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CRITERIA OF DIVERGENCE ALMOST EVERYWHERE IN ERGODIC THEORY

Abstract. In this expository paper, we survey nowadays classical tools or criteria used in problems of convergence everywhere to build counterexamples: the Stein continuity principle, Bourgain's entropy criteria and Kakutani–Rochlin lemma, most classical device for these questions in ergodic theory. First, we state a L^1-version of the continuity principle and give an example of its usefulness by applying it to some famous problem on divergence almost everywhere of Fourier series. Next we particularly focus on entropy criteria in L^p, $2 \leqslant p \leqslant \infty$, and provide detailed proofs. We also study the link between the associated maximal operators and the canonical Gaussian process on L^2. We further study the corresponding criterion in L^p, $1 < p < 2$, using properties of p-stable processes. Finally we consider Kakutani–Rochlin's lemma, one of the most frequently used tool in ergodic theory, by stating and proving a criterion for a.e. divergence of weighted ergodic averages.

§1. Introduction.

This is an expository paper on criteria of divergence almost everywhere in ergodic theory, and mainly Bourgain's entropy criteria in $L^p, 2 \leqslant p \leqslant \infty$. The paper is written in a self-contained and informative way: tools needed are presented, with (expected to be) helpful and sometimes historical comments, auxiliary results are included, as well as detailed and careful proofs of main theorems. The preparation of this paper is thus made in order to be also an efficient tool for investigating these questions. This is in fact our main objective. We do not study nor present applications. We refer for these to Bourgain [1–3]. We also refer to Rosenblatt and Wierdl monograph [26], to our monograph [33] devoted to the study of these criteria and to Chapters 5 and 6 of our book [32] where applications of the Stein continuity principle are also studied. We further refer to Lacey [18],

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Lesigne [20], Berkes and Weber [6] notably for other applications. In writing the present paper, we referred to Chapter 6 of [32]. We were able to improve and simplify some proofs and also complete it by new results. The entropy criterion in L^p, $1 < p < 2$, obtained in Weber [38] is stated and proved under a less restrictive commutation assumption, and we included the necessary material from the theory of α-stable processes (here $\alpha = p$) for the proof. The metric entropy method (first introduced by Strassen in the theory of Gaussian processes, see [8]) is briefly and concisely presented for the need of the study.

The paper is organized as follows. In Section 2, we start with what is certainly, by the probabilistic argument used in its proof, the basis of everything: the Stein continuity principle. A less known aspect of this principle is that it is also a tool for producing counterexamples to almost everywhere questions. That point is developed in this Section. Next, Section 3 is the central part of the paper and concerns Bourgain's entropy criteria and extensions of them. In Section 4, we present auxiliary results concerning L^p-isometries, stable random variables and processes, variants of Banach principle, a metric comparison lemma and basic Gaussian tools. Section 5 is completely devoted to proofs of the results stated in Section 3. We conclude the paper with Kakutani–Rochlin lemma, one of the most classical devices in ergodic theory. There are many applications of this result, also called Kakutani–Rochlin towers' lemma. We refer to Rosenblatt and Wierdl monograph [26]. We illustrate it by stating and proving a criterion for a.e. divergence of weighted ergodic averages, based on Deniel's construction [7].

§2. The Continuity Principle.

Let $(X, \mathcal{A}, \mu)$ be a probability space with a μ-complete σ-field $\mathcal{A}$. Throughout the paper S denotes, unless explicitly mentioned, a sequence of continuous operators $S_n\colon L^2(\mu) \to L^2(\mu)$, $n \geqslant 1$. Recall some basic facts. Let $1 \leqslant p \leqslant \infty$. By the Banach principle, the set

$$\mathcal{F}(S) = \big\{ f \in L^p(\mu) : \{S_n f, n \geqslant 1\} \text{ converges } \mu\text{-almost everywhere} \big\}$$

is closed in $L^p(\mu)$ if and only if:

There exists a non-increasing function $C\colon \mathbb{R}^+ \to \mathbb{R}^+$ *with* $\lim_{\alpha\to\infty} C(\alpha) = 0$, *and such that for any* $\alpha \geqslant 0$ *and any* $f \in L^p(\mu)$,

$$\mu\big\{S^* f > \alpha \|f\|_p\big\} \leqslant C(\alpha) \qquad \text{where} \quad S^* f = \sup_{n\geqslant 1} |S_n f|.$$

When the sequence S commutes with a sequence $\{\tau_j, j \geqslant 1\}$ of measurable transformations of X preserving μ and mixing in the following sense:

$$\forall\, A, B \in \mathcal{A},\ \forall\, \alpha > 1,\ \exists\, j \geqslant 1\colon \quad \mu(A \cap \tau_j^{-1} B) \leqslant \alpha\, \mu(A)\, \mu(B), \qquad (H)$$

and $1 \leqslant p \leqslant 2$, then by the continuity principle $C(\alpha) = \mathcal{O}(\alpha^{-p})$.

This is fulfilled if S commutes with an ergodic endomorphism of $(X,\mathcal{A},\mu)$. So that the study of the convergence almost everywhere of the sequence S amounts, under appropriate commutation assumptions, to establish a maximal inequality and to exhibit a dense subset of $L^p(\mu)$ for which the convergence almost everywhere already holds.

Before stating the Continuity Principle, recall that the topology of convergence in measure on $L^0(\mu)$ ($g_n \xrightarrow{\mu} g$ if $\mu\{|g_n - g| > \varepsilon\} \to 0$, for any $\varepsilon > 0$) is metrizable and, endowed with the metric $d(f,g) = \int_X \frac{|f-g|}{1+|f-g|}\, d\mu$, $(L^0(\mu), d)$ is a complete metric space. A mapping V from a Banach space B to $L^0(\mu)$ is said to be continuous in measure or d-continuous, if for any sequence $(f,\ f_n,\ n \geqslant 1) \subset B$, we have $d(Sf_n, Sf) \to 0$ whenever $\|f_n - f\| \to 0$.

Theorem 2.1. *Suppose that $\{S_n,\ n \geqslant 1\}$ is a sequence of operators, $S_n\colon L^p(\mu) \to L^0(\mu)$, $1 \leqslant p \leqslant 2$, which are continuous in measure and satisfy the commutation assumption (H). Then the following properties are equivalent:*

(i) $\forall\, f \in L^p(\mu), \quad \mu\{x : S^* f(x) < \infty\} = 1.$

(ii) $\exists\, 0 < C < \infty : \forall\, f \in L^p(\mu),$

$$\sup_{\lambda \geqslant 0} \lambda^p \mu\{x : S^* f(x) > \lambda\} \leqslant C \int_X |f|^p \, d\mu.$$

Remark 2.2. If $p > 2$, the same conclusion holds for positive operators ($S_n f \geqslant 0$, if $f \geqslant 0$). This was proved later by Sawyer in [27].

The proof combines quite subtely and remarkably, analysis and probability. The commutation property of the operators S_n is crucial, and makes the proof possible. Earlier, Kolmogorov used already in [13] the fact that the operators

$$H_n f(x) = \int_{|t|>1/n} f(x-t)\frac{dt}{t}, \qquad f \in L^1_{\mathrm{loc}}(\mathbb{R})$$

all commute with translations to prove the similar inequality: let $H^*f(x) = \sup\{|H_n f(x)|,\ n \geqslant 1\}$, then

$$\sup_{\lambda \geqslant 0} \lambda\, m\{x : H^*f(x) > \lambda\} \leqslant C \int_{\mathbb{R}} |f(x)|\, dx,$$

m denoting here the Lebesgue measure on $\mathbb{R}$. The setting considered in [29] is group theoretic: Ω is a commutative compact group, μ is the Haar measure and S_n are commuting with translations. Sawyer [27] showed that this setting is not necessary and that a general principle can be derived under the above assumptions. We refer to the nice monograph of Garsia [10].

The Continuity Principle is *not only* a tool for studying integrability of maximal operators S^*f, but also a device for producing counterexamples in problems of convergence almost everywhere. This was already observed and studied by Stein [29], but also by Burkholder [5] and Sawyer [27]. In [29], Stein has established other forms of this principle with quite striking applications, proving notably negative convergence results. One of these applications concerns a deep result of Kolmogorov [14, 15] showing the existence of an integrable function whose Fourier series diverges almost everywhere. The proof is known to be very difficult. Using a suitable form of his principle for the space $L^1(\mu)$, Stein could refine and also provide a simpler proof of Kolmogorov's result. Convergence criteria for this space are not frequent, and reveal crucial in many deep questions. We recall it now.

We assume here that X is a commutative compact group and denote by "+" the group operation. Let μ be the unique invariant measure, the Haar measure on X. Let $\mathcal{C}(X)$ be the space of continuous functions on X, with the supremum norm, and $\mathcal{B}(X)$ be the space of finite Borel measures on X with the usual norm. Let $\{S_n, n \geqslant 1\}$ be a sequence of operators. We assume:

(a) Each S_n is a bounded operator from $L^1(\mu)$ to $\mathcal{C}(X)$.

(b) Each S_n commutes with translations.

By Riesz's representation of bounded linear functionals on $L^1(\mu)$, conditions (a) and (b) are equivalent with

(c) $S_n f(x) = \int_X K_n(x-y)\, f(y)\, \mu(dy)$, where $K \in L^\infty(X)$.

Such an operator has a natural extension to a bounded operator from $\mathcal{B}(X)$ to $L^\infty(\mu)$, which we again denote by S_n. Notice that this extension still commutes with translations. Similarly, we also write $S^*\nu = \sup_{n\in\mathbb{N}} |S_n\nu|$.

Theorem 2.3. *Under assumptions (a) and (b), the following assertions are equivalent:*

$$\forall\, f \in L^1(\mu), \quad \mu\{x : S^* f(x) < \infty\} = 1, \tag{2.1}$$

$$\exists 0 < C < \infty\colon\ \forall \nu \in \mathcal{B}(X), \quad \sup_{\lambda\geqslant 0} \lambda\, \mu\Big\{x : S^*\nu(x) > \lambda \int_X |d\nu|\Big\} \leqslant C. \tag{2.2}$$

To give an idea of its strength, let us show how to recover Kolmogorov's theorem. Introduce the necessary notation. We denote throughout this article by $\mathbb{T}$ the circle $\mathbb{R}/\mathbb{Z} \sim [0,1[$.

Take $X = \mathbb{T}$ and let μ be the normalized Lebesgue measure on $\mathbb{T}$. Let $S_n(f)$ denote here the partial sum of order n of the Fourier series of f, and more generally let $S_n(\nu)$ be the partial sum of order n of the Fourier–Stieltjes expansion of a Borel measure ν. Recall that for any integrable f,

$$S_n f(x) - S_m f(x) = \mathcal{O}(\log(1 + |m-n|)), \qquad m, n \to \infty,$$

almost everywhere. Stein proved the following refinement:

Theorem 2.4. *Let $\varphi(n) > 0$ be any function tending to zero as n tends to infinity. Then there exists an integrable function $f(x)$ such that the more restrictive property*

$$S_n(f)(x) - S_m(f)(x) = \mathcal{O}(\varphi(|m-n|)\log(1 + |m-n|)) \tag{2.3}$$

is false for almost every x.

This of course implies Kolmogorov's theorem. For the proof, consider for $n \neq m$ the family of operators

$$\Delta_{(m,n)} f = \frac{S_n(f) - S_m(f)}{\varphi(|m-n|)\log(1+|m-n|)}.$$

These operators satisfy conditions (a) and (b) of Theorem 2.3. A lemma is necessary.

Lemma 2.5. *There exists an absolute constant C such that for any integer k, there exists a measure ν on $\mathbb{T}$ with $\int_{\mathbb{T}} |d\nu| = 1$ and*

$$\sup_{n,m:|n-m|=k} \big|S_n(\nu) - S_m(\nu)\big| \geqslant C \log k \qquad \textit{almost surely.}$$

Proof. Let $x_1, \ldots, x_N$ be some points of $\mathbb{T}$ to be specified later, and set $\nu = \frac{1}{N} \sum_{i=1}^{N} \delta_{x_i}$, where δ_x denotes the Dirac measure at point x. Then $\int_{\mathbb{T}} |d\nu| = 1$. Plainly,

$$S_n(\nu)(x) - S_m(\nu)(x) \\ = \frac{2}{N} \sum_{i=1}^{N} \frac{\cos \pi(n+m+1)(x-x_i) \sin \pi(n-m)(x-x_i)}{\sin \pi(x-x_j)}.$$

Write $k = n - m$, $\ell = n + m + 1$. Assume that k is odd. Then ℓ must be even, but this is the only restriction on ℓ. We choose the x_i to be linearly independent over $\mathbb{Q}$, and such that they are very close to i/N. It is easily seen then, that for almost every x, the $x - x_i$ are linearly independent over $\mathbb{Q}$. Choosing ℓ large enough, depending on x, we have

$$\sup_{n,m:|n-m|=k} |S_n(\nu)(x) - S_m(\nu)(x)| = \frac{2}{N} \sum_{i=1}^{N} \frac{|\sin \pi k(x-x_i)|}{|\sin \pi(x-x_j)|}.$$

The fact that x_i are very close to i/N and N is large enough, shows that the sum on the right is close to its integral counterpart, and so exceeds half of its value. Therefore,

$$\sup_{n,m:|n-m|=k} |S_n(\nu)(x) - S_m(\nu)(x)| \geqslant \frac{1}{2} \int_{\mathbb{T}} \frac{|\sin \pi k(x-y)|}{|\sin \pi(x-y)|}\, dy \geqslant C \log k,$$

as claimed. □

Now we prove Theorem 2.4. Suppose on the contrary that property (2.3) were true with positive probability, and this for any $f \in L^1(\mathbb{T})$. Let τ be an irrational rotation of $\mathbb{T}$, thereby an ergodic measure preserving transformation. Note that if $A = \{\sup_{n \neq m} |\Delta_{(m,n)} f| < \infty\}$, then $\tau^{-1}(A) \subset A$. By Birkhoff's theorem, this suffices to imply that $\mu(A) = 1$. So that the operators $\Delta_{(m,n)} f$ would satisfy condition (2.1). Consequently, the maximal operator

$$\nu \mapsto \Delta^*(\nu) := \sup_{n \neq m} \left| \frac{S_n(\nu)(x) - S_m(\nu)(x)}{\varphi(|m-n|) \log(1 + |m-n|)} \right|$$

would satisfy (2.2). Therefore this would imply the existence of a constant C_0 such that for any $\nu \in \mathcal{B}(M)$ with $\int_{\mathbb{T}} |d\nu| = 1$, and any $t \geqslant 0$, $t\mu\{x : \Delta^*\nu(x) > t\} \leqslant C_0$.

Let k be a positive integer, which we choose sufficiently large to ensure that $\log k > (2\,C_0)/C$, where C is the same constant as in Lemma 2.5. Apply this for $t = (C\ \log k)/2$; then,

$$\mu\Big\{x : \Delta^*\nu(x) > \frac{C}{2}\log k\Big\} \leqslant \frac{2\,C_0}{C\ \log k} < 1.$$

By Lemma 2.5, there exists $\nu \in \mathcal{B}(M)$ with $\int_{\mathbb{T}} |d\nu| = 1$ such that $\Delta^*\nu \geqslant C \log k$ almost surely. Hence a contradiction and condition (2.1) cannot hold. Therefore there exists an integrable function such that property (2.3) is false for almost every x.

For recent results related to Kolmogorov's theorem, see Lacey's very nice paper [17], Section 9.3. We refer to [29] (see also [32, Chapter 5]) for several other applications of this kind.

To $f \in L^2(\mu)$, associate the sequence in which we set $T_j f = f \circ \tau_j$,

$$F_{J,f} = \frac{1}{\sqrt{J}} \sum_{1 \leqslant j \leqslant J} g_j T_j f, \qquad (J \geqslant 1), \tag{2.4}$$

where $g_1, g_2, \dots$ are i.i.d. standard Gaussian random variables, defined on a common joint probability space $(\Omega, \mathcal{B}, \mathbf{P})$.

