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NONLINEAR FOKKER–PLANCK–KOLMOGOROV EQUATIONS IN HILBERT SPACES

Abstract. We study the Cauchy problem for nonlinear Fokker–Planck–Kolmogorov equations for probability measures on a Hilbert space, corresponding to stochastic partial differential equations. Sufficient conditions for the uniqueness of probability solutions for a cylindrical diffusion operator and for a possibly degenerate diffusion operator are given. A new general existence result is established without explicit growth restrictions on the coefficients.

§1. Introduction and main definitions

We study the following Cauchy problem for a nonlinear Fokker–Planck–Kolmogorov equation with respect to probability measures on a separable Hilbert space H:

$$\partial_t\mu_t = \partial^2_{e_i e_j}(a^{ij}(\mu,x,t)\mu_t) - \partial_{e_i}(b^i(\mu,x,t)\mu_t), \quad \mu_0=\nu, \tag{1}$$

where ν is a Borel probability measure on H. The definition of a solution will be given below (see (5)). Throughout, summation over all repeated indices is assumed. In typical applications, the drift coefficients have the following structure:

$$b^i(\mu,x,t) = -\lambda_i x_i + \Phi_i(\mu,x,t), \qquad a^{ij}(\mu,x,t)=\beta^j\delta^{ij},$$

where δ^{ij} is the Kronecker delta symbol. This structure corresponds to the Kolmogorov equation for a nonlinear stochastic partial differential equation (SPDE)

$$dX_t = \sqrt{2}dw_t + \Big(\Lambda X_t + \Phi(\mu,X_t,t)\Big)dt,$$

where Λ is a self-adjoint negative unbounded operator with domain $D(\Lambda)\subset H$ with eigenvalues λ_j and the corresponding orthonormal basis e_j; w_t is a Wiener process in H. Assume that it has the form $w_t=\sum\limits_{j=1}^{\infty}\sqrt{\beta_j}\zeta_t^j e_j$,

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where ζ_t^j, $j \in \mathbb{N}$, are one-dimensional independent standard Wiener processes. We assume that the Wiener process and Λ have the same orthogonal basis only for simplicity. Then Eq. (1) is written in this basis. A standard example is given by the stochastic heat equation: $\Lambda = \Delta$ and $H = D(\Lambda) = H_0^{1,2}$. Similarly to the finite-dimensional case, the transition probabilities of the solution to such an equation satisfy an appropriate Fokker–Planck–Kolmogorov (FPK) equation.

Equations (1) are usually called nonlinear FPK equations, see [5, 14.2.2]. The term "nonlinear" indicates that the coefficients of the equation depend on the solution. Such equations arise in many problems of mechanics, statistical physics, probability theory, and control of diffusion processes. Linear equations of this type appeared in the first half of the XX century in the works of Fokker [6], Planck [12], and Kolmogorov [7]. Even linear FPK equations remain a very popular area of research, and nonlinear FPK equations belong to the mainstream in PDEs (see, for instance, [4] and references therein). Infinite-dimensional equations have been studied less, but they are also of great importance, especially due to intensive studies of SPDEs. However, not much is known about the well-posedness of the Cauchy problem (1) for such equations in the general setting (for the linear case, see [2] and references therein). We can mention the work [3], where the existence of solutions was established for equations with the identically zero diffusion matrix A and under certain growth restrictions on the drift term b. The work [1] is concerned with the gradient flow structure of particular equations.

The goal of this paper is two-fold. First, we provide sufficient conditions for the existence and uniqueness of solutions to the nonlinear problem (1) in the space of probability measures on a Hilbert space H in a rather general setting. Our existence results are stronger than the ones mentioned above. Moreover, no uniqueness results for general nonlinear Fokker–Planck–Kolmogorov equations in infinite dimensions have been known so far. Our second goal is to apply some methods developed for the study of uniqueness in finite dimensions to the infinite-dimensional case, namely, a modification of Holmgren's principle, used earlier in [2, 8, 9, 11]. The main idea is very simple and can be illustrated by the following (finite-dimensional) toy example: suppose that all coefficients are smooth enough for the following computation to make sense. Assume that the diffusion coefficients a^{ij} do not depend on measures. Suppose that there are two solutions $\mu = \mu_t\, dt$ and $\sigma = \sigma_t\, dt$ to the Cauchy problem (1) with initial conditions μ_0 and σ_0, respectively. Let us solve the adjoint problems

