact, then*

$$\mathbf{E}[W_K^k] = 2^{-\frac{k}{2}} \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n+k}{2}\right)} \mathbf{E}\Big[\Big(\operatorname*{Range}_{t\in K}\ \Xi(t)\Big)^k\Big]. \tag{21}$$

If, moreover, the set K is symmetric with respect to the origin, then

$$\mathbf{E}[W_K^k] = 2^{\frac{k}{2}} \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n+k}{2}\right)} \mathbf{E}\Big[\Big(\sup_{t\in K} \Xi(t)\Big)^k\Big]. \tag{22}$$

Proof. The standard Gaussian vector η in $\mathbb{R}^n$ can be written as

$$\eta \stackrel{d}{=} RU,$$

where U and R^2 are such that

(1) U is a random vector with uniform distribution on the unit sphere in $\mathbb{R}^n$;
(2) R^2 is a random variable having χ^2-distribution with n degrees of freedom;
(3) U and R^2 are independent.

It follows that we have the distributional equality

$$\operatorname*{Range}_{t\in K}\ \Xi(t) = \sup_{t\in K}\langle \eta, t\rangle - \inf_{t\in K}\langle \eta, t\rangle \stackrel{d}{=} \sup_{t\in K}\langle RU, t\rangle - \inf_{t\in K}\langle RU, t\rangle = RW_K. \tag{23}$$

Taking k-th moments of both parts and noting that R and W_K are independent, we obtain that

$$\mathbf{E}\Big[\Big(\operatorname*{Range}_{t\in K}\ \Xi(t)\Big)^k\Big] = \mathbf{E}[R^k]\,\mathbf{E}[W_K^k].$$

The moments $\mathbf{E}[R^k]$ of the χ^2-distribution are known. Inserting the value of the moment, we obtain (21) (which holds without the symmetry assumption on K). If the set K is symmetric with respect to the origin, then $\operatorname{Range}_{t\in K}\Xi(t) = 2\sup_{t\in K}\Xi(t)$ and we obtain (22). □

Remark 3.2. In particular, taking $k = 1$ in Theorem 3.1 and noting that the first intrinsic volume is related to the mean width $\mathbf{E}[W_K]$ by (1), we recover from (21) Sudakov's [21] formula

$$V_1(K) = \sqrt{\frac{\pi}{2}}\,\mathbf{E}\Big[\operatorname*{Range}_{t\in K}\ \Xi(t)\Big] = \sqrt{2\pi}\,\mathbf{E}\Big[\sup_{t\in K}\Xi(t)\Big]. \tag{24}$$

Note that the symmetry assumption on K is not needed in the derivation of (24) because in the last equality we used only that $\sup_{t\in K}\Xi(t)$ has the same distribution as $-\inf_{t\in K}\Xi(t)$.

3.2. Applications to regular polytopes. Theorem 3.1 can be used to prove several conjectures on projections of regular polytopes which are due to Finch [11–13].

Example 3.3. Let $Q_n = [-\frac{1}{2}, +\frac{1}{2}]^n$ be the n-dimensional cube of unit volume. It is easy to see that Range $_{t\in Q_n} \Xi(t) = \sum_{i=1}^{n} |\eta_i|$. Therefore, by (21),

$$\mathbf{E}\,[W_{Q_n}^k] = 2^{-\frac{k}{2}} \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n+k}{2}\right)} \mathbf{E}\left[\Big(\sum_{i=1}^{n} |\eta_i|\Big)^k\right]. \tag{25}$$

In particular, taking $k = 1$ and noting that $\mathbf{E}|\eta_1| = \sqrt{\frac{2}{\pi}}$ we obtain that the mean width is

$$\mathbf{E}\,[W_{Q_n}] = \frac{\Gamma\left(\frac{n}{2}\right)}{\sqrt{2}\,\Gamma\left(\frac{n+1}{2}\right)}\, n\,\mathbf{E}\,|\eta_1| = \frac{n\,\Gamma\left(\frac{n}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{n+1}{2}\right)},$$

or, equivalently, $V_1(Q_n) = n$, which is well known. The second moment of the projection length is given by

$$\mathbf{E}\,[W_{Q_n}^2] = \frac{1}{n}\,\mathbf{E}\left[\Big(|\eta_1| + \cdots + |\eta_n|\Big)^2\right] = \mathbf{E}\,|\eta_1^2| + (n-1)\,\mathbf{E}\,|\eta_1\eta_2| = 1 + \frac{2}{\pi}\,(n-1),$$

where we have used that $\mathbf{E}\,|\eta_1^2| = 1$ and $\mathbf{E}\,|\eta_1\eta_2| = (\mathbf{E}\,|\eta_1|)^2 = \frac{2}{\pi}$. This formula has been conjectured by Finch [12, p. 9] who established it for $n = 2, 3$ by purely geometric arguments [13]. Using (25) it is possible to compute more moments of W_{Q_n}, for example

$$\mathbf{E}\,[W_{Q_n}^3] = \frac{\Gamma\left(\frac{n}{2}\right)}{2\,\Gamma\left(\frac{n+3}{2}\right)} \pi^{-\frac{3}{2}} n\Big(2\,n^2 + (3\pi - 6)\,n + 4 - \pi\Big),$$

$$\mathbf{E}\,[W_{Q_n}^4] = \frac{1}{(n+2)\pi^2}\,\big(4\,n^3 + (12\,\pi - 24)\,n^2 + (44 - 20\,\pi + 3\,\pi^2)\,n + 8\,\pi - 24\big)\,,$$

where we have used that $\mathbf{E}\,|\eta_1| = \sqrt{\frac{2}{\pi}}$, $\mathbf{E}\,|\eta_1^2| = 1$, $\mathbf{E}\,|\eta_1^3| = 2\sqrt{\frac{2}{\pi}}$, $\mathbf{E}\,|\eta_1^4| = 3$.