These random elements (with Rademacher weights instead of Gaussian's) are key tools in Stein's proof. The same elements (sometimes with stable weights) are also playing a central role in Bourgain's entropy criteria and extensions obtained by the author. The notation used in (2.4) will be later formalized to include these cases, see (4.4). Lifshits and Weber studied in [21, 22] and [35] their oscillations properties and the tightness properties of their laws.

The Continuity Principle is established in an indirect way in [29]. A direct proof with Gaussian weights (as in the proofs of Bourgain's entropy criteria) was given in [32].

We close this section with an interesting and somehow intriguing observation. The key point of the proof is contained in the following inequality (see [32, p. 211–212])

$$\frac{n\mu\{S^*(f) > M(1+n)^{1/p}\} - 2}{n\mu\left\{S^*(f) > M(1+n)^{1/p}\right\}} \leqslant 8\,\mathbf{E}\,\mu\{S^*(F_{n,f}) > cM\}, \tag{2.5}$$

which holds for any $M > 0$, any integer $n \geqslant 2$, and c is a numerical constant. Now by simply permuting the order of integration, we get

$$\frac{n\mu\{S^*(f) > M(1+n)^{1/p}\} - 2}{n\mu\{S^*(f) > M(1+n)^{1/p}\}} \leqslant 8 \int\limits_X \mathbf{P}\{S^*(F_{n,f}) > cM\}\; d\mu, \tag{2.6}$$

where this time, $S^*(f)$ is controlled by its random counterpart of $S^*(F_{n,f})$ for an appropriate choice of the integer n. Therefore a good control of the random counterpart also provides a good control of the initial sequence.

Notation. We reserve the letter g to denote throughout an $\mathcal{N}(0,1)$ distributed random variable. An index or a sub-index always denotes an infinite increasing sequence of positive integers.

§3. Metric Entropy Criteria

Using the theory of Gaussian processes, Bourgain has established in [2] two very useful criteria linking the regularity properties (boundedness, convergence almost everywhere) of the sequence S with the metric entropy properties of the sets C_f below.

The concept of entropy numbers (namely covering numbers) associated with a metric space is old; it was invented by Kolmogorov as a device for classifying functional spaces. See Kolmogorov [13], Kolmogorov and Tikhomirov [16], Lorentz [23]. In many situations, these numbers are computable (typical examples of sets are ellipsoids, see [9]); hence their interest. Recall that any compact set in a separable Hilbert space is included in some ellipsoid, see Raimi [25] and for relations between their entropy numbers, see Helemskiĭ and Henkin [11].

Bourgain also showed, by means of imaginative constructions, how to apply these criteria to several analysis problems, among them Marstrand's disproof of Khintchin's Conjecture, a problem posed by Bellow and a question raised by Erdös. This is a quite striking achievement, which adds a new chapter to Stein's Continuity Principle. We believe that Bourgain's approach goes beyond the setting explored in [1–3] and should deserve further investigations. The author has obtained in [6, 33, 38] extensions of these criteria and applied them to similar questions. He further studied in [34, 36, 37] the geometry of the sets C_f defined in (3.1), as well as and their natural extension $C(A) = \{S_n(f),\ n \geqslant 1,\ f \in A\}$, in which A is an arbitrary subset of $L^2(\mu)$. We also refer to Talagrand [31] where this question was investigated in a larger context.

Introduce the following commutation condition:

(C) There exists a sequence $\{T_j,\ j \geqslant 1\}$ of $L^1(\mu)$ positive isometries, such that $T_j 1 = 1$, commuting with S, $S_n(T_j f) = T_j(S_n f)$, and such that for any $f \in L^1(\mu)$,

$$\lim_{J\to\infty} \Big\| \frac{1}{J} \sum_{j\leqslant J} T_j f - \int f\, d\mu \Big\|_{1,\mu} = 0.$$

Set for any $f \in L^2(\mu)$,

$$C_f = \{S_n(f),\ n \geqslant 1\}. \tag{3.1}$$

Consider for $2 \leqslant p \leqslant \infty$, the following convergence property

($\mathcal{C}_p$) $\qquad \mu\big\{\{S_n(f),\ n \geqslant 1\} \text{ converges}\big\} = 1, \qquad$ for all $f \in L^p(\mu)$.

Bourgain's first criterion [2, Proposition 1] shows that if ($\mathcal{C}_p$) holds for some $2 \leqslant p < \infty$, the sets C_f cannot be too large. More precisely,

Theorem 3.1. *Let S be a sequence of $L^2(\mu)$ contractions satisfying condition* (C)*. Assume that* ($\mathcal{C}_p$) *holds for some $2 \leqslant p < \infty$. Then there exists a numerical constant C_0 such that for any $f \in L^p(\mu)$,*

$$\sup_{\varepsilon>0} \varepsilon \sqrt{\log N_f(\varepsilon)} \leqslant C_0 \|f\|_2,$$

where for any $\varepsilon > 0$, $N_f(\varepsilon)$ denotes the minimal number of $L^2(\mu)$ open balls of radius ε, centered in C_f and enough to cover C_f.

Remark 3.2. By using covering properties of ellipsoids, one can show that the above entropy estimate is optimal for convolutions on the circle; and thus admits no improvement. See [33, p. 47]. However, it can be far from optimal on typical examples. Let $S_n f = \frac{1}{n} \sum_{j\leqslant n} T^j f$, where T is some measure preserving transformation on $(X, \mathcal{A}, \mu)$. By a theorem of Talagrand $N_f(\varepsilon) \leqslant C \max(1, \|f\|_{2,\mu}^2/\varepsilon^2)$, $0 < \varepsilon \leqslant \|f\|_{2,\mu}$, where C is an absolute constant. See [31], [32, Theorem 1.4.1].

Bourgain's second criterion [2, Proposition 2] states as follows.

Theorem 3.3. *Let S be a sequence of $L^2(\mu)$ contractions satisfying condition* (C). *Assume that* $(\mathcal{C}_\infty)$ *is fulfilled. Then for any real $\delta > 0$,*

$$C(\delta) = \sup_{f \in L^\infty(\mu),\, \|f\|_2 \leqslant 1} N_f(\delta) < \infty.$$

A starting point of the proof is a version (see [2, (9)]) of the Banach principle for $L^\infty(\mu)$, namely the fact that the convergence property $(\mathcal{C}_\infty)$ implies that

$$\sup_{\|f\|_{\infty,\mu} \leqslant 1, \|f\|_{2,\mu} \leqslant \varepsilon} \int_X \frac{S^* f}{1 + S^* f}\, d\mu \;\to\; 0, \qquad \text{as} \quad \varepsilon \to 0. \tag{3.2}$$

This result was established few after by Bellow and Jones in [4]. The proof is however lenghty and indirect. It is possible to provide a direct and short proof, similar to the one of the standard Banach principle, see [32, Theorem 5.1.5].

Note that the integrability of $S^* f$, which is required in (3.2), is not ensured by the assumption made in Theorem 3.3. This is for instance guaranteed when S_n are $L^2(\mu)$-$L^\infty(\mu)$ contractions, which is the case of all applications given in [2]. Moreover, Bourgain's proof runs with no modification using (3.2) at the conclusion.

Given a separable Hilbert space H, recall that the canonical Gaussian (also called isonormal) process $Z = \{Z_h, h \in H\}$ on H is the centered Gaussian process with covariance function

$$\Gamma(h, h') = \langle h, h' \rangle, \qquad h, h' \in H.$$

Let $\{h_n, n \geqslant 1\}$ be a countable orthonormal basis of H. Let also $\{g_n, n \geqslant 1\}$ be a sequence of i.i.d. $\mathcal{N}(0,1)$ distributed random variables on a basic probability space $(\Omega, \mathcal{A}, \mathbf{P})$. Then Z can be defined as follows: for any $h \in H$,

$$Z_h = \sum_{n=1}^{\infty} g_n \langle h, h_n \rangle.$$

A subset A of H is a GB set (for Gaussian bounded) if the restriction of Z on A possesses a version which is sample bounded. Further, A is a GC set (for Gaussian continuous) if the restriction of Z on A possesses a version which is sample $\|\cdot\|$-continuous. These notions were introduced in Dudley [9].

A countable subset A of H is a GB set if $\mathbf{E}\sup_{h\in A}|Z(h)| < \infty$, or equivalently $\mathbf{E}\sup_{h\in A} Z(h) < \infty$, since as is well-known,

$$\mathbf{E}\sup_{h\in A} Z(h) \leqslant \mathbf{E}\sup_{h\in A}|Z(h)| \leqslant 2\,\mathbf{E}\sup_{h\in A} Z(h) + \inf_{h_0\in A}\mathbf{E}\,|Z(h_0)|.$$

Under assumptions of Theorem 3.1, Bourgain has also shown that the sets C_f are GB sets. Some remarks are in order. It is not necessary to assume that S_n are $L^2(\mu)$-contractions. Moreover, the conclusion remains true under a weaker condition than $(\mathcal{C}_p)$. Theorem 3.1 can be reformulated as follows.

Theorem 3.4. *Let S be satisfying assumption* (C). *Assume that for some $2\leqslant p<\infty$,*

$$(\mathcal{B}_p) \qquad \mu\{\sup_{n\geqslant 1}|S_n(f)|<\infty\} = 1, \qquad \textit{for all } f\in L^p(\mu).$$

Then for any $f\in L^p(\mu)$, the sets C_f are GB sets of $L^2(\mu)$. Further there exists a numerical constant C_1 and a constant C_2 such that for any $f\in L^p(\mu)$,

$$C_1\sup_{\varepsilon>0}\varepsilon\sqrt{\log N_f(\varepsilon)} \leqslant \mathbf{E}\sup_{n\geqslant 1} Z(S_n(f)) \leqslant C_2\,\|f\|_{2,\mu}.$$

The use of the fact that if $N(X)$ is a Gaussian semi-norm, then

$$\mathbf{P}\{N(X)\leqslant s\}>0 \qquad\Rightarrow\qquad \mathbf{E}\,N(X)\leqslant \frac{4s}{\mathbf{P}\{N(X)\leqslant s\}}, \tag{3.3}$$

slightly simplifies the proof, which otherwise is very similar ([32]).

Remark 3.5. One can naturally wonder whether property $(\mathcal{C}_p)$ analogously implies that the sets C_f are GC sets. This question was investigated in [33, § 5.2.2], where in Theorem 5.2.4 it is shown that the answer is positive when $X=\mathbb{T}$ and S_n are commuting with rotations.

Note before continuing that when $\int_X S^*f\,d\mu$ is finite, no explicit link with

$$\mathbf{E}\sup_{n\geqslant 1} Z(S_n(f))$$

can be drawn from Theorem 3.4. In Theorem 3.6 below, this is established. A general inequality valid for arbitrary partial maxima, can be directly indeed derived from condition (C) only. Before, we add further comments. First, say a few words on the way the commutation condition (C) links

Z and S. This explains easily. Let $f \in L^2(\mu)$ and let I be a finite set of integers. Then one derives from (C), that there exists an index $\mathcal{J}$ such that the two-sided inequalities

$$\begin{aligned}\frac{1}{2}\|S_n(f) - S_m(f)\|_{2,\mu} &\leqslant \|S_n(F_{J,f}) - S_m(F_{J,f})(x)\|_{2,\mathbf{P}} \\ &\leqslant 2\|S_n(f) - S_m(f)\|_{2,\mu},\end{aligned}$$

hold true for all $n, m \in I$ and all $J \in \mathcal{J}$, and for all x in a measurable set of positive measure. See Lemma 4.6. Theorem 3.1 is obtained as a straightforward application of the Banach principle, and Slepian's inequality combined with Sudakov's minoration (Lemma 4.9).

Bourgain essentially applied Theorem 3.3, and this in the case $X = \mathbb{T}$, and T_j are translation or dilation operators. The counter-examples are built on functions of the type

$$f = \frac{1}{\sqrt{\#(F)}} \sum_{n \in F} e_n \qquad (e_n(x) = e^{2i\pi nx}),$$

where F are specific arithmetic sets. These elements, as well as all $T_j f$, $j \geqslant 1$, not only belong to $L^p(\mu)$ but also to many more specific spaces. So that for Banach spaces B such that $B \subset L^2(\mu)$, a requirement on $f \in B$ like

$$T_j f \in B, \qquad j \geqslant 1,$$

is frequently non void. Call $\mathcal{R}(B)$ the set of these elements. Then $F_{J,f} \in B$ whenever $f \in \mathcal{R}(B)$. If $B = L^p(\mu)$ for instance, then by Corollary 4.3 and Lemma 4.4, $\mathcal{R}(B) = B$.

Theorem 3.6. *Let S be satisfying condition* (C). *Let additionally I be a finite set of integers and $0 < \varepsilon < 1$. Then there exists a partial index $\mathcal{J}$ such that for any $J \in \mathcal{J}$, any positive increasing convex function $G\colon \mathbb{R}^+ \to \mathbb{R}^+$, the following are true:*

(i) *Let $B \subset L^2(\mu)$ be a Banach space with norm $\|\cdot\|_B$. For any $f \in \mathcal{R}(B)$,*

$$\sqrt{1-\varepsilon}\ \mathbf{E} \sup_{n \in I} Z(S_n(f)) \leqslant \mathbf{E}\, \|F_{J,f}\|_B \sup_{\|h\|_B \leqslant 1} \int \sup_{n \in I} |S_n(h)|\; d\mu.$$

Moreover,

$$\mathbf{E}\, G\Big(\sqrt{1-\varepsilon}\sup_{n,m\in I}\big|Z(S_n(f))-Z(S_m(f))\big|\Big)$$
$$\leqslant \mathbf{E}\,\|F_{J,f}\|_B \sup_{\|h\|_B\leqslant 1}\mathbf{E}\int_X G\big(\sup_{n,m\in I}\big|(S_n-S_m)(h)\big|\big)\,d\mu.$$

(ii) *In particular, for any $f\in L^p(\mu)$ with $2\leqslant p<\infty$,*

$$\sqrt{1-\varepsilon}\sup_{\|f\|_{2,\mu}\leqslant 1}\mathbf{E}\sup_{n\in I} Z(S_n(f))\leqslant C_p\sup_{\|h\|_{p,\mu}\leqslant 1}\int\sup_{n\in I}|S_n(h)|\,d\mu,$$

where $C_p=\|g\|_p/\|g\|_2$, recalling the notation used. Further

$$\sup_{\|f\|_{2,\mu}\leqslant 1}\mathbf{E}\, G\Big(\sqrt{1-\varepsilon}\sup_{n,m\in I}\big|Z(S_n(f))-Z(S_m(f))\big|\Big)$$
$$\leqslant C_p\sup_{\|h\|_{p,\mu}\leqslant 1}\mathbf{E}\int_X G\big(\sup_{n,m\in I}\big|(S_n-S_m)(h)\big|\big)\,d\mu.$$

We have the following criterion providing a general form of Theorem 3.4.

Theorem 3.7. *Let S be satisfying assumption* (C). *Let $B\subset L^2(\mu)$ be a Banach space with norm $\|\cdot\|_B$. Assume that the following property is fulfilled:*

$$\mu\big\{\sup_{n\geqslant 1}|S_n(f)|<\infty\big\}=1,\qquad \forall\, f\in B.$$

Then there exists a constant K depending on S and B only such that

$$\mathbf{E}\sup_{n\geqslant 1} Z(S_n(f))\quad\leqslant\quad K\limsup_{H\to\infty}\mathbf{E}\,\|F_{H,f}\|_B,\qquad \forall\, f\in\mathcal{R}(B).$$

Let us derive a criterion which has been recently applied in [6] to show the optimality of a famous theorem of Koksma. Let $\{h_n,\ n\in\mathbb{Z}\}$ be a countable orthonormal basis of $L^2(\mu)$ and use the notation $f\sim\sum_{n\in\mathbb{Z}}a_n(f)\,h_n$, $\sum_{n\in\mathbb{Z}}a_n^2(f)<\infty$, if $f\in L^2(\mu)$. Given a sequence of positive reals $w=\{w_n,\ n\in\mathbb{Z}\}$ with $w_n\geqslant 1$, we recall that $L^2_w(\mu)$ is the sub-space of $L^2(\mu)$ consisting of functions such that

$$\sum_{n\in\mathbb{Z}}w_n\,a_n^2(f)<\infty.$$

This is a Hilbert space with scalar product defined by

$$\langle f, h\rangle = \sum_{n\in\mathbb{Z}} w_n\, a_n(f)\, a_n(h),$$

and norm

$$\|f\|_{2,w} = \Big(\sum_{n\in\mathbb{Z}} w_n\, a_n^2(f)\Big)^{1/2}.$$

The space $L^2(\mu)$ corresponds to the case $w_n \equiv 1$. Moreover, $L^2_w(\mu)$ trivially contains any f such that $a_n(f) = 0$ except for finitely many n.