$$\partial_s f + \Big(a^{ij}(x,s)\partial^2_{x_i x_j} + b^i(\mu_s,x,s)\partial_{x_i}\Big) f(x,s) = 0, \quad f|_{s=t} = \psi,$$

where $\psi \in C_0^\infty(\mathbb{R}^d)$. Formally testing the Eq. (1) with f and integrating by parts, we obtain

$$\int_{\mathbb{R}^d} \psi\, d(\mu_t - \sigma_t) = \int_{\mathbb{R}^d} \psi\, d(\mu_0 - \sigma_0) + \int_0^t \int_{\mathbb{R}^d} \langle b(\mu_t) - b(\sigma_t), \nabla f\rangle\, d\sigma_s\, ds.$$

Recall that the Kantorovich 1-metric is defined by

$$W_1(\mu_t,\sigma_t) = \sup\Big\{\int_{\mathbb{R}^d} \psi\, d(\mu_t - \sigma_t) \colon \psi \in C_0^\infty(\mathbb{R}^d),\ \ |\nabla\psi| \leqslant 1\Big\} \qquad (2)$$

on the subset of probability measures having finite first moments. Suppose that $|b(\mu,x,t) - b(\sigma,x,t)| \leqslant C_0 W_1(\mu_t,\sigma_t)$ and $|\nabla f|$ is bounded (this holds if the coefficients are regular enough). Passing to the supremum over ψ as in (2), we arrive at

$$W_1(\mu_t,\sigma_t) \leqslant W_1(\mu_0,\sigma_0) + C\int_0^t W_1(\mu_s,\sigma_s)\, ds,$$

and Gronwall's inequality yields $W_1(\mu_t,\sigma_t) \leqslant e^{Ct} W_1(\mu_0,\sigma_0)$. In particular, if $\mu_0 = \sigma_0$, then $W_1(\mu_t,\sigma_t) = 0$ and $\mu_t = \sigma_t$.

The paper is divided into three sections. Section 2 is devoted to the uniqueness of solutions in two essentially different cases: a possibly degenerate diffusion and a "cylindrical" diffusion operator (the term corresponds to the notion of cylindrical Wiener process, which is a process with unit covariance operator of infinite trace class). Section 3 is concerned with the existence of local and global solutions. Finally, at the end of Sec. 3 we provide an existence and uniqueness theorem for equations of a particular form.

We now proceed to introduce notation and exact definitions. Let H be a real separable Hilbert space with inner product $\langle\,\cdot\,,\,\cdot\,\rangle$ (generating the norm $|\cdot|$), and let $\{e_j\}$, $j \in \mathbb{N}$, be an orthonormal basis in H. Let P_N be the orthogonal projection of H onto $H_N = \operatorname{span}\{e_1,\dots,e_N\} \simeq \mathbb{R}^N$. For any vector $c \in H$, let c_N denote the orthogonal projection of c to $\mathbb{R}^N$, i.e., $c_N = P_N c$. Sometimes we will think of c_N as of a vector from

H, since this cannot lead to a misunderstanding. Given $x \in H$ and $\alpha = (\alpha_1, \dots, \alpha_j, \dots)$, $\alpha_j > 0$, set $\|x\|_\alpha^2 := \sum\limits_{j=1}^{\infty} \alpha_j \langle x, e_j \rangle^2$.

Let $C_0^\infty(\mathbb{R}^d)$ denote the set of infinitely smooth compactly supported functions on $\mathbb{R}^d$. Let $C(H)$ denote the space of continuous functions on H, and let $C^+(I)$ denote the space of positive continuous functions on an interval $I \subset \mathbb{R}$. Let Lip_γ be the set of γ-Lipschitz continuous functions, that is, all $f \in C(H)$ such that $|f(x) - f(y)| \leqslant \gamma |x - y|$ whenever $x, y \in H$. The class $\mathcal{FC}_0^\infty(H)$ of test functions on H consists of functions $\varphi(x) = \varphi_0(x_1, \dots, x_d)$ where $\varphi_0 \in C_0^\infty(\mathbb{R}^d)$ and $x_j = \langle x, e_j \rangle$.

