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ON THE CLASSIFICATION PROBLEM OF MEASURABLE FUNCTIONS IN SEVERAL VARIABLES AND ON MATRIX DISTRIBUTIONS

Abstract. We resume the results from [12] on the classification of measurable functions in several variables, with some minor corrections of purely technical nature. We give a partial solution of he characterization problem of so–called matrix distributions, which are the metric invariants of measurable functions introduced in [12]. Matrix distibutions considered as $\S_{\mathbb{N}} \times \S_{\mathbb{N}}$–invariant, ergodic measures on the space of matrices – this fact connects our problem with Aldous' and Hoover's theorem [2, 6],

To the memory of Michael I. Gordin

§1. Introduction and outline of the paper

The classification problem of measurable functions is the question whether a measurable function

$$f : X_1 \times X_2 \times \cdots \times X_n \longrightarrow Z,$$

defined in several variables from standard probability spaces $(X_i, \mathfrak{B}_i, \mu_i)$ and values in a Borel space Z, is isomorphic to another such function h with arguments taken from other probability spaces $(Y_i, \mathfrak{C}_i, \nu_i)$. The notion of isomorphism refers to the category of measure spaces: the existence of measure-preserving, invertible Borel maps

$$T_i : (X_i, \mathfrak{B}_i, \mu_i) \longrightarrow (Y_i, \mathfrak{C}_i, \nu_i)$$

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which carry the function f to h by seperate coordinate–wise application. In terms of commutative diagrams: the diagram

$$\begin{array}{ccc} X_1 \times X_2 \times \cdots \times X_n & \xrightarrow{\quad f \quad} & \mathbb{R} \\ {\scriptstyle \text{meas.-pres.}}\Big\downarrow {\scriptstyle T=(T_1,T_2,\ldots,T_n)} & & \Big\downarrow {\scriptstyle \mathrm{id}} \\ Y_1 \times Y_2 \times \cdots \times Y_n & \xrightarrow{\quad h \quad} & \mathbb{R} \end{array}$$

commutes on a set of full measure. The classical case of functions in one argument was solved by Rokhlin [7], and is nowadays found in many modern textbooks on measure theory such as [3]: roughly speaking, two functions are isomorphic if and only if the distribution as well as the multiplicities of the attained values, described by the metric types of the conditional measures $\mu_z(\,\cdot\,) = \mu(\,\cdot\,|f = z)$, coincide (we will give the precise statement in Section 2).

When considering the isomorphism problem for functions in several arguments one obviously needs an entirely different concept. This problem was posed in full generality by the first author in [12], and a first application of the idea of matrix distributions was in the context of classifying metric triples, i.e. Polish spaces with fully supported probability measure, initiated by M. Gromov, cf. [4] and [11]. The *tensor distribution* D_f of a measurable function f (or *matrix distribution* in the case of two variables only) introduced in [12] is a probability measure on the space of infinite tensors, i.e.

$$D_f \in \mathrm{Meas}_1\left(Z^{\mathbb{N}^n}\right),$$

which arises as the distribution of the tensors

$$(r_{i_1,i_2,\ldots,i_n}) = \left(f\left(x_1^{(i_1)}, x_2^{(i_2)}, \ldots, x_n^{(i_n)}\right)\right)_{i_1,i_2,\ldots,i_n=1}^{\infty}$$

determined by the f-values when the arguments are sampled independently and at random according to the given measures μ_i. This measure is invariant and ergodic with respect to action of the product

$$(S_{\mathbb{N}})^n = S_{\mathbb{N}} \times S_{\mathbb{N}} \times \cdots \times S_{\mathbb{N}},$$

of the infinite symmetric group $S_{\mathbb{N}}$, acting independently on the indices of the tensors. It is shown in [12] that D_f is a complete metric invariant for the isomorphism problem of measurable functions, provided that the functions under consideration are *pure*, which means that they do not admit non-trivial factors in the category of measurable functions (see Section 2 for a precise statement of that property).

The characterization problem of those invariant measures

$$\lambda \in \mathrm{Meas}_1\left(\mathbb{R}^{\mathbb{N}\times\mathbb{N}}\right)$$

which are the matrix distribution of a function $f(x,y)$ in two variables is closely related to Aldous' and Hoover's representation of exchangeable distributions on infinite arrays in two dimensions [2, 6]: *any array of random variables* $(X_{i,j})$*,* $1 \leqslant i,j < \infty$*, with an* $(S_{\mathbb{N}} \times S_{\mathbb{N}})$*-invariant joint distribution can be represented as function*

$$X_{i,j} = f\left(\alpha, \zeta_i, \eta_j, \xi_{i,j}\right)$$

of underlying i.i.d. random variables α*,* ζ_i*,* η_j*, and* $\xi_{i,j}$*.* The connection of Aldous' theorem with [12] consists in the fact that some $(S_{\mathbb{N}} \times S_{\mathbb{N}})$-invariant ergodic measures correspond to the matrix distribution of a measurable function, which is up to isomorphy unique – a property that doesn't follow from the approach in [2, 6]. Recently, the first author proved the same answer for arbitrary invariant ergodic measures which covers Aldous' representation in full generality [8], but we shall not touch this topic here.

In the present work we concetrate on the above mentioned partial solution to the characterization problem of matrix distributions, correcting the corresponding statement Theorem 3 in [12]: an $(S_N)^n$-invariant ergodic measure λ is the matrix distribution of a *completely pure* function, that is a pure function with trivial *congruence group*

$$K_f = \left\{ (T_i)_{i=1}^n \in \prod_{i=1}^n \mathrm{Aut}_0(X_i, \mu_i) : \right.$$
$$\left. f\left(T_1(x_1), \ldots, T_n(x_n)\right) = f(x_1, \ldots, x_n) \quad \text{a.e.} \right\},$$

if and only if it is a *simple measure*, i.e. the ergodic components of the seperate actions of $S_{\mathbb{N}}$ generate the entire sigma algebra in the space of tensors.

The paper is organized as follows: In Section 2 we recall important definitions and facts from [12] and restate Rokhlin's classification theorem for univalent functions. Section 3 revises basic properties of pure functions from [12] including self-contained proofs. We chose to repeat these elementary facts, as they are needed in Section 4, in which we present a slightly modified proof of the completeness theorem Theorem 2 from [12]. Finally, Section 5 elaborates the above mentioned partial characterization of matrix distributions via the so-called *general canonical model* for a measurable function. As the case of functions in more than two variables bears no additional obstacles from the conceptual point of view, we shall restrict ourselves throughout Sections 2–5 to the case of two variables

$$n = 2$$

only. The general case, which is then obtained by a straight-forward generalization of our methods, is briefly discussed in Section 6.

The present paper is a revised version of a chapter taken from the thesis [5], which originated from a discussion on the classification of matrix distributions during the course *Measure theoretic constructions and their applications in ergodic theory, asymptotics, combinatorics, and geometry* given by the first author in autumn 2002 at the Erwin Schrödinger Institute, Vienna.

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§2. Basic definitions and facts

Throughout the following we consider all spaces to be standard probability spaces, i.e. standard Borel spaces $(X, \mathfrak{B})$ equipped with a Borel probability measure μ. To avoid cumbersome notation we shall write (X, μ) (or just X if it is clear to what measure on X we refer) instead of $(X, \mathfrak{B}, \mu)$, whenever it is convient. All functions are considered to be measurable unless the contrary is explicitely stated.

2.1. Isomorphy, factors, and pure functions. We call two measurable functions $f : (X, \mu) \times (Y, \nu) \longrightarrow \mathbb{R}$ and $f' : (X', \mu') \times (Y', \nu') \longrightarrow \mathbb{R}$ to be *isomorphic* if we can find measure preserving isomorphisms $S : X \longrightarrow X'$ and $T : Y \longrightarrow Y'$ such that

$$f'\big(S(x), T(y)\big) = f(x, y) \quad \text{a.e.},$$

where 'a.e.' refers to the product measure $\mu \times \nu$. Whenever the transformations S and T are measure preserving projections (and not necessarily invertible mod 0), i.e. they map onto a set of full measure, we say that f' is a *factor* of the function f.

