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ON CONVEX HULL AND WINDING NUMBER OF SELF-SIMILAR PROCESSES

Abstract. It is well known that for a standard Brownian motion (BM) $\{B(t),\ t \geqslant 0\}$ with values in $\mathbf{R}^d$, its convex hull $V(t) = \operatorname{conv}\{\, B(s),\ s \leqslant t\}$ with probability 1 for each $t > 0$ contains 0 as an interior point (see Evans [3]). We also know that the winding number of a typical path of a two-dimensional BM is equal to $+\infty$. The aim of this article is to show that these properties aren't specifically "Brownian", but hold for a much larger class of d-dimensional self-similar processes. This class contains in particular d-dimensional fractional Brownian motions and (concerning convex hulls) strictly stable Lévy processes.

§1. Introduction

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a basic probability space. Consider a d-dimensional process $X = \{X(t),\ t \geqslant 0\}$ defined on Ω which is self-similar of index $H > 0$. It means that for each constant $c > 0$ the process $\{X(ct),\ t \geqslant 0\}$ has the same distribution as $\{c^H X(t),\ t \geqslant 0\}$.

Let $L = \{L(u),\ u \in \mathbf{R}^1\}$ be the strictly stationary process obtained from X by Lamperti transformation:

$$L(u) = e^{-Hu} X(e^u), \quad u \in \mathbf{R}^1. \tag{1}$$

Equivalently,

$$X(t) = t^H L(\log t), \quad t \in \mathbf{R}_*^+.$$

Let $\Theta = \{0,1\}^d$ be the set of all dyadic sequences of length d. Denote by $D_\theta,\ \theta \in \Theta$, the quadrant

$$D_\theta = \prod_{i=1}^{d} \mathbf{R}_{\theta_i},$$

where $\mathbf{R}_{\theta_i} = [0, \infty)$ if $\theta_i = 1$, and $\mathbf{R}_{\theta_i} = (-\infty, 0]$ if $\theta_i = 0$. The positive quadrant $D_{(1,1,\ldots,1)}$ for simplicity is denoted by D.

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We say that the process X is *non-degenerate* if for all $\theta \in \Theta$

$$\mathbf{P}\{X(1) \in D_\theta\} > 0.$$

Two important examples of self-similar processes are **fractional Brownian motion** and **stable Lévy process**.

Definition 1. *We call a self-similar (of index $H > 0$) process B^H fractional Brownian motion (FBM) if for each $e \in \mathbf{R}^d$ the scalar process $t \to \langle B^H(t), e\rangle$ is a standard one-dimensional FBM of index H up to a constant $c(e)$.*

It is easy to see that in this case $c^2(e) = \langle Qe, e\rangle$, where Q is the covariance matrix of $B^H(1)$, and hence

$$\mathbf{E}\langle B^H(t), e\rangle\langle B^H(s), e\rangle = \langle Qe, e\rangle \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}), \quad t, s \geqslant 0;\ e \in \mathbf{R}^d.$$

The process B^H is non-degenerate iff the rank of the matrix Q is equal to d. If $H = \frac{1}{2}, Q = I_d$, then B^H is a standard Brownian motion.

(See Xiao [8], Račkauskas and Suquet [6], Lavancier et al. [4] and references therein for more general definitions of operator self-similar FBM).

Definition 2. *We call $S = \{S(t),\ t \in \mathbf{R}_+\}$ α-strictly stable Lévy process (StS) if*

1) *$S(1)$ has a α-strictly stable distribution in $\mathbf{R}^d$;*
2) *it has independent and stationary increments;*
3) *it is continuous in probability.*

Then for each $t \in \mathbf{R}_+$ the random variable $S(t)$ has the same distribution as $t^{\frac{1}{\alpha}} S(1)$.