Let $\mathcal{P}_1(H)$ and $\mathcal{P}_2(H)$ denote the sets of probability measures on H with finite first and second moments, respectively. We shall use the following metrics on the space of measures. The total variation of a finite Radon (possibly signed) measure ρ on H is defined by

$$\|\rho\|_{TV} := \sup\Bigl\{\Bigl|\int f(x)\rho(dx)\Bigr|:\ f \in \mathcal{FC}_0^\infty(H),\ |f| \leqslant 1\Bigr\}. \tag{3}$$

Similarly to (2), the Kantorovich metric W_1 is defined on the space $\mathcal{P}_1(H)$ by

$$W_1(\mu, \sigma) := \sup\Bigl\{\int f(x)(\mu - \sigma)(dx):\ f \in \mathcal{FC}_0^\infty(H),\ |\nabla f| \leqslant 1\Bigr\}. \tag{4}$$

Note that usually W_1 is defined as the supremum over Lip_1-functions, but the integral of a Lip_1-function over a measure from $\mathcal{P}_1(H)$ can be approximated by the intergals over the projections of this function, therefore, we can pass to the supremum over the smaller class of functions $\mathcal{FC}_0^\infty(H) \cap \mathrm{Lip}_1$.

We shall say that μ is given by a family of probability measures $(\mu_t)_{t\in[0,T]}$ on H, and write $\mu = (\mu_t)_{t\in[0,T]}$, if $\mu(dx\,dt) = \mu_t(dx)dt$, which means that

$$\int\limits_{H\times[0,T]} f\,d\mu = \int\limits_0^T \int\limits_H f\,d\mu_t\,dt.$$

In the subsequent considerations, we will use the notion of the Lyapunov function V for a differential operator. The choice of this function is explained in § 3, but now we introduce some related notation. Given a continuous strictly positive function V on H and $T > 0$, set

$$M_T(V) := \Bigl\{\mu = (\mu_t)_{t\in[0,T]}:\ \sup_{t\in[0,T]} \int V(x)d\mu_t(x) < +\infty\Bigr\}.$$

Given two countable sets of mappings

$$a^{ij}(\mu,x,t)\colon\ M_{T_0}(V)\times H\times[0,T_0]\to\mathbb{R},$$
$$b^{i}(\mu,x,t)\colon\ M_{T_0}(V)\times H\times[0,T_0]\to\mathbb{R},\quad i,j\in\mathbb{N},$$

that are Borel measurable in (x,t), and a probability measure ν on H, we consider the Cauchy problem (1). Set

$$L_\mu\psi := \sum_{i,j=1}^{\infty} a^{ij}(\mu,x,t)\partial^2_{e_ie_j}\psi + \sum_{i=1}^{\infty} b^i(\mu,x,t)\partial_{e_i}\psi.$$

We shall say that $\mu=(\mu_t)_{t\in[0,T_0]}$ is a probability solution to the Cauchy problem (1) if μ_t are probability measures, $\mu\in M_{T_0}(V)$, and for every $t\in[0,T_0]$ and $\varphi\in\mathcal{FC}_0^\infty(H)$ one has

$$\int\varphi d\mu_t-\int\varphi d\nu=\int_0^t\int L_\mu\varphi d\mu_s ds. \tag{5}$$

Here we assume by definition that $a^{ij},b^i\in L^1(H\times[0,T_0],d\mu)$ for $i,j\in\mathbb{N}$, i.e., the integrand on the right-hand side is well defined.

Sometimes it is more convenient to use an equivalent definition: let a test function v depend on a finite set of variables $x_1,\dots,x_k$, vanish outside some ball in $H_k\cong\mathbb{R}^k$, and lie in $C^{2,1}(\mathbb{R}^k\times(0,T_0))\bigcap C(\mathbb{R}^k\times[0,T_0))$. Then for all $t\in[0,T_0]$ the following identity holds:

$$\int v(x,t)\,d\mu_t=\int v(x,0)\,d\nu+\int_0^t\int\left[\partial_s v+L_\mu v\right]d\mu_s\,ds. \tag{6}$$

We shall say that $\mu=(\mu_t)_{t\in[0,\tau]}$ is a local solution to (1) if (5) and our regularity assumptions hold with τ in place of T_0.