Notice that isomorphy as well as being a factor is a notion on the equivalence classes (mod 0) of functions. In terms of commutative diagrams, if f' is a factor of f, then the diagram

$$\begin{array}{ccc} X\times Y & \xrightarrow{\ f\ } & Z \\ {\scriptstyle \text{meas.pres. projections}}\big\downarrow S \quad \big\downarrow T & & \big\downarrow \mathrm{id} \\ X'\times Y' & \xrightarrow{\ f'\ } & Z \end{array}$$

commutes.

Definition 2.1. *A measurable function $f : (X,\mu)\times(Y,\nu) \longrightarrow Z$ is pure if it admits no true factor, by which we mean that every factor $f' : (X',\mu')\times(Y',\nu') \longrightarrow Z$ of f is already isomorphic to f.*

We denote by $\mathfrak{B}(X,Z)$ (or $\mathfrak{B}(Y,Z)$) the space of all equivalence classes mod 0 of measurable functions from X (or Y, respectively) into the standard Borel space Z, endowed with topology of convergence in measure with respect to any Polish topology generating the Borel structure of Z. Since f is measurable so are the mappings

$$f_X : X \longrightarrow \mathfrak{B}(Y,Z), \quad x \mapsto [f(x,\,\cdot\,)]_\nu$$

and

$$f_Y : Y \longrightarrow \mathfrak{B}(X,Z), \quad y \mapsto [f(\,\cdot\,,y)]_\mu\,,$$

where $[\,\cdot\,]_\nu$ and $[\,\cdot\,]_\mu$ denote the corresponding equivalence class. For brevity, we will omit the brackets in the sequel.

It is evident from Definition 2.1 that pureness of a function can be rephrased as follows.

Lemma 2.2. *A function $f : X\times Y \longrightarrow Z$ is pure if and only if both mappings $f_X : X \longrightarrow \mathfrak{B}(Y,Z)$, $x \mapsto f(x,\cdot)$, and $f_Y : Y \longrightarrow \mathfrak{B}(X,Z)$, $y \mapsto f(\,\cdot\,,y)$ are one-to-one on a set of full measure.*

2.2. Rokhlin's Theorem. It is evident that two univalent measurable functions $f : (X,\mu) \longrightarrow Z$ and $f' : (X',\mu') \longrightarrow Z$, i.e. one–to–one on a set of full measure, are isomorphic if and only if their distributions $D_f = \mu\circ f^{-1}$ and $D_{f'} = \mu'\circ f'^{-1}$ coincide. If the functions under consideration are not univalent one has to take in account the 'multiplicity' certain values are obtained. This is done by looking at the function

$$m_f : Z \longrightarrow \Sigma = \Big\{(c_i)_{i\geqslant 1} : 0 \leqslant c_i \leqslant 1, \sum_{i=1}^{\infty} c_i \leqslant 1\Big\}$$

which maps any value z to the *metric type* $m_f(z)$ of the conditional probability distribution

$$\mu_z(\,\cdot\,) = \mu(\,\cdot\,|f = z),$$

which is the sequence of weights $\big\{c_i = \mu_z(\{a_i\})\big\}_{i\geqslant 1}$ of the atoms a_i of the measure μ_z arranged in a non-increasing way. Note that since the conditional probability distributions are defined uniquely (mod 0) so is the function m_f.

Theorem 2.3 (Rokhlin, [7]). *Assume that $f_i : X_i \longrightarrow Z$ $(i = 1, 2)$ are two measurable functions defined on standard probability spaces (X_i, μ_i). Then there exists an isomorphism $T : X_1 \longrightarrow X_2$ of the measure spaces with the property that $\mu_1 \circ T^{-1} = \mu_2$ and $f_2 \circ T(x) = f_1(x)$ almost everywhere if and only if their extended functions*

$$f_i^e : X_i \longrightarrow Z \times \Sigma, \quad x \mapsto \big(f_i(x), m_{f_i} \circ f_i(x)\big)$$

have the same distribution, by which we mean that the measures $D_1 = \mu_1 \circ (f_1^e)^{-1}$ and $D_2 = \mu_2 \circ (f_2^e)^{-1}$ coincide.

2.3. Group actions and ergodic decompositions. The product $S_{\mathbb{N}} \times S_{\mathbb{N}}$ of the infinite symmetric group

$$S_{\mathbb{N}} = \bigcup_{n=1}^{\infty} S_{\{1,\dots,n\}}$$

of all finite permutations of $\mathbb{N}$ acts on the product space

$$(X \times Y)^{\mathbb{N}} = X^{\mathbb{N}} \times Y^{\mathbb{N}}$$

in the canonical way by acting independently on the indices of the sequences, i.e.

$$g \cdot \Big((x_i)_{i=1}^{\infty}, (y_j)_{j=1}^{\infty}\Big) = \Big(\Big(x_{g_1^{-1}(i)}\Big)_{i=1}^{\infty}, \Big(y_{g_2^{-1}(j)}\Big)_{j=1}^{\infty}\Big)$$

for every $((x_i), (y_i))$ from $X^{\mathbb{N}} \times Y^{\mathbb{N}}$ and $g = (g_1, g_2)$ from $S_{\mathbb{N}} \times S_{\mathbb{N}}$. Its action on the space of infinite matrices is that of permuting rows and columns separately:

$$g \cdot (r_{i,j}) = \Big(r_{g_1^{-1}(i), g_2^{-1}(j)}\Big)_{i,j}$$

for every $g = (g_1, g_2)$ from $S_{\mathbb{N}} \times S_{\mathbb{N}}$ and matrix $(r_{i,j})$.

Let G be a countable semigroup acting measurably on a standard Borel space, which will be in our case the space of matrices $Z^{\mathbb{N}\times\mathbb{N}}$. Every G-invariant probability measure D on $Z^{\mathbb{N}\times\mathbb{N}}$ is then decomposed into G-invariant ergodic measures D_r by a (mod 0) uniquely defined Borel mapping

$$\pi : Z^{\mathbb{N}\times\mathbb{N}} \longrightarrow \mathrm{Meas}_1^G\left(Z^{\mathbb{N}\times\mathbb{N}}\right), \quad r \mapsto D_r^G,$$

which maps into the standard Borel space of G-invariant probability measures, endowed with the topology of weak convergence, such that the formula

$$D(B) = \int\limits_{Z^{\mathbb{N}\times\mathbb{N}}} D_r^G(B) \cdot dD(r)$$

holds for every Borel set $B \subseteq Z^{\mathbb{N}\times\mathbb{N}}$.

2.4. De Finetti's theorem. We shall make use of de Finett's theorem on exchangeable distributions in the following form: *Every Borel probability measure m on the product $X^{\mathbb{N}}$ of a Borel space X, which is invariant and ergodic with respect to the action $S_{\mathbb{N}}$ is Bernoulli, i.e.*

$$m = \mu^{\mathbb{N}}$$

for a Borel probability measure μ on X. There is a very simple and elegant proof of de Finetti's theorem with help of the point–wise ergodic theorem with respect to the countable group $S_{\mathbb{N}}$, but we shall not dwell on this, cf. [14].