Corollary 3.8. *Let S be satisfying assumption* (C)*. Assume that the following property is fulfilled:*

$$\mu\{\sup_{n\geqslant 1} |S_n(f)| < \infty\} = 1, \quad \textit{for all } f \in L^2_w(\mu).$$

Then there exists a constant K depending on S and w only such that

$$\sup_{\varepsilon>0} \varepsilon\sqrt{\log N_f(\varepsilon)} \leqslant K \limsup_{J\to\infty} \mathbf{E}\,\|F_{J,f}\|_{2,w}, \quad \textit{for all } f \in \mathcal{R}(L^2_w(\mu)).$$

Remark 3.9. Let $X = \mathbb{T}$, μ the normalized Lebesgue measure and let T_j be dilation operators, $T_j f(x) = f(jx)$. Then any finite trigonometric sum belongs to $\mathcal{R}(L^2_w(\mu))$.

We refer to [32, Chapter 6] for a study of the link between the partial maximum operators (I being a set integers).

$$\sup_{\substack{\|h\|_{\infty,\mu}\leqslant 1\\ \|h\|_{2,\mu}\leqslant\varepsilon}} \int_X \sup_{n\in I} |S_n(h)|\, d\mu \qquad \text{and} \qquad \sup_{\|f\|_{2,\mu}\leqslant 1} \mathbf{E} \sup_{n\in I} Z(S_n(f)). \tag{3.4}$$

In the theorem below, we provide a quantitative link.

Theorem 3.10. *Let S_n, $n \geqslant 1$, be $L^2(\mu)$-$L^\infty(\mu)$ continuous operators verifying condition* (C)*. Let I be any set of integers with cardinality M. For any reals $A > 0$, $R > 0$, it is true that*

$$\begin{aligned}\sup_{\|f\|_{2,\mu}\leqslant 1} \mathbf{E} \sup_{n\in I} Z(S_n(f)) &\leqslant 6\sqrt{M}\, S_1(I)\, \exp\{-A^2/8\} + A(\sqrt{2})\, S_2(I)\, e^{-R^2/4} \\ &\qquad + A \sup_{\substack{\|h\|_{\infty,\mu}\leqslant 1\\ \|h\|_{2,\mu}\leqslant R/A}} \int_X \sup_{n\in I} |S_n(h)|\, d\mu,\end{aligned}$$

where $S_1(I) = \max_{n\in I} \|S_n\|_2$, $S_2(I) = \max_{n\in I} \|S_n\|_\infty$, *and*

$$\|S_n\|_2 = \sup_{\|f\|_2 \leqslant 1} \|S_n(f)\|_2 \qquad \|S_n\|_\infty = \sup_{\|f\|_\infty \leqslant 1} \|S_n(f)\|_\infty.$$

Remark 3.11. It is not complicate to derive from this bound Theorem 3.3, for $L^2(\mu)$-$L^\infty(\mu)$ contractions.

Now consider the spaces $L^p(\mu)$, $1 < p < 2$. A corresponding entropy criterion can be also established.

Theorem 3.12. *Let* $1 < p \leqslant 2$ *with conjugate number* q. *Consider a sequence* $S = \{S_n, n \geqslant 1\}$ *of continuous operators from* $L^p(\mu)$ *to* $L^p(\mu)$. *Assume that condition* (C) *is satisfied.*

Further assume that for some real $0 < r < p$, *property* $(\mathcal{B}_r)$ *is satisfied. Then there exists a constant* $C(r,p) < \infty$ *depending on* r *and* p *only, such that for any* $f \in L^p(\mu)$,

$$\sup_{\varepsilon>0} \varepsilon \left(\log N_f^p(\varepsilon)\right)^{1/q} \leqslant C(r,p)\, \|f\|_p,$$

where $N_f^p(\varepsilon)$ *is the minimal number of open* L^p*-balls of radius* ε, *centered in* C_f *and enough to cover it. Further* $C(r,p)$ *tends to infinity as* r *tends to* p.

The proof given in [33] relies on properties of p-stable processes; it is assumed that S commutes with an ergodic endomorphism of $(X, \mathcal{A}, \mu)$, which in fact is unnecessary. The restriction $p \neq 1$ is only used at the very end of the proof, but is then crucially necessary.

Remark 3.13. The pending question of a possible convergence criterion for the space $L^1(\mu)$ is of course very interesting. But its true nature is unknown, since we are not operating in a (strictly) stationary context. In particular, $\|S_n(f) - S_m(f)\|_{p,\mu}$, crucial in (5.11), does not even depend on $n - m$ only, in general. Moreover, we know (see Talagrand [30, § 8.1]), that a necessary condition for a 1-stable process to be sample bounded rather expresses in terms of majorizing measures. This important concept is however not relevant in the present context because of its difficulty of application.

As announced already, we have made the paper self-contained. We provide proofs of these theorems in Section 5.

§4. Auxiliary Results.

4.1. L^p-isometries. We first recall a classical result of Lamperti [19, Theorem 3.1]. Let μ be a σ-finite measure on $(X,\mathcal{A})$. Some basic properties of isometries of $L^p(\mu)$ are used in what follows. Recall that a regular set isomorphism of the measure space $(X,\mathcal{A},\mu)$ is a mapping Θ of $\mathcal{A}$ into itself such that

$$\begin{aligned} &\text{(i)} && \Theta(A^c) = \Theta X \setminus \Theta A \\ &\text{(ii)} && \Theta\Big(\bigcup_{n=1}^{\infty} A_n\Big) = \bigcup_{n=1}^{\infty} \Theta A_n \quad \text{for disjoint } A_n \\ &\text{(iii)} && \mu(\Theta A) = 0 \text{ if and only if } \mu(A) = 0, \end{aligned} \tag{4.1}$$

for all elements A, A_n of $\mathcal{A}$. Then Θ induces a linear transformation (noted again by Θ) on the space of measurable functions, defined as follows, $\Theta\chi_A = \chi_{\Theta A}$.

Remark 4.1. The question whether a measure preserving set transformation can be obtained from a point mapping has been already considered. By a result of von Neumann, so is the case if for instance X is a closed region in $\mathbb{R}^n$ and μ is equivalent to the Lebesgue measure, see [19, p. 463].

Lemma 4.2. *Let T be a linear operator on $L^p(\mu)$ where $1 \leqslant p < \infty$ and $p \neq 2$, and such that $\|Tf\|_p = \|f\|_p$, for all $f \in L^p(\mu)$. Then there exists a regular set-isomorphism Θ and a function $h(x)$ such that T is given by*

$$Tf(x) = h(x)\,\Theta\, f(x).$$

Define a measure μ^* by $\mu^*(A) = \mu(\Theta^{-1}A)$. Then

$$|h(x)|^p = \frac{d\mu^*}{d\mu}(x) \qquad \text{a.e. on } \Theta X\,.$$

Corollary 4.3. *Let μ be a probability measure. Let T be a positive isometry of $L^p(\mu)$ with $1 \leqslant p < \infty$ and $p \neq 2$, such that $T1 = 1$. Then $Tf(x) = \Theta\, f(x)$ with $\Theta\,1 = 1$ and Θ is a regular set-isomorphism. Moreover for any $f \in L^\infty(\mu)$, $|Tf|^a \overset{\text{a.e.}}{=} T|f|^a$, for any $0 \leqslant a < \infty$. Further $|Tf|^p \overset{\text{a.e.}}{=} T|f|^p$, if $f \in L^p(\mu)$.*

Proof. By Lemma 4.2, $Tf(x) = h(x)\,\Theta\, f(x)$. As $\mu(X) = 1$ and $T1 = 1$ it follows from the proof of Theorem 3.1 in [19] that $h(x) \overset{\text{a.e.}}{=} 1$, and $T = \Theta$. But as $\Theta\chi_A = \chi_{\Theta A}$, we get $|Tf|^a = T|f|^a$ for simple functions, for any $0 \leqslant a < \infty$. Hence by approximation $|Tf|^a \overset{\text{a.e.}}{=} T|f|^a$ holds for all $f \in L^\infty(\mu)$. Further by approximation again, since T is an isometry of $L^p(\mu)$, $|Tf|^p \overset{\text{a.e.}}{=} T|f|^p$, if $f \in L^p(\mu)$. $\square$

For the sake of completeness, we included the following lemma concerning the (simpler) case $p = 2$.

Lemma 4.4. *Let T be a positive isometry of $L^2(\mu)$ such that $T1 = 1$. Then $(Tf)^2 \overset{\text{a.e.}}{=} Tf^2$, for any $f \in L^2(\mu)$.*

Proof. Let $A \in \mathcal{A}$ with $0 < \mu(A) < 1$. Trivially $f, g \in L^2_+(\mu)$ have disjoint supports if and only if $\|f + g\|^2_{2,\mu} = \|f\|^2_{2,\mu} + \|g\|^2_{2,\mu}$. Hence it follows that $T\mathbf{1}_A$ and $T\mathbf{1}_{A^c}$ have disjoint supports. Let $E = \{0 < T\mathbf{1}_A < 1\} = \{0 < T\mathbf{1}_{A^c} < 1\}$. As $E \subset \operatorname{supp}(T\mathbf{1}_A) \cap \operatorname{supp}(T\mathbf{1}_{A^c})$, we conclude that $T\mathbf{1}_A$ and $T\mathbf{1}_{A^c}$ are indicator functions. Thus any simple function is mapped by T into a simple function. For these functions we have $(Tf)^2 = Tf^2$. Now let $f \in L^2_+(\mu)$; there exists a sequence $(f_n) \subset L^\infty(\mu)$ such that $\|f - f_n\|_2 \to 0$ as $n \to \infty$. First observe by applying twice Hölder's inequality that

$$\begin{aligned}\int_X (T|f_n^2 - f^2|)^{1/2}\, d\mu &\leqslant \Big(\int_X T|f_n^2 - f^2|\, d\mu\Big)^{1/2} \\ &= \|f_n^2 - f^2\|_1^{1/2} \leqslant \big(\|f_n - f\|_2 \cdot \|f_n + f\|_2\big)^{1/2}.\end{aligned}$$

Consequently,

$$\begin{aligned}\|Tf - \sqrt{Tf^2}\|_1 &\leqslant \|Tf - Tf_n\|_1 + \|Tf_n - \sqrt{Tf_n^2}\|_1 + \|\sqrt{Tf_n^2} - \sqrt{Tf^2}\|_1 \\ &= \|f - f_n\|_1 + \|\sqrt{Tf_n^2} - \sqrt{Tf^2}\|_1 \\ &\leqslant \|f - f_n\|_1 + \big(\|f_n - f\|_2 \cdot \|f_n + f\|_2\big)^{1/2} \to 0,\end{aligned}$$

as $n \to \infty$. Hence $(Tf)^2 = f^2$ a.e.. As $f = f^+ - f^-$, we deduce that this holds for any $f \in L^2(\mu)$. $\square$

4.2. Stable processes. This part was essentially written for the ergodician reader not necessarily familiar with stable processes. We use very few from the theory. We refer to [24]. We also refer the interested reader to the very nice book of Talagrand [30] for a thorough study of the regularity of stable processes. For the same reason, the last part of the proof of Theorem 3.12 is detailed and we refer to [24]. The stable processes we consider are simple, of finite rank. They are however *not* strongly stationary. Recall and briefly explain some basic facts and properties of stable random variables and stable processes.

Let $0 < \alpha \leqslant 2$. A real valued random variable θ is symmetric α-stable of parameter σ if

$$\mathbf{E}\, e^{it\theta} = e^{-\sigma^\alpha |t|^\alpha}, \qquad \forall\, t \in \mathbb{R}. \tag{4.2}$$

Then for all $0 < r < \alpha$, $(\mathbf{E}\,|\theta|^r)^{1/r} = \delta(r,\alpha)\,\sigma$, where $\delta(r,\alpha)$ depends only on r and α. Stable random variables are mixtures of Gaussian random variables. Indeed, as is well-known the function $f(\lambda) = e^{-\lambda^\alpha}$ is completely monotone on $\mathbb{R}^+$, for each $0 < \alpha \leqslant 1$. Consequently, there exists a random variable $v(\alpha)$ such that $\mathbf{E}\, e^{-\lambda v(\alpha)} = f(\lambda)$, for all $\lambda \geqslant 0$. Let $\eta(\alpha) := (2\,v(\alpha/2))^{1/2}$. Let g be standard Gaussian independent from η. By taking Fourier transforms $\mathbf{E}\, e^{it\eta(\alpha)\cdot g} = \mathbf{E}\, e^{-t^2\eta(\alpha)^2/2} = \mathbf{E}\, e^{-t^2 v(\alpha/2)} = e^{-|t|^\alpha}$. Whence it follows that $\theta \overset{\mathcal{D}}{=} \eta(\alpha)\cdot g$. Let $\theta_1, \ldots, \theta_J$ be i.i.d. α-stable real valued random variables, and let $c_1, \ldots, c_J$ be real numbers. From (4.2) we get

$$\sum_{j=1}^{J} c_j \theta_j \overset{\mathcal{D}}{=} \theta_1 \Big(\sum_{j=1}^{J} |c_j|^\alpha \Big)^{1/\alpha}. \tag{4.3}$$

A stochastic process $\{X(t), t \in T\}$ is a real valued α-stable if any finite linear combination $\sum_j c_j X(t_j)$ is an α-stable real valued random variable.

From now on, we extend the notation used in (2.4) in the following way. To any $f \in L^p(\mu)$, $1 < p \leqslant \infty$, we associate the random element,

$$F_{J,f}(\omega, x) = \frac{1}{J^{1/p}} \sum_{1\leqslant j\leqslant J} \theta_j(\omega)\, T_j f(x), \qquad \omega \in \Omega,\ x \in X. \tag{4.4}$$

Remark 4.5. As long as entropy criteria are studied in $L^p(\mu)$ with $2 \leqslant p \leqslant \infty$, the relevant random elements $F_{J,f}$ are Gaussian ($\alpha = 2$). When $1 < p < 2$, we choose them p-stable ($\alpha = p$).

Clearly (4.4) defines a real valued α-stable process. It follows in particular that for any $x \in X$,

$$F_{J,f}(\,\cdot\,, x) \overset{\mathcal{D}}{=} \theta_1 \Big(\frac{1}{J} \sum_{1\leqslant j\leqslant J} |T_j f(x)|^\alpha \Big)^{1/\alpha}. \tag{4.5}$$

Let $\{\eta_j, j = 1, \ldots, J\}$ be a sequence of i.i.d. random variables with the same law than $\eta(\alpha)$, and let $\{g_j, j = 1, \ldots, J\}$ be a sequence of i.i.d. Gaussian standard random variables. We assume that these sequences are respectively defined on joint probability spaces $(\Omega', \mathcal{B}', \mathbf{P}')$ and $(\Omega'', \mathcal{B}'', \mathbf{P}'')$.

Then the process

$$\mathcal{F}_{J,f}(\omega',\omega'',x)=\sum_{j\leqslant J}\eta_j(\omega')\,g_j(\omega'')\,T_jf(x),\qquad x\in X,$$

has the same distribution as $\{F_{J,f}(x),x\in X\}$.

4.3. A comparison Lemma. In the next lemma, we denote the norms corresponding to the spaces $L^r(\mu)$ and $L^r(\mathbf{P})$ respectively by $\|\cdot\|_{r,\mu}$ and $\|\cdot\|_{r,\mathbf{P}}$.

Lemma 4.6. *Let $1\leqslant p\leqslant 2$. Let $S_n\colon L^p(\mu)\to L^p(\mu)$, $n=1,2,\dots,$ be continuous operators verifying assumption* (C).

