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ON ESTIMATE OF THE NORM OF THE HOLOMORPHIC COMPONENT OF A MEROMORPHIC FUNCTION IN FINITELY CONNECTED DOMAINS

Abstract. In this paper we extend Gonchar-Grigorjan type estimate of the norm of holomorphic part of meromorphic functions in finitely connected Jordan domains with C^2 smooth boundary when the poles are in a compact set. A uniform estimate for Cauchy type integral is also given.

§1. Introduction

Landau investigated holomorphic functions in the unit disk $\mathbb{D}$ with $\|f\|_{\partial\mathbb{D}} \leqslant 1$ where $\|.\|_{\partial\mathbb{D}}$ denotes the sup norm over the boundary $\partial\mathbb{D}$ of $\mathbb{D}$. He showed that the absolute value of the sum of first n coefficients of Maclaurin series for such functions has order of growth $\log n$ (see [11], pp. 26-28). L.D. Grigorjan generalized this in the following sense, see [7]. Consider meromorphic functions in the unit disk with poles in some fixed compact subset of the unit disk and with total order n. Then the growth of the norm on the unit circle of sum of the principal parts is $\log n$. It is easy to see that the case when the origin is the only pole yields Landau's result. More generally, on simply connected domains with smooth boundary, when there is no restriction on the location of the poles, then we get linear growth for the norm (instead of $\log n$; see [6]).

Let us introduce the sup norm of meromorphic functions f on a domain D as follows:

$$\|f\|_{\partial D} := \sup\left\{\limsup_{\zeta\to z} |f(\zeta)| : \ z \in \partial D\right\}.$$

In [5] A.A. Gonchar and L.D. Grigorjan proved the following theorem.

Theorem. *Let $D \subset \mathbf{C}$ be a simply connected domain and its boundary be C^1 smooth. Let $f : D \to \mathbf{C}_\infty$ be a meromorphic function on D such that it has m different poles. Denote by f_r the sum of principal parts of f (with*

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$f_r(\infty)=0$) and let f_h be the holomorphic part of f in D. Denote the total order of the poles of f by n. Then $f=f_r+f_h$ and there exists $C_1(D)>0$ depending on D only such that

$$\|f_h\|_{\partial D} \leqslant C_1(D)\, m\,(1+\log n)\,\|f\|_{\partial D}.$$

Later, it was proved in [8] that on finitely connected domains if the poles can be anywhere, then the growth of the norm is linear again.

The results mentioned above have several applications in e.g. Padé approximation (see e.g. [1, 12]), estimating Faber polynomials (see [17] and [10]) or polynomial inequalities (see, e.g. [9]).

We are going to extend this Theorem on finitely connected domains when the poles are in a compact set (see also [8] and [7]).

§2. Auxiliary tools

We put $D(z,r) := \{w \in \mathbf{C} : |z-w|<r\}$. We denote Green's function of domain $G \subset \mathbf{C}_\infty$ with pole at a by $g_G(.,a)$, for potential theory we refer to [15] and [16]. If $v=(v_1,v_2)\in \mathbf{R}^2$, then we use $\|v\| := \sqrt{v_1^2+v_2^2} = |v_1+iv_2|$. If Γ is a Jordan curve or union of finitely many Jordan curves, then $\mathrm{Ext}\Gamma$ denotes the unbounded component of $\mathbf{C}\setminus\Gamma$. If $H\subset\mathbf{C}$ is a compact set, then the exterior boundary of H is the boundary of the unbounded component of $\mathbf{C}\setminus H$. If w is a complex number, then $\arg w := w/|w|$ (if $w\neq 0$) and $\arg 0 := 0$.

Lemma 1. *Let $G\subset\mathbf{C}_\infty$ be a finitely connected domain and its boundary $\Gamma := \partial G$ be finite union of C^2 smooth Jordan curves. Let $Z\subset G$ be a closed set.*

Then there exist $\rho_1>0$, $C_2>0$ such that for all $a\in Z$ and $\rho\in(0,\rho_1)$ the set $\{g_G(z,a)=\rho\}$ is finite union of smooth Jordan curves and if z is such that $g_G(z,a)=\rho$, then $\mathrm{grad}\, g_G(z,a)\neq 0$ *and*

$$\frac{1}{C_2}\mathrm{dist}(z,\Gamma) \leqslant g_G(z,a) \leqslant C_2 \mathrm{dist}(z,\Gamma). \tag{1}$$

Furthermore, there exists $C_3>0$ such that for all $a,b\in Z$ and $z\in G$ with $g_G(z,a), g_G(z,b)<\rho_1$, we have

$$\frac{1}{C_3} g_G(z,b) \leqslant g_G(z,a) \leqslant C_3 g_G(z,b). \tag{2}$$

Proof. Let $r_0>0$ be so small that for all $0<r\leqslant r_0$ we have that $D(z,r)\cap\Gamma$ is a single Jordan arc and $D(z,r)\cap G$ is a simply connected domain for all $z\in\Gamma$, and $r_0<\frac{1}{2}\mathrm{dist}(Z,\Gamma)$ and r_0 is less than 1/4 times the distance between the different components of Γ and we also require that the normal vectors $n(z')$ to Γ at $z' \in D(z,r) \cap \Gamma$ pointing inward with unit length satisfy

$$|n(z') - n(z)| < \frac{\pi}{16}. \tag{3}$$

Since G is finitely connected, any $g_G(z,a)$ has finitely many critical points (see [2], p. 76 and [3], p. 410). Moreover, since ∂G is also C^2 smooth, the union of these critical points for $a \in Z$ stays away from ∂G at positive distance. Indeed, suppose indirectly that: $z_n \to z_\infty$ (∂G is compact), $a_n \to a_\infty$ ($a_\infty \in Z$ since Z is closed on $\mathbf{C}_\infty$) and $\mathrm{grad} g_G(z_n, a_n) = 0$. Then, choosing a suitable subsequence, $g_G(z, a_n)$ converges locally uniformly to $g_G(z, a_\infty)$ in a neighborhood of z_∞, say $D(z_\infty, r) \cap G$. We also know that $\mathrm{grad} g_G(z, a_n)$ converges locally uniformly to $\mathrm{grad} g_G(z, a_\infty)$ on $D(z_\infty, r) \cap G$, they extend continuously to $D(z_\infty, r) \cap \Gamma$ and they are uniformly bounded (for all n and $z \in D(z_\infty, r) \cap G$).