Example 3.4. For the regular crosspolytope $C_n = \mathrm{conv}(\pm e_1, \ldots, \pm e_n)$ we have $\sup_{t\in C_n} \Xi(t) = \max\{|\eta_1|, \ldots, |\eta_n|\}$ and therefore Theorem 3.1 yields

$$\mathbf{E}\,[W_{C_n}^k] = 2^{\frac{k}{2}} \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n+k}{2}\right)} \mathbf{E}\left[\Big(\max_{1\leqslant i\leqslant n} |\eta_i|\Big)^k\right], \quad k \in \mathbb{N}.$$

For $k = 2$, this formula was conjectured by Finch in [11, p. 3]; see also [12].

Example 3.5. For the regular $(n-1)$-dimensional simplex

$$S_{n-1} = \operatorname{conv}(e_1, \dots, e_n) \subset \mathbb{R}^n,$$

Theorem 3.1 yields

$$\mathbf{E}\,[W^k_{S_{n-1}}] = 2^{-\frac{k}{2}} \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n+k}{2}\right)}\, \mathbf{E}\left[\Big(\max_{1\leqslant i\leqslant n} \eta_i - \min_{1\leqslant i\leqslant n} \eta_i\Big)^k\right].$$

Note that in this formula, S_{n-1} is projected onto a random direction in $\mathbb{R}^n$, even though S_{n-1} is $(n-1)$-dimensional.

It is more natural to state the corresponding formula for T_{n-1} (which is a regular simplex with n vertices inscribed in $\mathbb{S}^{n-2} \subset \mathbb{R}^{n-1}$) projected onto a random direction in $\mathbb{R}^{n-1}$. As a realization of T_{n-1} we take the points

$$v_i := \sqrt{\frac{n}{n-1}} \Big(e_i - \frac{e_1 + \dots + e_n}{n}\Big), \quad i = 1, \dots, n,$$

in the hyperplane $L := \{x_1 + \dots + x_n = 0\} \subset \mathbb{R}^n$ (which can be identified with $\mathbb{R}^{n-1}$). By (20), the isonormal process Ξ on L satisfies

$$(\Xi(v_i))_{i=1,\dots,n} \stackrel{d}{=} \sqrt{\frac{n}{n-1}} \Big(\eta_i - \frac{\eta_1 + \dots + \eta_n}{n}\Big)_{i=1,\dots,n},$$

so that for its range on T_{n-1} we have

$$\operatorname*{Range}_{t\in T_{n-1}} \Xi(t) \stackrel{d}{=} \sqrt{\frac{n}{n-1}} \Big(\max_{1\leqslant i\leqslant n} \eta_i - \min_{1\leqslant i\leqslant n} \eta_i\Big).$$

Therefore, for the projection length of T_{n-1} onto a uniformly chosen random direction in the hyperplane L we obtain

$$\mathbf{E}[W^k_{T_{n-1}}] = 2^{-\frac{k}{2}} \frac{\Gamma\left(\frac{n-1}{2}\right)}{\Gamma\left(\frac{n-1+k}{2}\right)} \left(\frac{n}{n-1}\right)^{k/2} \mathbf{E}\left[\Big(\max_{1\leqslant i\leqslant n} \eta_i - \min_{1\leqslant i\leqslant n} \eta_i\Big)^k\right].$$

For $k = 2$, this formula was conjectured by Finch [12, p. 4] who verified it for small values of n.

3.3. Limit distribution for the random width. What is the asymptotic distribution of the projection length of a high-dimensional regular polytope onto a random line? The next two theorems answer this question. The proofs are postponed to Section 6.

Theorem 3.6. *The random width of the cube* $Q_n = [-\frac{1}{2}, \frac{1}{2}]^n$ *satisfies the following central limit theorem:*

$$W_{Q_n} - \sqrt{\frac{2n}{\pi}} \xrightarrow[n\to\infty]{d} \mathcal{N}\Big(0, \frac{\pi-3}{\pi}\Big).$$

Theorem 3.7. *For the random width of the simplex* $S_{n-1} = \operatorname{conv}(e_1, \dots, e_n)$ *and the crosspolytope* $C_n = \operatorname{conv}(\pm e_1, \dots, \pm e_n)$ *we have*

$$\sqrt{2n\log n}\,\Big(W_{S_{n-1}} - \frac{2u_n}{\sqrt{n}}\Big) \xrightarrow[n\to\infty]{d} G_1 + G_2, \tag{26}$$

$$\sqrt{2n\log n}\,\Big(W_{C_n} - \frac{2u_{2n}}{\sqrt{n}}\Big) \xrightarrow[n\to\infty]{d} 2G_1, \tag{27}$$

where G_1, G_2 *are independent random variables with the Gumbel double exponential distribution* $\mathbf{P}[G_1 \leqslant x] = \mathbf{P}[G_2 \leqslant x] = \mathrm{e}^{-\mathrm{e}^{-x}}$, $x \in \mathbb{R}$.

Remark 3.8. It is easy to check that the density of $G_1 + G_2$ equals

$$2\,\mathrm{e}^{-x} K_0(2\,\mathrm{e}^{-x/2}), \quad x \in \mathbb{R},$$

where

$$K_0(z) = \int_0^\infty \mathrm{e}^{-z\cosh t}\,\mathrm{d}t, \quad z > 0,$$

is the modified Bessel function of the second kind.