(i) *Let $1\leqslant p<2$. Let $f\in L^p(\mu)$ and $0<\varepsilon<1$. Let also I be a finite set of integers such that*

$$\|S_n(f)-S_m(f)\|_{p,\mu}\neq 0,\qquad \textit{for all } n\neq m,\ n,m\in I.$$

Then given any index $\mathcal{J}_0$, there exists a sub-index $\mathcal{J}$ and a measurable set A with $\mu\{A\}\geqslant 1-\varepsilon$, and such that for all $x\in A$, we have

$$(1-\varepsilon)^{1/p}\leqslant\frac{\big\|(S_n-S_m)(F_{J,f})(x)\big\|_{r,\mathbf{P}}}{c(r)\,\|S_n(f)-S_m(f)\|_{p,\mu}}\leqslant(1+\varepsilon)^{1/p},\tag{4.6}$$

for all $J\in\mathcal{J}$, all $n,m\in I$, $m\neq n$, and $r<p$. Moreover, $c(r)=\|\theta_1\|_r$.

(ii) *Let $p=2$. We have in place of* (4.6),

$$(1-\varepsilon)^{1/2}\leqslant\frac{\big\|(S_n-S_m)(F_{J,f})(x)\big\|_{2,\mathbf{P}}}{\|S_n(f)-S_m(f)\|_{2,\mu}}\leqslant(1+\varepsilon)^{1/2},\tag{4.7}$$

for all $J\in\mathcal{J}$, all $n,m\in I$, $m\neq n$. Further, for any positive increasing convex function G on $\mathbb{R}^+$, any $J\in\mathcal{J}$,

$$\mathbf{E}\,G\Big(\sqrt{1-\varepsilon}\sup_{n,m\in I}\ Z(S_n(f))-Z(S_m(f))\Big)\leqslant\mathbf{E}\int_X G\big(\sup_{n,m\in I}\ S_n(F_{J,f})-S_m(F_{J,f})\big)\,d\mu.$$

In particular for any $J\in\mathcal{J}$,

$$\sqrt{1-\varepsilon}\ \mathbf{E}\sup_{n\in I}\ Z(S_n(f))\ \leqslant\ \mathbf{E}\int_X\sup_{n\in I}\ S_n(F_{J,f})\ d\mu.\tag{4.8}$$

Proof. We assume $\mathcal{J}_0 = \mathbb{N}$, the case of an arbitrary index $\mathcal{J}_0$ being treated identically.

Proof of (i): Let $f \in L^p(\mu)$. By the commutation assumption,

$$S_n(F_{J,f}) = \frac{1}{J^{\frac{1}{p}}} \sum_{j \leqslant J} \theta_j S_n(T_j(f)) = \frac{1}{J^{1/p}} \sum_{j \leqslant J} \theta_j T_j(S_n(f)), \qquad \forall\, n \geqslant 1,\ \forall\, J \geqslant 1.$$

Hence by (4.3), for any fixed $x \in X$,

$$\frac{1}{J^{\frac{1}{p}}} \sum_{j \leqslant J} \theta_j T_j(S_n(f) - S_m(f))(x) \overset{\mathcal{D}}{=} \theta_1 \Big(\frac{1}{J} \sum_{j \leqslant J} \big| T_j(S_n(f) - S_m(f))(x) \big|^p \Big)^{1/p}.$$

Using the fact that $|T_j h|^p \overset{\text{a.e.}}{=} T_j |h|^p$ if $h \in L^p(\mu)$, it follows that

$$\begin{aligned} \mathbf{E}\, \big|(S_n - S_m)(F_{J,f})(x)\big|^r &= (\mathbf{E}\, |\theta_1|^r) \Big(\frac{1}{J} \sum_{j \leqslant J} \big| T_j(S_n(f) - S_m(f))(x) \big|^p \Big)^{r/p} \\ &= (\mathbf{E}\, |\theta_1|^r) \Big(\frac{1}{J} \sum_{j \leqslant J} T_j(\big| S_n(f) - S_m(f) \big|^p)(x) \Big)^{r/p}, \end{aligned} \tag{4.9}$$

for almost all x.

Let I be a finite set of integers such that

$$\|S_n(f) - S_m(f)\|_{p,\mu} \geqslant \delta > 0, \qquad \text{for all } n \neq m,\ n, m \in I.$$

Let $0 < \varepsilon < 1$ and choose an integer L sufficiently large so that $2^{-L} \leqslant \varepsilon$ and $\delta \geqslant 2^{-L-1}/\varepsilon$. Assumption (C) implies that

$$\lim_{J \to \infty} \Big\| \frac{1}{J} \sum_{j \leqslant J} T_j(\big| S_n(f) - S_m(f) \big|^p) - \|S_n(f) - S_m(f)\|_{p,\mu}^p \Big\|_{1,\mu} = 0,$$

for all $n, m \in I$. By extraction, we can find an index $\mathcal{J} = \{J_k, k > L\}$ (depending on I and ε), such that

$$\Big\| \frac{1}{J_k} \sum_{j \leqslant J_k} T_j(\big| S_n(f) - S_m(f) \big|^p) - \|S_n(f) - S_m(f)\|_{p,\mu}^p \Big\|_{1,\mu} \leqslant \frac{1}{\#(I)^2 2^{2k}},$$

for all $n, m \in I$ and all $k > L$. Put

$$A_k = \Big\{\exists\, n, m \in I :$$
$$\Big|\frac{1}{J_k}\sum_{j\leqslant J_k} T_j(\big|S_n(f) - S_m(f)\big|^p) - \|S_n(f) - S_m(f)\|_{p,\mu}^p\Big| \geqslant 2^{-k}\Big\},$$
$$k > L.$$

By Chebyshev's inequality, we have $\mu(A_k) \leqslant 2^{-k}$. Let

$$\begin{aligned} &A_\varepsilon(n, m, J) \\ &= \Big\{(1-\varepsilon)\|S_n(f) - S_m(f)\|_{p,\mu}^p \leqslant \frac{1}{J}\sum_{j\leqslant J} T_j\Big(\big|S_n(f) - S_m(f)\big|^p\Big) \\ &\qquad\qquad \leqslant (1+\varepsilon)\|S_n(f) - S_m(f)\|_{p,\mu}^p\Big\}, \end{aligned}$$

and

$$A_{I,\varepsilon} = \bigcap_{k>L}\ \bigcap_{n,m\in I} A_\varepsilon(n, m, J_k).$$

Then,

$$\mu\{A_{I,\varepsilon}\} \geqslant \mu\Big\{\bigcap_{k>L} A_k^c\Big\} \geqslant 1 - \sum_{k>L} 2^{-k} = \quad 1 - 2^{-L} \geqslant 1 - \varepsilon.$$

As by (4.9), for any $r < p$,

$$\big\|(S_n - S_m)(F_{J,f})(x)\big\|_{r,\mathbf{P}} \quad = \quad \|\theta_1\|_r\Big(\frac{1}{J}\sum_{j\leqslant J} T_j(\big|S_n(f) - S_m(f)\big|^p)(x)\Big)^{1/p},$$

it follows that for every $x \in A_{I,\varepsilon}$, we have

$$(1-\varepsilon)^{1/p} \leqslant \frac{\big\|(S_n - S_m)(F_{J,f})(x)\big\|_{r,\mathbf{P}}}{\|\theta_1\|_r\, \|S_n(f) - S_m(f)\|_{p,\mu}} \leqslant (1+\varepsilon)^{1/p},$$

for all $J \in \mathcal{J}$, all $n, m \in I$, $m \neq n$, and $r < p$.

Proof of (ii): The proof is the first inequality is identical and so we omit it. Let $f \in L^2(\mu)$. Let $0 < \varepsilon < 1$ be fixed. Let I be a finite set of integers such that

$$\|S_n(f) - S_m(f)\|_{p,\mu} \neq 0, \qquad \text{for all } n \neq m,\ n, m \in I.$$

Now notice that $\mu\{\ \mathbf{E}\sup_{n\in I}\ S_n(F_{J,f}) \geqslant 0\ \} = 1$. Using (4.7), next Slepian comparison lemma, we have along the index $\mathcal{J}$,

$$\begin{aligned}\int_X \mathbf{E}\sup_{n\in I}\ S_n(F_{J,f})\,d\mu &\geqslant \int_A \mathbf{E}\sup_{n\in I}\ S_n(F_{J,f})\,d\mu\\ &\geqslant \sqrt{1-\varepsilon}\,\mu(A)\,\mathbf{E}\sup_{n\in I}\ Z(S_n(f))\\ &\geqslant (1-\varepsilon)\,\mathbf{E}\sup_{n\in I}\ Z(S_n(f)).\end{aligned}$$

Similarly,

$$\begin{aligned}&\int_X \mathbf{E}\,G\big(\sup_{n,m\in I}\ S_n(F_{J,f})-S_m(F_{J,f})\big)\,d\mu\\ &\quad\geqslant \int_A \mathbf{E}\,G\big(\sup_{n,m\in I}\ S_n(F_{J,f})-S_m(F_{J,f})\big)\,d\mu\\ &\quad\geqslant \int_A \mathbf{E}\,G\big(\sqrt{1-\varepsilon}\sup_{n,m\in I}\ Z(S_n(f))-Z(S_m(f))\big)\,d\mu\\ &\quad\geqslant \sqrt{1-\varepsilon}\,\mathbf{E}\,G\big(\sqrt{1-\varepsilon}\sup_{n,m\in I}\ Z(S_n(f))-Z(S_m(f))\big)\\ &\quad\geqslant \sqrt{1-\varepsilon}\,\mathbf{E}\,G\big(\sqrt{1-\varepsilon}\sup_{n,m\in I}\ Z(S_n(f))-Z(S_m(f))\big).\end{aligned}$$

This completes the proof of Lemma 4.6. □

4.4. Banach Principle. Let

$$\mathcal{Y} = \{f \in L^\infty(\mu) : \|f\|_\infty \leqslant 1\}.$$

A mapping $V\colon (\mathcal{Y}, d) \to L^0(\mu)$ is said to be continuous at 0, if V is d-continuous at 0 on $\mathcal{Y}$. When V is linear, then V is continuous at 0 if and only if V is d-continuous on $L^\infty(\mu)$.

Lemma 4.7 ([4]). *Let $\{S_n,\ n \geqslant 1\}$ be a sequence of linear operators of $L^\infty(\mu)$ in $L^0(\mu)$. Assume that the following conditions are realized:*

(i) *Each S_n is continuous at 0,*
(ii) *For any $f \in L^\infty(\mu)$, $\mu\{x : \{S_n(f)(x), n \geqslant 1\}$ converges$\} = 1$.*

Then $S^*\colon \mathcal{Y} \to L^0(\mu)$ is continuous at 0.

For a short proof, we refer to [32, p. 205]. The next lemma is used repeatedly.

Lemma 4.8. *Let $(B, \|\cdot\|_B)$ be a Banach space and let $S_n\colon B \to L^0(\mu)$, $n \geqslant 1$, be continuous in measure operators. Assume that*

$$\mu\{\sup_{n\geqslant 1} |S_n(f)| < \infty\} = 1, \qquad \text{for all } f \in B.$$

Then there exists a non-increasing function $C\ :]0,1] \to \mathbb{R}_+$ such that for any $0 < \varepsilon < 1$, any $J \geqslant 1$ and any $f \in \mathcal{R}(B)$, there exists a measurable set $X_{\varepsilon,J,f}$ with $\mu(X_{\varepsilon,J,f}) \geqslant 1 - \varepsilon$, such that

$$\mathbf{P}\Big\{\omega : \sup_{n\geqslant 1} |S_n(F_{J,f}(\omega,\cdot))(x)| \leqslant C(\varepsilon)\mathbf{E}\,\|F_{J,f}\|_B\Big\} \geqslant 1 - \varepsilon,$$

for any $x \in X_{\varepsilon,J,f}$, recalling that $F_{J,f}$ are defined in (4.4).

Proof. By the Banach principle, there exists a non-increasing function $\delta\ :]0,1] \to \mathbb{R}_+$ such that

$$\mu\Big\{\sup_{n} |S_n(h)| \geqslant \delta(\varepsilon)\,\|h\|_B\Big\} \leqslant \varepsilon^2/2, \qquad \forall\, 0 < \varepsilon \leqslant 1,\ \forall\, h \in B.$$

Let $f \in \mathcal{R}(B)$, then $F_{J,f} \in \mathcal{R}(B)$ almost surely. Taking $h = F_{J,f}$ and using Fubini's theorem, gives

$$\int_X \mathbf{P}\Big\{\sup_{n\geqslant 1} |S_n(F_{J,f})| \geqslant \delta(\varepsilon)\,\|F_{J,f}\|_B\Big\}\,d\mu \leqslant \varepsilon^2/2.$$

Now we bound as follows

$$\begin{aligned}
&\int_X \mathbf{P}\Big\{\sup_{n\geqslant 1} |S_n(F_{J,f})| \geqslant \frac{2\,\delta(\varepsilon)}{\varepsilon^2}\,\mathbf{E}\,\|F_{J,f}\|_B\Big\}\,d\mu \\
&\leqslant \int_X \mathbf{P}\Big\{\sup_{n\geqslant 1} |S_n(F_{J,f})| \geqslant \frac{2\,\delta(\varepsilon)}{\varepsilon^2}\,\mathbf{E}\,\|F_{J,f}\|_B, \|F_{J,f}\|_B \leqslant \frac{2}{\varepsilon^2}\,\mathbf{E}\,\|F_{J,f}\|_B\Big\}\,d\mu \\
&+ \qquad \mathbf{P}\Big\{\|F_{J,f}\|_B > \frac{2}{\varepsilon^2}\,\mathbf{E}\,\|F_{J,f}\|_B\Big\} \\
&\leqslant \int_X \mathbf{P}\Big\{\sup_{n\geqslant 1} |S_n(F_{J,f})| \geqslant \delta(\varepsilon)\,\|F_{J,f}\|_B\Big\}\,d\mu + \varepsilon^2/2 \leqslant \varepsilon^2/2 + \varepsilon^2/2 \ = \varepsilon^2.
\end{aligned}$$

Hence,

$$\mu\Big\{x \in X : \mathbf{P}\Big\{\omega : \sup_{n\geqslant 1} |S_n(F_{J,f}(\omega,\cdot))(x)| \geqslant \frac{2\,\delta(\varepsilon)}{\varepsilon^2}\,\mathbf{E}\,\|F_{J,f}\|_B\Big\} \geqslant \varepsilon\Big\} \leqslant \varepsilon,$$

or

$$\mu\Big\{x \in X : \mathbf{P}\Big\{\omega :$$
$$\sup_{n\geqslant 1} |S_n(F_{J,f}(\omega, \cdot))(x)| \leqslant \frac{2\,\delta(\varepsilon)}{\varepsilon^2}\,\mathbf{E}\,\|F_{J,f}\|_B\Big\} \geqslant 1-\varepsilon\Big\} \geqslant 1-\varepsilon.$$

By letting $C(\varepsilon) = \frac{2\delta(\varepsilon)}{\varepsilon^2}$, we easily conclude. □

4.5. Some Gaussian tools. The next lemma is well-known in the theory of Gaussian processes. We refer for instance to [32, Chapter 10].

Lemma 4.9. *Let $X = \{X_t, t \in T\}$ and $Y = \{Y_t, t \in T\}$ be two centered Gaussian processes defined on a finite set T.*

(a) [Slepian's Lemma] *Assume that for any $s, t \in T$,*

$$\|X_s - X_t\|_2 \leqslant \|Y_s - Y_t\|_2.$$

Then for any positive increasing convex function f on $\mathbb{R}^+$,

$$\mathbf{E}\,f\big[\sup_{T\times T}(X_s - X_t)\big] \leqslant \mathbf{E}\,f\big[\sup_{T\times T}(Y_s - Y_t)\big].$$

In particular,

$$\mathbf{E}\sup_{t\in T} X_t \leqslant \mathbf{E}\sup_{t\in T} Y_t.$$

(b) [Sudakov's minoration] *There exists a universal constant B such that for any Gaussian process $X = \{X_t, t \in T\}$*

$$\mathbf{E}\sup_{t\in T} X_t \geqslant B \inf_{\substack{s,t\in T\\ s\neq t}} \|X_s - X_t\|_{2,\mathbf{P}} \sqrt{\log \#(T)}.$$

(c) [Lower bound for Gaussian norms] *Let X be a Gaussian vector and N a non-negative semi-norm. Then*

$$\mathbf{P}\{N(X) < \infty\} = 1 \qquad \Rightarrow \qquad \mathbf{P}\Big\{N(X) \geqslant \frac{1}{2}\,\mathbf{E}\,N(X)\Big\} \geqslant c$$

where $0 < c < 1$ is a universal constant.

