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ON THE SPECTRAL DENSITY OF STATIONARY PROCESSES AND RANDOM FIELDS

Abstract. In this note we show that a stationary sequence obtained by applying a fixed deterministic function to the shifts of a stationary sequence (satisfying a mild regularity condition) has a spectral density. In the multiparametric setting, we obtain a similar result for a function of a shifted i.i.d. field.

§1. Introduction

Stationary processes are an important tool for modelling time series appearing in theoretical probability theory and also in real life evolutions. In many situations, the correlations between variables could be viewed as a measure of dependence and, in the Gaussian setting, they determine the distribution. The condensed information about the correlation structure of a stochastic process is contained in the so called "*spectral measure*" and, when it exists, in its density called the "*spectral density function*". Then, the covariances between variables are obtained as the Fourier coefficients of this function. Because the spectral density function encapsulates all information about covariances of a stochastic process, its study occupies a central place in their theory. In this note, our investigation is centered around the existence of spectral density.

Let $(X_n)_{n\in\mathbb{Z}}$ be a sequence of complex-valued mean zero random variables defined on a probability space $(\Omega, \mathcal{K}, \mathbf{P})$. We call this sequence *weakly stationary* (or *second order stationary*) if there exist complex numbers $\gamma(n)$, $n \in \mathbb{Z}$, such that for all $j, k \in \mathbb{Z}$

$$\mathrm{cov}(X_j, X_k) = \mathbf{E}(X_j \overline{X_k}) = \gamma(j-k).$$

Note that $\gamma(-n) = \overline{\gamma(n)}$.

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By the Birkhoff–Herglotz Theorem (see e.g. Brockwell and Davis [3]), there exists a unique measure on the unit circle, or equivalently a non-decreasing function F, called the *spectral distribution function* on $[0, 2\pi)$, such that

$$\gamma(n) = \int_0^{2\pi} \mathrm{e}^{\mathrm{i}nt} F(\mathrm{d}t), \quad \text{for all } n \in \mathbb{Z}. \tag{1}$$

If F is absolutely continuous with respect to the Lebesgue measure λ on $[0, 2\pi)$, then the Radon–Nikodym derivative f of F with respect to the Lebesgue measure is called the *spectral density;* in other words $F(\mathrm{d}t) = f(t)\,\mathrm{d}t$ and

$$\gamma(n) = \int_0^{2\pi} \mathrm{e}^{\mathrm{i}nt} f(t)\,\mathrm{d}t, \quad \text{for all } n \in \mathbb{Z}.$$

The most common situation where the existence of the spectral density may be established is the case of a *regular process*, cf. e.g. [4, Chapter 7]. Recall that a process $(X_n)_{n\in\mathbb{Z}}$ is called *regular*, if the tail space

$$G^X_{-\infty} := \bigcap_{n\in\mathbb{Z}} G^X_n$$

is trivial, where G^X_n is the closed linear span of $\{X_k\}_{k\leqslant n}$.

Regularity of the process is equivalent (cf. [3, Chapter 5] or [4, Chapter 7, Theorem 13]) to the existence of Wold representation, i.e.,

$$X_k = \sum_{j=0}^{\infty} a_j \eta_{k-j}$$

where $\{a_j\}_{j\geqslant 0}$ is a square summable deterministic sequence of complex numbers and $\{\eta_n\}_{n\in\mathbb{Z}}$ is an uncorrelated zero mean unit variance sequence of random variables such that $G^\eta_n = G^X_n$. In this case $(X_k)_{k\in\mathbb{Z}}$ has the same scalar product (covariance) structure in $\mathbb{L}_2(\Omega, \mathcal{K}, \mathbf{P})$ as the sequence of functions $(x_k)_{k\in\mathbb{Z}}$ in $\mathbb{L}_2([0, 2\pi), \lambda)$, where

$$x_k(t) := (2\pi)^{-1/2} \sum_{j=0}^{\infty} a_j \mathrm{e}^{\mathrm{i}(k-j)t} = \mathrm{e}^{\mathrm{i}kt} x_0(t),$$

therefore

$$\gamma(k) = \int_0^{2\pi} x_k(t)\,\overline{x_0(t)}\,\mathrm{d}t = \int_0^{2\pi} \mathrm{e}^{\mathrm{i}kt}\,|x_0(t)|^2\,\mathrm{d}t.$$

It follows that X has the spectral density

$$f(t) = |x_0(t)|^2 = \frac{1}{2\pi}\Big|\sum_{j=0}^{\infty} a_j \mathrm{e}^{-\mathrm{i}jt}\Big|^2, \qquad t \in [0, 2\pi),$$

cf. [4, Chapter 7, Corollary 5]. Moreover, by Kolmogorov criterion [4, Chapter 7, Theorem 15], the process $(X_n)_{n\in\mathbb{Z}}$ is regular iff it has a spectral density f satisfying condition

$$\int_0^{2\pi} \ln f(t)\, \mathrm{d}t > -\infty.$$

It is not clear however what can we say about the density existence when regularity condition is not necessarily satisfied, as, for example, in the case of functions of a two-sided sequence of i.i.d. random variables.