§4. Proof of Theorem 2.1

Proof. As already mentioned, the first estimate in Theorem 2.1 is a consequence of the Slepian lemma. Therefore, we concentrate on proving the inequality

$$\mathbf{E}\max\big\{|\eta_1|, \dots, |\eta_n|\big\} \leqslant \sqrt{\frac{2n}{2n-1}}\,\mathbf{E}\max\big\{\eta_1, \dots, \eta_{2n}\big\}.$$

The idea of the proof of goes back to the work of Chatterjee (see [7] or [1, p. 50]). For $t \in [0,1]$ consider a centered Gaussian vector

$$\xi(t) = \big(\xi_1(t), \dots, \xi_{2n}(t)\big)$$

with correlations defined by

$$\mathbf{E}\left[\xi_i^2(t)\right] = \frac{2n}{t+2n-1}, \quad i = 1, \dots, 2n,$$

$$\mathbf{E}\left[\xi_{2i-1}(t)\,\xi_{2i}(t)\right] = -\frac{2nt}{t+2n-1}, \quad i = 1, \dots, n,$$

and $\mathbf{E}\left[\xi_i(t)\,\xi_j(t)\right] = 0$ otherwise. We have

$$\xi(0) \stackrel{d}{=} \sqrt{\frac{2n}{2n-1}}\,(\eta_1,\dots,\eta_{2n}), \quad \xi(1) \stackrel{d}{=} (\eta_1,-\eta_1,\eta_2,-\eta_2,\dots,\eta_n,-\eta_n).$$

Hence it is sufficient to show that the function

$$\varphi(t) := \mathbf{E}\,\max\left\{\xi_1(t),\dots,\xi_{2n}(t)\right\}$$

is non-increasing on $[0,1]$. Consider the function

$$F_\beta(x_1,\dots,x_{2n}) := \frac{1}{\beta}\log\Big(\sum_{i=1}^{2n} \mathrm{e}^{\beta x_i}\Big).$$

It is immediate that

$$\max\{x_1,\dots,x_{2n}\} \leqslant F_\beta(x_1,\dots,x_{2n}) \leqslant \frac{\log 2n}{\beta} + \max\{x_1,\dots,x_{2n}\}.$$

Therefore we only need to show that for any $\beta > 0$ the function

$$\varphi_\beta(t) := \mathbf{E}\,F_\beta(\xi(t))$$

is non-increasing on $[0,1]$.

In what follows, $\mathbf{x}$ stands for $(x_1,\dots,x_{2n})$. Set $\sigma_{ij}(t) := \mathbf{E}\left[\xi_i(t)\,\xi_j(t)\right]$ and let us denote by $f(t,\mathbf{x})$ the probability density function of $\xi(t)$. It is a well-known property of f that

$$\frac{\partial f}{\partial \sigma_{ii}} = \frac{1}{2}\,\frac{\partial^2 f}{\partial x_i^2}, \qquad \frac{\partial f}{\partial \sigma_{ij}} = \frac{\partial^2 f}{\partial x_i \partial x_j}, \quad i \neq j.$$

Therefore,

$$\frac{\partial \varphi_\beta}{\partial t} = \int\limits_{\mathbb{R}^{2n}} F_\beta(\mathbf{x})\frac{\partial f}{\partial t}(\mathbf{x})\,\mathrm{d}\mathbf{x} = \int\limits_{\mathbb{R}^{2n}} F_\beta(\mathbf{x}) \sum_{i,j=1}^{2n} \frac{\partial f}{\partial \sigma_{ij}}(\mathbf{x})\frac{\partial \sigma_{ij}}{\partial t}\,\mathrm{d}\mathbf{x}.$$

We have

$$\begin{aligned} \frac{\partial \sigma_{ii}}{\partial t} &= -\frac{2n}{(t+2n-1)^2}, \quad i = 1,\dots,2n, \\ \frac{\partial \sigma_{2i-1,2i}}{\partial t} &= -\frac{2n(2n-1)}{(t+2n-1)^2}, \quad i = 1,\dots,n, \end{aligned}$$

and $\partial\sigma_{ij}/\partial t = 0$ otherwise. Thus we obtain

$$
\begin{aligned}
\frac{\partial \varphi_\beta}{\partial t} &= -\frac{4\,n}{(t+2\,n-1)^2} \int_{\mathbb{R}^{2n}} F_\beta(\mathbf{x}) \sum_{i=1}^{n} \Big[\frac{\partial^2 f}{\partial x_{2i-1}^2}(\mathbf{x}) \\
&\qquad + 2\,(2\,n-1) \frac{\partial^2 f}{\partial x_{2i-1} \partial x_{2i}}(\mathbf{x}) + \frac{\partial^2 f}{\partial x_{2i}^2}(\mathbf{x}) \Big] \,\mathrm{d}\mathbf{x} \\
&= -\frac{4\,n}{(t+2\,n-1)^2} \sum_{i=1}^{n} \int_{\mathbb{R}^{2n}} f(\mathbf{x}) \Big[\frac{\partial^2 F_\beta}{\partial x_{2i-1}^2}(\mathbf{x}) \\
&\qquad + 2\,(2\,n-1) \frac{\partial^2 F_\beta}{\partial x_{2i-1} \partial x_{2i}}(\mathbf{x}) + \frac{\partial^2 F_\beta}{\partial x_{2i}^2}(\mathbf{x}) \Big] \,\mathrm{d}\mathbf{x} \\
&= -\frac{4\,n}{(t+2\,n-1)^2} \sum_{i=1}^{n} \mathbf{E} \Big[\frac{\partial^2 F_\beta}{\partial x_{2i-1}^2}(\mathbf{x}) \\
&\qquad + 2\,(2\,n-1) \frac{\partial^2 F_\beta}{\partial x_{2i-1} \partial x_{2i}}(\mathbf{x}) + \frac{\partial^2 F_\beta}{\partial x_{2i}^2}(\mathbf{x}) \Big].
\end{aligned}
$$