The cadlag version of S on $[0, 1]$ can be obtained with the help of LePage series representation (see [7] for more details). If $\alpha \in (0, 1)$ or if $\alpha \in (1, 2)$ and $\mathbf{E}\, X(1) = 0$, then we have:

$$\{S(t),\ t \in [0,1]\} \stackrel{\mathcal{L}}{=} \{c \sum_1^\infty \Gamma_k^{-1/\alpha} \varepsilon_k \mathbf{1}_{[0,t]}(\eta_k),\ \ t \in [0,1]\}, \tag{2}$$

where $\stackrel{\mathcal{L}}{=}$ means equality in law, c is a constant, $\Gamma_k = \sum_1^k \gamma_j$, $\{\gamma_j\}$ is a sequence of i.i.d. random variables with common standard exponential distribution, $\{\varepsilon_k\}$ is a sequence of i.i.d. random variables with common distribution σ concentrated on unit sphere $\mathbf{S}^{d-1}$, $\{\eta_k\}$ is a sequence of

$[0,1]$-uniformly distributed i.i.d. random variables, and the three sequences $\{\gamma_j\}$, $\{\varepsilon_k\}$, $\{\eta_k\}$ are supposed to be independent.

The measure σ is called spectral measure of S. It is easy to see that if (2) takes place, the process X is non-degenerate iff $\operatorname{vect}\{\operatorname{supp}\sigma\} = \mathbf{R}^1$.

In Section 2 the object of our interest is the convex hull process $V = \{V(t)\}$ associated with X. We show that under very sharp conditions with probability 1 for all $t > 0$ the convex set $V(t)$ contains 0 as its interior point. From this result some interesting corollaries are deduced.

Section 3 is devoted to studying the winding numbers of two-dimensional self-similar processes. As a corollary of our main result we show that for the typical path of a standard two-dimensional FBM the number of its clockwise and anti-clockwise winds around 0 in the neighborhood of zero or at infinity is equal to ∞.

§2. Convex hulls

For a Borel set $A \subset \mathbf{R}^d$ we denote by $\operatorname{conv}(A)$ the closed convex hull of A and define the convex hull process related to X:

$$V(t) = \operatorname{conv}\{\, X(s), \quad s \leqslant t\}.$$

Theorem 1. *Let X be a non-degenerate self-similar process such that the strictly stationary process L generating X is ergodic. Then with probability 1 for all $t > 0$ the point 0 is an interior point of $V(t)$.*

Application to FBM. Let B^H be a FBM with index H. The next properties follow from the definition without difficulties.

1) **Continuity.** The process X has a continuous version.
   Below we always suppose B^H to be continuous.
2) **Reversibility.** If the process Y is defined by
   $$Y(t) = B^H(1) - B^H(1-t), \quad t \in [0,1],$$
   then $\{Y(t),\ t \in [0,1]\} \stackrel{\mathcal{L}}{=} \{B^H(t),\ t \in [0,1]\}$.
3) **Ergodicity.** Let $L = \{L(u),\ u \in \mathbf{R}^1\}$ be the strictly stationary Gaussian process obtained from B^H by Lamperti transformation (1).
   Then L is ergodic (see Cornfeld et al. [1], Chapter 14, § 2, Theorem 1, Theorem 2).

It is supposed below that the process B^H is non-degenerate.

Corollary 1. *Let V be the convex hull process related to B^H. Then with probability* 1 *for all* $t > 0$ *the point* 0 *is an interior point of* $V(t)$.

This follows immediately from Theorem 1.

Corollary 2. *Let V be the convex hull process related to B^H. Then for each* $t > 0$ *with probability* 1 *the point* $B^H(t)$ *is an interior point of* $V(t)$.

Proof of Corollary 2. Denote by A° the interior of A. By self-similarity of the process B^H it is sufficient to state this property for $t = 1$. Then, due to the reversibility of B^H by Theorem 1, a.s.

$$0 \in [\mathrm{conv}\{\, B^H(1) - B^H(1-t), \quad t \in [0,1]\}]^\circ. \tag{3}$$

As

$$\mathrm{conv}\{\, B^H(1) - B^H(1-t),\ t \in [0,1]\} = B^H(1) - \mathrm{conv}\{\, B^H(1-s),\ s \in [0,1]\},$$

the relation (3) is equivalent to

$$B^H(1) \in [\mathrm{conv}\{\, B^H(s),\ s \in [0,1]\}]^\circ,$$

which concludes the proof. □

Let $\mathcal{K}_d$ be the family of all compact convex subsets of $\mathbf{R}^d$. It is well known that $\mathcal{K}_d$ equipped with Hausdorff metric is a Polish space.