More generally, we shall also study the existence of spectral density for random fields. For simplicity, we shall discuss only the $\mathbb{Z}^2$-indexed random fields. Extension to the index set $\mathbb{Z}^d$ with $d > 2$ is easy.

In the sequel, where necessary, we use the standard coordinate notation, e.g. $\mathbf{k} = (k_1, k_2)$ for $\mathbf{k} \in \mathbb{Z}^2$ and $\mathbf{k} \cdot \mathbf{t} = k_1 t_1 + k_2 t_2$ for $\mathbf{k} \in \mathbb{Z}^2$, $\mathbf{t} \in \mathbb{R}^2$.

We call the collection of complex-valued mean zero random variables $(X_{\mathbf{k}})_{\mathbf{k}\in\mathbb{Z}^2}$ *weakly stationary* (or *second order stationary*) if there exist complex numbers $\gamma(\mathbf{n})$, $\mathbf{n} \in \mathbb{Z}^2$, such that for all $\mathbf{j}, \mathbf{k} \in \mathbb{Z}^2$

$$\operatorname{cov}(X_{\mathbf{j}}, X_{\mathbf{k}}) = \mathbf{E}(X_{\mathbf{j}}\overline{X_{\mathbf{k}}}) = \gamma(\mathbf{j} - \mathbf{k}).$$

In the context of weakly stationary random fields it is known that there exists a unique measure F on $[0, 2\pi)^2$, such that

$$\operatorname{cov}(X_{\mathbf{k}}, X_{\mathbf{0}}) = \int_{[0,2\pi)^2} \mathrm{e}^{\mathrm{i}\,\mathbf{k}\cdot\mathbf{t}} F(\mathrm{d}t_1, \mathrm{d}t_2), \quad \text{for all } \mathbf{k} \in \mathbb{Z}^2 .$$

If F is absolutely continuous with respect to Lebesgue measure λ^2 on $[0, 2\pi)^2$, then there exists the Radon–Nikodym derivative f of F with respect to λ^2, i.e., $F(\mathrm{d}t_1, \mathrm{d}t_2) = f(t_1, t_2)\, \mathrm{d}t_1\, \mathrm{d}t_2$. This function f is called *spectral density* and we have

$$\operatorname{cov}(X_{\mathbf{k}}, X_{\mathbf{0}}) = \int_{[0,2\pi)^2} \mathrm{e}^{\mathrm{i}\,\mathbf{k}\cdot\mathbf{t}} f(t_1, t_2)\, \mathrm{d}t_1\, \mathrm{d}t_2, \quad \text{for all } \mathbf{k} \in \mathbb{Z}^2 .$$

For the sake of clarity we shall treat separately processes and then random fields.

1.1. Results for stationary processes. We start by pointing out a well known characterization of the existence of spectral density.

Theorem 1. *Let $X := (X_k)_{k\in\mathbb{Z}}$ be a mean zero complex-valued second order stationary stochastic process. Then the following statements are equivalent:*

1) *X has a spectral density.*

2) *There are complex numbers $(a_j)_{j\in\mathbb{Z}}$ with $\sum_{j\in\mathbb{Z}} |a_j|^2 < \infty$ such that*

$$\gamma(k) := \operatorname{cov}(X_k, X_0) = \sum_{j\in\mathbb{Z}} a_j \overline{a_{j+k}}, \qquad k \in \mathbb{Z}.$$

3) *There exists a stationary process $\widetilde{X} := (\widetilde{X}_k)_{k\in\mathbb{Z}}$ equidistributed with X such that $\widetilde{X}$ admits a representation*

$$\widetilde{X}_k = \sum_{j\in\mathbb{Z}} a_j\, \eta_{j+k}, \quad \textit{for all } k \in \mathbb{Z}, \tag{2}$$

where $(a_j)_{j\in\mathbb{Z}}$ satisfies $\sum_{j\in\mathbb{Z}} |a_j|^2 < \infty$ and $(\eta_j)_{j\in\mathbb{Z}}$ is a sequence of mean zero unit variance uncorrelated random variables. In this case the spectral density is

$$f(t) = \frac{1}{2\pi}\Big|\sum_{j\in\mathbb{Z}} a_j\, \mathrm{e}^{\mathrm{i}jt}\Big|^2.$$

Remark 2. If the second order stationary stochastic process $(X_k)_{k\in\mathbb{Z}}$ is real valued, Theorem 1 holds with a sequence $(a_n)_{n\in\mathbb{Z}}$ of real numbers and the density f is a symmetric function.