It follows using standard steps that $\mathrm{grad} g_G(z,a)$ is continuous when $z \in G \cup \Gamma \setminus Z$, $a \in Z$. Indeed, continuity is obvious if $z \in G \setminus Z$, $a \in Z$. If $z \in \Gamma$ and $a_n \in Z$ arbitrary, $a_n \to a_\infty$, we do the following. Let Γ_0 be the component of Γ containing z and Γ_1 be a C^2 smooth Jordan curve in G such that $\Gamma_1 \subset \{\zeta \in G : \ \mathrm{dist}(\zeta, \Gamma_0) < r_0\}$. Let G_2 be the domain determined by Γ_0 and Γ_1, i.e. if $\Gamma_1 \subset \mathrm{Int}\Gamma_0$, then $G_2 = \mathrm{Int}\Gamma_0 \cap \mathrm{Ext}\Gamma_1$ and let $G_2^+ := \mathrm{Int}\Gamma_0$, otherwise $G_2 = \mathrm{Int}\Gamma_1 \cap \mathrm{Ext}\Gamma_0$ and let $G_2^+ := \mathrm{Ext}\Gamma_0 \cup \{\infty\}$. Now applying Riemann mapping theorem and the Kellogg-Warschawski theorem (see e.g. [14], Theorem 3.6, p. 49), we obtain a conformal map φ from G_2^+ onto $\mathbb{D}$ such that $\varphi(\Gamma_0) = \partial\mathbb{D}$, $\varphi(\Gamma_1) \subset \mathbb{D}$ is a Jordan curve, and φ is a conformal map from G_2 onto $\varphi(G_2)$ and φ is C^2 smooth on the closure of G_2. Consider $\psi_n(w) := g_G\left(\varphi^{-1}[w], a_n\right)$ and $\psi_\infty(w) := g_G\left(\varphi^{-1}[w], a_\infty\right)$. They are harmonic on $w \in \varphi(G_2)$ and have zero value on the unit circle, so we can extend all these functions by reflection principle, to some fixed domain G_3 where $\partial\mathbb{D} \subset G_3$. We know that $\psi_n(w) - \psi_\infty(w) \to 0$ uniformly when $w \in \varphi(\Gamma_1) \subset \partial G_2$ and by reflection principle. this holds on $\partial G_3 \setminus \varphi(\Gamma_1)$ too, hence on the whole ∂G_3. Since $\partial\mathbb{D}$ is compact subset of G_3, $\mathrm{grad}\left(\psi_n(w) - \psi_\infty(w)\right) \to 0$ uniformly in $w \in \partial\mathbb{D}$ and the C^2 smoothness of φ (and φ^{-1}) shows $\mathrm{grad} g_G(z, a_n) \to \mathrm{grad} g_G(z, a_\infty)$ as $n \to \infty$, uniformly in $z \in \Gamma_0$, hence for all $z \in \Gamma$.

These imply that $\mathrm{grad} g_G(z_\infty, a_\infty) = 0$, which contradicts that ∂G is C^2 smooth.

Therefore, there exists $r_1 > 0$ (we may assume that $r_1 < r_0$) such that for any $z \in \Gamma$, the closure of $D(z, r_1)$ does not contain any critical points of $g_G(\cdot, a)$, $a \in Z$.

Consider the infimum and supremum of

$$\{\|\mathrm{grad} g_G(\zeta, a)\| : a \in Z, \zeta \in G, \operatorname{dist}(\zeta, \Gamma) < r_1\},$$

and it is easy to see that they are finite and positive. Hence there exist $C_2 > 0$, $r_2 > 0$ such that for all $z \in G$, $\operatorname{dist}(z, \Gamma) < r_2$, $a \in Z$, we have (1).

If we apply this step twice and take $C_3 = C_2^2$, then we obtain (2). □

In the following Lemma, for definiteness, we assume that imaginary part of logarithm (of a nonzero complex number) is in $[0, 2\pi)$.

Lemma 2. *Let now G be a bounded, simply connected domain with C^2 smooth boundary, and φ be a conformal mapping from G onto $\mathbb{D}$. We define the following conformal projection: if $\zeta \in G$, $\varphi(\zeta) \neq 0$, then let*

$$\zeta^* = \zeta^*(\varphi; \zeta) := \varphi^{-1}\left[\exp i \operatorname{Im} \log \varphi(\zeta)\right].$$