§2. Uniqueness of probability solutions

We start with conditions for uniqueness of probability solutions to the Cauchy problem

$$\partial_t\mu_t=\beta^j\partial_{jj}\mu_t-\partial_j(b^j(x,t,\mu)\mu_t),\quad \mu_t|_{t=0}=\mu_0 \tag{7}$$

with a constant diagonal diffusion operator $A=\operatorname{diag}(\beta^j)_{j=1}^\infty$ with $\beta^j\geqslant 0$. For each $N\in\mathbb{N}$, set $A_N=\operatorname{diag}(\beta^j)_{j=1}^N$. Throughout this section, we assume that the drift term has the following structure:

$$b^i(\mu,x,t)=-\lambda_i x_i+\Phi_i(\mu,x,t),\quad \lambda_i\geqslant 0. \tag{8}$$

In this section we provide sufficient conditions for the uniqueness of probability solutions to (7) in two essentially different cases: that of a degenerate operator A and the cylindrical case with $\beta^j \geqslant \beta_0 > 0$. This latter case corresponds to the cylindrical Wiener process w_t.

We start with the first case. Fix a positive function $V \in C^2(H)$. Assume that

(F0) $A(x,t) = \operatorname{diag}\{\beta^j\}_{j=1}^{\infty}, \quad \beta^j \geqslant 0.$

In particular, one can consider fully or partially degenerate matrices (with $\beta^j = 0$).

We consider only solutions to (7) from the class $\mathcal{K}_1 = \mathcal{P}_1(H) \cap M_T(V)$ such that

(F1) For every $\varepsilon > 0$, every $d \in \mathbb{N}$, and every $\mu \in \mathcal{K}_1$, there is $N \geqslant d$ and a function $\Phi_{\mu,N} \in C^\infty(\mathbb{R}^N \times [0,T_0])$ such that $\Phi_{\mu,N} \in L^1(H, \mu + \sigma)$ for every $\sigma \in \mathcal{K}_1$,

$$\int_0^T \int_H |\Phi_N(x,t,\mu) - \Phi_{\mu,N}(P_N x, t)|(\mu_t + \sigma_t)(dx)dt < \varepsilon, \tag{9}$$

and, in addition,

$$\sup_{x \in H} |\Phi_{\mu,N}(\mu, x, t)| \cdot (1 + |x|)^{-1} \leqslant C_N(\mu) < \infty. \tag{10}$$

(F2) For every solution $\mu \in \mathcal{K}_1$ there exists a constant $\theta = \theta(\mu)$ such that

$$\langle \Phi_{\mu,N}(\mu,x,t) - \Phi_{\mu,N}(\mu,y,t), x - y\rangle \leqslant \theta|x-y|^2 + \|x-y\|^2_{\lambda_N} \tag{11}$$

for all $x, y \in H$ and $t \in [0, T_0]$.

(F3) There exists a continuous increasing function G on $[0, +\infty)$ such that $G(0) = 0$ and

$$|\Phi(\mu,x,t) - \Phi(\sigma,x,t)| \leqslant V(x) G(W_1(\mu_t, \sigma_t)) \tag{12}$$

for all $(x,t) \in H \times [0,T_0]$ and $\mu, \sigma \in \mathcal{K}_1$.