Furthermore, if the process $(X_k)_{k\in\mathbb{Z}}$ is Gaussian, then the variables $(\eta_j)_{j\in\mathbb{Z}}$ in (2) are i.i.d. standard normal. For this latter statement see also Varadhan lectures [6, Chapter 6, Section 6.6].

Let $(\xi_j)_{j\in\mathbb{Z}}$ be a strictly stationary sequence of random variables defined on $(\Omega, \mathcal{K}, \mathbf{P})$ and for $g : \mathbb{R}^{\mathbb{Z}} \to \mathbb{C}$ construct

$$X_0 = g(\ldots, \xi_{-1}, \xi_0, \xi_{-1} \ldots), \quad X_k = X_0 \circ T^k, \tag{3}$$

where T is the shift operator on $\mathbb{R}^{\mathbb{Z}}$.

Define

$$\mathcal{F}_k = \sigma(\xi_j : j \leqslant k),\ \mathcal{F}_{-\infty} = \cap_{k\in\mathbb{Z}}\mathcal{F}_k,\ \text{and } \mathcal{F} = \sigma((\xi_j)_{j\in\mathbb{Z}}). \tag{4}$$

We shall assume the following regularity condition

$$\mathbf{E}(X_0|\mathcal{F}_{-\infty}) = 0 \quad \text{a.s.}, \tag{5}$$

which implies that $\mathbf{E}(X_0) = 0$.

Theorem 3. *Define the strictly stationary sequence (X_k) by (3). Assume that condition (5) is satisfied and $\mathbf{E}|X_0|^2 < \infty$. Then the sequence $(X_k)_{k\in\mathbb{Z}}$ has spectral density.*

Let us mention that condition (5) is satisfied when the left tail sigma field $\mathcal{F}_{-\infty}$ of $(\xi_k)_{k\in\mathbb{Z}}$ is trivial. This happens for instance when $(\xi_k)_{k\in\mathbb{Z}}$ is a sequence of i.i.d. random variables. Other examples are provided by conditions imposed on mixing coefficients.

The strong mixing coefficient is defined in the following way:

$$\alpha(\mathcal{A},\mathcal{B}) = \sup\{|\mathbf{P}(A\cap B) - \mathbf{P}(A)\,\mathbf{P}(B)| : A \in \mathcal{A},\ B \in \mathcal{B}\},$$

where $\mathcal{A}$ and $\mathcal{B}$ are two sigma fields.

The ρ-mixing coefficient, also known as maximal coefficient of correlation, is defined as

$$\rho(\mathcal{A},\mathcal{B}) = \sup\{\mathbf{E}(XY)/\|X\|_2\|Y\|_2 : X \in \mathbb{L}^2(\mathcal{A}),\ Y \in \mathbb{L}^2(\mathcal{B}),\ \mathbf{E}X = \mathbf{E}Y = 0\}.$$

For the stationary sequence of real valued random variables $(\xi_j)_{j\in\mathbb{Z}}$, $\mathcal{F}^n$ denotes the σ-field generated by ξ_j with indices $j \geqslant n$, and $\mathcal{F}_k$, as before, denotes the σ-field generated by ξ_j with indices $j \leqslant k$. Then we define the sequences of mixing coefficients

$$\alpha_n = \alpha(\mathcal{F}_0, \mathcal{F}^n) \ \text{ and } \rho_n = \rho(\mathcal{F}_0, \mathcal{F}^n)\,.$$

A sequence is called strongly mixing if $\alpha_n \to 0$. It is well-known that for strongly mixing sequences the left tail sigma field is trivial; see Claim 2.17a in Bradley [2]. Examples of this type include Harris recurrent Markov chains.

If $\rho_n < 1$ for some $n \geqslant 1$, then the tail sigma field is also trivial according to Section 2.5 in Bradley [1].

Therefore the result of Theorem 3 holds for functions of a sequence $(\xi_j)_{j\in\mathbb{Z}}$ if it is strongly mixing or satisfies $\rho_n < 1$ for some $n \geqslant 1$.