We say that a function $f : [0,1] \to \mathcal{K}_d$ is *increasing*, if $f(t) \subset f(s)$ for $0 \leqslant t < s \leqslant 1$.

We say that a function $f : [0,1] \to \mathcal{K}_d$ is *almost everywhere constant*, if f is such that for almost every $t \in [0,1]$ there exists an interval $(t-\varepsilon, t+\varepsilon)$ where f is constant.

We say that a function $f : [0,1] \to \mathcal{K}_d$ is a *Cantor-staircase* (C-S), if f is continuous, increasing and almost everywhere constant.

The next statement follows easily from Corollary 2.

Corollary 3. *Let V be the convex hull process related to B^H. Then with probability* 1 *the paths of the process* $t \to V(t)$ *are C-S functions.*

Remark 1. Let $h : \mathcal{K} \to \mathbf{R}^1$ be an increasing continuous function. Then almost all paths of the process $t \to h(V(t))$ are C-S real functions. This obvious fact may be applied to all reasonable geometrical characteristics of $V(t)$, such as volume, surface area, diameter, ...

Application to StS. Let now S be an StS process with exponent $\alpha < 2$. The following properties are more or less known.

1) **Right continuity.** The process S has a *cadlag* version (see remark above just after the definition).

2) **Reversibility.** Let
$$Y(t) = S(1) - S(1-t), \quad t \in [0,1].$$
Then $\{Y(t),\ t \in [0,1]\} \stackrel{\mathcal{L}}{=} \{S(t),\ t \in [0,1]\}$.

3) **Self-similarity.** The process S is self-similar of index $H = \frac{1}{\alpha}$.

4) **Ergodicity.** Let $L = \{L(u),\ u \in \mathbf{R}^1\}$ be the strictly stationary process obtained from S by the Lamperti transformation (1). Then L is ergodic.

We suppose that the law of $S(1)$ is non-degenerate.

Corollary 4. *Let V be the convex hull process related to S. Then with probability 1 for all $t > 0$ the point 0 is an interior point of $V(t)$.*

Corollary 5. *Let V be the convex hull process related to S. Then for each $t > 0$ with probability 1 the point $X(t)$ is an interior point of $V(t)$.*

Corollary 6. *Let V be the convex hull process related to S. Then with probability 1 the paths of the process $t \to V(t)$ are right continuous almost everywhere constant functions.*

We omit proofs of these statements as they are similar to proofs of Corollaries 1–3.

Proof of Theorem 1. We first show that
$$p \stackrel{\text{def}}{=} \mathbf{P}\{\,\exists\, t \in (0,1] \mid X(t) \in D^\circ\} = 1. \tag{4}$$
Remark that p is strictly positive:
$$p \ \geqslant\ \mathbf{P}\{X(1) \in D^\circ\} > 0 \tag{5}$$
due to the hypothesis that the law of $X(1)$ is non-degenerate.

By self-similarity
$$\mathbf{P}\big\{D^\circ \cap \{X(t),\ t \in [0,T]\} = \varnothing\big\} = 1 - p$$
for every $T > 0$. The sequence of events $(A_n)_{n\in\mathbf{N}}$,
$$A_n = \big\{D^\circ\ \cap \{X(t), t \in [0,n]\} = \varnothing\big\},$$

being decreasing, it follows that

$$1-p=\lim \mathbf{P}(A_n)=\mathbf{P}(\cap_n A_n)=\mathbf{P}\{X(t)\notin D^\circ,\ \ \forall\, t\geqslant 0\}.$$

In terms of the stationary process L from the Lamperti representation it means that

$$\mathbf{P}\{L(s)\notin D^\circ,\ \forall\, s\in\mathbf{R}^1\}=1-p.$$

As this event is invariant, by ergodicity of L and due to (5) we see that the value $p=1$ is the only one possible.