2.5. Matrix distribution of a measurable function. Let us recall the definition of matrix distributions from [12]:

Definition 2.4. *Let $f : (X, \mu) \times (Y, \nu) \longrightarrow Z$ be a measurable function in two variables. Its matrix distribution D_f is the pushforward measure*

$$D_f = (\mu^{\mathbb{N}} \times \nu^{\mathbb{N}}) \circ F_f^{-1}$$

of the Bernoulli measure $\mu^{\mathbb{N}} \times \nu^{\mathbb{N}}$ under the evaluation function

$$F_f : X^{\mathbb{N}} \times Y^{\mathbb{N}} \longrightarrow Z^{\mathbb{N}\times\mathbb{N}}, \quad \big((x_i), (y_j)\big) \mapsto \big(f(x_i, y_j)\big)_{i,j}.$$

This definition generalizes the notion of matrix distributions for Gromov triples, i.e. Polish spaces with probability measure introduced in [13]. Note that D_f is a probability measure on the space of infinite matrices $Z^{\mathbb{N}\times\mathbb{N}}$, and that F_f is equivariant with respect to the action of $S_{\mathbb{N}} \times S_{\mathbb{N}}$, i.e.

$$F_f\big(g \cdot \big((x_i)\,,(y_j)\,\big)\big) = g \cdot F_f\big(\,(x_i)\,,(y_j)\,\big).$$

Hence invariance and ergodicity of the Bernoulli measure $\mu^{\mathbb{N}} \times \nu^{\mathbb{N}}$ yields invariance and ergodicity of the matrix distribution.

2.6. The congruence group of a measurable function. The *congruence group* of a function in two arguments is the group of measure–preserving symmetries

$$K_f = \Big\{(S,T) \in \mathrm{Aut}_0\,(X,\mu) \times \mathrm{Aut}_0\,(Y,\nu) : \\ f\big(S(x),T(y)\big) = f(x,y) \quad \text{a.e.}\Big\}.$$

This group plays an important role for our partial solution to the characterization problem of matrix distributions, Theorem 5.4. It is remarkable that the congruence group is compact, when endowed with the weak topology. This fact is shown in [10], but we will not make use of it in the sequel.

As in [9] we shall call any pure function f with trivial congruence group simply *completely pure function.*

§3. Reduction of the classification problem to that of pure functions

Any measurable function f has a pure factor $\bar{f}$ defined in a natural way: As the functions f_X and f_Y are Borel the equivalence relations defined by

$$\zeta_X = \big\{\,(x_1,x_2) \in X \times X : f(x_1,\,\cdot\,) = f(x_2,\,\cdot\,)\,(\mathrm{mod}\,\nu)\big\}$$

and

$$\zeta_Y = \big\{\,(y_1,y_2) \in Y \times Y : f(\,\cdot\,,y_1) = f(\,\cdot\,,y_2)\,(\mathrm{mod}\,\mu)\big\}$$

are partitions of the respective spaces into measurable components. In order to stay within the category of standard measure spaces we define[1] the factor spaces

$$X/\zeta_X \quad \text{and} \quad Y/\zeta_Y$$

[1] It is no good choice to define the factor X/ζ_X to be the set of all equivalence classes with respect to ζ_X and its sigma algebra the algebra of all ζ_X-saturated $\mathfrak{S}$-measurable sets; this space need not to be standard Borel.

as the standard Borel space $\mathfrak{B}(Y,Z)$ and $\mathfrak{B}(X,Z)$ with the factor mappings

$$\pi_X : X \longrightarrow X/\zeta_X, \quad \pi_X = f_X,$$

and

$$\pi_Y : Y \longrightarrow Y/\zeta_Y, \quad \pi_Y = f_Y,$$

and the projected measures $\mu \circ \pi_X^{-1}$ and $\nu \circ \pi_Y^{-1}$ as measures, respectively. The mapping f_X as function from X into the space $\mathfrak{B}(Y,Z)$ translates under the projection π_X to the identity mapping regarded as function from X/ζ_X to $\mathfrak{B}(Y,Z)$, and by the measurability it originates from a jointly measurable function

$$f' : X/\zeta_X \times Y \longrightarrow Z$$

which by construction satisfies that

$$f'\big(\pi_X(x), y\big) = f(x,y) \quad \text{a.e.},$$

and the corresponding map f'_{X/ζ_X} is one-to-one on a set of full measure. In the same manner we proceed with the second argument and finally end with a measurable function

$$\bar{f} : X/\zeta_X \times Y/\zeta_Y \longrightarrow Z$$

such that

$$\bar{f}\big(\pi_X(x), \pi_Y(y)\big) = f(x,y) \quad \text{a.e.}.$$

This shows that the diagram

$$\begin{array}{ccc} X \times Y & \xrightarrow{\quad f \quad} & Z \\ \text{meas.-pres.}\Big\downarrow{\scriptstyle \pi=(\pi_X,\pi_Y)} & & \Big\downarrow{\scriptstyle \mathrm{id}} \\ X/\zeta_X \times Y/\zeta_Y & \xrightarrow{\quad f' \quad} & Z \end{array}$$

is commutative on a set of full measure, and since the corresponding mappings $\bar{f}_{X/\zeta_X}$ and $\bar{f}_{Y/\zeta_Y}$ are now one–to–one on a set of full measure, the function $\bar{f}$ is consequently pure. It is not difficult to see that this pure factor $\bar{f}$ is uniquely determined up to isomorphy. We will sometimes refer to it as the *purification* or the *unique pure factor* of f.

The reduction of the isomorphism problem is done in the spirit of Rokhlin's Theorem 2.3. Instead of f we consider its extended pure factor

$$\bar{f}^e : X/\zeta_X \times Y/\zeta_Y \longrightarrow Z \times \Delta^2, \quad (x,y) \mapsto \big(\bar{f}(x,y), m(\mu_x), m(\nu_y)\big)\,,$$

where $m(\mu_x)$ and $m(\nu_y)$ denote the metric types of the respective (mod 0 uniquely determined) conditional measures $\mu(\,\cdot\,|\pi_X = x)$ and $\nu(\,\cdot\,|\pi_Y = y)$.

Theorem 3.1. *Two not necessarily pure functions $f : (X, \mu) \times (Y, \nu) \longrightarrow Z$ and $g : (X', \nu') \times (Y', \nu') \longrightarrow Z$ are isomorphic if and only if their extended pure factors*

$$\bar{f}^e : X/\zeta_X \times Y/\zeta_Y \longrightarrow Z \times \Delta^2, \quad (x, y) \mapsto \big(\bar{f}(x, y), m(\mu_x), m(\nu_y)\big),$$

and

$$\bar{g}^e : X'/\zeta_{X'} \times Y'/\zeta_{Y'} \longrightarrow Z \times \Delta^2, \quad (x', y') \mapsto \big(\bar{g}(x', y'), m(\mu'_{x'}), m(\nu'_{y'})\big),$$

both defined as above, are isomorphic.

Proof. The statement of the theorem is clear since every measure preserving isomorphism $\bar{T} : X/\zeta_X \longrightarrow X'/\zeta'_X$ which carries the function $x \mapsto m(\mu_x)$ to the function $x' \mapsto m(\mu'_{x'})$ (mod 0) can be lifted to a measure preserving isomorphism $T : X \longrightarrow X'$ so that $T = \bar{T} \circ \pi_X$ (mod 0) and the same is true for the second coordinate spaces Y and Y'. □

§4. The matrix distribution as complete invariant for pure functions

In this section we resume the individual canoncial model and the completeness theorem for measurable functions in two variables. Are results and proofs, apart from some technical details which are elaborated in greater detail, can be also found in [12].

Let us start with an auxiliary lemma on pure functions. We say that a sigma algebra $\mathfrak{S} \subseteq \mathfrak{B}_X$ equals modulo a measure μ (or simply 'mod 0', whenever it is clear to which measure we refer) the whole Borel algebra $\mathfrak{B}_X$, if its measure algebra $\mathfrak{S}_\mu = \{[A]_\mu : A \in \mathfrak{S}\}$, with $[A]_\mu = \{B \in \mathfrak{B}_X : \mu(B \Delta A) = 0\}$, coincides with the measure algebra defined by $\mathfrak{B}_X$ itself.

Lemma 4.1. *Suppose $f : X \times Y \to Z$ is a pure function. Then the following properties also hold:*

(i) *For any Borel set $Y' \subseteq Y$ which is of full ν-measure the set of functions $\{f(\,\cdot\,, y) : y \in Y'\}$ generates* (mod 0) *the Borel algebra of X.*
(ii) *For $\nu^{\mathbb{N}}$-almost every sequence $(y_j)_{j=1}^{\infty}$ the countable collection of functions $\{f(\,\cdot\,, y_j) : j \geqslant 1\}$ generates* (mod 0) *the Borel algebra of X.*

By symmetry the same statements hold when interchanging the role of Y *and* X.