(d) [Mill's ratio] *The Mill's ratio $R(x) = e^{x^2/2}\int_x^\infty e^{-t^2/2}\,dt$ verifies for any $x \geqslant 0$,*

$$\frac{2}{\sqrt{x^2+4}+x} \leqslant R(x) \leqslant \frac{2}{\sqrt{x^2+\frac{8}{\pi}}+x} \leqslant \sqrt{\frac{\pi}{2}}\,.$$

It follows that for any standard Gaussian random variable g, any $T > 0$,

$$\mathbf{E}\, g^2 \chi\{|g| \geqslant T\} \leqslant 6\, e^{-T^2/4}.$$

§5. Proofs.

As clarified in Remark 4.5, we use the random elements $F_{J,f}$ introduced in (4.4) differently, according to the cases $2 \leqslant p \leqslant \infty$, in which they are Gaussian, and $1 < p < 2$, where we choose them p-stable.

5.1. Proof of Theorem 3.7. Let $0 < \varepsilon < 1/2$. Let $f \in \mathcal{R}(B)$. By Lemma 4.8, there exists a non-increasing function $C :]0,1] \to \mathbb{R}_+$ and a set $X_{\varepsilon,J,f}$ of measure greater than $1 - \varepsilon$ such that for all $x \in X_{\varepsilon,J,f}$,

$$\mathbf{P}\Big\{\omega : \sup_{n \geqslant 1} |S_n(F_{J,f}(\omega, \cdot)(x)| \leqslant C(\varepsilon)\, \mathbf{E}\, \|F_{J,f}\|_B\Big\} \geqslant 1 - \varepsilon.$$

Estimate (3.3) implies

$$\mathbf{E} \sup_{n \geqslant 1} |S_n(F_{J,f}(\omega, \cdot))(x)| \quad \leqslant \quad \frac{4C(\varepsilon)}{1-\varepsilon}\, \mathbf{E}\, \|F_{J,f}\|_B, \qquad \forall\, x \in X_{\varepsilon,J,f}.$$

Recall that $B \subset L^2(\mu)$. Let I be a finite set of integers such that $\|S_n(f) - S_m(f)\|_2 \neq 0$, for all $m, n \in I$, $m \neq n$. By Lemma 4.6(ii), taking $\mathcal{J}_0 = \mathbb{N}$, there exists a sub-index $\mathcal{J}$ such that if

$$A(I) = \Big\{\forall J \in \mathcal{J},\, \forall n, m \in I,\, m \neq n,\ \frac{\|S_n(F_{J,f}) - S_m(F_{J,f})\|_{2,\mathbf{P}}}{\|S_n(f) - S_m(f)\|_{2,\mu}} \geqslant \sqrt{1-\varepsilon}\Big\},$$

then $\mu\{A(I)\} \geqslant \sqrt{1-\varepsilon}$.

By integrating on $X_{\varepsilon,J,f} \cap A(I)$, next using the fact that

$$\mathbf{E} \sup_{n \in I} S_n(F_{J,f}) \geqslant 0,$$

and Lemma 4.9(a), we get for any $J \in \mathcal{J}$,

$$\begin{aligned}
\int_{X_{\varepsilon,J,f}} \mathbf{E} \sup_{n \in I} S_n(F_{J,f})\, d\mu \quad &\geqslant \quad \int_{X_{\varepsilon,J,f} \cap A(I)} \mathbf{E} \sup_{n \in I} S_n(F_{J,f})\, d\mu \\
&\geqslant \quad \sqrt{1-\varepsilon}\, \mu\{X_{\varepsilon,J,f} \cap A(I)\}\, \mathbf{E} \sup_{n \in I} Z(S_n(f)) \\
&\geqslant \quad \sqrt{1-\varepsilon}\, (\sqrt{1-\varepsilon} - \varepsilon)\, \mathbf{E} \sup_{n \in I} Z(S_n(f)) \\
&\geqslant \quad (1 - 2\varepsilon)\, \mathbf{E} \sup_{n \in I} Z(S_n(f)).
\end{aligned}$$

By combining, for any $J \in \mathcal{J}$,

$$\mathbf{E} \sup_{n\in I} Z(S_n(f)) \leqslant \frac{1}{(1-2\varepsilon)} \int\limits_{X_{\varepsilon,J,f}} \mathbf{E} \sup_{n\in I} S_n(F_{J,f})\, d\mu \tag{5.1}$$
$$\leqslant K(\varepsilon)\, \mathbf{E}\, \|F_{J,f}\|_B,$$

with

$$K(\varepsilon) = \frac{4C(\varepsilon)}{(1-2\varepsilon)(1-\varepsilon)}\,.$$

Therefore, for any $f \in \mathcal{R}(B)$, any finite set I,

$$\mathbf{E} \sup_{n\in I} Z(S_n(f)) \leqslant K(\varepsilon) \inf_{J\in\mathcal{J}} \sup_{H\geqslant J} \mathbf{E}\, \|F_{H,f}\|_B = K(\varepsilon) \limsup_{H\to\infty} \mathbf{E}\, \|F_{H,f}\|_B.$$

Taking $I = [1, N]$ and letting next N tends to infinity, gives

$$\mathbf{E} \sup_{n\geqslant 1} Z(S_n(f)) \leqslant K(\varepsilon) \limsup_{H\to\infty} \mathbf{E}\, \|F_{H,f}\|_B.$$

5.2. Proof of Theorem 3.4. Let $f \in L^\infty(\mu)$. Fubini's theorem and Lemma 4.4 allow us to write,

$$\mathbf{E} \int |F_{J,f}|^p d\mu \leqslant C_p^p \int \big(\mathbf{E}\, |F_{J,f}|^2\big)^{p/2} d\mu$$
$$= C_p^p \int \Big(\frac{1}{J} \sum_{j\leqslant J} T_j f^2(x)\Big)^{p/2} d\mu(x).$$

By assumption

$$\lim_{J\to\infty} \Big\| \frac{1}{J} \sum_{j\leqslant J} T_j f^2 - \|f\|_{2,\mu}^2 \Big\|_{1,\mu} = 0.$$

By proceeding by extraction, this convergence also holds almost surely along some subsequence $\mathcal{J}_0$. As $\frac{1}{J} \sum\limits_{j\leqslant J} T_j f^2(x) \leqslant \|f\|_\infty^2$, we further deduce from the dominated convergence theorem,

$$\lim_{\mathcal{J}_0\ni J\to\infty} \mathbf{E} \int \Big(\frac{1}{J} \sum_{j\leqslant J} T_j f^2(x)\Big)^{p/2} d\mu = \|f\|_{2,\mu}^p.$$

Let $0 < \varepsilon < 1$. Extracting if necessary from $\mathcal{J}_0$ a sub-index which we call again $\mathcal{J}_0$, we thus conclude that

$$\mathbf{E}\, \|F_{J,f}\|_{p,\mu} \leqslant (1+\varepsilon)\, C_p\, \|f\|_{2,\mu}, \qquad \forall\, J \in \mathcal{J}_0.$$

Next the proof is exactly the same as before except that we replace everywhere the norm $\|.\|_B$ by the norm $\|.\|_{p,\mu}$. Let I be a finite set of integers.

From Lemma 4.6, we can extract from $\mathcal{J}_0$ a partial index $\mathcal{J}$ such that the analog of (5.1) holds, namely for any $J \in \mathcal{J}$,

$$\mathbf{E} \sup_{n \in I} Z(S_n(f)) \quad \leqslant \quad K(\varepsilon)\, \mathbf{E}\, \|F_{J,f}\|_{p,\mu} \ \leqslant\ C_p\, K(\varepsilon)\, (1+\varepsilon)\, \|f\|_{p,\mu}.$$

A simple approximation argument allows to get the same inequality for all $f \in L^p$. Sudakov's minoration implies

$$\sup_{\varrho > 0} \varrho \sqrt{\log N_f(\varrho)} \ \leqslant\ C_p\, K(\varepsilon)\, (1+\varepsilon)\, \|f\|_p.$$

5.3. Proof of Theorem 3.6. (i) By Lemma 4.6-(b), given any index $\mathcal{J}_0$, there exists an index $\mathcal{J} \subseteq \mathcal{J}_0$ such that for any $J \in \mathcal{J}$,

$$(1-\varepsilon)\, \mathbf{E} \sup_{n \in I} Z(S_n(f)) \leqslant \mathbf{E} \int \sup_{n \in I} S_n(F_{J,f})\ d\mu.$$

Moreover, for any positive increasing convex function G on $\mathbb{R}^+$, any $J \in \mathcal{J}$,

$$\mathbf{E}\, G\Big(\sqrt{1-\varepsilon} \sup_{n,m \in I} Z(S_n(f)) - Z(S_m(f))\Big)$$
$$\leqslant \mathbf{E} \int_X G\big(\sup_{n,m \in I} (S_n - S_m)(F_{J,f})\big)\, d\mu.$$

In the following calculation we put

$$L = \sup_{\|g\|_B \leqslant 1} \int \sup_{n \in I} |S_n(g)|\ d\mu,$$

and we let $u_0 = 0$, $u_n = \varepsilon(1+\varepsilon)^{n-1}$ $n \geqslant 1$. Then

$$\begin{aligned}
&\mathbf{E} \int \sup_{n \in I} |S_n(F_{J,f})|\ d\mu \\
&= \sum_{k=1}^{\infty} \mathbf{E} \left(\mathbf{1}_{u_{k-1} \leqslant \|F_{J,f}\|_B < u_k} \cdot \int \sup_{n \in I} |S_n(F_{J,f})|\ d\mu \right) \\
&\leqslant \sum_{k=1}^{\infty} \mathbf{P}\big\{u_{k-1} \leqslant \|F_{J,f}\|_B < u_k\big\} \sup_{u_{k-1} \leqslant \|g\|_B < u_k} \int \sup_{n \in I} |S_n(g)|\ d\mu \\
&\leqslant L \sum_{k=1}^{\infty} u_k \mathbf{P}\big\{u_{k-1} \leqslant \|F_{J,f}\|_B < u_k\big\} \\
&\leqslant L\big(u_1 \mathbf{P}\{\|F_{J,f}\|_B < u_1\} + (1+\varepsilon)\, \mathbf{E}\, \|F_{J,f}\|_B \cdot \mathbf{1}_{u_1 \leqslant \|F_{J,f}\|_B}\big) \\
&\leqslant L(\varepsilon + (1+\varepsilon)\, \mathbf{E}\, \|F_{J,f}\|_B).
\end{aligned}$$

By combining, and letting next ε tends to 0, we get for any $f \in L^2(\mu)$,

$$\mathbf{E} \sup_{n\in I} Z(S_n(f)) \leqslant \mathbf{E}\,\|F_{J,f}\|_B \sup_{\|g\|_B\leqslant 1} \int \sup_{n\in I} |S_n(g)|\, d\mu.$$

Similarly

$$\mathbf{E}\, G\Big(\sqrt{1-\varepsilon} \sup_{n,m\in I} \big|Z(S_n(f)) - Z(S_m(f))\big|\Big)$$
$$\leqslant \mathbf{E}\,\|F_{J,f}\|_B \sup_{\|g\|_B\leqslant 1} \mathbf{E} \int_X G\big(\sup_{n,m\in I} \big|(S_n - S_m)(g)\big|\big)\, d\mu.$$

(ii) Let $B = L^p(\mu)$. We have seen that there exists an index $\mathcal{J}_0$ such that

$$\mathbf{E}\,\|F_{J,f}\|_{p,\mu} \leqslant (1+\varepsilon) C_p \|f\|_{2,\mu}, \qquad \forall\, J \in \mathcal{J}_0.$$

Therefore

$$\sup_{\|f\|_{2,\mu}\leqslant 1} \mathbf{E} \sup_{n\in I} Z(S_n(f)) \leqslant C_p \sup_{\|g\|_{p,\mu}\leqslant 1} \int \sup_{n\in I} |S_n(g)|\, d\mu.$$

Moreover,

$$\sup_{\|f\|_{2,\mu}\leqslant 1} \mathbf{E}\, G\Big(\sqrt{1-\varepsilon} \sup_{n,m\in I} \big|Z(S_n(f)) - Z(S_m(f))\big|\Big)$$
$$\leqslant C_p \sup_{\|g\|_{p,\mu}\leqslant 1} \mathbf{E} \int_X G\big(\sup_{n,m\in I} \big|(S_n - S_m)(g)\big|\big)\, d\mu.$$

5.4. Proof of Theorem 3.12. Let $f \in L^p(\mu)$. Let J be any positive integer and $x \in X$. By (4.5),

$$\frac{1}{J^{\frac{1}{p}}} \sum_{j\leqslant J} \theta_j T_j f(x) \overset{\mathcal{D}}{=} \theta_1 \Big(\frac{1}{J} \sum_{j\leqslant J} |T_j f(x)|^p\Big)^{1/p}.$$

Thus for any $r < p$,

$$\mathbf{E}\, |F_{J,f}(x)|^r = (\mathbf{E}\, |\theta_1|^r) \Big(\frac{1}{J} \sum_{j\leqslant J} |T_j f(x)|^p\Big)^{r/p}.$$

By Corollary 4.3, $|T_j f(x)|^p \overset{\text{a.e.}}{=} T_j |f|^p(x)$, so that we have

$$\mathbf{E}\, |F_{J,f}(x)|^r = (\mathbf{E}\, |\theta_1|^r) \Big(\frac{1}{J} \sum_{j\leqslant J} T_j |f|^p(x)\Big)^{r/p}. \tag{5.2}$$

for almost all x and all $J \geqslant 1$. As trivially $T_j|f|^p \in L^1(\mu)$, we deduce

$$\mathbf{E} \int_X |F_{J,f}(x)|^r \, d\mu(x) = (\mathbf{E}\, |\theta_1|^r) \int_X \Big(\frac{1}{J} \sum_{j \leqslant J} T_j|f|^p(x)\Big)^{\frac{r}{p}} \, d\mu(x). \tag{5.3}$$

Hence,

$$\begin{aligned}
&\Big| \mathbf{E} \int_X |F_{J,f}(x)|^r \, d\mu(x) - (\mathbf{E}\, |\theta_1|^r) \|f\|_{p,\mu}^r \Big| \\
&= (\mathbf{E}\, |\theta_1|^r) \Big| \int_X \Big(\frac{1}{J} \sum_{j \leqslant J} T_j|f|^p(x)\Big)^{\frac{r}{p}} \, d\mu(x) - (\|f\|_{p,\mu}^p)^{\frac{r}{p}} \Big| \\
&\leqslant (\mathbf{E}\, |\theta_1|^r) \int_X \Big| \Big(\frac{1}{J} \sum_{j \leqslant J} T_j|f|^p(x)\Big)^{\frac{r}{p}} - (\|f\|_{p,\mu}^p)^{\frac{r}{p}} \Big| \, d\mu(x) \\
&\leqslant (\mathbf{E}\, |\theta_1|^r) \int_X \Big| \frac{1}{J} \sum_{j \leqslant J} T_j|f|^p(x) - \|f\|_{p,\mu}^p \Big|^{\frac{r}{p}} \, d\mu(x) \\
&\leqslant (\mathbf{E}\, |\theta_1|^r) \Big(\int_X \Big| \frac{1}{J} \sum_{j \leqslant J} T_j|f|^p(x) - \|f\|_{p,\mu}^p \Big| \, d\mu(x) \Big)^{r/p} \to 0,
\end{aligned}$$

as J tends to infinity by assumption (C). Therefore,

$$\lim_{J \to \infty} \mathbf{E} \int_X |F_{J,f}(x)|^r \, d\mu(x) = (\mathbf{E}\, |\theta_1|^r) \, \|f\|_{p,\mu}^r, \qquad \forall\, 0 < r < p.$$

By using Hölder's inequality, we deduce that

$$\mathbf{E}\, \|F_{J,f}\|_{r,\mu} \leqslant \Big(\mathbf{E} \int |F_{J,f}(x)|^r \, d\mu \Big)^{1/r} \leqslant 2\, \|\theta_1\|_r \, \|f\|_{p,\mu}, \tag{5.4}$$

for all $J \geqslant J_0$, say.