1.2. Results for stationary random fields. Similar results hold for random fields. Below the indexes are in $\mathbb{Z}^2$, but we can easily formulate the results for indexes in $\mathbb{Z}^d$ with d integer. Here is a generalization of Theorem 1 for random fields:

Theorem 4. *A second order stationary complex valued random field $(X_{\mathbf{k}})_{\mathbf{k}\in\mathbb{Z}^2}$ has spectral density if and only if there are numbers $(a_{\mathbf{k}})_{\mathbf{k}\in\mathbb{Z}^2}$ satisfying condition $\sum_{\mathbf{k}\in\mathbb{Z}^2} |a_{\mathbf{k}}|^2 < \infty$ such that* $\operatorname{cov}(X_{\mathbf{k}}, X_{\mathbf{0}}) = \sum_{\mathbf{j}\in\mathbb{Z}^2} a_{\mathbf{j}}\, \overline{a_{\mathbf{j}+\mathbf{k}}}$.

Our Remark 2 can be extended to random fields in an obvious way, just replacing the indexes in $\mathbb{Z}$ with indexes in $\mathbb{Z}^2$. The extension of Theorem 3 is more delicate, because, in the multi-index setting, there is no unique interpretation of past and future. Here we restrict our considerations to the functions of an i.i.d. random field. As the reader will see, this setting provides some additional useful commutativity properties for related projection operators.

Let $(\xi_{\mathbf{k}})_{\mathbf{k}\in\mathbb{Z}^2}$ be an i.i.d. random field defined on a probability space $(\Omega, \mathcal{K}, \mathbf{P})$ and define a random variable

$$X_{\mathbf{0}} = g((\xi_{\mathbf{k}})_{\mathbf{k}\in\mathbb{Z}}).$$

where $g : \mathbb{R}^{\mathbb{Z}^2} \to \mathbb{C}$ is a measurable function.

Moreover, define two translation operators on $\mathbb{R}^{\mathbb{Z}^2}$

$$T_1((x_{\mathbf{u}})_{\mathbf{u}\in\mathbb{Z}^2}) = (x_{u_1+1,u_2})_{\mathbf{u}\in\mathbb{Z}^2}$$

and

$$T_2((x_{\mathbf{u}})_{\mathbf{u}\in\mathbb{Z}^2}) = (x_{u_1,u_2+1})_{\mathbf{u}\in\mathbb{Z}^2}.$$

Finally, let

$$X_{\mathbf{u}} = g(T_1^{u_1} T_2^{u_2}(\xi_{\mathbf{k}})) = g((\xi_{\mathbf{k}+\mathbf{u}})_{\mathbf{k}\in\mathbb{Z}^2}). \tag{6}$$

Theorem 5. *Let the stationary sequence $(X_{\mathbf{k}})_{\mathbf{k}\in\mathbb{Z}^2}$ be defined by* (6) *and assume $\mathbf{E}(X_{\mathbf{0}}) = 0$ and $\mathbf{E}|X_{\mathbf{0}}|^2 < \infty$. Then the sequence $(X_{\mathbf{k}})_{\mathbf{k}\in\mathbb{Z}^2}$ has spectral density.*

This theorem has immediate applications, for example, to Volterra-type random fields which play an important role in the nonlinear system theory. For any $\mathbf{k} \in \mathbb{Z}^2$, define the Volterra-type expansion as follows:

$$X_{\mathbf{k}} = \sum_{\mathbf{u},\mathbf{v}\in\mathbb{Z}^2} b_{\mathbf{u},\mathbf{v}}\, \xi_{\mathbf{k}-\mathbf{u}}\, \xi_{\mathbf{k}-\mathbf{v}}\,,$$

where $b_{\mathbf{u},\mathbf{v}}$ are real numbers satisfying

$$b_{\mathbf{u},\mathbf{v}} = 0 \text{ if } \mathbf{u} = \mathbf{v}, \qquad \sum_{\mathbf{u},\mathbf{v}\in\mathbb{Z}^2} b^2_{\mathbf{u},\mathbf{v}} < \infty\,,$$

and $(\xi_{\mathbf{k}})_{\mathbf{k}\in\mathbb{Z}^2}$ is an i.i.d. random field of centered and square integrable random variables. Under the above conditions, the random field $(X_{\mathbf{k}})_{\mathbf{k}\in\mathbb{Z}^2}$ exists, is stationary, zero mean and square integrable. By Theorem 5, this random field has spectral density since it is a function of i.i.d. field.