Proof. As the restriction of f to $X \times Y'$ is also a pure function, it is sufficient to prove (i) for the case $Y' = Y$. Let $\mathfrak{F}_Y$ denote the sigma algebra generated by the set of functions $\{f_y = f(\,\cdot\,, y) : y \in Y\}$. Using standard arguments one sees that the function f is measurable with respect to the sigma algebra $\mathfrak{F}_Y \times \mathfrak{B}_Y$, where $\mathfrak{B}_Y$ is the Borel algebra of Y, after a modification on a set of measure zero if necessary. Thus the mapping

$$f_X : X \longrightarrow \mathfrak{B}(Y, Z), \quad x \mapsto f(x, \,\cdot\,),$$

is measurable $(\mathrm{mod}\, 0)$ with respect to the sigma algebra $\mathfrak{F}_Y$.[2] Hence injectivity of the map f_X on a set of full measure implies that $\mathfrak{F}_Y$ coincides modulo null sets with the entire Borel algebra of X.

To prove (ii) let us choose a countable base $\{O_n\}_{n \geqslant 1}$ for the topology in $\mathfrak{B}(Y, Z)$. Now $\nu^{\mathbb{N}}$-almost every sequence (y_j) is such that for any set O_n of positive measure $\mu \circ f_X^{-1}$, its intersection with $\{f_{y_j} : j \geqslant 1\}$ is non-empty. For every such sequence (y_j) the sigma algebra $\mathfrak{F}_{(y_j)}$ generated by the functions $\{f_{y_j} : j \geqslant 1\}$ contains $(\mathrm{mod}\, 0)$ the sigma algebra generated by the collection $\{f_y : y \in Y'\}$ with Y' being the preimage of the support of the measure $\nu \circ f_Y^{-1}$ under the map f_Y. As this set has full measure it follows from (i) that $\mathfrak{F}_{(y_j)}$ coincides $(\mathrm{mod}\, 0)$ with the entire Borel algebra of X. □

4.1. Individual canonical model of a measurable function. In the sequel we assume that that $f : X \times Y \longrightarrow Z$ takes values in the interval $Z = [0, 1]$. This means no loss in generality, as any Borel space is measurably isomorphic to $[0, 1]$ or to an – at most – countable subset. It is an immediate consequence of Lemma 4.1 that for $\mu^{\mathbb{N}}$-almost every sequence (x_i) and $\nu^{\mathbb{N}}$-almost every sequence (y_j) the mappings

$$\begin{aligned} L_{(y_j)} &: X \longrightarrow [0,1]^{\mathbb{N}}, \quad x \mapsto \big(f(x, y_j)\big)_j, \\ L_{(x_i)} &: Y \longrightarrow [0,1]^{\mathbb{N}}, \quad y \mapsto \big(f(x_i, y)\big)_i, \end{aligned}$$

[2]By which we mean that it coincides on a set of full measure with an $\mathfrak{F}_Y$-measurable function.

are one-to-one on a set of full measure, and therefore they are isomorphisms between the measure spaces (X,μ), (Y,ν) and the spaces

$$\big(X_{(y_j)},\mu_{(y_j)}\big) = \Big([0,1]^{\mathbb{N}},\ \mu\circ L^{-1}_{(y_j)}\Big)\,,$$
$$\big(Y_{(x_i)},\nu_{(x_i)}\big) = \Big([0,1]^{\mathbb{N}},\ \nu\circ L^{-1}_{(x_i)}\Big)\,,$$

respectively. These spaces together with the function

$$f_{(x_i),(y_j)} = f\circ\big(L_{(y_j)}\times L_{(x_i)}\big)^{-1}$$

form the *canonical representation* (or *canonical model*) of our measurable function f. We shall call this model *individual canonical model*, as both measures $\mu_{(y_j)}$, $\nu_{(x_j)}$ and the function $f_{(x_i),(y_j)}$, being the density of the absolutely continuous measure

$$m_{(y_j),(x_i)} = f_{(x_i),(y_j)}\cdot d(\mu_{(y_j)}\times\nu_{(x_i)}) = m\circ(L_{(y_j)}\times L_{(x_i)})^{-1},$$

are uniquely determined by the values of the single infinite matrix

$$r = (r_{i,j}) = \big(f(x_i,y_j)\big)\in Z^{\mathbb{N}\times\mathbb{N}}.$$

The construction of the model is done by ergodic arguments with respect to the two-dimensional shift on $X^{\mathbb{N}}\times Y^{\mathbb{N}}$ defined by

$$\sigma^{(k,l)}\big((x_i),(y_j)\big) = \big((x_{i+k}),(y_{j+l})\big),$$

for every $k,l\geqslant 1$ and $\big((x_i),(y_j)\big)$ from $X^{\mathbb{N}}\times Y^{\mathbb{N}}$, but can as well be performed with respect to the action of $S_{\mathbb{N}}\times S_{\mathbb{N}}$. For every $n\geqslant 1$ let us choose a countable algebra $\mathfrak{A}_n$ which generates the sigma algebra of all n–cylinders, i.e. the Borel sets formulated in the first n coordinates only. Using the ergodic theorem for the two–dimensional shift, we conclude that almost every choice of sequences $(x_i)_{i\in\mathbb{N}}$ and $(y_j)_{j\in\mathbb{N}}$ the following relations hold for every A_1 and A_2 from $\mathfrak{A}=\bigcup_n\mathfrak{A}_n$:

$$\mu_{(y_j)}(A_1) = \lim_{m\to\infty}\frac{1}{m}\cdot\sum_{k=1}^{m}\delta_{(r_{k,i})_{i=1}^{\infty}}(A_1), \tag{1}$$
$$\nu_{(x_j)}(A_2) = \lim_{m\to\infty}\frac{1}{m}\cdot\sum_{l=1}^{m}\delta_{(r_{i,l})_{i=1}^{\infty}}(A_2), \tag{2}$$

and

$$m_{(x_i),(y_j)}(A_1 \times A_2) = \lim_{m\to\infty} \frac{1}{m^2} \cdot \sum_{k,l=1}^{m} r_{k,l} \cdot \delta_{(r_{i,k})_{i=1}^{\infty}}(A_1) \cdot \delta_{(r_{l,j})_{j=1}^{\infty}}(A_2). \quad (3)$$

In fact, given finitely many points $\{x_i : 1 \leqslant i \leqslant n\}$ from X and $\{y_j : 1 \leqslant j \leqslant n\}$ from Y, and introducing the notation

$$f_{(x_1,\ldots,x_n)}(y) = \big(f(x_1,y),\ldots,f(x_n,y)\big),$$

and

$$f_{(y_1,\ldots,y_n)}(x) = \big(f(x,y_1),\ldots,f(x,y_n)\big),$$

then almost every continuations $(x_i)_{i=n+1}^{\infty}$ and $(y_j)_{j=n+1}^{\infty}$ of the finite sequences $(x_i)_{i=1}^{n}$ and $(y_j)_{j=1}^{n}$ are such that for any choice of sets A_1 and A_2 from $\mathfrak{A}_n$,

$$\mu \circ f^{-1}_{(y_1,\ldots y_n)}(A_1) = \lim_{m\to\infty} \frac{1}{m} \cdot \Big| \big\{ n+1 \leqslant k \leqslant m : f_{(y_1,\ldots,y_n)}(x_k) \in A_1 \big\} \Big|,$$

$$\nu \circ f^{-1}_{(x_1,\ldots x_n)}(A_2) = \lim_{m\to\infty} \frac{1}{m} \cdot \Big| \big\{ n+1 \leqslant k \leqslant m : f_{(x_1,\ldots,x_n)}(y_k) \in A_2 \big\} \Big|,$$

and

$$m\big(f^{-1}_{(y_1,\ldots y_n)}(A_1) \times f^{-1}_{(x_1,\ldots,x_n)}(A_2)\big) = \lim_{m\to\infty} \frac{1}{m^2} \cdot \sum_{(k,l)\in W\cap[n+1,m]^2} f(x_k,y_l).$$