This mapping is uniformly continuous away from $\varphi^{-1}[0]$. Furthermore, there exists $C_4 = C_4(G) > 0$ such that for any $\zeta \in G$ with $\varphi(\zeta) \neq 0$ and $\eta \in \partial G$ we have the following "reverse triangle" inequality:

$$|\zeta - \zeta^*| + |\zeta^* - \eta| \leqslant C_4 |\zeta - \eta|. \tag{4}$$

Proof. The Kellogg-Warschawski theorem implies that φ and φ' extend continuously to G. Denote by

$$M_1 := \inf\{|\varphi'(\zeta)| : |\zeta| < 1\}, \quad M_2 := \sup\{|\varphi'(\zeta)| : |\zeta| < 1\},$$

hence $0 < M_1 \leqslant M_2 < \infty$. The mapping $\zeta \mapsto \zeta^*$ is well defined (if $\varphi(\zeta) \neq 0$), and $\exp i \operatorname{Im} \log \varphi(\zeta)$ is continuous (when $\zeta \in G \setminus \varphi^{-1}(0)$). Therefore the uniform continuity follows. As for the "reverse triangle" inequality, let $\xi \in \mathbb{D}$, $\xi \neq 0$, $\xi^* := \arg \xi = \xi / |\xi|$ and $|\eta_1| = 1$ be arbitrary. It is easy to see that $|\xi^* - \eta_1| \leqslant 2|\xi - \eta_1|$ and $|\xi - \xi^*| \leqslant |\xi - \eta_1|$. Let us note that if $\xi = 0$ and $|\xi^*| = 1$, $|\eta_1| = 1$, then $|\xi^* - \eta_1| \leqslant 2|\xi - \eta_1|$ and $|\xi - \xi^*| = |\xi - \eta_1|$. In any case, we have $|\xi - \xi^*| + |\xi^* - \eta_1| \leqslant 3|\xi - \eta_1|$. Now we use the conformal mapping φ and the substitutions $\xi = \varphi(\zeta)$, $\xi^* = \varphi(\zeta^*)$ and $\eta_1 = \varphi(\eta)$. Obviously, $|\zeta - \zeta^*| \leqslant M_2 |\xi - \xi^*|$, $|\zeta^* - \eta| \leqslant M_2 |\xi^* - \eta_1|$ and $M_1 |\xi - \eta_1| \leqslant |\zeta - \eta|$. Therefore, $|\zeta - \zeta^*| + |\zeta^* - \eta| \leqslant M_2 (|\xi - \xi^*| + |\xi^* - \eta_1|) \leqslant 3M_2 |\xi - \eta_1| \leqslant \frac{3M_2}{M_1} |\zeta - \eta|$. We established the "reverse triangle" inequality. □

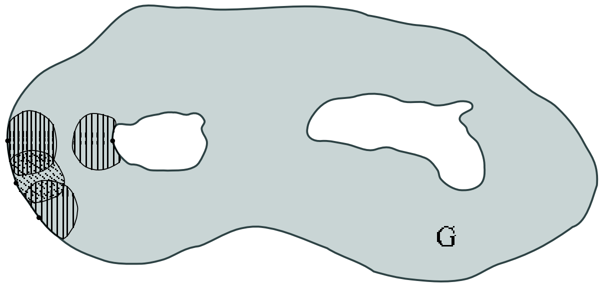

Fig. 1. G and some of the attached, simply connected sets $E(\zeta, r_3)$.

Furthermore, it follows from the proof that (4) holds when $\varphi(\zeta) = 0$, ζ^* is any point from ∂G and $\eta \in \partial G$.

§3. Main results

Main tool is an estimate for a Cauchy type integral. Its importance is mentioned in [5] and similar estimates were also established by Kővári and Pommerenke in [10] (see also [17], p. 185).

Proposition 1. *Let $G \subset \mathbf{C}_\infty$ be a finitely connected domain and its boundary $\Gamma := \partial G$ be finite union of C^2 smooth Jordan curves. Let $Z \subset G$ be a closed set. Then there exists $\rho_2 > 0$ such that for all $0 < \rho < \rho_2$, $\gamma_\rho(a) = \{w \in G : g_G(w, a) = \rho\}$ is finite union of C^2 smooth Jordan curves (for any $a \in Z$) and*

$$C_5 := \sup \left\{ |\log(\rho)|^{-1} \int\limits_{\gamma_\rho(a)} \frac{|dw|}{|w - z|} : a \in Z,\, z \in \Gamma,\, \rho_2 > \rho > 0 \right\} < \infty.$$

Proof. We use r_0, r_1, r_2 introduced in the proof of Lemma 1.

There exists $r_3 > 0$ such that $r_3 < r_0$ and for every $\zeta \in \Gamma$ and $r > 0$, $r < r_3$ there exists a simply connected domain $E(\zeta, r)$ such that $E(\zeta, r) \subset D(\zeta, r) \cap G$, $\partial E(\zeta, r)$ is a C^2 smooth Jordan curve, $D(\zeta, 0.99r) \cap G \subset \overline{E(\zeta, r)}$ and the boundaries coincide in the sense: $\partial E(\zeta, r) \cap \Gamma = \partial E(\zeta, r) \cap \overline{D(\zeta, 0.99r)}$ where $\overline{D(\zeta, 0.99r)}$ means the closed disk here. We may assume that $r_3 < r_1, r_2$. Sometimes we call $E(\zeta, r)$'s attached domains.