§2. Proofs

Proof of Theorem 1. 1) $\curvearrowright$ 2). Let f be the spectral density of X. Since $\sqrt{f(x)}$ is square integrable, by Carleson Theorem (cf. [5]), we have an expansion

$$\sqrt{f(t)} = \frac{1}{\sqrt{2\pi}} \sum_{j\in\mathbb{Z}} a_j \mathrm{e}^{\mathrm{i}jt} \qquad \text{a.s. and in } L_2([0,2\pi),\lambda),$$

with Fourier coefficients

$$a_j := \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} \sqrt{f(t)}\mathrm{e}^{-\mathrm{i}jt}\,\mathrm{d}t, \qquad j\in\mathbb{Z},$$

satisfying $\sum\limits_{j\in\mathbb{Z}} |a_j|^2 < \infty$. Therefore, by (1)

$$\begin{aligned}\gamma(k) &= \int_0^{2\pi} \mathrm{e}^{\mathrm{i}kt} f(t)\,\mathrm{d}t = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{e}^{\mathrm{i}kt} \Big| \sum_{j\in\mathbb{Z}} a_j \mathrm{e}^{\mathrm{i}jt}\Big|^2 \mathrm{d}t \\ &= \frac{1}{2\pi}\int_0^{2\pi} \Big(\sum_{j_1\in\mathbb{Z}} a_{j_1}\,\mathrm{e}^{\mathrm{i}(j_1+k)t}\Big)\Big(\sum_{j_2\in\mathbb{Z}} \overline{a_{j_2}}\,\mathrm{e}^{-\mathrm{i}j_2 t}\Big)\,\mathrm{d}t = \sum_{j\in\mathbb{Z}} a_j\,\overline{a_{j+k}}\;,\end{aligned}$$

as required in 2).

2) $\curvearrowright$ 1). Let

$$f(t) = \frac{1}{2\pi}\Big|\sum_{j\in\mathbb{Z}} a_j \mathrm{e}^{\mathrm{i}jt}\Big|^2.$$

Then

$$\int_0^{2\pi} \mathrm{e}^{\mathrm{i}kt} f(t)\,\mathrm{d}t = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{e}^{\mathrm{i}kt}\Big|\sum_{j\in\mathbb{Z}} a_j \mathrm{e}^{\mathrm{i}jt}\Big|^2\,\mathrm{d}t$$
$$= \frac{1}{2\pi}\int_0^{2\pi}\Big(\sum_{j_1\in\mathbb{Z}} a_{j_1}\mathrm{e}^{\mathrm{i}(j_1+k)t}\Big)\Big(\sum_{j\in\mathbb{Z}} \overline{a_{j_2}}\mathrm{e}^{-\mathrm{i}j_2 t}\Big)\,\mathrm{d}t$$
$$= \sum_{j\in\mathbb{Z}} a_j\,\overline{a_{j+k}} = \gamma(k),$$

as required in the definition of spectral density.

3) $\curvearrowright$ 2) is obvious.

For 1) $\curvearrowright$ 3) see [4, Chapter 7, Theorem 10]. □

Proof of Theorem 3. For every $\ell\in\mathbb{Z}$ we define the projection operator $\mathcal{P}_\ell$ by letting

$$\mathcal{P}_\ell X = \mathbf{E}(X|\mathcal{F}_\ell) - \mathbf{E}(X|\mathcal{F}_{\ell-1})$$

for any integrable random variable $X\in\mathbb{L}_1(\Omega,\mathcal{K},\mathbf{P})$.

Since we assumed that $\mathbf{E}(X_0|\mathcal{F}_{-\infty}) = 0$ a.s., by stationarity for all $k\in\mathbb{Z}$, $\mathbf{E}(X_k|\mathcal{F}_{-\infty}) = 0$. Furthermore, since all X_k are $\mathcal{F}$-measurable, we have the representation

$$X_k = \sum_{\ell\in\mathbb{Z}} \mathcal{P}_\ell X_k.$$

Let us compute the covariances. We have

$$\operatorname{cov}(X_k, X_0) = \sum_{\ell_1,\ell_2\in\mathbb{Z}} \operatorname{cov}(\mathcal{P}_{\ell_1}X_k, \mathcal{P}_{\ell_2}X_0).$$

Since the projections are orthogonal, we have

$$\operatorname{cov}(X_k, X_0) = \sum_{\ell\in\mathbb{Z}}\operatorname{cov}(\mathcal{P}_\ell X_k, \mathcal{P}_\ell X_0) = \sum_{\ell\in\mathbb{Z}}\operatorname{cov}(\mathcal{P}_0 X_{k-\ell}, \mathcal{P}_0 X_{-\ell}) \tag{7}$$

where, in the last inequality, we used the fact that (X_k) is strictly stationary.