Here W denotes the set

$$W = \{(k,l) \in \mathbb{N}\times\mathbb{N} : f_{(y_1,\ldots,y_n)}(x_k) \in A_1 \text{ and } f_{(x_1,\ldots,x_n)}(y_l) \in A_2\}.$$

Hence the concatenated sequences $(x_i)_{i=1}^{\infty}$ and $(y_j)_{j=1}^{\infty}$ obviously satisfy the equations (1), (2) and (3). Integrating over all choices of $((x_i)_{i=1}^{n}, (y_j)_{j=1}^{n})$ from $X^n \times Y^n$ yields the sets

$$\mathcal{E}_f(\mathfrak{A}_n) = \Big\{ \big((x_i),(y_j)\big) \in X^{\mathbb{N}} \times Y^{\mathbb{N}} : L_{(y_j)} \text{ and } L_{(x_i)} \text{ are one-to-one } (\mathrm{mod}\, 0) \text{ and (1), (2), (3) hold for all } A_1, A_2 \in \mathfrak{A}_n \Big\}$$

and hence their intersection

$$\mathcal{E}_f(\mathfrak{A}) = \bigcap_{n\geqslant 1} \mathcal{E}_f(\mathfrak{A}_n)$$

is of full measure.

We shall call any $\big((x_i),(y_j)\big)$ from the set $\mathcal{E}_f(\mathfrak{A})$ a pair of *typical sequences*. It is well known that the image $F_f\big(\mathcal{E}_f(\mathfrak{A})\big)$ is then measurable with respect to the D_f-completion of the Borel algebra in the space $Z^{\mathbb{N}\times\mathbb{N}}$ thus we can find a Borel set

$$\mathcal{T}_f(\mathfrak{A}) \subseteq F_f\big(\mathcal{E}_f(\mathfrak{A})\big)$$

which is of full measure. We shall call this set $\mathcal{T}_f(\mathfrak{A})$ the set of *typical matrices* with respect to the countable algebra $\mathfrak{A}$ and the function f.

Summarizing these facts we have shown that for any choice of

$$\big((x_i),(y_j)\big)$$

from the set $\mathcal{E}_f(\mathfrak{A})$ the canonical model depends only on their matrix $(f(x_i,y_j))_{i,j}$. Likewise, for any matrix

$$r = (r_{i,j})_{i,j\in\mathbb{N}}$$

from the set of typical matrices $\mathcal{T}_f(\mathfrak{A})$ we can construct spaces

$$(X_r,\mu_r) = \big([0,1]^{\mathbb{N}},\mu_r\big) \tag{4}$$

and

$$(Y_r,\nu_r) = \big([0,1]^{\mathbb{N}},\nu_r\big), \tag{5}$$

the measures μ_r and ν_r determined by the limits (1) and (2), and a measurable function

$$f_r : X_r\times Y_r \longrightarrow [0,1], \quad f_r = \frac{dm_r}{d(\mu_r\times\nu_r)}, \tag{6}$$

i.e. the Radon–Nikodým derivative of the measure m_r determined by the limit in (3) with respect to $\mu_r\times\nu_r$, which is isomorphic to the original function f. Note that μ_r and ν_r are the empirical distributions of the rows and columns of the matrix r, and $m_r(A_1\times A_2)$ for two n-dimensional Borel sets A_1 and A_2 equals the average value $r_{k,l}$ which occurs when observing simultaneously the row segment $(r_{k,1},\dots,r_{k,n})$ belonging to A_1 and the column segment $(r_{1,l},\dots,r_{n,l})$ belonging to A_2.

In particular, the so constructed function f_r is unique: Any matrix r from the set $\mathcal{T}_f(\mathfrak{A})$ determines – up to isomorphism – the same function. For general ergodic $(S_{\mathbb{N}}\times S_{\mathbb{N}})$-invariant measures D this construction might fail: Of course the existence of the measures μ_r and ν_r follows from stationarity of the two-dimensional shift action, or of the separate actions of $S_{\mathbb{N}}\times\{1\}$ and $\{1\}\times S_{\mathbb{N}}$ (cf. Section 5), but it is by no means clear that the limit in (3) defines a measure m_r, nor that the so constructed function f_r is unique up to isomorphism, contrary to what was stated in [12].

With the above reconstruction of the canonical model we are able to show the main result of this section:

Theorem 4.2 (Completeness Theorem)**.** *Suppose $f : X \times Y \longrightarrow Z$ and $g : X' \times Y' \longrightarrow Z$ are pure functions. There exist measure preserving isomorphisms $S : (X, \mu) \longrightarrow (X', \mu')$ and $T : (Y, \nu) \longrightarrow (Y', \nu')$ of the corresponding standard probability spaces such that*

$$g\big(S(x), T(y)\big) = f(x, y) \pmod 0,$$

if and only if their matrix distributions D_f and D_g coincide.

Proof. One direction is trivial: If there exist such isomorphisms S and T then the corresponding product transformation which maps $\big((x_i)_i, (y_j)_j\big)$ to $\big((S(x_i))_i, (T(y_j))_j\big)$ is a measure preserving isomorphism of the spaces $X^{\mathbb{N}} \times Y^{\mathbb{N}}$ and $X'^{\mathbb{N}} \times Y'^{\mathbb{N}}$ which carries the matrix valued functions F_f to F_g. Hence their push forward measures D_f and D_g coincide.

Now assume that $D_f = D_g$. As in the preceding discussion we consider $Z = [0, 1]$, fix countable algebras $\mathfrak{A}_n$, $n \geqslant 1$, which generate the Borel structure of $[0, 1]^n$, and put $\mathfrak{A} = \bigcup_n \mathfrak{A}_n$. As $D_f = D_g$ the intersection of the sets of typical matrices

$$\mathcal{T}_f(\mathfrak{A}) \cap \mathcal{T}_g(\mathfrak{A})$$

is still of full measure and therfore it is non-empty. But any matrix $r = (r_{i,j})$ from this intersection determines a function

$$f_r : (X_r, \mu_r) \times (Y_r, \nu_r) \longrightarrow [0, 1]$$

which is an isomorphic model *simultaneously* for both functions f and g. This proves the existence of the claimed isomorphisms S and T. □

Note that an explicit form of the isomorphisms from Theorem 4.2 is

$$S = L^{-1}_{(y'_j)} \circ L_{(y_j)} \text{ and } T = L^{-1}_{(x'_i)} \circ L_{(x_i)},$$

the pairs $\big((x_i), (y_j)\big)$ and $\big((x'_i), (y'_j)\big)$ being any two pairs of typical sequences, i.e. from

$$\mathcal{E}_f(\mathfrak{A}) \cap \mathcal{E}_g(\mathfrak{A}),$$

which define the same matrix: $\big(f(x_i, y_j)\big) = \big(g(x'_i, y'_j)\big)$. This observation will be useful in the proof of the following corollary, which will be needed in Section 5.

Corollary 4.3. *Let $f : X \times Y \longrightarrow Z$ be pure. Then the map $F_f : X^{\mathbb{N}} \times Y^{\mathbb{N}} \longrightarrow Z$ is one-to-one* (mod 0) *if and only if its congruence group K_f is trivial.*

Proof. As before we assume that $Z = [0,1]$. Assume that F_f is not one–to–one (mod 0). If $\mathfrak{A}$ is any arbitrary countable algebra generating the Borel structure of $[0,1]^{\mathbb{N}}$ then the map F_f cannot be injective on $\mathcal{E}_f(\mathfrak{A})$, since this set is of full measure. Thus there exist two different pairs $\big((x_i),(y_j)\big)$ and $\big((x_i'),(y_j')\big)$ from $\mathcal{E}(\mathfrak{A})$ which have the same image under F_f. Therefore the mappings

$$S = L^{-1}_{(y_j')} \circ L_{(y_j)} \text{ and } T = L^{-1}_{(x_i')} \circ L_{(x_i)},$$

are measure preserving automorphisms of (X,μ) and (Y,ν) respectively for which $f(Sx,Ty) = f(x,y)$ (mod 0). Moreover these automorphisms satisfy $f(Sx,y_j) = f(x,y_j')$ for a.e. x in X and $f(x_i,Ty) = f(x_i',y)$ for a.e. y in Y, i and j being arbitrary. This proves that either S or T must be non-trivial since we may assume that both mappings f_X and f_Y (defined in Section 2) are one-to-one.