Theorem 2.1. *Assume that conditions* (F0), (F1), (F2), (F3) *hold. If* $\mu_0 \in \mathcal{P}_1(H)$, $V \in L^1(\mu_0)$, *and* G *is an Osgood function, i.e.,*

$$\int_{0+} \frac{du}{G(u)} = +\infty,$$

then a solution to the Cauchy problem (7) *in the class* $\mathcal{K}_1$ *is unique, provided it exists. Moreover, for every two solutions* $(\mu_t)_{t\in[0,T_0]}$ *and* $(\sigma_t)_{t\in[0,T_0]}$

from $\mathcal{K}_1$ one has

$$W_1(\mu_t,\sigma_t) \leqslant F^{-1}\Big(F(2e^{2\theta T_0}W_1(\mu_0,\sigma_0)) - Ct\Big),$$

where F^{-1} is the inverse function to $F(v) := \int_v^1 G(u)^{-1}du$. In particular, if $G(u)=u$, we obtain

$$W_1(\mu_t,\sigma_t) \leqslant 2e^{2\theta T_0}W_1(\mu_0,\sigma_0)e^{Ct}.$$

Example 2.1. Consider

$$b^i = -\lambda_i x_i + f^i(x)\int \varphi(y)d\mu_t(y).$$

Assume that there exist a sequence of bounded smooth functions $f_n(x,t)$ such that $\lim_{n\to\infty} f_n(x,t) = f(x,t)$ for any x,t and constants $C_1, C_2 > 0$, independent of n, such that

$$|f_n(x,t)| \leqslant C_1(1+|x|), \quad \langle f_n(x,t) - f_n(y,t), x-y\rangle \leqslant C_2|x-y|^2.$$

Assume also that $|\varphi(x)| \leqslant C_2(1+|x|)$ and $\mu_0 \in \mathcal{P}_1(H)$. Then the Lebesgue dominated convergence theorem implies that all assumptions of Theorem 2.1 are fulfilled with $V(x) = 1+|x|$. Hence the problem (7) has at most one solution in $M_{T_0}(|x|)$.

Proof. Consider solutions $\mu = (\mu_t)_{t\in[0,T_0]}$ and $\sigma = (\sigma_t)_{t\in[0,T_0]}$ to (7) from the class $\mathcal{K}_1$ with initial conditions $\mu_0 \in \mathcal{P}_1(H)$ and $\sigma_0 \in \mathcal{P}_1(H)$, respectively. Assume that $V \in L^1(\mu_0+\sigma_0)$. Fix a function $\psi \in \mathcal{FC}_0^\infty(H)$ such that $|\nabla\psi(x)| \leqslant 1$. Fix d such that $\psi(x) = \psi_0(P_d x)$ (which exists by the definition of $\mathcal{FC}_0^\infty(H)$).

Fix $\varepsilon > 0$. According to assumption (F1), there exists a smooth finite-dimensional approximating sequence $\Phi_{\mu,N}$, $n \geqslant d$, such that (9) and (10) hold. Set

$$b_{\mu,N}(x,t) := (-\lambda_1 x_1, \ldots, -\lambda_N x_N) + \Phi_{\mu,N}(\mu,x,t),$$
$$Cl_N(\mu) = \max_{1\leqslant i\leqslant N}\lambda_i + C_N(\mu).$$

Fix a cut-off function $\varphi \in C_0^\infty(\mathbb{R}^1)$ such that $0 \leqslant \varphi \leqslant 1$, $\varphi(x) = 1$ for $|x| < 1$, and $\varphi(x) = 0$ for $|x| > 2$; moreover, assume that for some $C > 0$ and all $x \in \mathbb{R}$ one has $|\varphi''(x)|^2 + |\varphi'(x)|^2 \leqslant C\varphi(x)$. For each $K \geqslant 1$ set $\varphi_K(t,x) := \varphi(t/K)\cdot\varphi(|x/K|)$.

We split the proof into several steps.

Step 1. “The adjoint problem.” We extend $b^i_{\mu,N}$ to the whole space $\mathbb{R}^{N+1}$ as follows: $b^i_{\mu,N}(x,t) = b^i_{\mu,N}(x,T_0)$ for $t > T_0$ and $b^i_{\mu,N}(x,t) = b^i_{\mu,N}(x,0)$ for $t < 0$. Consider the problem

$$\partial_s f + \widetilde{L}f = 0, \quad f|_{s=t} = \psi, \quad \widetilde{L}f := \sum_{j=1}^{N} \beta^j \partial^2_{x_j x_j} f + b^j_{\mu,N} \partial_{x_j} f \tag{13}$$

in $\mathbb{R}^N$. This problem has a solution $f = f_N$ of class $C^{2,1}(\mathbb{R}^N \times [0,t])$. Indeed, according to [13], the stochastic differential equation in $\mathbb{R}^N$