It is easy to check that

$$
\frac{\partial^2 F_\beta}{\partial x_i^2}(\mathbf{x}) = \beta\,(p_i(\mathbf{x}) - p_i^2(\mathbf{x})), \qquad \frac{\partial^2 F_\beta}{\partial x_i \partial x_j}(\mathbf{x}) = -\beta\, p_i(\mathbf{x})\, p_j(\mathbf{x}), \quad i \neq j,
$$

where

$$
p_i(\mathbf{x}) := \frac{\partial F_\beta}{\partial x_i}(\mathbf{x}) = \frac{\mathrm{e}^{\beta x_i}}{\sum\limits_{i=1}^{2n} \mathrm{e}^{\beta x_i}}.
$$

Thus,

$$
\begin{aligned}
&-\frac{(t+2\,n-1)^2}{4\,n\,\beta} \cdot \frac{\partial \varphi_\beta}{\partial t} = \sum_{i=1}^{2n} \mathbf{E}\left[p_i(\xi(t))\right] \\
&- \sum_{i=1}^{2n} \mathbf{E}\left[p_i^2(\xi(t))\right] - 2\,(2\,n-1) \sum_{i=1}^{n} \mathbf{E}\left[p_{2i-1}(\xi(t))\, p_{2i}(\xi(t))\right] \qquad (28) \\
&= 1 - \sum_{i=1}^{2n} \mathbf{E}\left[p_i^2(\xi(t))\right] - 2\,(2\,n-1) \sum_{i=1}^{n} \mathbf{E}\left[p_{2i-1}(\xi(t))\, p_{2i}(\xi(t))\right].
\end{aligned}
$$

For $i = 1, \ldots, n$ the random variables $\xi_{2i-1}(t)$ and $\xi_{2i}(t)$ are non-positively correlated. Therefore it follows from [16] that they are *negatively* *associated* (see [6] for more details), which implies

$$\mathbf{E}\left[p_{2i-1}(\xi(t))\,p_{2i}(\xi(t))\right] \leqslant \mathbf{E}\left[p_i(\xi(t))\right]\mathbf{E}\left[p_j(\xi(t))\right].$$

Hence for all $k \neq l$ we have

$$\mathbf{E}\left[p_{2i-1}(\xi(t))\,p_{2i}(\xi(t))\right] \leqslant \mathbf{E}\left[p_k(\xi(t))\right]\mathbf{E}\left[p_l(\xi(t))\right].$$

We thus get

$$\begin{aligned}\sum_{i=1}^{n}\Big(\mathbf{E}\left[p_{2i-1}^2(\xi(t))\right] + \mathbf{E}\left[p_{2i}^2(\xi(t))\right] + 2\,(2\,n-1)\mathbf{E}\left[p_{2i-1}(\xi(t))\,p_{2i}(\xi(t))\right]\Big) \\ \leqslant \sum_{i=1}^{n}\Big(\mathbf{E}\left[p_{2i-1}^2(\xi(t))\right] + \sum_{j\neq 2i-1}\mathbf{E}\left[p_{2i-1}(\xi(t))\right]\mathbf{E}\left[p_j(\xi(t))\right] \\ + \sum_{j\neq 2i-1}\mathbf{E}\left[p_{2i}(\xi(t))\right]\mathbf{E}\left[p_j(\xi(t))\right]\Big) = \Big(\mathbf{E}\Big[\sum_{i=1}^{2n}p_i(\xi(t))\Big]\Big)^2 = 1.\end{aligned}$$

Combining this with (28) yields $\partial\varphi_\beta/\partial t \leqslant 0$, which completes the proof. $\square$

§5. Proof of Theorem 2.3

Proof. Recall that both $A_n := \mathbf{E}\,\max_{1\leqslant i\leqslant n}|\eta_i|$ and $B_n := \mathbf{E}\,\max_{1\leqslant i\leqslant 2n}\eta_i$ are asymptotically equivalent to $\sqrt{2\log n}$. Therefore, in order to prove the theorem, we need to show that

$$\lim_{n\to\infty} 4\,n\,\sqrt{2\log n}\,\Big(\mathbf{E}\max_{1\leqslant i\leqslant n}|\eta_i| - \mathbf{E}\max_{1\leqslant i\leqslant 2n}\eta_i\Big) = 1. \tag{29}$$

Denote the tail function of the standard normal distribution by

$$\overline{\Phi}(t) = \int_t^\infty \mathrm{e}^{-s^2/2}\,\frac{\mathrm{d}s}{\sqrt{2\pi}}.$$

It is well known [1, p. 9] or [8, p. 137] that for $t > 0$ one has

$$\frac{1}{\sqrt{2\pi}}\Big(\frac{1}{t} - \frac{1}{t^3}\Big)\,\mathrm{e}^{-t^2/2} \leqslant \overline{\Phi}(t) \leqslant \frac{1}{\sqrt{2\pi}\,t}\,\mathrm{e}^{-t^2/2}. \tag{30}$$