Fix $\zeta \in \Gamma$ arbitrarily. Let $\varphi = \varphi(\zeta; z) = \varphi(\zeta, r_3; z)$ be a conformal map from $E(\zeta, r_0)$ onto $\mathbb{D}$. Note that φ, φ' extend continuously to $\partial E(\zeta, r)$, this follows from the Kellogg-Warschawski theorem. Since Γ is compact,

$$\{z \in G \cup \Gamma : \operatorname{dist}(z, \Gamma) \leqslant r_3/4\}$$

is also compact therefore disks with centers from this set and with radii $r_3/2$ cover this set. Because of compactness, there is a finite set $\mathcal{E}$, $\mathcal{E} \subset \Gamma$ such that

$$\bigcup \{D(\zeta, r_3/2) : \zeta \in \mathcal{E}\} \supset \{z \in G \cup \Gamma : \operatorname{dist}(z, \Gamma) \leqslant r_3/4\}$$

and since the length of Γ is finite, we may require that each (open) arc from $\Gamma \setminus \mathcal{E}$ has length $r_3/2$ at most. Then

$$\bigcup \{E(\zeta, r_3) : \zeta \in \mathcal{E}\} \supset \{z \in G \cup \Gamma : \operatorname{dist}(z, \Gamma) \leqslant r_3/4\}$$

and for all $z \in \Gamma$ there exists $\mathcal{E}_1 \subset \mathcal{E}$ consisting of at most two points such that $D(z, r_3/4)$ can be covered with disks with radii $r_3/2$ with those centers, $D(z, r_3/4) \subset \bigcup\{D(\zeta, r_3/2) : \zeta \in \mathcal{E}_1\}$, and the disk can be also covered with the corresponding simply connected domains: $D(z, r_3/4) \subset \bigcup\{E(\zeta, r_3) : \zeta \in \mathcal{E}_1\}$.

Let

$$C_6 := \inf\{|\varphi'(\zeta; z)| : \zeta \in \mathcal{E}, z \in E(\zeta, r_3)\},$$

$$C_7 := \sup\{|\varphi'(\zeta; z)| : \zeta \in \mathcal{E}, z \in E(\zeta, r_3)\}.$$

It is easy to see that $0 < C_6 \leqslant C_7 < \infty$.

We use the conformal maps $\varphi(\zeta; z)$, where $\zeta \in \mathcal{E}$, to compare any point on the Green level lines with the boundary Γ of G as follows.

Consider the "sectors"

$$\{w \in \mathbb{D} : 0 \leqslant |w| < 1, \arg w = \arg \varphi(\zeta; z), z \in \partial E(\zeta, r_3) \cap \Gamma\}$$

where $\zeta \in \mathcal{E}$.

These are closed sets in $\mathbb{D}$ and we take inverse images of the "semi-open sectors":

$$H := \cup_{\zeta \in \mathcal{E}} H_\zeta,$$

$$H_\zeta := \Big\{\varphi^{-1}[\zeta; w] : 0 < |w| \leqslant 1, \arg w = \arg \varphi(\zeta; z), \\ z \in \operatorname{Int}_\Gamma(\partial E(\zeta, r_3) \cap \Gamma)\Big\},$$

where $\operatorname{Int}_\Gamma(.)$ means the relative interior to Γ. By construction, $H_\zeta \cap G$ is an open set, H covers Γ ($\Gamma \subset H$) and $H \cap G$ is an open set too. Therefore there exists $r_4 > 0$ such that $\operatorname{dist}(\Gamma, G \setminus H) > r_4$ where r_4 depends on G

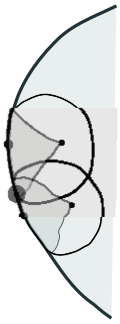

Fig. 2. Two "sectors" in the conformal projection.

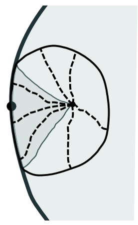

Fig. 3. Conformal projection.

only. We may assume that $r_4 < r_3/4$. We obtain the following conformal projection property:

$$\text{if } z \in G, \operatorname{dist}(z, \Gamma) < r_4, \text{ then } \exists \zeta \in \mathcal{E} : \ z^* = \varphi^{-1}[\zeta, \arg \varphi(\zeta; z)] \in \Gamma. \quad (5)$$

Note that the choice of ζ is local: if z can be projected conformally using $\varphi^{-1}[\zeta, \arg \varphi(\zeta; .)]$, then the same mapping is defined and can be applied in a neighborhood of z. Obviously, this projection $z \mapsto z^*$ is continuous (with fixed ζ). This conformal projection is depicted on Figure 3.

We show that there exists $r_5 > 0$ such that for all $z \in \Gamma$ there exists $\zeta \in \mathcal{E}$ such that $D(z, r_5) \cap G \subset H_\zeta$, in other words, the same projection can be applied in a uniformly large neighborhood of arbitrary boundary point. Let $h_\zeta(z) := \operatorname{dist}(z, (G \cup \Gamma) \setminus H_\zeta)$ $(z \in \mathbf{C},\ \zeta \in \mathcal{E})$, this $h_\zeta(.)$ is continuous, hence $D(z, h_\zeta(z)) \subset H_\zeta$. Put $h(z) := \max(h_\zeta(z) : \zeta \in \mathcal{E})$ which is continuous too. Since H_ζ's cover Γ, for all $z \in \Gamma$ there exists $\zeta \in \mathcal{E}$ such that $h_\zeta(z) > 0$. Hence $h(z) > 0$, and is continuous on the compact Γ, therefore $\inf\{h(z) : z \in \Gamma\} > 0$. Let r_5 be the minimum of this inf and r_4, obviously r_5 depends only on G and is independent of ρ.