Applying the similar arguments to another quadrants D_θ, $\theta\in\Theta$, we get that with probability 1 there exists points $t_\theta\in(0,1]$, such that $X(t_\theta)\in D_\theta^\circ$, $\theta\in\Theta$. Now, to end the proof it is sufficient to remark that

$$V(1)^\circ=\operatorname{conv}\{X(t),\ t\in[0,1]\}^\circ\supset\operatorname{conv}\{X(t_\theta),\ \theta\in\Theta\}^\circ$$

and that the last set evidently contains 0. □

§3. Winding numbers

Let now $X=\{X(t),\ t\geqslant 0\}$ be a two-dimensional self-similar process. It is supposed that the following properties are fulfilled:

1) Process X is continuous.
2) Process X is non-degenerate.
3) Process X is symmetric: X and $-X$ have the same law.
4) The stationary process L associated with X is ergodic.
5) Starting from 0 the process X with probability 1 never come back:

$$\mathbf{P}\{X(t)\neq 0,\ \forall\, t>0\}=1. \tag{6}$$

Due to the last hypothesis, considering $\mathbf{R}^2$ as complex plane, we can define the winding numbers (around 0) $\nu[s,t]$, $0<s<t$, by the usual way (see [5], Chapter 5):

$$\nu[s,t]=\arg\left(X(t)\right)-\arg\left(X(s)\right).$$

We set

$$\nu_+(0,t]=\limsup_{s\downarrow 0}\nu[s,t],\quad \nu_-(0,t]=\liminf_{s\downarrow 0}\nu[s,t],$$

$$\nu_+[s,\infty)=\limsup_{t\to\infty}\nu[s,t],\quad \nu_-[s,\infty)=\liminf_{t\to\infty}\nu[s,t].$$

The values $\nu_+(0,t]$, $-\nu_-(0,t]$ represent respectively the number of clockwise and anti-clockwise winds around 0 in the neighborhood of the starting point, while $\nu_+[s,\infty)$, $-\nu_-[s,\infty)$ are the similar winding numbers at infinity.

Theorem 2. *Let X be a two-dimensional self-similar process with the properties* 1)–5) *mentioned above. Then with probability one for all $t > 0$*

$$\nu_+(0,t] = \nu_+[t,\infty) = -\nu_-(0,t] = -\nu_-[t,\infty) = +\infty. \tag{7}$$

Corollary 7. *Let B^H be a two-dimensional standard FBM and assume that $H \in [1/2, 1)$. Then with probability one for all $t > 0$ the equalities* (7) *take place.*

Proof. Case $H = 1/2$ is well known, see [5, Chapter 5], which contains exhaustive information on Brownian winding numbers.

If $H \in (1/2, 1)$, we apply Theorem 2 as all hypothesis 1)–5) are fulfilled: indeed, the properties 1)–3) are obvious; the ergodicity of L, $L(t) = (L_1(t), L_2(t))$, follows from the fact that $\mathbf{E}\, L_1(t)L_1(0) \to 0$ when $t \to \infty$ (see [1, Chapter 14, Section 2, Theorem 2]); The property 5) can be deduced from Theorem 11 of [8] (see also [9, Theorem 4.2] and [10, Theorem 2.6]). $\square$

Remark 2. If $H \in (0, \frac{1}{2})$, the process $t \to \arg B^H(t) - \arg B^H(0)$ is not continuous with positive probability as the set $\{t \in (0,1] \mid B^H(t) = 0\}$ is not empty (see [8, Theorem 11)]). It means that in this case the winding numbers could be defined only for the excursions of B^H, and we need for its study more sophisticated methods.

Proof of Theorem 2. By 5) we have

$$\mathbf{P}\big\{L(t) \neq 0,\ \forall\, t \in \mathbf{R}^1\big\} = 1.$$

Hence as above we can define for L the winding numbers $\nu^L_{\pm}(-\infty, t]$, $\nu^L_{\pm}[t, \infty)$, and besides we have

$$\nu^L_{\pm}(-\infty, t] = \nu_{\pm}(0, e^t], \quad \nu^L_{\pm}[t, \infty) = \nu_{\pm}[e^t, \infty).$$

Therefore from now on we can work with the process L and will omit the index L in the notation of winding numbers.