The distribution functions of the maxima we are interested in are given by

$$F_n(t) := \mathbf{P}\Big[\max_{1\leqslant i\leqslant n} |\eta_i| \leqslant t\Big] = \Big(\int\limits_{-t}^{t} \mathrm{e}^{-s^2/2} \frac{\mathrm{d}s}{\sqrt{2\pi}}\Big)^n = (1-2\,\overline{\Phi}(t))^n, \quad t\geqslant 0, \tag{31}$$

$$G_n(t) := \mathbf{P}\Big[\max_{1\leqslant i\leqslant 2n} \eta_i \leqslant t\Big] = \Big(\int\limits_{-\infty}^{t} \mathrm{e}^{-s^2/2} \frac{\mathrm{d}s}{\sqrt{2\pi}}\Big)^{2n} = (1-\overline{\Phi}(t))^{2n}, \quad t\in\mathbb{R}. \tag{32}$$

It follows that

$$A_n = \mathbf{E} \max_{1\leqslant i\leqslant n} |\eta_i| = \int\limits_0^\infty (1-F_n(t))\,\mathrm{d}t,$$

$$B_n = \mathbf{E} \max_{1\leqslant i\leqslant 2n} \eta_i = \int\limits_0^\infty (1-G_n(t))\,\mathrm{d}t - \int\limits_0^\infty (\overline{\Phi}(t))^{2n}\,\mathrm{d}t.$$

To prove the second equality, note that for $M := \max_{1\leqslant i\leqslant 2n} \eta_i$ we have $M = M_+ - M_-$ with $M_+ = \max(M,0)$, $M_- = \max(-M,0)$, and

$$\mathbf{P}\,[M_+ > t] = 1 - G_n(t), \quad t>0,$$
$$\mathbf{P}\,[M_- > t] = \mathbf{P}\,[M < -t] = (1-\overline{\Phi}(-t))^{2n} = (\overline{\Phi}(t))^{2n}, \quad t>0.$$

In fact, the second integral in the formula for B_n is negligible. Indeed, noting that $\overline{\Phi}(0) = 1/2$ and using (30) we obtain

$$\int\limits_0^\infty (\overline{\Phi}(t))^{2n}\,\mathrm{d}t \leqslant (\overline{\Phi}(0))^{2n} + \int\limits_1^\infty (\overline{\Phi}(t))^{2n}\,\mathrm{d}t \leqslant 2^{-2n} + \int\limits_1^\infty (2\pi)^{-n}\, t^{-2n} \mathrm{e}^{-t^2 n} \mathrm{d}t$$
$$\leqslant 2^{-2n} + (2\pi\mathrm{e})^{-n} \leqslant 2^{-n}.$$

In view of the above considerations, in order to prove (29) it suffices to show that

$$\lim_{n\to\infty} 4n\sqrt{2\log n} \int\limits_0^\infty (G_n(t) - F_n(t))\,\mathrm{d}t = 1.$$

After a change of variable $t := t_n + \frac{a}{t_n}$, $a\in\mathbb{R}$, where $t_n \sim \sqrt{2\log n}$ is the solution to the equation

$$\overline{\Phi}(t_n) = \frac{1}{2n}, \tag{33}$$

our task reduces to proving that

$$\lim_{n\to\infty} \int\limits_{-t_n^2}^{\infty} 4\,n\Big(G_n\Big(t_n+\frac{a}{t_n}\Big)-F_n\Big(t_n+\frac{a}{t_n}\Big)\Big)\,\mathrm{d}a = 1. \qquad (34)$$

First we prove the pointwise convergence of the function under the integral sign. If $a\in\mathbb{R}$ is fixed and $n\to\infty$, then by (30) and (33),

$$r_n := \overline{\Phi}\Big(t_n+\frac{a}{t_n}\Big) \sim \frac{1}{\sqrt{2\pi}\,t_n}\,\mathrm{e}^{-\frac{1}{2}(t_n+\frac{a}{t_n})^2} \sim \frac{1}{\sqrt{2\pi}\,t_n}\,\mathrm{e}^{-\frac{1}{2}t_n^2}\,\mathrm{e}^{-a} \sim \frac{\mathrm{e}^{-a}}{2\,n}. \qquad (35)$$

Recalling the formulas for F_n and G_n, see (31), (32), we can write

$$F_n\Big(t_n+\frac{a}{t_n}\Big) = (1-2\,r_n)^n = \mathrm{e}^{n\log(1-2\,r_n)},$$
$$G_n\Big(t_n+\frac{a}{t_n}\Big) = (1-r_n)^{2n} = \mathrm{e}^{2n\log(1-r_n)}.$$

Using (35) and the Taylor series for the logarithm and the exponent, we obtain

$$F_n\Big(t_n+\frac{a}{t_n}\Big)=\exp\Big\{-n\Big(2\,r_n+2\,r_n^2+o\Big(\frac{1}{n^2}\Big)\Big)\Big\}=\mathrm{e}^{-2nr_n}\Big(1-2\,n\,r_n^2+o\Big(\frac{1}{n}\Big)\Big),$$
$$G_n\Big(t_n+\frac{a}{t_n}\Big)=\exp\Big\{-2\,n\Big(r_n+\frac{r_n^2}{2}+o\Big(\frac{1}{n^2}\Big)\Big)\Big\}=\mathrm{e}^{-2nr_n}\Big(1-n\,r_n^2+o\Big(\frac{1}{n}\Big)\Big).$$

Subtracting both expansions and using (35) twice, we obtain

$$4\,n\Big(G_n\Big(t_n+\frac{a}{t_n}\Big)-F_n\Big(t_n+\frac{a}{t_n}\Big)\Big)=4\,n\,\mathrm{e}^{-2nr_n}\Big(n\,r_n^2+o\Big(\frac{1}{n}\Big)\Big)\underset{n\to\infty}{\longrightarrow}\mathrm{e}^{-\mathrm{e}^{-a}}\mathrm{e}^{-2a}.$$

If we allow for a moment interchanging the limit and the integral, the limit in (34) equals

$$\text{LHS of (34)} = \int\limits_{-\infty}^{+\infty} \mathrm{e}^{-\mathrm{e}^{-a}}\mathrm{e}^{-2a}\,\mathrm{d}a = \int\limits_{0}^{\infty}\mathrm{e}^{-y}y\,\mathrm{d}y = 1,$$

where we used the change of variable $y=\mathrm{e}^{-a}$.