Now we show that the level lines of $\operatorname{Re}\log\varphi(\zeta;\cdot)$ and $g_G(\cdot,a)$, $a \in Z$ are "almost parallel" if we are close to Γ. We need to estimate the angles made by the level lines of $\operatorname{Re}\log\varphi(\zeta;\cdot)$ and $g_G(\cdot,a)$. It is well known that if f and g are holomorphic functions, then the level lines of $\operatorname{Re} f$ and $\operatorname{Re} g$ make angle

$$\arccos\frac{\langle \operatorname{grad}\operatorname{Re} f, \operatorname{grad}\operatorname{Re} g\rangle}{\|\operatorname{grad}\operatorname{Re} f\|\,\|\operatorname{grad}\operatorname{Re} g\|} = \arccos\frac{\operatorname{Re}(f'\overline{g'})}{|f'||g'|}. \tag{6}$$

We need the following two assertions: the ρ-level lines of g_G converge uniformly to Γ as $\rho \to 0$, and similar uniform convergence holds for the tangents of those. More precisely,

$$\sup\{\operatorname{dist}(z,\Gamma) :\, g_G(z,a) = \rho\} \to 0$$

uniformly in $a \in Z$, and if $n(z_0)$ denotes the normal vector to Γ at $z_0 \in \Gamma$ pointing inward with unit length, then $\forall\varepsilon\ \exists\rho_3 > 0\ \forall a \in Z$, $\forall z \in G$, $\exists z_1 \in \Gamma$, $g_G(z,a) < \rho_3$, $|z - z_1| < \rho_3$ we have

$$\left|\frac{\operatorname{grad} g_G(z,a)}{\|\operatorname{grad} g_G(z,a)\|} - n(z_1)\right| < \varepsilon.$$

This first assertion follows from (1).

For the second assertion, consider $\frac{\operatorname{grad} g_G(z,a)}{\|\operatorname{grad} g_G(z,a)\|}$ close to Γ ($\operatorname{dist}(z,\Gamma) <$ r_4). It is a continuous function in z (for any fixed $a \in Z$) and can be extended continuously to Γ, because Γ is C^2-smooth. As $z \to z_1$ where $z_1 \in \Gamma$ is fixed, this function will tend to $n(z_1)$, because the gradient of Green's function on the boundary is pointing inward. The uniformity in $a \in Z$ follows using the continuity in a and the compactness of Z.

This second assertion, with the conformal projection z^* gives that for all $\varepsilon > 0$ and $\zeta \in \mathcal{E}$ there exists $\rho_4 = \rho_4(\zeta) > 0$ such that for all $a \in Z$, $z \in G$ with $g_G(z,a) < \rho_4$, $z \in E(\zeta, r_0)$ and $z^*(z) = \varphi^{-1}[\zeta, \arg\varphi(\zeta;z)]$ we have

$$\left|\frac{\operatorname{grad} g_{G_1}(z,a)}{\|\operatorname{grad} g_{G_1}(z,a)\|} - n(z^*(z))\right| < \varepsilon. \tag{7}$$

Similar argument can also be applied for the conformal map $\varphi(\zeta;z)$, because $\operatorname{Re}\log\varphi(\zeta;z)$ is a Green's function of $E(\zeta, r_3)$. This yields that for all $\varepsilon > 0$ and $\zeta \in \mathcal{E}$ there exists $\rho_5 = \rho_5(\zeta) > 0$ such that for all $z \in G$

with $z \in E(\zeta, r_3)$, $\operatorname{dist}(z, \Gamma) < \rho_5$ we have

$$\left| \frac{\operatorname{grad} \operatorname{Re} \log \varphi(\zeta; z)}{\|\operatorname{grad} \operatorname{Re} \log \varphi(\zeta; z)\|} - n(z^*(z)) \right| < \varepsilon. \tag{8}$$

Now we combine (1), (7) and (8) with $\varepsilon = 1/16$. Whence there exists $\rho_6 > 0$ (actually, $\rho_6 = \min(\rho_5(\zeta), C_2\rho_4(\zeta) : \zeta \in \mathcal{E})$) such that for all $a \in Z$, $\zeta \in \mathcal{E}$, $z \in G$ with $\operatorname{dist}(z, \Gamma) < \rho_6$, $z \in E(\zeta, r_3)$ we have

$$\left| \frac{\operatorname{grad} \operatorname{Re} \log \varphi(\zeta; z)}{\|\operatorname{grad} \operatorname{Re} \log \varphi(\zeta; z)\|} - \frac{\operatorname{grad} g_G(z, a)}{\|\operatorname{grad} g_G(z, a)\|} \right| < \frac{1}{8}. \tag{9}$$

Now we are going to estimate the integral in the Proposition. Fix $z \in \Gamma$ arbitrarily. There exists $\zeta \in \mathcal{E}$ such that $D(z, r_5) \cap G \subset H_\zeta$. If ρ is small ($\rho < \rho_6$), then $\gamma_\rho(a) \cap D(z, r_5)$ is a single Jordan arc (if not empty) and $\gamma_\rho(a) \setminus D(z, r_5)$ is union of finitely many Jordan arcs and curves. Moreover, since the length $|\gamma_\rho(a)|$ of $\gamma_\rho(a)$ tends to the length $|\Gamma|$ of Γ as $\rho \to 0$ uniformly in $a \in Z$ (see (7)), the gradients of Green's functions close to Γ are bounded. In particular, there exists $C_8 > 0$ such that for all $a \in Z$, $0 < \rho \leqslant \rho_6$, we have $|\gamma_\rho(a)| \leqslant C_8 |\Gamma|$.

We split the integral in the Proposition into two integrals as follows: denote by $\gamma^{(1)}$ the Jordan arc $\gamma_\rho(a) \cap D(z, r_5)$ and by $\gamma^{(2)}$ the remaining part of $\gamma_\rho(a)$, $\gamma^{(2)} = \gamma_\rho(a) \setminus D(z, r_5)$. On $\gamma^{(2)}$, the estimate is easy: $\zeta \in \gamma^{(2)}$, so $|z - w| > r_5$ and

$$\int\limits_{\gamma^{(2)}} \frac{|dw|}{|z - w|} \leqslant C_8 |\Gamma| \frac{1}{r_5}$$

which is bounded from above for all small ρ ($0 < \rho \leqslant \rho_6$).