The other direction is trivial: Assume that there exists a non–trivial automorphism (S,T) in K_f. Then every set $B \subseteq X^{\mathbb{N}} \times Y^{\mathbb{N}}$ of full measure is almost invariant under $(S,T)^{\mathbb{N}}$. But at the same time the automorphism $(S,T)^{\mathbb{N}}$ leaves F_f almost invariant and therefore we can find two different points in B with the same image under F_f. □

§5. Absence of symmetry - simple measures

In this section we characterize those matrix distributions which originate from a function with no symmetries, i.e. with trivial congruence group. Its main result, Theorem 5.4 corrects the corresponding statement Theorem 3 in [12].

Definition 5.1. *Let D be a $(S_{\mathbb{N}} \times S_{\mathbb{N}})$-invariant and ergodic measure on the space of matrices. We say that D is simple if the invariant algebras $\mathfrak{B}^{S_{\mathbb{N}}\times\{1\}}$ and $\mathfrak{B}^{\{1\}\times S_{\mathbb{N}}}$ generate* (mod 0) *the whole Borel algebra $\mathfrak{B}$ of $Z^{\mathbb{N}\times\mathbb{N}}$, i.e.*

$$\mathfrak{B}^{S_{\mathbb{N}}\times\{1\}} \vee \mathfrak{B}^{\{1\}\times S_{\mathbb{N}}} = \mathfrak{B} \pmod{D}.$$

5.1. Simplicity and decomposition of the action of $S_{\mathbb{N}} \times S_{\mathbb{N}}$. Simplicity of an $(S_{\mathbb{N}} \times S_{\mathbb{N}})$-invariant and ergodic measure D means that the dynamical system

$$\big(Z^{\mathbb{N}\times\mathbb{N}}, D, S_{\mathbb{N}} \times S_{\mathbb{N}}\big)$$

is isomorphic to the direct product of its factors

$$(\mathfrak{X}, \mu_{\mathfrak{X}}, G_1) = \left(Z^{\mathbb{N}\times\mathbb{N}}/G_2, D, G_1\right),$$

and

$$(\mathfrak{Y}, \mu_{\mathfrak{Y}}, G_2) = \left(Z^{\mathbb{N}\times\mathbb{N}}/G_1, D, G_2\right),$$

i.e. the systems of the ergodic components with respect to the subgroups

$$G_2 = \{1\} \times S_{\mathbb{N}} \quad \text{and} \quad G_1 = S_{\mathbb{N}} \times \{1\}.$$

As model of these factors we choose the standard Borel spaces

$$\mathfrak{X} = \mathrm{Meas}_1^{G_2}\left(Z^{\mathbb{N}\times\mathbb{N}}\right),$$
$$\mathfrak{Y} = \mathrm{Meas}_1^{G_1}\left(Z^{\mathbb{N}\times\mathbb{N}}\right),$$

of respective invariant probability measures, together with the isomorphism

$$\pi : Z^{\mathbb{N}\times\mathbb{N}} \longrightarrow \mathfrak{X} \times \mathfrak{Y}, \quad r \mapsto \left(D_r^{G_2}, D_r^{G_1}\right),$$

given by the decomposition of D into its G_2-ergodic and G_1-ergodic measures, respectively. We assume without any loss in generality the mappings $\pi_{:} r \mapsto D_r^{G_2}$ and $\pi_2 = r \mapsto D_r^{G_1}$ invariant with respect to G_2 and G_1 respectively, whence the equations

$$g_1 \cdot D_r^{G_2} = D_{g_1 \cdot r}^{G_2},$$
$$g_2 \cdot D_r^{G_1} = D_{g_1 \cdot r}^{G_1},$$

with $g_1 \in G_1$ and $g_2 \in G_2$, define actions of the permutation groups G_1 and G_2 on $\mathfrak{X}$ and $\mathfrak{Y}$, respectively. By invariance and ergodicity of D, the pushforward measures

$$\mu_{\mathfrak{X}} = D \circ \pi_1^{-1},$$
$$\nu_{\mathfrak{Y}} = D \circ \pi_2^{-1},$$

are invariant and ergodic with respect to the actions defined. Note that the points $\mathfrak{x}$ and $\mathfrak{y}$ of the factor spaces $\mathfrak{X}$ and $\mathfrak{Y}$ are represented by permutation–invariant and ergodic measures $D_r^{G_2}$ and $D_r^{G_1}$ which are therefore Bernoulli measures by de Finetti's theorem. Concretely, regarding

$$Z^{\mathbb{N}\times\mathbb{N}} = (Z^{\{1\}\times\mathbb{N}})^{\mathbb{N}},$$

the 'space of sequences of rows' or on the other hand

$$Z^{\mathbb{N}\times\mathbb{N}} = (Z^{\mathbb{N}\times\{1\}})^{\mathbb{N}},$$

the 'space of sequences of columns', then

$$D_r^{G_2} = \nu_r^{\mathbb{N}} \quad \text{and} \quad D_r^{G_1} = \mu_r^{\mathbb{N}}$$

the measures ν_r and μ_r being a probability measure on the space of columns and rows, respectively. For almost every matrix r, the measures ν_r and μ_r are the empirical distributions of its columns and rows.

Note that if the measure D is not simple then the direct product of $(\mathfrak{X}, \mu_{\mathfrak{X}}, G_1)$ and $(\mathfrak{Y}, \nu_{\mathfrak{Y}}, G_2)$ is a non–trivial factor of $\left(Z^{\mathbb{N}\times\mathbb{N}}, D, G_1 \times G_2\right)$. This gives us the following characterisation of simple measures:

Proposition 5.2. *A measure D is simple if and only if the* (*almost everywhere uniquely defined*) *mapping which sends a matrix r to the pair (ν_r, μ_r) of its empirical distribution of columns and rows respectively, is one-to-one* (mod 0).

With the help of the decomposition of the group action we are able to proof the following lemma.

Lemma 5.3. *Assume that $D = D_f$ is the matrix distribution of a pure function $f : (X, \mu) \times (Y, \nu) \longrightarrow Z$. Then D is simple if and only if the congruence group K_f is trivial.*

Proof. First of all note that by Corollary 4.3, we only have to show that a measure D is simple if and only if the map

$$F_f : (X^{\mathbb{N}}, \mu^{\mathbb{N}}) \times (Y^{\mathbb{N}}, \nu^{\mathbb{N}}) \longrightarrow (Z^{\mathbb{N}\times\mathbb{N}}, D)$$

is one–to–one on a set of full measure. One direction is trivial: if F_f is one-to-one, it is an equivariant isomorphism between the measure spaces. Hence

$$\mathfrak{B}^{S_{\mathbb{N}}\times\{1\}} \vee \mathfrak{B}^{\{1\}\times S_{\mathbb{N}}} = \mathfrak{B} \quad (\operatorname{mod} D),$$

as the same assertion is true for the corresponding invariant algebras in the space $X^{\mathbb{N}} \times Y^{\mathbb{N}}$.