To complete the proof we need to justify the use of the Lebesgue dominated convergence theorem. It suffices to show that for some integrable function $g(a)$,

$$0\leqslant n\Big(G_n\Big(t_n+\frac{a}{t_n}\Big)-F_n\Big(t_n+\frac{a}{t_n}\Big)\Big)\leqslant g(a), \quad a>-\frac{1}{4}\,t_n^2, \quad n\in\mathbb{N}, \qquad (36)$$

and

$$\lim_{n\to\infty} n \int_{-t_n^2}^{-t_n^2/4} \Big(G_n\Big(t_n+\frac{a}{t_n}\Big)-F_n\Big(t_n+\frac{a}{t_n}\Big)\Big)\,\mathrm{d}a = 0. \tag{37}$$

The non-negativity of G_n-F_n is a consequence of the inequality $(1-z)^2 \geqslant 1-2z$; see (31), (32). Now we prove the upper bound in (36). Using the inequality

$$y^n - x^n \leqslant n\,(y-x)\,y^{n-1}$$

for $0\leqslant x\leqslant y$, we obtain that

$$\begin{aligned} G_n(t)-F_n(t) &= (1-2\,\overline{\Phi}(t)+\overline{\Phi}^2(t))^n-(1-2\,\overline{\Phi}(t))^n \\ &\leqslant n\,\overline{\Phi}^2(t)\,(1-\overline{\Phi}(t))^{2n-2}. \end{aligned} \tag{38}$$

In the following, $C, C_1, \ldots > 0$ denote absolute constants. Let first $a > -\frac{1}{4}t_n^2$ so that $t_n+\frac{a}{t_n} > \frac{3}{4}t_n$. By (30) and (33),

$$\overline{\Phi}\Big(t_n+\frac{a}{t_n}\Big) \leqslant \frac{C_1}{t_n+\frac{a}{t_n}}\,\mathrm{e}^{-\frac{1}{2}\left(t_n+\frac{a}{t_n}\right)^2} \leqslant \frac{4C_1}{3\,t_n}\,\mathrm{e}^{-\frac{1}{2}t_n^2}\,\mathrm{e}^{-a} \leqslant \frac{C_2}{n}\,\mathrm{e}^{-a}. \tag{39}$$

On the other hand, if $a\in[-\frac{1}{4}t_n^2, 0]$, then again using (30) and (33) we obtain

$$\overline{\Phi}\Big(t_n+\frac{a}{t_n}\Big) \geqslant \frac{C_1'}{t_n+\frac{a}{t_n}}\,\mathrm{e}^{-\frac{1}{2}\left(t_n+\frac{a}{t_n}\right)^2} \geqslant \frac{C_1'}{t_n}\,\mathrm{e}^{-\frac{1}{2}t_n^2}\,\mathrm{e}^{-a}\mathrm{e}^{-\frac{a^2}{2t_n^2}} \geqslant \frac{C_2'}{n}\,\mathrm{e}^{-\frac{7}{8}a}, \tag{40}$$

where in the last estimate we used that $-\frac{a^2}{2t_n^2} \geqslant \frac{1}{8}a$.

Note that because of $-a \leqslant \frac{1}{4}t_n^2 \sim \frac{1}{2}\log n$, the right-hand side of (39) can be estimated above by $Cn^{-1/4}$. Using the inequality $1-x\leqslant \mathrm{e}^{-\frac{1}{2}x}$ (which is valid for sufficiently small $x>0$) together with (40), we obtain that for $a\in[-\frac{1}{4}t_n^2, 0]$,

$$\Big(1-\overline{\Phi}\Big(t_n+\frac{a}{t_n}\Big)\Big)^{2n-2} \leqslant \mathrm{e}^{-(n-1)\frac{C_2'}{n}\mathrm{e}^{-\frac{7}{8}a}} \leqslant \mathrm{e}^{-C'\mathrm{e}^{-\frac{7}{8}a}}.$$

It follows from this and (38), (39) that for all $a > -\frac{1}{2}t_n^2$,

$$\begin{aligned} n(G_n-F_n)\Big(t_n+\frac{a}{t_n}\Big) &\leqslant n^2\overline{\Phi}^2\Big(t_n+\frac{a}{t_n}\Big)\Big(1-\overline{\Phi}\Big(t_n+\frac{a}{t_n}\Big)\Big)^{2n-2} \\ &\leqslant C''\mathrm{e}^{-2a}\mathrm{e}^{-C'\mathrm{e}^{-\frac{7}{8}a}}, \end{aligned}$$

where in the case $a > 0$ we used the trivial estimate $1 - \overline{\Phi}(t) \leqslant 1$. The function on the right-hand side is integrable in a, thus completing the proof of (36).