On $\gamma^{(1)}$, we use the "conformal projection" (on H_ζ) to change the integration from $w \in \gamma^{(1)}$ to $w^* \in \Gamma$ and the comparison of the angles between gradients (see (9)) to transfer the arc length measure on $\gamma^{(1)}$ onto Γ and we estimate it there as follows. First,

$$\int\limits_{\gamma^{(1)}} \frac{|dw|}{|w - z|} \leqslant \int\limits_{\gamma^{(1)}} C_4 \frac{|dw|}{|w - w^*| + |w^* - z|} \leqslant C_4 \int\limits_{\gamma^{(1)}} \frac{|dw|}{C_2^{-1}\rho + |w^* - z|}, \tag{10}$$

where we used the "reverse triangle" inequality (4), and by (1), $|w - w^*| \geqslant \operatorname{dist}(w, \Gamma) \geqslant \frac{1}{C_2}\rho$. We will continue this estimate later by applying the substitution $w = w(w^*)$.

We may assume that $\gamma^{(1)}$ is parametrized by t with respect to the arc length, $w = w(t)$, $|dw| = dt$, and we may assume that the direction of $iw'(t)$ and the direction of the gradient of $g_G(\cdot, a)$ at $w(t)$ coincide, i.e.

$$\begin{aligned}&(\mathrm{Re}(iw'(t)), \mathrm{Im}(iw'(t))\\ &\quad= \frac{1}{\|\mathrm{grad} g_G\left(w\left(t\right), a\right)\|}\left(\frac{\partial}{\partial x} g_G\left(w\left(t\right), a\right), \frac{\partial}{\partial y} g_G\left(w\left(t\right), a\right)\right).\end{aligned} \tag{11}$$

We need an upper estimate of the modulus of the derivative of w as a function of w^*, that is, a lower estimate on the modulus of the derivative of $w^*\left(w\left(t\right)\right)$.

$$\begin{aligned}\frac{d}{dt} w^*(w\left(t\right)) &= \frac{d}{dw}\varphi^{-1}\left[\zeta;\ \exp i \,\mathrm{Im} \log \varphi\left(\zeta; w\right)\right]\\ &= \frac{1}{\varphi'\left(\zeta; w^*\right)} \cdot \exp\left(i \,\mathrm{Im} \log \varphi\left(\zeta; w\right)\right) i \cdot \frac{d}{dt} \mathrm{Im} \log \varphi\left(\zeta; w(t)\right).\end{aligned} \tag{12}$$

Here, the modulus of the first factor is bounded from below by $1/C_7$, the second factor has modulus one. To estimate the third factor from below, we write

$$\begin{aligned}\frac{d}{dt} \mathrm{Im} \log \varphi\left(\zeta; w\left(t\right)\right) &= \mathrm{Im} \frac{d}{dt} \log \varphi\left(\zeta; w\left(t\right)\right)\\ &= -\,\mathrm{Re}\left(i \frac{\varphi'\left(\zeta; w\left(t\right)\right)}{\varphi\left(\zeta; w\left(t\right)\right)} \cdot w'(t)\right).\end{aligned} \tag{13}$$

Here we compare $iw'(t)$ with $\mathrm{grad} g_{G_1}(\cdot, a)$ and $\overline{\left(\frac{\varphi'}{\varphi}\right)}$ with $\mathrm{grad}\,\mathrm{Re} \log \varphi$ as follows. If $\varphi = u + iv$, then

$$\frac{\varphi'}{\varphi} = \frac{u_x + iv_x}{u + iv} = \frac{u_x u + v_x v}{u^2 + v^2} + i\frac{uv_x - vu_x}{u^2 + v^2} = \frac{u_x u + v_x v}{u^2 + v^2} - i\frac{u_y u + v_y v}{u^2 + v^2},$$

and

$$\mathrm{grad}\,\mathrm{Re} \log \varphi = \mathrm{grad}\frac{1}{2} \log(\varphi\bar{\varphi}) = \left(\frac{u_x u + v_x v}{u^2 + v^2}, \frac{u_y u + v_y v}{u^2 + v^2}\right).$$

Now using (11) we can continue (13)

$$\begin{aligned} &= -\operatorname{Re}\left((iw'(t))\overline{\left(\frac{u_x u + v_x v}{u^2+v^2} + i\frac{u_y u + v_y v}{u^2+v^2}\right)}\right) \\ &= -\left\langle \left(\operatorname{Re}(iw'(t)), \operatorname{Im}(iw'(t))\right), (\operatorname{grad}\operatorname{Re}\log\varphi)(w(t))\right\rangle \\ &= -\left\|(\operatorname{grad}\operatorname{Re}\log\varphi)(w(t))\right\| \\ &\qquad \times\left\langle \left(\operatorname{Re}(iw'(t)), \operatorname{Im}(iw'(t))\right), \frac{(\operatorname{grad}\operatorname{Re}\log\varphi)(w(t))}{\left\|(\operatorname{grad}\operatorname{Re}\log\varphi)(w(t))\right\|}\right\rangle. \end{aligned}$$

Here, the factor in front of the scalar product is bounded from below:

$$\left\|(\operatorname{grad}\operatorname{Re}\log\varphi)(w(t))\right\| = \left|\frac{\overline{\varphi'(w(t))}}{\varphi(w(t))}\right| = \left|\frac{\varphi'(w(t))}{\varphi(w(t))}\right| \geqslant C_6$$

and C_6 is positive.