Let us show that

$$\mathbf{P}\big\{|\nu_{\pm}[t,\infty)| = \infty,\ \forall\, t \in \mathbf{R}^1\big\} = 1. \tag{8}$$

By symmetry (property 3) it is sufficient to state that

$$\mathbf{P}\big\{\nu_+[t,\infty) = \infty,\ \forall\, t \in \mathbf{R}^1\big\} = 1. \tag{9}$$

Using the arguments from the proof of Theorem 1 we remark that the process L visits infinitely often each of four basic quadrants. It follows by continuity that at least one of two events A, B,

$$A = \{\exists\, t > 0, \text{ such that } \arg X(t) - \arg X(0) > \pi/2\},$$

$$B = \{\exists\, t > 0, \text{ such that } \arg X(t) - \arg X(0) < \pi/2\},$$

has probability 1. By symmetry (property 3)) $\mathbf{P}(A) = \mathbf{P}(B)$. Thus,

$$\mathbf{P}\{\exists\, t > 0, \text{ such that } \arg X(t) - \arg X(0) > \pi/2\} = 1.$$

From this follows by stationarity that for all $s \in \mathbf{R}^1$,

$$\mathbf{P}\{\exists\, t > s, \text{ such that } \arg X(t) - \arg X(s) > \pi/2\} = 1.$$

The set

$$E = \{(s,\omega) \in \mathbf{R}^1 \times \Omega \mid \exists\, t > s, \text{ such that } \arg X(t) - \arg X(s) > \pi/2\}$$

is measurable as the process $s \to \sup_{t>s}(\arg X(t) - \arg X(s))$ is continuous.

Based on the aforementioned and due to the Fubini theorem, the set E is such that $\lambda \times \mathbf{P}(E^{\complement}) = 0$, λ being the Lebesgue measure. Therefore there exists $\Omega' \subset \Omega$, $\mathbf{P}(\Omega') = 1$, such that for each $\omega \in \Omega'$, for almost all $s \in \mathbf{R}^1$, there exists $t > s$ for which $\arg X(t) - \arg X(s) > \pi/2$. Take $\omega \in \Omega'$. Let us denote E_ω the corresponding ω-section of E. Without loss of generality, we can suppose that for each $\omega \in \Omega'$, the point 0 belongs to E_ω. As $\lambda(E_\omega^{\complement}) = 0$, E_ω is dense in $\mathbf{R}^1$. Let $u > 0$ be such that $\arg X(u) - \arg X(0) > \pi/2$. By continuity, $\arg X(t) - \arg X(0) > \pi/2$ for all t in a sufficiently small neighborhood of u and, therefore, there exists $t_1 \in E_\omega$ for which $\arg X(t_1) - \arg X(0) > \pi/2$. Repeating this reasoning, we can build an increasing sequence (t_n) such that $t_1 = 0$ and $t_n \in E_\omega$. Since for each n, $\arg X(t_n) - \arg X(t_{n-1}) > \pi/2$, we get

$$\sup_{t>0}(\arg X(t) - \arg X(0)) = \infty.$$

Thus, it is proved that for each t

$$\mathbf{P}\{\nu_+[t,\infty) = \infty\} = 1. \tag{10}$$

Now to show that

$$\mathbf{P}\{\nu_+[t,\infty) = \infty,\ \forall\, t \in \mathbf{R}^1\} = 1$$

it is sufficient to remark that for each ω from Ω' the ω-section $E_\omega = \mathbf{R}^1$. Indeed, supposing that there exists $u \in E_\omega^{\complement}$ we should have

$$\arg X(s) - \arg X(u) \leqslant \pi/2$$

for each $s > t$, but that is in contradiction with the existence of $t \in E_\omega$, $t > u$, for which (10) holds. Thus (9) is proved. Applying the previous reasonings to the process $\{L(-t),\ t \in \mathbf{R}^1\}$, we prove the remaining equalities of (7). $\square$

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