Let us denote $Y_\ell = \mathcal{P}_0 X_\ell$. Note that by stationarity and orthogonality of the projections it is true that

$$\sum_{\ell\in\mathbb{Z}} \mathbf{E}\,|Y_\ell|^2 = \sum_{\ell\in\mathbb{Z}} \mathbf{E}\,|\mathcal{P}_0 X_\ell|^2 = \sum_{\ell\in\mathbb{Z}} \mathbf{E}\,|\mathcal{P}_{-\ell} X_0|^2 = \mathbf{E}\,|X_0|^2 < \infty. \tag{8}$$

Consider the function

$$f(t) = \frac{1}{2\pi}\, \mathbf{E} \Big| \sum_{\ell\in\mathbb{Z}} Y_{-\ell}\, \mathrm{e}^{\mathrm{i}\ell t} \Big|^2, \qquad t \in [0, 2\pi).$$

By the Fubini theorem and (8) we have

$$\int_0^{2\pi} f(t)\, \mathrm{d}t = \frac{1}{2\pi}\, \mathbf{E} \int_0^{2\pi} \Big| \sum_{\ell\in\mathbb{Z}} Y_{-\ell}\, \mathrm{e}^{\mathrm{i}\ell t} \Big|^2 \mathrm{d}t = \mathbf{E} \sum_{\ell\in\mathbb{Z}} |Y_{-\ell}|^2 < \infty.$$

Let us now compute the Fourier coefficients of f. For every $k \in \mathbb{Z}$ we have

$$\begin{aligned} \int_0^{2\pi} \mathrm{e}^{\mathrm{i}kt} f(t)\, \mathrm{d}t &= \frac{1}{2\pi}\, \mathbf{E} \int_0^{2\pi} \Big(\sum_{\ell_1\in\mathbb{Z}} Y_{-\ell_1}\, \mathrm{e}^{\mathrm{i}(k+\ell_1)t} \Big) \Big(\sum_{\ell_2\in\mathbb{Z}} \overline{Y_{-\ell_2}}\, \mathrm{e}^{-\mathrm{i}\ell_2 t} \Big)\, \mathrm{d}t \\ &= \sum_{\ell_1,\ell_2\in\mathbb{Z}} \mathbf{E} \left(Y_{-\ell_1} \overline{Y_{-\ell_2}} \right) \mathbf{1}_{\{k+\ell_1=\ell_2\}} = \sum_{\ell\in\mathbb{Z}} \mathbf{E} \left(Y_{k-\ell} \overline{Y_{-\ell}} \right). \end{aligned}$$

By comparing this expression with (7) we see that f is the spectral density for $(X_k)_{k\in\mathbb{Z}}$. □

Proof of Theorem 4. is completely identical to that of Theorem 1 and therefore is omitted. We only notice that the spectral density for the process satisfying

$$\operatorname{cov}(X_{\mathbf{k}}, X_{\mathbf{0}}) = \sum_{\mathbf{j}\in\mathbb{Z}^2} a_{\mathbf{j}}\, \overline{a_{\mathbf{j}+\mathbf{k}}}$$

has the form

$$f(\mathbf{t}) = \frac{1}{(2\pi)^2} \Big| \sum_{\mathbf{j}\in\mathbb{Z}^2} a_{\mathbf{j}}\, \mathrm{e}^{\mathrm{i}\,\mathbf{j}\cdot\mathbf{t}} \Big|^2, \qquad \mathbf{t} \in [0, 2\pi)^2.$$

□

Proof of Theorem 5. Define the sigma fields

$$\mathcal{F}_{k_1,k_2} = \sigma(\xi_{\mathbf{j}} : j_1 \leqslant k_1,\ j_2 \leqslant k_2).$$

Next, for $k \in \mathbb{Z}$, denote $\mathcal{F}_{k,\infty} = \vee_{k_2\in\mathbb{Z}} \mathcal{F}_{k,k_2}$ and $\mathcal{F}_{\infty,k} = \vee_{k_1\in\mathbb{Z}} \mathcal{F}_{k_1,k}$.

We introduce the projection operators by letting

$$\mathcal{P}_{u,\infty} X = \mathbf{E}(X|\mathcal{F}_{u,\infty}) - \mathbf{E}(X|\mathcal{F}_{u-1,\infty})$$

and

$$\mathcal{P}_{\infty,u} X = \mathbf{E}(X|\mathcal{F}_{\infty,u}) - \mathbf{E}(X|\mathcal{F}_{\infty,u-1})$$

for any integrable random variable $X \in \mathbb{L}_1(\Omega, \mathcal{K}, \mathbf{P})$. Furthermore, we define the iterated operator by

$$\mathcal{P}_{u_1,u_2} X = (\mathcal{P}_{u_1,\infty} \circ \mathcal{P}_{\infty,u_2})\, X.$$