Conversely, assume that D is simple. Whe may thus regard F_f as mapping

$$(X^{\mathbb{N}}, \mu^{\mathbb{N}}, G_1) \times (Y^{\mathbb{N}}, \nu^{\mathbb{N}}, G_2) \longrightarrow (\mathfrak{X}, \mu_{\mathfrak{X}}, G_1) \times (\mathfrak{Y}, \nu_{\mathfrak{Y}}, G_2).$$

By equivariance, the preimage $F_f^{-1}(B)$ of any G_i-invariant Borel set $B \subseteq \mathfrak{X} \times \mathfrak{Y}$ is also G_i-invariant, for every $i = 1, 2$. Thus F_f must be of the form

$$F_f = (\phi_1, \phi_2),$$

with $\phi_1 : X^{\mathbb{N}} \longrightarrow \mathfrak{X}$ and $\phi_1 : Y^{\mathbb{N}} \longrightarrow \mathfrak{Y}$. By Lemma 4.1 we know that for $\nu^{\mathbb{N}}$-almost every sequence (y_j) the restriction $F_f(\,\cdot\,, (y_j))$ is one-to-one (mod 0) and so is ϕ_1. For the same reasoning the function ϕ_2 is one-to-one (mod 0). This proves that $F_f = (\phi_1, \phi_2)$ is one-to-one on a set of full measure. □

5.2. General canonical model given the matrix distribution of a completely pure function. Assume that D is the matrix distribution of a *completely pure function* f, that is a pure function f with trivial congruence group K_f. Then by Lemma 5.3 the measure D is simple and hence the decomposition

$$(Z^{\mathbb{N}\times\mathbb{N}}, D, S_{\mathbb{N}} \times S_{\mathbb{N}}) = \big(\mathfrak{X}, \mu_{\mathfrak{X}}, G_1\big) \times \big(\mathfrak{Y}, \nu_{\mathfrak{Y}}, G_2 \times S_{\mathbb{N}}\big)$$

described in the previous section, together with its isomorphism

$$\pi : Z^{\mathbb{N}\times\mathbb{N}} \longrightarrow \mathfrak{X} \times \mathfrak{Y}$$

gives us another possibility to reconstruct the function f: We simply set

$$\mathfrak{f} : \mathfrak{X} \times \mathfrak{Y} \longrightarrow Z, \quad \mathfrak{f} = r_{1,1} \circ \pi^{-1},$$

and claim that it is isomorphic to the function

$$f : X^{\mathbb{N}} \times Y^{\mathbb{N}} \longrightarrow Z$$

regarded as function on the first coordinates x_1 and y_1. In fact, the mapping

$$\Phi = \pi \circ F_f : X^{\mathbb{N}} \times Y^{\mathbb{N}} \longrightarrow \mathfrak{X} \times \mathfrak{Y}$$

is one-to-one, measure preserving and obviously carries the function $f : X^{\mathbb{N}} \times Y^{\mathbb{N}} \longrightarrow Z$ to $\mathfrak{f}$. Thus the only thing we need to check to is that Φ is of product type, i.e. $\Phi = (\Phi_1, \Phi_2)$ with measurable functions $\Phi_1 : X^{\mathbb{N}} \longrightarrow \mathfrak{X}$ and $\Phi_2 : Y^{\mathbb{N}} \longrightarrow \mathfrak{Y}$. But this is clear from the proof of Lemma 5.3.

We shall call the above constructed function $\mathfrak{f}$ the *general canonical model* of f, since it does not depend on the choice of a particular matrix r as the individual canonical model from Section 4. In contrast to the individual canonical model the function $\mathfrak{f}$ is never pure as it is a model for f as function on $X^{\mathbb{N}} \times Y^{\mathbb{N}}$ rather than as function $X \times Y \longrightarrow Z$. Nevertheless its purification

$$\bar{\mathfrak{f}} : \mathfrak{X}/\zeta_{\mathfrak{X}} \times \mathfrak{Y}/\zeta_{\mathfrak{Y}} \longrightarrow Z$$

as described in Section 3 is clearly isomorphic to the original pure function

$$f : X \times Y \longrightarrow Z$$

by the uniqueness of pure factors. The direct connection between both models becomes clear in the proof of Theorem 5.4.

Of course the above construction of $\mathfrak{f}$ makes sense for any $(S_{\mathbb{N}} \times S_{\mathbb{N}})$-invariant simple measure on $Z^{\mathbb{N}\times\mathbb{N}}$. This observation is the key for the following theorem.

Theorem 5.4 (Characterisation of simple measures using the general canonical model $\mathfrak{f}$)**.** *Let D be an $(S_{\mathbb{N}}\times S_{\mathbb{N}})$-invariant and ergodic probability measure on the space of matrices $Z^{\mathbb{N}\times\mathbb{N}}$. Then D is a matrix distribution of a function $f : (X,\mu)\times(Y,\nu) \longrightarrow Z$ with trivial congruence group K_f if and only if D is a simple measure.*

Remark 5.5. This theorem corrects Theorem 2 from [12], which states that an $(S_{\mathbb{N}} \times S_{\mathbb{N}})$-invariant and ergodic measure is a matrix distribution if and only if it is simple.

Proof. One direction is already content of Lemma 5.3. Conversely, let us assume that D is simple. Again, the dynamical system $(\mathbb{Z}^{\mathbb{N}\times\mathbb{N}}, D, S_{\mathbb{N}} \times S_{\mathbb{N}})$ decomposes via the isomorphism

$$\pi : r \mapsto \left(D_r^{G_2}, D_r^{G_1}\right)$$

into the product

$$(\mathfrak{X}, \mu_{\mathfrak{X}}, G_1) \times (\mathfrak{Y}, \nu_{\mathfrak{Y}}, G_2)$$

of the spaces of ergodic components

$$\mathfrak{X} = \mathrm{Meas}_1^{G_2}\left(Z^{\mathbb{N}\times\mathbb{N}}\right),$$
$$\mathfrak{Y} = \mathrm{Meas}_1^{G_1}\left(Z^{\mathbb{N}\times\mathbb{N}}\right),$$

the measures $\mu_{\mathfrak{X}}$, $\nu_{\mathfrak{Y}}$, and group actions as defined in Section 5.1. Taking a closer look on these measures it is obvious that e.g. $\pi_1 : r \mapsto D_r^{G_2}$ regarded as G_1-equivariant mapping

$$\pi_1 : \left(\mathbb{Z}^{\{1\}\times\mathbb{N}}\right)^{\mathbb{N}} \longrightarrow \mathfrak{X}$$

maps each of the G_1-invariant ergodic measures $D_r^{G_1}$ onto an G_1-invariant ergodic measure on $\mathfrak{X}$. Since $\mu_{\mathfrak{X}}$ is also an G_1-invariant ergodic measure, being the integral convex combination

$$\mu_{\mathfrak{X}}(B) = \int\limits_r D_r^{G_1} \circ \pi_1^{-1}(B) \cdot dD(r)$$

of such measures, it therefore must equal to at least[3] one such pushforward measures $D_r^{G_1} \circ \pi_1^{-1}$. As noted in Section 5.1, we may regard the latter measure $D_r^{G_1}$ as Bernoulli measure $\mu_r^{\mathbb{N}}$ on the space of sequences of rows $\left(\mathbb{Z}^{\{1\}\times\mathbb{N}}\right)^{\mathbb{N}}$, and π_1 as measure preserving isomorphism between the space of sequences of rows and $\mathfrak{X}$. We thus may consider

$$(\mathfrak{X}, \mu_{\mathfrak{X}}) = \left(\left(\mathbb{Z}^{\{1\}\times\mathbb{N}}\right)^{\mathbb{N}}, \mu_r^{\mathbb{N}}\right),$$

and in the same way one sees that

$$(\mathfrak{Y}, \mu_{\mathfrak{Y}}) = \left(\left(\mathbb{Z}^{\mathbb{N}\times\{1\}}\right)^{\mathbb{N}}, \nu_r^{\mathbb{N}}\right),$$

that is the space of sequences of columns with Bernoulli measure $\nu_r^{\mathbb{N}}$. We claim that the matrix distribution of the function

$$\mathfrak{f} : \mathfrak{X} \times \mathfrak{Y} \longrightarrow Z, \quad \mathfrak{f} = r_{1,1} \circ \pi^{-1},$$

equals our measure D. For brevity, we write $X = Z^{\{1\}\times\mathbb{N}}$ and $Y = Z^{\mathbb{N}\times\{1\}}$ for the space of rows and the space of columns, respectively. As the matrix valued function

$$\mathfrak{F} : \mathfrak{X} \times \mathfrak{Y} \longrightarrow Z^{\mathbb{N}\times\mathbb{N}}, \quad \mathfrak{F} = \mathrm{id} \circ \pi^{-1},$$

is equivariant with respect to the action of $G_1 \times G_2$ so it is regarding it as function from $X^{\mathbb{N}} \times Y^{\mathbb{N}}$ to $Z^{\mathbb{N}\times\mathbb{N}}$. From this it follows easily that $\mathfrak{F}$ is of the form

$$\mathfrak{F}\big((x_i), (y_j)\big) = \big(\mathfrak{F}_{1,1}(x_i, x_j)\big) \pmod 0,$$

as is shown in the postponed Lemma 5.7. This proves that the distribution of the matrix valued function $\mathfrak{F}$, which by definition equals D, is the matrix distribution of $\mathfrak{f}$. Moreover, the function $\mathfrak{F} = F_{\mathfrak{f}}$ is by construction one-to-one and we conclude from Corollary 4.3 that the congruence group of $\mathfrak{f}$ is trivial. □