By assumption, property $(\mathcal{B}_r)$ holds for some $1 < r < p$. From Lemma 4.8 follows that there exists a non-increasing function $C :]0,1] \to \mathbb{R}_+$ such that for any $f \in L^r(\mu)$, for any $J \geqslant 1$, any $0 < \varepsilon < 1$, there exists a measurable set $\widetilde{X} = \widetilde{X}_{\varepsilon,J,f}$ of measure greater than $1 - \sqrt{\varepsilon}$, such that for all $x \in \widetilde{X}$,

$$\mathbf{P}\big\{\omega : \sup_{n \geqslant 1} \, |S_n(F_{J,f}(\omega, \cdot))(x)| > C(\varepsilon) \|F_{J,f}\|_{r,\mu}\big\} \leqslant \varepsilon. \tag{5.5}$$

We assume $0 < \varepsilon < 1/6$ in what follows. Let $\delta(\varepsilon) = C(\varepsilon)/\varepsilon$. Let also $x \in \widetilde{X}$, $J \geqslant J_0$. Using Chebyshev's inequality and (5.4), we get

$$\begin{aligned}
\mathbf{P}\big\{\omega : \sup_{n\geqslant 1} |S_n(F_{J,f}(\omega,\,\cdot\,))(x)| > 2\,\delta(\varepsilon)\|\theta_1\|_r\,\|f\|_{p,\mu}\big\} \\
\leqslant \mathbf{P}\big\{\omega : \sup_{n\geqslant 1} |S_n(F_{J,f}(\omega,\,\cdot\,))(x)| > \delta(\varepsilon)\,\mathbf{E}\,\|F_{J,f}\|_{r,\mu}\big\} \\
\leqslant \mathbf{P}\big\{\|F_{J,f}\|_{r,\mu} > \mathbf{E}\,\|F_{J,f}\|_{r,\mu}/\varepsilon\big\} \\
+ \mathbf{P}\Big\{\omega : \sup_{n\geqslant 1} |S_n(F_{J,f}(\omega,\,\cdot\,))(x)| > \delta(\varepsilon)\,\mathbf{E}\,\|F_{J,f}\|_{r,\mu}, \|F_{J,f}\|_{r,\mu} \\
\leqslant \mathbf{E}\,\|F_{J,f}\|_{r,\mu}/\varepsilon\Big\} \\
\leqslant \varepsilon + \mathbf{P}\big\{\omega : \sup_{n\geqslant 1} |S_n(F_{J,f}(\omega,\,\cdot\,))(x)| > C(\varepsilon)\|F_{J,f}\|_{r,\mu}\big\} \\
\leqslant 2\,\varepsilon.
\end{aligned}$$

Therefore,

$$\mathbf{P}\big\{\omega : \sup_{n\geqslant 1} |S_n(F_{J,f}(\omega,\,\cdot\,))(x)| \leqslant 2\,\delta(\varepsilon)\,\|\theta_1\|_r\,\|f\|_{p,\mu}\big\} \geqslant 1 - 2\,\varepsilon, \quad \forall\, x \in \widetilde{X},\ \forall\, J \geqslant J_0. \tag{5.6}$$

Let δ be some fixed positive real. Let I be a finite set of positive integers and let $M = \#\{I\}$. Assume that $\|S_n(f) - S_m(f)\|_{p,\mu} \geqslant \delta$ if $n \neq m$, $n, m \in I$. By Lemma 4.6-(i), there exists an index $\mathcal{J}$ and a measurable set $A = A_{\varepsilon,I,f}$ such that $\mu\{A\} \geqslant 1 - \varepsilon$, and further, for all $x \in A$, the following inequalities

$$\begin{aligned}
(1-\varepsilon)^{1/p}\|\theta_1\|_r\,\|S_n(f) - S_m(f)\|_{p,\mu} &\leqslant \big\|(S_n - S_m)(F_{J,f})(x)\big\|_{r,\mathbf{P}} \\
&\leqslant (1+\varepsilon)^{1/p}\,\|\theta_1\|_r\,\|S_n(f) - S_m(f)\|_{p,\mu},
\end{aligned}$$

are satisfied for all $J \in \mathcal{J}$, all $n, m \in I$ and all $r < p$. Set

$$Y = Y_{\varepsilon,I,J,f} = \widetilde{X} \cap A.$$

For each x fixed, the process

$$S_{J,f,x}(\omega, n) = \frac{1}{J^{1/p}} \sum_{j\leqslant J} \theta_j(\omega)\, T_j S_n(f)(x), \qquad n \geqslant 1,$$

is a p-stable random function. Further, the process

$$\mathcal{S}_{J,f,x}(\omega', \omega'', n) = \frac{1}{J^{1/p}} \sum_{1\leqslant j\leqslant J} \eta_j(\omega')\, g_j(\omega'')\, T_j S_n f(x), \qquad n \geqslant 1,$$

has the same distribution as $\{S_{J,f,x}(\,\cdot\,,n),\ n \geqslant 1\}$. Recall (sub-section 4.2) that we have underlying joint probability spaces $(\Omega', \mathcal{B}', \mathbf{P}')$ and $(\Omega'', \mathcal{B}'', \mathbf{P}'')$ on which the sequence $\{\eta_j,\ j \geqslant 1\}$ and the sequence $\{g_j,\ j \geqslant 1\}$ of i.i.d. Gaussian standard random variables are respectively defined. Here we take both sequences infinite.

Thus (5.6) reads: for all $x \in \widetilde{X}$, and all $J \geqslant J_0$,

$$\mathbf{P}' \times \mathbf{P}'' \Big\{ (\omega', \omega'') : \sup_{n \geqslant 1} |\mathcal{S}_{J,f,x}(\omega', \omega'', n)| \leqslant 2\,\delta(\varepsilon) \|\theta_1\|_r \|f\|_{p,\mu} \Big\} \geqslant 1 - 2\varepsilon. \tag{5.7}$$

Let

$$H(\omega') = \mathbf{P}'' \{ \omega'' : \sup_{n \geqslant 1} |\mathcal{S}_{J,f,x}(\omega', \omega'', n)| \leqslant 2\,\delta(\varepsilon)\, \|\theta_1\|_r \, \|f\|_{p,\mu} \}.$$

By Fubini's theorem, the left-term in (5.7) also writes

$$\begin{aligned} \int_{\Omega'} H(\omega')\, d\mathbf{P}'(\omega') &= \int_{\omega' : H(\omega') \leqslant \varepsilon} H(\omega')\, d\mathbf{P}'(\omega') + \int_{\omega' : H(\omega') > \varepsilon} H(\omega')\, d\mathbf{P}'(\omega') \\ &\leqslant \varepsilon + \mathbf{P}' \{ \omega' : H(\omega') > \varepsilon \}. \end{aligned}$$

Hence

$$\mathbf{P}' \Big\{ \omega' : \mathbf{P}'' \{ \omega'' : \sup_{n \geqslant 1} |\mathcal{S}_{J,f,x}(\omega', \omega'', n)| \leqslant 2\,\delta(\varepsilon)\, \|\theta_1\|_r \, \|f\|_{p,\mu} \} \geqslant \varepsilon \Big\} \geqslant 1 - 3\,\varepsilon. \tag{5.8}$$

For each fixed $\omega' \in \Omega'$, $\{\mathcal{S}_{J,f,x}(\omega', \,\cdot\,, n), n \geqslant 1\}$ is a Gaussian process. Let $\mathbf{E}_{\mathbf{P}''}$ denote the expectation symbol with respect to $\mathbf{P}''$. By using estimate (3.3), for every $x \in X_{\varepsilon,J,f}$,

$$1 - 3\,\varepsilon \leqslant \mathbf{P}' \Big\{ \omega' : \mathbf{E}_{\mathbf{P}''} \sup_{n \geqslant 1} |\mathcal{S}_{J,f,x}(\,\cdot\,, \omega', x))| \leqslant \tfrac{8\delta(\varepsilon)}{\varepsilon} \, \|\theta_1\|_r \, \|f\|_{p,\mu} \Big\}. \tag{5.9}$$

Write for a while

$$\begin{aligned} D(\omega, n, m) = D_{J,f,x}(\omega, n, m) &= S_{J,f,x}(\omega, n) - S_{J,f,x}(\omega, m) \\ \mathcal{D}(\omega', \omega'', n, m) = \mathcal{D}_{J,f,x}(\omega', \omega'', n, m) &= \mathcal{S}_{J,f,x}(\omega', \omega'', n) - \mathcal{S}_{J,f,x}(\omega', \omega'', m) \\ \Delta(n, m) = \Delta_{J,f,x}(n, m) &= \Big(\frac{1}{J} \sum_{1 \leqslant j \leqslant J} \big| T_j(S_n - S_m)\, f(x) \big|^p \Big)^{1/p}. \end{aligned}$$

By (4.3),

$$\mathbf{E}_{\mathbf{P}'}\mathbf{E}_{\mathbf{P}''}e^{it\mathcal{D}(\omega',\omega'',n,m)} = \mathbf{E}_{\mathbf{P}}e^{it(S_{J,f,x}(.,n)-S_{J,f,x}(.,m))}$$
$$= \mathbf{E}_{\mathbf{P}}e^{it\theta_1\Delta(n,m)} = e^{-|t|^p\Delta(n,m)^p}.$$

As $\mathbf{E}\,e^{itg} = e^{-t^2\tau^2/2}$ where $\tau = (\mathbf{E}\,g^2)^{1/2}$, we get from (4.2),

$$\mathbf{E}_{\mathbf{P}'}\mathbf{E}_{\mathbf{P}''}e^{it\mathcal{D}(\omega',\omega'',n,m)} = \mathbf{E}_{\mathbf{P}'}e^{-t^2\|\mathcal{D}(\omega',\,\cdot\,,n,m)\|^2_{2,\mathbf{P}''}/2} = e^{-|t|^p\Delta(n,m)^p}.$$

Put for each $\omega' \in \Omega'$,

$$d_{J,\omega',x}(n,m) = \|\mathcal{D}_{J,f,x}(\omega',\,\cdot\,,n,m)\|_{2,\mathbf{P}''}.$$

Moreover, let

$$d_{J,x}(n,m) = \Big(\frac{1}{J}\sum_{j\leqslant J} T_j|S_n(f) - S_m(f)|^p(x)\Big)^{1/p}.$$

We note that $d_{J,x}(n,m) = \Delta_{J,f,x}(n,m)$ for almost all $x \in X$. Further

$$\mathbf{E}_{\mathbf{P}'}e^{-t^2 d_{J,\omega',x}(n,m)^2/2} = e^{-|t|^p d_{J,x}(n,m)^p}.$$

Then

$$\mathbf{P}\big\{\exists\, n,m \in I : d_{J,\omega',x}(n,m) < \varepsilon d_{J,x}(n,m)\big\}$$
$$\leqslant \sum_{n,m\in I}\mathbf{P}\big\{e^{-t^2 d_{J,\omega',x}(n,m)/2} > e^{-t^2\varepsilon^2 d^2_{J,x}(n,m)}\big\}$$
$$\leqslant M^2 e^{t^2\varepsilon^2 d^2_{J,x}(n,m) - |t|^p d_{J,x}(n,m)^p},$$

and so,

$$\mathbf{P}\big\{\exists\, n,m \in I : d_{J,\omega',x}(n,m) < \varepsilon\, d_{J,x}(n,m)\big\}$$
$$\leqslant M^2 \inf_{t>0}\, e^{t^2\varepsilon^2 d^2_{J,x}(n,m) - |t|^p d_{J,x}(n,m)^p}.$$

The function $\varphi(t) = e^{t^2 a - t^p b}$ has an extremum at the value $t^* = \big(\frac{pb}{2a}\big)^{\frac{1}{2-p}}$, and

$$\varphi(t^*) = \exp\Big\{a^{-\frac{p}{2-p}}b^{\frac{2}{2-p}}\big[(p/2)^{\frac{2}{2-p}} - (p/2)^{\frac{p}{2-p}}\big]\Big\}.$$

Applying this with $a = \varepsilon^2 d^2_{J,x}(n,m)$, $b = d_{J,x}(n,m)^p$, we get

$$\mathbf{P}\big\{\exists\, n,m \in I : d_{J,\omega',x}(n,m) < \varepsilon\, d_{J,x}(n,m)\big\}$$
$$\leqslant M^2 \exp\big\{\varepsilon^{-\frac{2p}{2-p}}\,(d_{J,x}(n,m))^{-\frac{2p}{2-p}}\, d_{J,x}(n,m)^{\frac{2p}{2-p}}\,\big[(p/2)^{\frac{2}{2-p}} - (p/2)^{\frac{p}{2-p}}\big]\big\}$$
$$:= M^2 \exp\big\{-\varepsilon^{-\frac{2p}{2-p}}\, C(p)\big\}.$$

with $C(p) = (p/2)^{\frac{p}{2-p}} - (p/2)^{-\frac{2}{2-p}} > 0$. Choose $\varepsilon = (\tau \log M)^{-\frac{2-p}{2p}}$. We get

$$\begin{aligned}\mathbf{P}\big\{\exists\, n,m \in I : d_{J,\omega',x}(n,m) < (\tau \log M)^{-\frac{2-p}{2p}} d_{J,x}(n,m)\big\} \\ \leqslant M^{2-\tau C(p)} \leqslant \frac{1}{2},\end{aligned} \tag{5.10}$$

for $\tau = \tau(p)$ depending on p only, and small enough.

Now if $x \in Y$, we have

$$\big\|(S_n - S_m)(F_{J,f})(x)\big\|_{r,\mathbf{P}'\times\mathbf{P}''} \geqslant c(\varepsilon, r, p)\, \|S_n(f) - S_m(f)\|_{p,\mu},$$

for all $J \in \mathcal{J}$, all $n, m \in I$, $m \neq n$, and all $r < p$. As $(S_n - S_m)(F_{J,f})(x) \overset{\mathcal{D}}{=} (\mathcal{S}_{J,f,x}(n) - (\mathcal{S}_{J,f,x}(m))$, we have

$$\big\|(S_n - S_m)(F_{J,f})(x)\big\|_{r,\mathbf{P}'\times\mathbf{P}''} = \|\theta_1\|_r\, d_{J,x}(n,m),$$

whence

$$d_{J,x}(n,m) \quad \geqslant \quad c(\varepsilon, r, p)\, \|S_n(f) - S_m(f)\|_{p,\mu}, \tag{5.11}$$

for all $J \in \mathcal{J}$, all $n, m \in I$, $m \neq n$.

Putting together (5.10) and (5.9) implies that there exists a measurable set Ω_0' with $\mathbf{P}'(\Omega_0') > 0$, such that for any $\omega' \in \Omega_0'$, and all $n, m \in I$,

$$d_{J,\omega',x}(n,m) \geqslant c(\varepsilon, r, p) \frac{d_{J,x}(n,m)}{(\log \#\{I\})^{1/p-1/2}} \geqslant c(\varepsilon, r, p) \frac{\delta}{(\log \#\{I\})^{1/p-1/2}}$$

By Sudakov's inequality,

$$\begin{aligned}\|f\|_{p,\mu} &\geqslant c(r,p)\, \mathbf{E}_{\mathbf{P}''} \sup_{n\in I} |\mathcal{S}_{J,f,x}(\,\cdot\,,\omega',x))| \\ &\geqslant c(r,p)\, \delta\, (\log \#\{I\})^{1/2+1/2-1/p}.\end{aligned} \tag{5.12}$$

A routine argument together with (5.9) now easily leads to

$$\|f\|_{p,\mu} \geqslant c(r,p) \sup_{\delta>0}\, \delta\, \big(\log N_f^p(\delta)\big)^{1/q},$$

where $c(r,p) > 0$ depends on r and p only. It is only at this last stage that the fact that $p > 1$ is necessary.