Since the variables $(\xi_{\mathbf{k}})$ are independent, for all $-\infty \leqslant p_1, p_2, u_1, u_2 \leqslant \infty$ it is true that

$$\mathbf{E}\left(\mathbf{E}(X|\mathcal{F}_{p_1,p_2})|\mathcal{F}_{u_1,u_2}\right) = \mathbf{E}(X|\mathcal{F}_{p_1\wedge u_1, p_2 \wedge u_2}) \quad \text{a.s.}$$

By using this property and the definition of the iterated operator, we see that for all $u_1, u_2 \in \mathbb{Z}$, almost surely,

$$\mathcal{P}_{u_1,u_2} X = \mathbf{E}(X|\mathcal{F}_{u_1,u_2}) - \mathbf{E}(X|\mathcal{F}_{u_1,u_2-1}) - \mathbf{E}(X|\mathcal{F}_{u_1-1,u_2}) + \mathbf{E}(X|\mathcal{F}_{u_1-1,u_2-1}).$$

We also obtain the same expression for $(\mathcal{P}_{\infty,u_2} \circ \mathcal{P}_{u_1,\infty})\, X$, thus we see that the operators $\mathcal{P}_{u_1,\infty}$ and $\mathcal{P}_{\infty,u_2}$ commute.

Next, we borrow an idea from Volný and Wang [7, Lemma 2.4(ii)] by claiming that $(u_1, u_2) \neq (p_1, p_2)$ yields

$$\operatorname{cov}(\mathcal{P}_{u_1,u_2} X, \mathcal{P}_{p_1,p_2} Y) = \mathbf{E}[(\mathcal{P}_{u_1,u_2} X)(\overline{\mathcal{P}_{p_1,p_2} Y})] = 0.$$

for all mean zero X and Y in $\mathbb{L}_2(\Omega, \mathcal{K}, \mathbf{P})$. Indeed, assume, without loss of generality, that $p_1 < u_1$. For any X the variable $\mathcal{P}_{u_1,\infty} X$ is orthogonal to the space $H = \mathbb{L}_2(\Omega, \mathcal{F}_{u_1-1,\infty}, \mathbf{P})$. Hence, $\mathcal{P}_{u_1,u_2} X$ is also orthogonal to H, while $\mathcal{P}_{p_1,p_2} Y$ belongs to H due to assumption $p_1 < u_1$.

Note that for all $u \in \mathbb{Z}$, the corresponding tail sigma fields defined as $\mathcal{F}_{u,-\infty} = \cap_{u_2 \in \mathbb{Z}} \mathcal{F}_{u,u_2}$, $\mathcal{F}_{-\infty,u} = \cap_{u_1 \in \mathbb{Z}} \mathcal{F}_{u_1,u}$ and $\mathcal{F}_{-\infty,-\infty} = \cap_{u \in \mathbb{Z}} \mathcal{F}_{u,-\infty}$ are trivial. Therefore, we have $\mathbf{E}(X|\mathcal{F}_{u,-\infty}) = 0$ a.s., $\mathbf{E}(X|\mathcal{F}_{-\infty,u}) = 0$ a.s., and $\mathbf{E}(X|\mathcal{F}_{-\infty,-\infty}) = 0$ a.s. It follows that for any mean zero X in $\mathbb{L}_2(\Omega, \mathcal{K}, \mathbf{P})$ we have the following orthogonal representation,

$$X = \sum_{u_1 \in \mathbb{Z}} \mathcal{P}_{u_1,\infty} X = \sum_{u_1 \in \mathbb{Z}} \mathcal{P}_{u_1,\infty}\Big(\sum_{u_2 \in \mathbb{Z}} \mathcal{P}_{\infty,u_2} X\Big) = \sum_{u_1,u_2 \in \mathbb{Z}} \mathcal{P}_{u_1,u_2} X \quad \text{a.s.} \tag{9}$$

Let us compute the covariances of $X_{\mathbf{k}}$ and $X_{\mathbf{0}}$. By using the above projection decomposition written for both $X_{\mathbf{k}}$ and $X_{\mathbf{0}}$, together with the orthogonality of the projections and stationarity, we have for all $\mathbf{k} \in \mathbb{Z}^2$,