We turn now to (37). Using again (38), the trivial estimate $\overline{\Phi}(t) \leqslant 1$, and the increasing property of $1 - \overline{\Phi}(t)$, we obtain that

$$I_n := n \int_{-t_n^2}^{-t_n^2/4} (G_n - F_n)\Big(t_n + \frac{a}{t_n}\Big)\,\mathrm{d}a \leqslant n^2 t_n^2 \Big(1 - \overline{\Phi}\Big(\frac{3}{4}\,t_n^2\Big)\Big)^{2n-2}.$$

Recall now that $t_n^2 \sim 2\log n$ and use (40) which implies that for some $\varepsilon \in (0,1)$,

$$\overline{\Phi}\Big(\frac{3}{4}\,t_n^2\Big) \geqslant Cn^{-\varepsilon}, \text{ but } \lim_{n\to\infty} \overline{\Phi}\Big(\frac{3}{4}\,t_n^2\Big) = 0.$$

Again using inequality $1 - x \leqslant \mathrm{e}^{-\frac{1}{2}x}$ (valid for small $x > 0$), we obtain

$$I_n \leqslant C\,n^2(\log n)\,\mathrm{e}^{-Cn^{-\varepsilon}(n-1)} \underset{n\to\infty}{\longrightarrow} 0,$$

thus proving (37). $\square$

§6. Proofs of Theorems 3.6 and 3.7

Both proofs rely on the observation that a random vector U distributed uniformly on $\mathbb{S}^{n-1}$ can be represented as

$$U \overset{d}{=} \Big(\frac{\eta_1}{\sqrt{\eta_1^2 + \cdots + \eta_n^2}}, \ldots, \frac{\eta_n}{\sqrt{\eta_1^2 + \cdots + \eta_n^2}}\Big). \tag{41}$$

Proof of Theorem 3.6. It follows from the definition of W_{Q_n}, see (19), and from the central symmetry of the cube that

$$W_{Q_n} = 2\sup_{t\in Q_n} \langle t, U\rangle \overset{d}{=} \frac{|\eta_1| + \cdots + |\eta_n|}{\sqrt{\eta_1^2 + \cdots + \eta_n^2}}. \tag{42}$$

Consider a random vector (X_n, Y_n) with

$$X_n := \frac{|\eta_1| + \cdots + |\eta_n| - n\mu}{\sigma\sqrt{n}}, \quad Y_n := \frac{\eta_1^2 + \cdots + \eta_n^2 - n}{v\sqrt{n}}, \tag{43}$$

where

$$\mu := \mathbf{E}|\eta_1| = \sqrt{2/\pi}, \tag{44}$$

$$\sigma^2 := \operatorname{Var}|\eta_1| = \mathbf{E}[\eta_1^2] - (\mathbf{E}|\eta_1|)^2 = (\pi - 2)/\pi, \tag{45}$$

$$v^2 := \operatorname{Var}(\eta_1^2) = \mathbf{E}[\eta_1^4] - (\mathbf{E}[\eta_1^2])^2 = 2. \tag{46}$$

Note that $\mathbf{E}X_n = \mathbf{E}Y_n = 0$ and $\operatorname{Var} X_n = \operatorname{Var} Y_n = 1$, while

$$r := \operatorname{Cov}(X_n, Y_n) = \frac{n \operatorname{Cov}(|\eta_1|, \eta_1^2)}{\sigma v n} = \frac{1}{\sqrt{\pi - 2}}, \tag{47}$$

where we used that $\mathbf{E}|\eta_1^3| = 2\sqrt{2/\pi}$. By the bivariate central limit theorem,

$$(X_n, Y_n) \xrightarrow[n\to\infty]{d} (X, Y), \tag{48}$$

where (X, Y) is a bivariate Gaussian vector with standard margins and covariance r. It follows from (43) that

$$W_{Q_n} \stackrel{d}{=} \frac{|\eta_1| + \cdots + |\eta_n|}{\sqrt{\eta_1^2 + \cdots + \eta_n^2}} = \frac{\mu n + \sigma\sqrt{n} X_n}{\sqrt{n + v\sqrt{n} Y_n}} = \frac{\mu n \left(1 + \frac{\sigma X_n}{\mu\sqrt{n}}\right)}{\sqrt{n}\sqrt{1 + \frac{vY_n}{\sqrt{n}}}}.$$

Letting $n \to \infty$, expanding into a Taylor series around 0 and noting that $X_n = O_P(1)$, $Y_n = O_P(1)$, we get

$$\frac{|\eta_1| + \cdots + |\eta_n|}{\sqrt{\eta_1^2 + \cdots + \eta_n^2}} = \mu\sqrt{n}\Big(1 + \frac{1}{\sqrt{n}}\Big(\frac{\sigma X_n}{\mu} - \frac{vY_n}{2}\Big) + O_P\Big(\frac{1}{n}\Big)\Big), \quad n \to \infty.$$

It follows that

$$\frac{|\eta_1| + \cdots + |\eta_n|}{\sqrt{\eta_1^2 + \cdots + \eta_n^2}} - \mu\sqrt{n} = \sigma X_n - \frac{1}{2}\,\mu\, v\, Y_n + O_P\Big(\frac{1}{\sqrt{n}}\Big), \quad n \to \infty.$$