5.5. Proof of Theorem 3.3. Let $f \in L^\infty(\mu)$ such that $\|f\|_2 = 1$. Let I be a finite subset of $\mathbb{N}$ and let $M = \#\{I\}$. Write for a while $N = N_\omega = |S_n(F_{J,f}(\omega, \cdot))|$, $\beta(\omega) = \mu\big\{x : N_\omega(x) \geqslant \frac{1}{2}\,\mathbf{E}\,N_\omega(x)\big\}$. By Lemma 4.9(c), for each x,

$$\mathbf{P}\Big\{N_\omega(x) \geqslant \frac{1}{2}\,\mathbf{E}\,N_\omega(x)\Big\} \geqslant c.$$

And so,

$$\mu \otimes \mathbf{P}\Big\{(\omega, x) : N_\omega(x) \geqslant \frac{1}{2}\,\mathbf{E}\,N_\omega(x)\Big\} \geqslant c.$$

We have

$$c \leqslant \mathbf{E}\,\beta = \mathbf{E}\,\beta\big(\chi_{\{\beta \geqslant c/2\}} + \chi_{\{\beta \leqslant c/2\}}\big) \leqslant c/2 + \mathbf{P}\{\beta \geqslant c/2\}.$$

Hence $\mathbf{P}\{\beta \geqslant c/2\} \geqslant c/2$, and using the previous notation, we deduce that for each $J \geqslant 1$, there exists a measurable set D_J of probability larger than $c/2$, such that we have

$$\mu\big\{x : |S_n(F_{J,f}(\omega, \cdot))(x)| \geqslant \frac{1}{2}\,\mathbf{E}\,|S_n(F_{J,f})(x)|\big\} \geqslant c/2, \quad \forall\, \omega \in D_J. \tag{5.13}$$

Let $0 < \gamma < 1$ be fixed. By Lemma 4.6(ii), there exists an index $\mathcal{J}$ and a measurable set A with $\mu\{A\} \geqslant \gamma^2$, and such that for all $x \in A$, we have

$$\gamma\,\mathbf{E} \sup_{n \in I} Z(S_n(f)) \quad \leqslant \quad \mathbf{E} \int \sup_{n \in I} S_n(F_{J,f})\, d\mu \quad \forall\, J \in \mathcal{J}. \tag{5.14}$$

Hence,

$$\mu\big\{x : |S_n(F_{J,f}(\omega, \cdot))(x)| \geqslant \frac{\gamma}{2}\mathbf{E} \sup_{n \in I} Z(S_n(f))\big\} \geqslant c/3, \quad \forall \omega \in D_J, \tag{5.15}$$

assuming γ sufficiently close to 1 and all $J \in \mathcal{J}$ greater than some sufficiently large number, which we do.

We simplify the notation in what follows and write $F_J = F_{J,f}$. Put for any $A > 0$,

$$E_A = \big\{(\omega, x) \in \Omega \times X : |F_J(\omega, x)| \leqslant A\big\}, \quad E_{A,\omega} = \big\{x \in X : (\omega, x) \in E_A\big\},$$

and let for any $\omega \in \Omega$, $x \in X$,

$$\begin{aligned} F_{A,J}(x) &= F_{A,J,\omega}(x) = F_J(\omega, x) \cdot \mathbf{1}_{E_{A,\omega}}(x), \\ F^{A,J}(x) &= F^{A,J,\omega}(x) = F_J(\omega, x) \cdot \mathbf{1}_{E^c_{A,\omega}}(x). \end{aligned}$$

Obviously,

$$\begin{aligned}\mathbf{E}\int \sup_{n\in I} S_n(F_{J,f})\, d\mu \leqslant \mathbf{E}\int \sup_{n\in I} |S_n(F^{A,J})|\, d\mu \\ +\mathbf{E}\int \sup_{n\in I} |S_n(F_{A,J})|\, d\mu.\end{aligned} \tag{5.16}$$

By definition $F_{A,J,\omega}(\,\cdot\,)$ (resp. $F^{A,J,\omega}(\,\cdot\,)$) is $\mathcal{A}$-measurable. As $f \in L^\infty(\mu)$, we have

$$\mathbf{P}\big\{\omega : F_{A,J,\omega}(\,\cdot\,) \text{ and } F^{A,J,\omega}(\,\cdot\,) \in L^\infty(\mu)\big\} = 1.$$

As $\max_{i\leqslant n} x_i \leqslant (\sum_{i\leqslant n} x_i^2)^{1/2}$ for any nonnegative real numbers, by using twice Cauchy–Schwarz's inequality, next Fubini's inequality, we get

$$\begin{aligned}\mathbf{E}\int \sup_{n\in I} |S_n(F^{A,J})|\, d\mu &\leqslant \mathbf{E}\Big(\sum_{n\in I}\int |S_n(F^{A,J})|^2\, d\mu\Big)^{1/2} \\ &\leqslant \Big(\sum_{n\in I}\int \mathbf{E}\,|S_n(F^{A,J})|^2\, d\mu\Big)^{1/2} \\ &\leqslant \sqrt{M}\,\mathbf{E}\,\|F^{A,J}\|_{2,\mu}.\end{aligned} \tag{5.17}$$

We have to estimate $\|F^{A,J}\|_{2,\mu}$. By Fubini's theorem, next Lemma 4.9(d) applied with $g = F_{J,f}/\|F_{J,f}\|_{2,\mathbf{P}}$ and $T = A/\|F_{J,f}\|_{2,\mathbf{P}}$, it follows that

$$\begin{aligned}\mathbf{E}\ \|F^{A,J}\|^2_{2,\mu} &= \int_X \mathbf{E}\ |F_{J,f}(x)|^2 \cdot \mathbf{1}_{(|F_{J,f}(x)|\geqslant A)}\, d\mu(x) \\ &\leqslant 6\int_X \|F_{J,f}(x)\|^2_{2,\mathbf{P}}\ \exp\Big\{-\frac{A^2}{4\,\|F_{J,f}(x)\|^2_{2,\mathbf{P}}}\Big\}\, d\mu(x).\end{aligned}$$

We have $\|F_{J,f}(x)\|^2_{2,\mathbf{P}} \overset{\text{a.e.}}{=} \frac{1}{J}\sum_{j\leqslant J} T_j(f^2)(x)$. By assumption (C), $\frac{1}{J}\sum_{j\leqslant J} T_j f^2$ converges to 1 in $L^1(\mu)$, along some subsequence extracted from $\mathcal{J}$, we can make this convergence almost everywhere too. The requirement that $f \in L^\infty(\mu)$, together with the dominated convergence theorem, then implies that

$$\int_X \|F_{J,f}(x)\|^2_{2,\mathbf{P}}\ \exp\Big\{-\frac{A^2}{4\,\|F_{J,f}(x)\|^2_{2,\mathbf{P}}}\Big\}\, d\mu(x) \to \exp\{-A^2/4\},$$

along this index.

Extracting again if necessary we obtain that

$$\mathbf{E}\,\|F^{A,J}\|_{2,\mu}^2 \leqslant 2\,\exp\{-A^2/4\},$$

along some index, which we still denote by $\mathcal{J}$. Choose now $A = \sqrt{8\log M}$. We get

$$\mathbf{E}\int \sup_{n\in I}\ |S_n(F^{A,J,\omega})|\ d\mu \leqslant 9\sqrt{M}\,\exp\{-A^2/8\} \leqslant 9\,M^{-1/2}. \tag{5.18}$$

Assume that

$$\min_{\substack{n,m\in I\\ n\neq m}} \|S_n(f) - S_m(f)\|_{2,\mu} \geqslant \delta. \tag{5.19}$$

Using Lemma 4.9(b), we get

$$\mu\Big\{x : \sup_{n\in I}\ |S_n(F_{A,J,\omega})(x)| \geqslant \frac{\gamma B\delta}{2}\sqrt{\log M} - 9\,M^{-1/2}\Big\} \geqslant c/3, \qquad \forall\,\omega\in D_J, \tag{5.20}$$

for all $J\in\mathcal{J}$. Let

$$\phi_{I,J,\omega} = \frac{F_{A,J,\omega}}{A}.$$

It follows that

$$\mu\big\{x : \sup_{n\in I}\ |S_n(\phi_{I,J,\omega})(x)| \geqslant c'\delta\big\} \geqslant c/3, \quad \forall\,\omega\in D_J, \tag{5.21}$$

where c' is a positive universal constant. Suppose that for some $\delta > 0$, $C(\delta) = \infty$. This means that we can select sets I verifying (5.19) with cardinality M as large as we wish. But

$$\begin{aligned} d(S^*(\phi_{I,J,\omega}),0) &\geqslant d(\sup_{n\in I}\ |S_n(\phi_{I,J,\omega}),0) \\ &\geqslant \int_{\sup_{n\in I}\ |S_n(\phi_{I,J,\omega})\geqslant c'\delta} \frac{\sup_{n\in I}\ |S_n(\phi_{I,J,\omega})|}{1+\sup_{n\in I}\ |S_n(\phi_{I,J,\omega})|}\ d\mu \\ &\geqslant (c/3)\frac{c'\delta}{1+c'\delta}. \end{aligned} \tag{5.22}$$

Moreover, we have

$$\mathbf{E}\,\|\phi_{I,J,\omega}\|_{2,\mu}^2 \leqslant \frac{1}{8\,\log M}\mathbf{E}\int |F_{j,f}|^2\,d\mu \leqslant \frac{1}{8\,\log M}.$$

Hence on a subset D'_J of D_J of positive measure, we have

$$\|\phi_{I,J,\omega}\|_{\infty,\mu} \leqslant 1, \qquad \|\phi_{I,J,\omega}\|_{2,\mu} \leqslant K/\sqrt{\log M}.$$

Moreover, K depend on c only. Picking ω in D'_J, J varying, we deduce that S^* cannot be continuous at 0. Hence a contradiction with (3.2). This achieves the proof.

5.6. Proof of Theorem 3.10. We start as in the proof of Theorem 3.3. By using exactly the same arguments for proving (5.17), we get here

$$\begin{aligned}\mathbf{E}\int \sup_{n\in I} |S_n(F^{A,J})|\, d\mu &\leqslant \mathbf{E}\Big(\sum_{n\in I}\int |S_n(F^{A,J})|^2\, d\mu\Big)^{1/2}\\ &\leqslant \Big(\sum_{n\in I}\int \mathbf{E}\,|S_n(F^{A,J})|^2\, d\mu\Big)^{1/2}\\ &\leqslant \sqrt{M} S_1(I)\,\mathbf{E}\,\|F^{A,J}\|_{2,\mu}. \end{aligned}\tag{5.23}$$

Next estimate (5.18) is modified as follows. Let $\alpha > 1$ be some fixed real. By extracting we obtain that $\mathbf{E}\,\|F^{A,J}\|^2_{2,\mu} \leqslant \alpha \exp\{-A^2/4\}$, along some index, still denoted $\mathcal{J}$. Thus with (5.23),

$$\begin{aligned}\mathbf{E}\int \sup_{n\in I} |S_n(F^{A,J,\omega})|\, d\mu &\leqslant \sqrt{M} S_1(I)\,\mathbf{E}\,\|F^{A,J}\|_{2,\mu}\\ &\leqslant 6\sqrt{\alpha M} S_1(I)\, \exp\{-A^2/8\}. \end{aligned}\tag{5.24}$$

Let $\delta = \min\big\{(\alpha-1)\, e^{-A^2/4\alpha}, 1)\big\}$ and $\delta_k = \delta\, 2^{-k}$, $k \geqslant 1$. We can extract from $\mathcal{J}$ a subsequence $\mathcal{J}^* = \{J_k,\ k \geqslant 1\}$ depending on f and α, such that

$$\mu\Big\{\Big|\frac{1}{J_k}\sum_{j\leqslant J_k} T_j f^2 - 1\Big| > \delta_k\Big\} \leqslant \delta_k, \quad \text{for all } k \geqslant 1.$$

Put

$$B = \Big\{\forall\, k \geqslant 1,\ \Big|\frac{1}{J_k}\sum_{j\leqslant J_k} T_j f^2 - 1\Big| \leqslant \delta_k\Big\}.$$

Plainly,

$$\begin{aligned}\mathbf{E}\int \sup_{n\in I} |S_n(F_{A,J})|\, d\mu &\leqslant \mathbf{E}\int_B \sup_{n\in I} |S_n(F_{A,J})|\, d\mu\\ &\quad + \mathbf{E}\int_{B^c} \sup_{n\in I} |S_n(F_{A,J})|\, d\mu. \end{aligned}\tag{5.25}$$

The first integral in the right-hand side of (5.25) can be bounded for any $R > 0$ by

$$\begin{aligned}&\int_B \mathbf{E}\big(\sup_{n\in I} |S_n(F_{A,J})|\ \mathbf{1}_{\{\|F_{A,J}\|_{2,\mu}>R\}}\big)\ d\mu \\ &\qquad + \int_B \mathbf{E}\big(\sup_{n\in I} |S_n(F_{A,J})|\ \mathbf{1}_{\{\|F_{A,J}\|_{2,\mu}\leqslant R\}}\big)\ d\mu.\end{aligned} \tag{5.26}$$

Consider the first integral in (5.26). The fact that S_n is continuous on $L^\infty(\mu)$ and Chebyshev's inequality allow to write

$$\begin{aligned}&\int_B \mathbf{E}\Big(\sup_{n\in I} |S_n(F_{A,J})|\ \mathbf{1}_{\{\|F_{A,J}\|_{2,\mu}>R\}}\Big)\, d\mu \\ &\qquad\leqslant \mathbf{E}\Big(\Big\|\sup_{n\in I} |S_n(F_{A,J})|\Big\|_{\infty,\mu}\cdot \mathbf{1}_{\{\|F_{A,J}\|_{2,\mu}>R\}}\Big) \\ &\qquad\leqslant A\, S_2(I)\, \mathbf{P}\big\{\|F_{A,J}\|_{2,\mu} > R\big\} \\ &\qquad\leqslant A\, S_2(I)\, e^{-R^2/4\alpha}\mathbf{E}\ \exp\Big\{\frac{1}{4\,\alpha}\|F_{A,J}\|^2_{2,\mu}\Big\}.\end{aligned} \tag{5.27}$$

We claim that for any $J \in \mathcal{J}^*$,

$$\mathbf{E}\ \exp\Big\{\frac{1}{4\,\alpha}\|F_{A,J}\|^2_{2,\mu}\Big\} \leqslant \sqrt{2}+\alpha-1. \tag{5.28}$$

Admit this for a while. We get

$$\begin{aligned}&\int_B \mathbf{E}\Big(\sup_{n\in I} |S_n(F_{A,J})|\ \mathbf{1}_{\{\|F_{A,J}\|_{2,\mu}>R\}}\Big)\, d\mu \\ &\qquad\qquad \leqslant A\, S_2(I)\, e^{-R^2/4\alpha}(\sqrt{2}+\alpha-1).\end{aligned} \tag{5.29}$$

Now we prove (5.28). Let $a = \frac{1}{4\alpha}$. At first by using Jensen's inequality,

$$\begin{aligned}\mathbf{E}\ \exp\big\{a\, \|F_{A,J}\|^2_{2,\mu}\big\} &= \mathbf{E}\ \exp\Big\{a\int_X F^2_{A,J}\ d\mu\Big\} \leqslant \mathbf{E}\int_X \exp\{a\, F^2_{A,J}\}\ d\mu \\ &\leqslant \mathbf{E}\int_B \exp\big\{a\, F^2_{A,J}\big\}\ d\mu + e^{aA^2}\mu(B^c_\alpha).\end{aligned}$$

Next on B, we have $\frac{1}{J}\sum_{j\leqslant J} T_j f^2 \leqslant 1+\delta < \alpha$, so that

$$1 - 2\,a\Big(\frac{1}{J}\sum_{j\leqslant J} T_j f^2\Big) > 1 - 2\,a\,\alpha = \frac{1}{2} \qquad \text{for all } J \in \mathcal{J}^*.$$