$$\begin{aligned}
\operatorname{cov}(X_{\mathbf{k}}, X_{\mathbf{0}}) &= \operatorname{cov}\Big(\sum_{\mathbf{j}\in\mathbb{Z}^2} \mathcal{P}_{j_1,j_2} X_{\mathbf{k}}, \sum_{\mathbf{u}\in\mathbb{Z}^2} \mathcal{P}_{u_1,u_2} X_{\mathbf{0}}\Big) \\
&= \sum_{\mathbf{j}\in\mathbb{Z}^2} \operatorname{cov}(\mathcal{P}_{j_1,j_2} X_{\mathbf{k}}, \mathcal{P}_{j_1,j_2} X_{\mathbf{0}}) \\
&= \sum_{\mathbf{j}\in\mathbb{Z}^2} \operatorname{cov}(\mathcal{P}_{0,0} X_{\mathbf{k}-\mathbf{j}}, \mathcal{P}_{0,0} X_{-\mathbf{j}}) \\
&= \sum_{\mathbf{j}\in\mathbb{Z}^2} \operatorname{cov}(Y_{\mathbf{k}-\mathbf{j}}, Y_{-\mathbf{j}}), \qquad (10)
\end{aligned}$$

where we used the notation $Y_{\mathbf{u}} = \mathcal{P}_{0,0} X_{\mathbf{u}}$. Observe also that, by taking into account (9) and stationarity, we have

$$\sum_{\mathbf{u}\in\mathbb{Z}^2} \mathbf{E}|Y_{\mathbf{u}}|^2 = \sum_{\mathbf{u}\in\mathbb{Z}^2} \mathbf{E}|\mathcal{P}_{u_1,u_2} X_{\mathbf{0}}|^2 = \mathbf{E}|X_{\mathbf{0}}|^2 < \infty. \qquad (11)$$

Consider the function

$$f(\mathbf{t}) = \frac{1}{(2\pi)^2}\, \mathbf{E}\Big|\sum_{\mathbf{j}\in\mathbb{Z}^2} Y_{-\mathbf{j}}\, \mathrm{e}^{\mathrm{i}\,\mathbf{j}\cdot\mathbf{t}}\Big|^2, \qquad \mathbf{t} \in [0, 2\pi)^2.$$

By Fubini theorem and (11) we have

$$\begin{aligned}
\int_{[0,2\pi)^2} f(\mathbf{t})\, \mathrm{d}t_1\, \mathrm{d}t_2 &= \frac{1}{(2\pi)^2}\, \mathbf{E} \int_{[0,2\pi)^2} \Big|\sum_{\mathbf{j}\in\mathbb{Z}^2} Y_{-\mathbf{j}}\, \mathrm{e}^{\mathrm{i}\,\mathbf{j}\cdot\mathbf{t}}\Big|^2 \mathrm{d}t_1\, \mathrm{d}t_2 \\
&= \mathbf{E} \sum_{\mathbf{j}\in\mathbb{Z}^2} |Y_{-\mathbf{j}}|^2 < \infty.
\end{aligned}$$

Let us now compute the Fourier coefficients of f. For every $\mathbf{k} \in \mathbb{Z}^2$ we have

$$\int_{[0,2\pi)^2} \mathrm{e}^{\mathrm{i}\,\mathbf{k}\cdot\mathbf{t}} f(\mathbf{t})\, \mathrm{d}t_1\, \mathrm{d}t_2 = \frac{1}{(2\pi)^2}\, \mathbf{E} \int_{[0,2\pi)^2} A_1(\mathbf{t}) A_2(\mathbf{t})\ \mathrm{d}t_1\, \mathrm{d}t_2,$$

where

$$A_1(\mathbf{t}) = \sum_{\mathbf{j}\in\mathbb{Z}^2} Y_{-\mathbf{j}}\ \mathrm{e}^{\mathrm{i}\,(\mathbf{k}+\mathbf{j})\cdot\mathbf{t}}, \quad A_2(\mathbf{t}) = \sum_{\mathbf{u}\in\mathbb{Z}^2} \overline{Y_{-\mathbf{u}}}\ \mathrm{e}^{-\mathrm{i}\,\mathbf{u}\cdot\mathbf{t}}.$$

By using orthogonality of the exponential functions, we obtain

$$\int\limits_{[0,2\pi)^2} e^{i\,\mathbf{k}\cdot\mathbf{t}} f(\mathbf{t})\, dt_1\, dt_2 = \sum_{\mathbf{j},\mathbf{u}\in\mathbb{Z}^2} \mathbf{E}\left(Y_{-\mathbf{j}}\overline{Y_{-\mathbf{u}}}\right) \mathbf{1}_{\{\mathbf{k}+\mathbf{j}=\mathbf{u}\}}$$

$$= \sum_{\mathbf{u}\in\mathbb{Z}^2} \mathbf{E}\left(Y_{\mathbf{k}-\mathbf{u}}\overline{Y_{-\mathbf{u}}}\right).$$

By comparing this expression with (10) we see that f is the spectral density for $(X_{\mathbf{k}})_{\mathbf{k}\in\mathbb{Z}^2}$. □

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