Remark 5.6. Observe that the characterization of matrix distributions in the case of functions in just one argument becomes essentially de Finetti's theorem: In this context, the *row distribution* (instead of matrix distribution) D_f of $f : X \longrightarrow Z$ is defined as the distribution of the process $(f(x_i))$ sampling its arguments independently according to the given measure μ on X. Thus D_f is simply the Bernoulli measure $D = \left(\mu \circ f^{-1}\right)^{\mathbb{N}}$ on the space

[3]it is not difficult to see that the set of all mesures $D_r^{G_1}$ which equal the ergodic measure $\mathfrak{X}$ is of full measure.

$Z^{\mathbb{N}}$. De Finetti's theorem states that the $S_{\mathbb{N}}$-invariant ergodic measure on $Z^{\mathbb{N}}$ are exactly the Bernoulli measures, and hence all such meausres are the row distribution of a function in one variable.

We close this section with the auxiliary lemma that we used in the proof of Theorem 5.4.

Lemma 5.7. *Suppose that the map F from $(X^{\mathbb{N}}, \mu^{\mathbb{N}}) \times (Y^{\mathbb{N}}, \nu^{\mathbb{N}})$ to $Z^{\mathbb{N}\times\mathbb{N}}$ is equivariant under the action of $S_{\mathbb{N}} \times S_{\mathbb{N}}$. Then there exists a measurable function $f : X \times Y \longrightarrow Z$ such that*

$$F = (F_{i,j}) = \big(f(x_i, x_j)\big) \pmod 0.$$

Proof. Using equivariance we know that for any g_1 and g_2 from the subgroup

$$S_{\mathbb{N}}^{(1)} = \big\{g \in S_{\mathbb{N}} : g(1) = 1\big\}$$

the following identity holds.

$$\mathfrak{F}_{1,1}\big(g_1(x_i), g_2(y_j)\big) = \mathfrak{F}_{g_1^{-1}(1), g_2^{-1}(1)}\big((x_i), (y_j)\big) = \mathfrak{F}_{1,1}\big((x_i), (y_j)\big).$$

Being invariant with respect to the action of $S_{\mathbb{N}}^{(1)} \times S_{\mathbb{N}}^{(1)}$ the function $\mathfrak{F}_{1,1}$ is $(\operatorname{mod} 0)$ measurable with respect to the first coordinates x_1 and y_1 of $X_r^{\mathbb{N}} \times Y_r^{\mathbb{N}}$ which means that $\mathfrak{F}_{1,1}\big((x_i), (y_j)\big) = f(x_1, y_1) \pmod 0$ for some function f defined on $X \times Y$. Using once more equivariance we conclude that

$$\begin{aligned} &F_{g_1^{-1}(1), g_2^{-1}(1)}\big((x_i), (y_j)\big) \\ &\qquad = F_{1,1}\left(\left(x_{g_1^{-1}(i)}\right), \left(y_{g_2^{-1}(j)}\right)\right) = f\left(x_{g_1^{-1}(1)}, y_{g_2^{-1}(1)}\right) \end{aligned}$$

$(\operatorname{mod} 0)$ for every g_1 and g_2 from $S_{\mathbb{N}}$, which proves the assertion of the lemma. □

§6. The case of functions in more than two arguments

Let us shortly discuss the case of Borel functions

$$f : \prod_{i=1}^{n} (X_i, \mu_i) \longrightarrow Z$$

in more than two arguments. Its *tensor distribution* D_f is defined as the distribution of the tensor-valued functional

$$F_f : \prod_{i=1}^{n} X_i^{\mathbb{N}} \longrightarrow Z^{\mathbb{N}^n},$$

$$\left(\left(x_i^{(1)}\right), \left(x_i^{(2)}\right), \ldots, \left(x_i^{(n)}\right)\right) \mapsto \left(f\left(x_{i_1}^{(1)}, \ldots, x_{i_n}^{(n)}\right)\right)_{i_1, \ldots, i_n},$$

under the measure $\prod_{i=1}^{n} \mu_i^{\mathbb{N}}$, i.e.

$$D_f = \left(\prod_{i=1}^{n} \mu_i^{\mathbb{N}}\right) \circ F_f^{-1}.$$

In this situation D_f is a measure on the space of infinite tensors $Z^{\mathbb{N}^n}$ which is ergodic and invariant with respect to the (analogously defined action) of

$$S_{\mathbb{N}}^n = S_{\mathbb{N}} \times S_{\mathbb{N}} \times \cdots \times S_{\mathbb{N}}$$

acting independently on the indices of the tensors. The notion of factors in the category of measurable functions in several variables is analogous to that of functions in two arguments, and so is the definition of pureness: *a function $f(x_1, x_2, \ldots, x_n)$ is said to be pure, if it admits no true factor.* The reduction of the isomorphism problem to pure functions is then proved in exactly the same way as Theorem 3.1. With help of an extended version of Lemma 2.2 it is also obvious how to prove the higher-dimensional analogue of Theorem 4.2: *Two pure measurable functions are isomorphic if and only if their tensor distributions coincide.*

The results from Section 5 are also extended easily: Let G_i be the permutation group which acts on the i-th index of the tensors only. An $(S_{\mathbb{N}}^n)$-invariant measure D on the space of tensors $Z^{\mathbb{N}^n}$ is said to be simple, if the invariant algebras $\mathfrak{B}^{G^{(i)}}$ with respect to the groups

$$G^{(i)} = G_1 \times \cdots \times G_{i-1} \times \{1\} \times G_{i+1} \times \cdots \times G_n$$

keeping the i-th index fixed, generate the whole Borel algebra $\mathfrak{B}$ of $Z^{\mathbb{N}^n}$, i.e.

$$\bigvee_{i=1}^{n} \mathfrak{B}^{G^{(i)}} = \mathfrak{B} \pmod{D}.$$

This again means that the dynamical system $(Z^{\mathbb{N}^n}, D, \prod_{i=1}^{n} G_i)$ is the direct product of the systems $(Z^{\mathbb{N}^n}/G^{(i)}, D, G_i)$, $1 \leqslant i \leqslant n$, the quotients interpreted as ergodic decompositions. With help of this decomposition, the general canonical model is defined analogously and Theorem 5.4 can be proved similary: *An $S_{\mathbb{N}}^n$-invariant and ergodic measure on the space of tensors $Z^{\mathbb{N}^n}$ is the tensor distribution of measurable function*

$$f : \prod_{i=1}^{n} (X_i, \mu_i) \longrightarrow Z$$

with trivial congruence group

$$K_f = \Big\{ (T_i)_{i=1}^n \in \prod_{i=1}^{n} \operatorname{Aut}_0(X_i, \mu_i) : \\ f\big(T_1(x_1), \ldots, T_n(x_n)\big) = f(x_1, \ldots, x_n) \quad a.e. \Big\}$$

if and only it is simple.

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