In the scalar product there are two unit vectors and by (9), their distance is at most 1/8 (small). Therefore the scalar product is at least $1-\frac{1}{128}$. Summarizing these lower estimates for the factors appearing in (12), we can write

$$\left|\frac{d}{dt}w^*(w(t))\right| \geqslant \frac{1}{C_7}C_6\left(1-\frac{1}{128}\right).$$

Therefore we can use this estimate in (10) and continue it with

$$\leqslant C_4\frac{C_7}{C_6}\left(1-\frac{1}{128}\right)^{-1}\int\limits_{\gamma_*^{(1)}}\frac{|dw^*|}{C_2^{-1}\rho+|w^*-z|},$$

where w^* runs through $\gamma_*^{(1)} = \left\{w^* = w^*(\zeta; w) : w\in\gamma^{(1)}\right\}\subset\Gamma$. For sake of convenience, we change notation $\eta = w^*$ (and $|d\eta| = |dw^*|$), this way we have to estimate

$$= C_4\frac{C_7}{C_6}\frac{128}{127}\int\limits_{\gamma_*^{(1)}}\frac{|d\eta|}{C_2^{-1}\rho+|\eta-z|}.$$

Now we use that $\gamma_*^{(1)}\subset D(z,r_5)\cap\Gamma\subset\partial E(\zeta,r_3)\cap\Gamma$ and (3) so the tangents of Γ at $z'\in\partial E(\zeta,r_3)\cap\Gamma$ and at z_1 differ at most $\frac{\pi}{16}$, and if we use arc length parametrization of $\gamma_*^{(1)}$, $\eta=\eta(s)$, with $z=\eta(0)$, then

$$\cos\left(\frac{\pi}{16}\right)|s| \leqslant |\eta(s)-z| \leqslant |s|.$$

We also have an upper estimate for the length of $\gamma_*^{(1)}$: $\left|\gamma_*^{(1)}\right| \leqslant \frac{2r_5}{\cos(\pi/16)}$. Therefore we can continue the estimate again (with $C_9 = C_4 \frac{C_7}{C_6} \frac{128}{127}$)

$$\leqslant 2C_9 \int\limits_0^{2r_5/\cos(\pi/16)} \frac{ds}{C_2^{-1}\rho + \cos(\pi/16)\, s} = \frac{2C_9}{\cos\frac{\pi}{16}} \log\left(1 + C_2 \frac{2r_5}{\rho}\right)$$
$$\leqslant |\log\rho| \frac{2C_9}{\cos\frac{\pi}{16}} + \frac{2C_9}{\cos\frac{\pi}{16}} \log\left((1+C_2)\, 2r_5\right).$$

So summing up the estimates on $\gamma^{(2)}$ and on $\gamma^{(1)}$, the proposition is proved. □

Using this proposition we can prove the main result of this paper.

Theorem 1. *Let $D \subset \mathbf{C}_\infty$ be a finitely connected domain and its boundary $\Gamma := \partial D$ be finite union of C^2 smooth Jordan curves. Let $Z \subset D$ be a closed set. Let $f : D \to \mathbf{C}_\infty$ be a meromorphic function on D such that all its poles are in Z. Denote by f_r the sum of principal part of f (with $f_r(\infty) = 0$) and let f_h be the holomorphic part of f in D. Denote the total order of the poles of f by n. Then $f = f_r + f_h$ and there exists $C = C(D, Z) > 0$ depending on D and Z only such that if $n \geqslant 2$ we have*

$$\|f_r\|_{\partial D}\,,\ \|f_h\|_{\partial D} \leqslant C\, \log(n)\, \|f\|_{\partial D}\,. \tag{14}$$

Proof. We may assume that D is bounded domain. We consider the level lines of $g_D(.,a)$: $\{w \in D : g_D(w,a) = \rho\}$ (where $a \in Z$) and by Proposition 1, if ρ is small enough, or, with $\rho = 1/n$, and $n > 1/\rho_2$ then these are finite union of smooth Jordan curves. It is easy to see that there is an outer curve and all the other curves are lying inside. Fix the orientation of the Jordan curves such way that the outer curve is directed counterclockwise and the other curves lying inside it are directed clockwise. Therefore the interior of this contour is contained in D.

Moreover, there is $\rho_7 > 0$ such that if $\operatorname{dist}(z,\Gamma) < \rho_7$, $z \in G$, then for all $a \in Z$, $g_G(z,a) < \rho_1$. This follows from the upper (right) estimate in (1), and ρ_7 depends only on D and Z.

Fix $a \in Z$ and consider $\gamma := \{w \in D : g_D(w,a) = 1/n\}$.

We use the Bernstein–Walsh estimate for meromorphic functions (for the polynomial case, see e.g. [15] p. 156, or on p. 624 of the english translation of [4]), so we write for $w \in \gamma$

$$|f(w)| \leqslant \|f\|_\Gamma \exp\left(\sum_b g_D(w,b)\right)$$

where the sum is taken for all poles b of f counting order of the poles. We assume that $\frac{1}{n} < C_2\rho_7$, therefore $g_G(z,b) < \rho_1$ (for all $b \in Z$), hence we can apply (2) to estimate $g_D(w,b)$ with $g_D(w,a)$ and continue the estimate

$$\leqslant \|f\|_\Gamma \exp(nC_3 g_D(w,a)) = \|f\|_\Gamma e^{C_3}.$$

Now we apply Cauchy integral formula for f as follows: we use $f = f_h + f_r$ decomposition and if z is on the outer boundary of Γ, then we apply Cauchy integral formula on unbounded domain for f_r (see e.g. [3], p. 223 or [13], volume I, p. 318) and in other cases, we apply Cauchy integral formula for holomorphic functions. This way we can write for $z \in \Gamma$

$$f_r(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w-z}dw.$$

This f_r is a rational function with $f_r(\infty) = 0$ and it is easy to see that f_r coincide with the sum of principal parts. We can estimate f_r as follows at z using Proposition 1

$$|f_r(z)| \leqslant \frac{1}{2\pi}\|f\|_\gamma \int_\gamma \frac{1}{|w-z|}|dw| \leqslant \frac{e^{C_3}}{2\pi}\|f\|_\Gamma \cdot C_5 \log n.$$