As $\mathbf{E}\, e^{bg^2} = \frac{1}{\sqrt{1-2b}}$ if $0 \leqslant b < \frac{1}{2}$, we get

$$\begin{aligned}\int_B \mathbf{E}\,\exp\big\{a\,F_{A,J}^2\big\}\,d\mu &\leqslant \int_B \mathbf{E}\,\exp\big\{a\,F_J^2\big\}\,d\mu \\ &= \int_B \frac{d\mu}{\sqrt{1-2\,a\,\big(\frac{1}{J}\sum_{j\leqslant J} T_j f^2\big)}} \leqslant \sqrt{2}.\end{aligned}$$

Hence for any $J \in \mathcal{J}^*$,

$$\mathbf{E}\,\exp\big\{a\,\|F_{A,J}\|_{2,\mu}^2\big\} \leqslant \sqrt{2} + e^{aA^2}\mu(B^c) \leqslant \sqrt{2} + \delta\, e^{A^2 a} \leqslant \sqrt{2} + \alpha - 1.$$

For the second integral in (5.26), we have the straightforward bound

$$\begin{aligned}&\int_B \mathbf{E}\,\big(\sup_{n\in I}\,|S_n(F_{A,J})|\,\mathbf{1}_{\{\|F_{A,J}\|_{2,\mu}\leqslant R\}}\big)\,d\mu \\ &\qquad \leqslant A \sup_{\substack{\|h\|_{\infty,\mu}\leqslant 1\\ \|h\|_{2,\mu}\leqslant R/A}} \int_X \sup_{n\in I}\,|S_n(h)|\,d\mu.\end{aligned} \tag{5.30}$$

By substituting estimates (5.29), (5.30) into (5.26), we can bound the first integral in the right-term of (5.25) as follows,

$$\begin{aligned}\mathbf{E}\int_B \sup_{n\in I}\,|S_n(F_{A,J})|\,d\mu &\leqslant AS_2(I)\,e^{-R^2/4\alpha}(\sqrt{2}+\alpha-1) \\ &\quad + A \sup_{\substack{\|h\|_{\infty,\mu}\leqslant 1\\ \|h\|_{2,\mu}\leqslant R/A}} \int_X \sup_{n\in I}\,|S_n(h)|\,d\mu.\end{aligned} \tag{5.31}$$

Consider the second integral in the right-term of (5.25). We use Cauchy-Schwarz's inequality and the facts that $\mu(B^c) \leqslant \delta$,

$$\mathbf{E}\,\|F_{A,J}\|_{2,\mu} \leqslant \mathbf{E}\,\|F_J\|_{2,\mu} \leqslant 1,$$

to get

$$\begin{aligned}\mathbf{E}\int_{B^c}\sup_{n\in I}\,|S_n(F_{A,J})|\,d\mu &\leqslant \sqrt{\mu(B^c)}\mathbf{E}\,\big\|\sup_{n\in I}\,|S_n(F_{A,J})|\big\|_{2,\mu}\\ &\leqslant \sqrt{\delta}\,\sqrt{M}\,S_1(I)\,\mathbf{E}\,\|F_{A,J}\|_{2,\mu}\\ &\leqslant \sqrt{\alpha-1}\;e^{-A^2/8\alpha}\sqrt{M}\,S_1(I).\end{aligned}\tag{5.32}$$

By inserting estimates (5.31), (5.32) into (5.25), we next arrive to

$$\begin{aligned}&\mathbf{E}\int\sup_{n\in I}\,|S_n(F_{A,J})|\,d\mu \leqslant A\,S_2(I)\,e^{-R^2/4\alpha}(\sqrt{2}+\alpha-1)\\ &+A\sup_{\substack{\|h\|_{\infty,\mu}\leqslant 1\\ \|h\|_{2,\mu}\leqslant R/A}}\int_X\sup_{n\in I}\,|S_n(h)|\,d\mu+\sqrt{\alpha-1}\;e^{-A^2/8\alpha}\sqrt{M}\,S_1(I).\end{aligned}\tag{5.33}$$

Now we insert (5.24), (5.33) into (5.14), and next use estimate (5.18). Picking J arbitrarily in $\mathcal{J}^*$, we get

$$\begin{aligned}&\gamma\,\mathbf{E}\sup_{n\in I}\,Z(S_n(f))\\ &\leqslant 6\,\sqrt{\alpha M}\,S_1(I)\,\exp\{-A^2/8\}+A\,S_2(I)\,e^{-R^2/4\alpha}(\sqrt{2}+\alpha-1)\\ &+\;A\sup_{\substack{\|h\|_{\infty,\mu}\leqslant 1\\ \|h\|_{2,\mu}\leqslant R/A}}\int_X\sup_{n\in I}\,|S_n(h)|\,d\mu+\sqrt{\alpha-1}\;e^{-A^2/8\alpha}\sqrt{M}\,S_1(I)\,.\end{aligned}\tag{5.34}$$

But $\alpha>1$ and γ can be chosen arbitrarily close to 1. We finally obtain,

$$\begin{aligned}\mathbf{E}\sup_{n\in I}\,Z(S_n(f))\quad&\leqslant\quad 6\,\sqrt{M}\,S_1(I)\,\exp\{-A^2/8\}+A\,(\sqrt{2})\,S_2(I)\,e^{-R^2/4}\\ &+A\sup_{\substack{\|h\|_{\infty,\mu}\leqslant 1\\ \|h\|_{2,\mu}\leqslant R/A}}\int_X\sup_{n\in I}\,|S_n(h)|\,d\mu.\end{aligned}\tag{5.35}$$

This last inequality being satisfied for any $f\in L^\infty(\mu)$ such that $\|f\|_{2,\mu}=1$, we easily deduce the claimed result by continuity in quadratic mean of Z.

§6. Kakutani–Rochlin's lemma

We conclude with this extremely useful tool in ergodic theory.

Lemma 6.1. *If T is aperiodic, then for every $\varepsilon>0$ and for every $n\geqslant 1$ there exists $F\in\mathcal{A}$ such that the sets F, $T^{-1}(F),\dots,T^{-(n-1)}(F)$ are mutually disjoint, and such that we have,*

$$\mu\big(F\cup T^{-1}(F)\cup\cdots\cup T^{-(n-1)}(F)\big)>1-\varepsilon.$$

Any set $F \in \mathcal{A}$ satisfying the conclusions of Lemma 6.1 is called an (ε, n)-Kakutani–Rochlin set.

We illustrate its usefulness by establishing two divergence criteria for ergodic summation methods. The proof is based on an argument due to Deniel (see [7]). Let $\{w_{n,k},\ 1 \leqslant k \leqslant n,\ n \geqslant 1\}$ be a triangular array of nonnegative reals, and set $W_n = \sum_{k=1}^{n} w_{n,k}$, $n \geqslant 1$. Consider an automorphism τ from a probability space $(X, \mathcal{A}, \mu)$. Put for $f \in L^0(\mu)$,

$$T_n f(x) = \frac{1}{W_n} \sum_{h=1}^{n} w_{n,h}\, f(\tau^h x).$$

Theorem 6.2. *Let $\varphi : \mathbb{N} \to \mathbb{N}$ be such that $\lim_{n\to\infty} \varphi(n) = \infty$. Assume that there exist $\rho > 0$, an infinite sequence $\mathcal{N}$ of integers such that for any $n \in \mathcal{N}$*

$$\min_{\varphi(n) \leqslant j \leqslant n-\varphi(n)} \Big(\frac{1}{W_{n-j}} \sum_{k=n-j-\varphi(n)}^{n-j-1} w_{n-j,k} \Big) \geqslant \rho, \tag{6.1}$$

and further that the series $\sum_{n\in\mathcal{N}} \varphi(n)/n$ converges. Let $0 < \eta < \rho$. Then there exists $B \in \mathcal{A}$ with $0 < \mu(B) \leqslant \eta$ such that $\limsup_{\mathcal{N} \ni n \to \infty} T_n \chi_B \geqslant \rho$ almost surely.

Remark 6.3. Suppose there exists a countable dense class $\mathcal{D}$ of functions from $L^1(\mu)$ such that $\{T_n f, n \in \mathcal{N}\}$ converges almost everywhere to $\int f\, d\mu$ for any $f \in \mathcal{D}$. Then if condition (6.1) is satisfied, there is no maximal inequality for the sequence $\{T_n,\ n \in \mathcal{N}\}$. Indeed, otherwise by the Banach principle, we would have that $\{T_n f,\ n \in \mathcal{N}\}$ converges almost everywhere to $\int f d\mu$ for any $f \in L^1(\mu)$. Taking $f = \chi_B$ where B is in the proposition above provides a contradiction.

Now let $\{w_k,\ k \geqslant 1\}$ be a sequence of non-negative reals and consider the ergodic sums

$$A_n f(x) = \sum_{h=1}^{n} w_h\, f(\tau^h x).$$

Theorem 6.4. *Let $\varphi : \mathbb{N} \to \mathbb{N}$ be such that $\lim_{n\to\infty} \varphi(n) = \infty$. Assume that there exist $\rho > 0$, an infinite sequence $\mathcal{N}$ of integers such that for any*

$n \in \mathcal{N}$

$$\Delta_n := \min_{1\leqslant h\leqslant n-\varphi(n)} \Big(\sum_{k=h}^{h+\varphi(n)} w_k \Big) \to \infty, \tag{6.2}$$

as $n \to \infty$ *along* $\mathcal{N}$, *and further that the series* $\sum_{n\in\mathcal{N}} \varphi(n)/n$ *converges. Let* $0 < \eta < \rho$. *Then there exists* $B \in \mathcal{A}$ *with* $0 < \mu(B) \leqslant \eta$ *such that* $\limsup_{\mathcal{N}\ni n\to\infty} A_n\,(\chi_B) = \infty$ *almost surely.*

Proof of Theorem 6.2. There is no loss of generality to assume

$$\sum_{n\in\mathcal{N}} \varphi(n)/n \leqslant \eta.$$

By Rochlin's lemma, for any $\varepsilon > 0$, any integer N, there exists $A \in \mathcal{A}$ such that $A, TA, \dots, T^{N-1}A$, are pairwise disjoint and $1-\varepsilon \leqslant N\mu(A) \leqslant 1$. By applying it for $N = n$, $\varepsilon = \varphi(n)/n$, we obtain that for each $n \in \mathcal{N}$, there exists $A_n \in \mathcal{A}$ such that $A_n, \tau A_n, \dots, \tau^{n-1}A_n$ are mutually disjoint and $\mu\big(\sum_{u=0}^{n-1} \tau^u A_n\big) = n\mu(A_n) \geqslant 1 - \varphi(n)/n$. Let

$$B_n = \sum_{n-\varphi(n)\leqslant u<n} \tau^u A_n, \qquad D_n = \sum_{\varphi(n)\leqslant j<n-\varphi(n)} \tau^j A_n.$$

Then we have

$$\begin{aligned}
\mu(B_n) &\leqslant \varphi(n)\,\mu(A_n) \leqslant \frac{\varphi(n)}{n},\\
\mu(D_n) &\geqslant \frac{n-2\,\varphi(n)}{n}\Big(1-\frac{\varphi(n)}{n}\Big) \geqslant \Big(1-2\,\frac{\varphi(n)}{n}\Big)^2 \geqslant 1-4\,\frac{\varphi(n)}{n}.
\end{aligned}$$

Now let $0 \leqslant \ell < n-\varphi(n)$. As $\tau^\ell x \in B_n$ iff $x \in \tau^{u-\ell}A_n$ for some $n-\varphi(n) \leqslant u < n$, we can write

$$\chi_{B_n}(\tau^\ell x) = \sum_{n-\varphi(n)\leqslant u<n} \chi_{\{\tau^{u-\ell}A_n\}}(x) = \sum_{n-\varphi(n)-\ell\leqslant v<n-\ell} \chi_{\{\tau^v A_n\}}(x).$$

Let $\ell = n - \varphi(n) - \lambda$ with $1 \leqslant \lambda < n - \varphi(n)$. We have

$$\chi_{B_n}(\tau^{n-\varphi(n)-\lambda}x) = \sum_{\lambda\leqslant v<\lambda+\varphi(n)} \chi_{\{\tau^v A_n\}}(x).$$

As $\varphi(n)/n \to 0$ when $n \to \infty$ along $\mathcal{N}$, we have $2\,\varphi(n) \leqslant n$ once n is large. Fix some $\varphi(n) \leqslant j < n - \varphi(n)$ and pick $x \in \tau^j A_n$. If we choose λ so that $\lambda \leqslant j < \lambda + \varphi(n)$, by letting $v = j$ in the equation above we see that $\tau^{n-\varphi(n)-\lambda}x \in B_n$.

Thus $x \in \tau^j A_n$ and $\lambda \in \{j-\varphi(n)+1, j-\varphi(n)+2, \dots, j\}$ imply

$$\tau^{n-\varphi(n)-\lambda}x \in B_n.$$

Consequently, if $x \in \tau^j A_n$

$$\begin{aligned} T_{n-j}\chi_{B_n}(x) &= \sum_{k=1}^{n-j} w_{n-j,n-j-k}\ \chi_{B_n}(\tau^{n-j-k}x) \\ &\geqslant \sum_{k=1}^{\varphi(n)} w_{n-j,n-j-k}\ \chi_{B_n}(\tau^{n-j-k}x) \\ (k=\varphi(n)+\lambda-j) &= \sum_{\lambda=j-\varphi(n)+1}^{j} w_{n-j,n-\varphi(n)-\lambda}\ \chi_{B_n}(\tau^{n-\varphi(n)-\lambda}x) \\ &= \sum_{\lambda=j-\varphi(n)+1}^{j} w_{n-j,n-\varphi(n)-\lambda} = \sum_{k=1}^{\varphi(n)} w_{n-j,n-j-k}. \end{aligned} \tag{6.3}$$

By the assumption made,

$$\begin{aligned} &\frac{1}{W_{n-j}} \sum_{k=1}^{\varphi(n)} w_{n-j,n-j-k} \\ &\qquad\geqslant \min_{\varphi(n)\leqslant j\leqslant n-\varphi(n)} \Big(\frac{1}{W_{n-j}} \sum_{k=1}^{\varphi(n)} w_{n-j,n-j-k}\Big) \geqslant \rho. \end{aligned} \tag{6.4}$$

Note that $n-j > \varphi(n)$. Thus on D_n,

$$\sup_{m>\varphi(n)} T_m(\chi_{B_n}) \geqslant \rho.$$

Set

$$E = \bigcup_{n\in\mathcal{N}} B_n, \qquad F_N = \bigcap_{\substack{n\in\mathcal{N}\\ n\geqslant N}} D_n.$$

We observe that $\mu(F_N) \geqslant 1 - 4 \sum_{\substack{n\in\mathcal{N}\\ n\geqslant N}} \varphi(n)/n \to 1$ as $N\to\infty$. Thus on F_N,

$$\limsup_{\mathcal{N}\ni n\to\infty} T_n(\chi_E) \geqslant \rho. \tag{6.5}$$

Further $\mu(E) \leqslant \sum_{n\in\mathcal{N}} \varphi(n)/n < \eta$. This establishes Theorem 6.2. □

Proof of Theorem 6.4. We start with (6.3) which here becomes

$$A_{n-j}\chi_{B_n}(x) \quad \geqslant \quad \sum_{k=1}^{\varphi(n)} w_{n-j-k},$$

and next modify the previous proof as follows:

$$\begin{aligned}\sum_{k=1}^{\varphi(n)} w_{n-j-k} &\geqslant \min_{\varphi(n)\leqslant j\leqslant n-\varphi(n)} \Big(\sum_{k=1}^{\varphi(n)} w_{n-j-k}\Big)\\ &\geqslant \min_{1\leqslant h\leqslant n-\varphi(n)} \Big(\sum_{k=h}^{h+\varphi(n)} w_k\Big) = \Delta_n.\end{aligned}$$

Thus $\sup_{m>\varphi(n)} \ A_m(\chi_{B_n}) \geqslant \Delta_n$ on D_n. Therefore on F_N,

$$\limsup_{\mathcal{N}\ni n\to\infty} T_n\chi_E = \infty.$$

Further $\mu(E) \leqslant \sum_{n\in\mathcal{N}} \varphi(n)/n < \eta$. □

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