Note that by the bivariate central limit theorem (48), the sequence $\sigma X_n - \frac{1}{2}\,\mu\, v\, Y_n$ has limiting normal distribution with mean zero and variance

$$\operatorname{Var}\Big[\sigma X_n - \frac{1}{2}\,\mu\, v\, Y_n\Big] = \sigma^2 + \frac{1}{4}\,\mu^2\, v^2 - \sigma\,\mu\, v\, r = \frac{\pi - 3}{\pi},$$

where we used (44), (45), (46), (47). Recalling (42) we obtain

$$W_{Q_n} - \sqrt{\frac{2\,n}{\pi}} \stackrel{d}{=} \frac{|\eta_1| + \cdots + |\eta_n|}{\sqrt{\eta_1^2 + \cdots + \eta_n^2}} - \mu\sqrt{n} \xrightarrow[n\to\infty]{d} \mathcal{N}\Big(0, \frac{\pi - 3}{\pi}\Big),$$

which proves the claim. □

Remark 6.1. Self-normalized or studentized sums of the form

$$R_n := \frac{\zeta_1 + \cdots + \zeta_n}{\sqrt{\zeta_1^2 + \cdots + \zeta_n^2}} \quad \text{or} \quad T_n := \frac{\zeta_1 + \cdots + \zeta_n}{\sqrt{(\zeta_1 - \overline{\zeta}_n)^2 + \cdots + (\zeta_n - \overline{\zeta}_n)^2}},$$

where $\zeta_1, \zeta_2, \ldots$ are i.i.d. random variables and $\overline{\zeta}_n = \frac{1}{n}(\zeta_1 + \cdots + \zeta_n)$, have been extensively studied in the literature; see, e.g., [10], with main emphasis on the central case $\mathbf{E}[\zeta_i]=0$. The non-central case $\mathbf{E}[\zeta_i]\neq 0$ has been analyzed by [5] (who studied T_n) and by [18] (who studied $1/R_n^2$ and related quantities). After some calculations, our central limit theorem for W_{Q_n} could be deduced from [18, Theorem 3.1(v)], but since these authors studied $1/R_n^2$ instead of R_n it was easier to provide a direct proof.

Proof of Theorem 3.7. We prove (26). Using representation (41), we obtain

$$W_{S_{n-1}} \stackrel{d}{=} \frac{\max_{1\leqslant i\leqslant n}\eta_i-\min_{1\leqslant i\leqslant n}\eta_i}{\sqrt{\eta_1^2+\cdots+\eta_n^2}}. \tag{49}$$

It is known from extreme-value theory that the range of the standard normal sample satisfies

$$Z_n := u_n\Big(\max_{1\leqslant i\leqslant n}\eta_i-\min_{1\leqslant i\leqslant n}\eta_i-2\,u_n\Big)\xrightarrow[n\to\infty]{d} G_1+G_2, \tag{50}$$

where $u_n\sim\sqrt{2\log n}$ is as in (16). In fact, this follows from the asymptotic independence [17, Theorem 1.8.3, p. 28] of $\max_{1\leqslant i\leqslant n}\eta_i$ and $-\min_{1\leqslant i\leqslant n}\eta_i$ which both have limiting Gumbel distribution as in (14). Define Y_n as in (43) and observe that Y_n has limiting standard normal distribution by the central limit theorem. We have

$$\frac{\max_{1\leqslant i\leqslant n}\eta_i-\min_{1\leqslant i\leqslant n}\eta_i}{\sqrt{\eta_1^2+\cdots+\eta_n^2}}=\frac{2\,u_n+\frac{Z_n}{u_n}}{\sqrt{n+\sqrt{2\,n}\,Y_n}}=\frac{2\,u_n}{\sqrt{n}}\,\frac{1+\frac{Z_n}{2\,u_n^2}}{\sqrt{1+\sqrt{2/n}\,Y_n}}.$$

Noting that both Z_n and Y_n are $O_P(1)$ and expanding into a Taylor series, we obtain

$$\frac{\max_{1\leqslant i\leqslant n}\eta_i-\min_{1\leqslant i\leqslant n}\eta_i}{\sqrt{\eta_1^2+\cdots+\eta_n^2}}=\frac{2\,u_n}{\sqrt{n}}\Big(1+\frac{Z_n}{2\,u_n^2}+O_P\Big(\frac{1}{u_n^4}\Big)\Big),$$

where we have used that $u_n\sim\sqrt{2\log n}$ and hence, the term with Y_n is negligible. It follows from (50) that

$$u_n\sqrt{n}\Big(\frac{\max_{1\leqslant i\leqslant n}\eta_i-\min_{1\leqslant i\leqslant n}\eta_i}{\sqrt{\eta_1^2+\cdots+\eta_n^2}}-\frac{2\,u_n}{\sqrt{n}}\Big)\xrightarrow[n\to\infty]{d} G_1+G_2,$$

which, in view of (49), implies (26).

The proof of (27) is analogous but instead of (42) it uses the representation

$$W_{C_n}\stackrel{d}{=}\frac{2\,\max_{1\leqslant i\leqslant n}|\eta_i|}{\sqrt{\eta_1^2+\cdots+\eta_n^2}}. \tag{51}$$

together with the limit relation

$$Z'_n := u_n\Big(\max_{1\leqslant i\leqslant n} |\eta_i| - u_{2n}\Big) \xrightarrow[n\to\infty]{d} G_1 \tag{52}$$

following from the asymptotic independence of the maximum and the minimum. $\square$

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Institut für Mathematische Statistik,
Universität Münster, Orléans–Ring 10,
48149 Münster, Germany
E-mail: `zakhar.kabluchko@uni-muenster.de`

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Department of Mathematical
and Statistical Sciences,
University of Alberta,
Edmonton, AB, T6G 2G1
E-mail: `aelitvak@gmail.com`

St.Petersburg Department
of the Steklov Institute of Mathematics,
Fontanka 27,
191011 St.Petersburg, Russia
E-mail: `zap1979@gmail.com`