Using $f_h = f - f_r$ and the assumption $n \geqslant 2$, there exists $C_{10} > 0$ independent of n such that

$$\frac{e^{C_3}}{2\pi}C_5(\log n) + 1 \leqslant C_{10}\log n.$$

Setting $n_1 := \max\left(2, \rho_2^{-1}, (C_2\rho_7)^{-1}\right)$, estimate (14) is proved for f_r and f_h when $n \geqslant n_1$.

If $2 \leqslant n < n_1$, then fix any $a_0 \in Z$. Denote the order of the pole of f at a_0 by n_0 (if f is holomorphic at a_0, then we let $n_0 = 0$). Consider

$$f^*(\varepsilon; z) := f(z) + \frac{\varepsilon}{(z-a_0)^{n_0+n_1-n}}.$$

Then $f^*(\varepsilon;z)$ is a meromorphic function such that sum of principal parts is $f_r^*(\varepsilon;z) = f_r(z) + \frac{\varepsilon}{(z-a_0)^{n_0+n_1-n}}$, holomorphic part is the same ($f_h^*(\varepsilon;z) = f_h(z)$) and as $\varepsilon \to 0$, then $\|f^*(\varepsilon;.)\|_{\partial D} \to \|f\|_{\partial D}$, $\|f_r^*(\varepsilon;.)\|_{\partial D} \to \|f_r\|_{\partial D}$. Applying the previous case for $f^*(\varepsilon;.)$ and then letting $\varepsilon \to 0$, we obtain the theorem (with $C = C_{10}\log(n_1)/\log 2$). □

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References

1. J. Cacoq, B. de la Calle Ysern, and G. López Lagomasino, *Direct and inverse results on row sequences of Hermite-Padé approximants.* — Constr. Approx. **38**, No. 1, (2013), 133–160.
2. J. B. Conway, *Functions of One Complex Variable, II.* In: *Graduate Texts in Mathematics.* **159** Springer-Verlag, New York (1995).
3. T. W. Gamelin, *Complex Analysis.* In: *Undergraduate Texts in Mathematics.* Springer-Verlag, New York (2001).
4. A. A. Gončar, *The problems of E. I. Zolotarev which are connected with rational functions.* — Mat. Sb. (N.S.) **78 (120)** (1969), 640–654.
5. A. A. Gončar and L. D. Grigorjan, *Estimations of the norm of the holomorphic component of a meromorphic function.* — Mat. Sb. (N.S.) **99(141)** No. 4 (1976), 634–638.
6. L. D. Grigorjan, *Estimates of the norm of holomorphic components of meromorphic functions in domains with a smooth boundary.* — Mat. Sb. (N.S.) **100(142)** No. 1 (1976), 156–164, 166.
7. L. D. Grigorjan, *A generalization of a theorem of E. Landau.* — Izv. Akad. Nauk Armjan. SSR Ser. Mat. **12** No. 3 (1977), 229–233, 242.
8. L. D. Grigorjan, *On the order of growth for the norm of the holomorphic component of a meromorphic function.* — In: *Analytic functions, Kozubnik 1979 (Proc. Seventh Conf., Kozubnik, 1979)*, Lecture Notes Math. **798** (1980), pp. 165–168.
9. S. Kalmykov, B. Nagy, *Polynomial and rational inequalities on analytic Jordan arcs and domains.* — J. Math. Anal. Appl. **2 (430)** (2015), 874–894.

10. T. Kövari, Ch. Pommerenke, *On Faber polynomials and Faber expansions.* — Math. Z. **99** (1967), 193–206.
11. E. Landau, D. Gaier, *Darstellung und Begründung Einiger Neuerer Ergebnisse der Funktionentheorie.* Springer-Verlag, Berlin (1986).
12. D. S. Lubinsky, *On the diagonal Padé approximants of meromorphic functions.* — Indag. Math. (N.S.) **7** (1996), 97–110.
13. A. I. Markushevich, *Theory of functions of a complex variable. Vol. I, II, III.* Translated and edited by Richard A. Silverman. 2nd English ed., Chelsea Publishing Co., New York (1977).
14. Ch. Pommerenke, *Boundary Behaviour of Conformal Maps.* Chelsea Publishing Co., New York (1977). In: *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]* **299**, Springer-Verlag, Berlin (1992).
15. T. Ransford, *Potential theory in the complex plane.* In: *London Mathematical Society Student Texts.* **28**, Cambridge University Press, Cambridge. Appendix B by Thomas Bloom (1995).
16. E. B. Saff, V. Totik, *Logarithmic potentials with external fields.* In: *Grundlehren der Mathematischen Wissenschaften. Fundamental Principles of Mathematical Sciences.* **316**, Springer-Verlag, Berlin. Appendix B by Thomas Bloom (1997).
17. P. K. Suetin, *Series of Faber Polynomials.* In: *Analytical Methods and Special Functions* **1**, Amsterdam (1998).

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