$$dX^N_t = \sqrt{2\beta^j}\, dW^j_t + b_{\mu,N}(X^N_t)dt, \quad X^N_0 = x$$

has a solution $X^N_t, t \geqslant 0$, and the function $f(x,s) = \mathbb{E}\Big(\psi(X^N_t)\Big|X^N_s = x\Big)$ solves (13). The smoothness follows from [14, Theorems 3.2.4, 3.2.6]. Clearly, $\sup|f| \leqslant \max|\psi| =: C(\psi)$.

Step 2. Plugging $v = \varphi_K f$ into identities (6) for solutions $\mu = (\mu_t)_{t\in[0,T_0]}$ and $\sigma = (\sigma_t)_{t\in[0,T_0]}$, we obtain

$$\begin{aligned}
\int \varphi_K(t,x)\psi(x)\, d\mu_t(x) &= \int \varphi_K(0,x) f(0,x)\, d\mu_0 \\
&\quad + \int_0^t \int \big[\varphi_K \langle B(\mu) - b_{\mu,N}, \nabla f\rangle + 2\langle A\nabla\varphi_K, \nabla f\rangle_N + fL\varphi_K\big]\, d\mu_s\, ds, \\
\int \varphi_K(t,x)\psi(x)\, d\sigma_t(x) &= \int \varphi_K(0,x) f(0,x)\, d\sigma_0 \\
&\quad + \int_0^t \int \big[\varphi_K \langle B(\sigma) - b_{\mu,N}, \nabla f\rangle + 2\langle A\nabla\varphi_K, \nabla f\rangle_N + fL\varphi_K\big]\, d\sigma_s\, ds.
\end{aligned}$$

Subtracting the second identity from the first one, we have

$$\int \varphi_K(t)\psi\, d(\mu_t-\sigma_t)\leqslant\int_0^t\!\!\int \varphi_K|B(\mu)-b_{\mu,N}||\nabla f|\, d(\sigma_s+\mu_s)\, ds$$
$$+2\int_0^t\!\!\int |A\nabla\varphi_K||\nabla f|+|f||L\varphi_K|d(\mu_s+\sigma_s)ds \tag{14}$$
$$+\int_0^t\!\!\int \varphi_K|B(\sigma)-B(\mu)||\nabla f|\, d\sigma_s\, ds+\Big(\int \varphi_K f\, d\mu_0-\int \varphi_K f\, d\sigma_0\Big).$$

Step 3. A bound on ∇f. Let us obtain a bound on $|\nabla f|$. We observe that (11) can be rewritten as

$$\langle b_{\mu,N}(\mu,x,t)-b_{\mu,N}(\mu,y,t),x-y\rangle\leqslant\theta|x-y|^2,$$

hence

$$\langle\mathcal{H}(x,t)y,y\rangle\leqslant\theta(\mu)|y|^2,\quad\text{where } \mathcal{H}=(\partial_{x_j}b^i_{\mu,N})_{i,j\leqslant N}.$$

Set $u=2^{-1}\sum\limits_{k=1}^{N}|\partial_{x_k}f|^2$. Differentiating Eq. (13) with respect to x_k and multiplying by $\partial_{x_k}f$, we obtain

$$\partial_t u+\widetilde{L}u+\langle\mathcal{H}\nabla f,\nabla f\rangle-\sum_{k=1}^{N}\beta^j(\partial^2_{x_jx_k}f)^2=0.$$

Since $\langle\mathcal{H}\nabla f,\nabla f\rangle\leqslant 2\theta u$ and the last summand is nonnegative,

$$\partial_t u+\widetilde{L}u+(2\theta)u\geqslant 0.$$

Then the maximum principle (see [14, Theorem 3.1.1]) yields that $|u(x,s)|\leqslant e^{2\theta(t-s)}|u(x,t)|$, i.e.,

$$|\nabla f|=2|u(x,s)|\leqslant 2e^{2\theta(t-s)}|\nabla\phi^S(x)|\leqslant 2e^{2\theta T_0}=:C_1. \tag{15}$$

Step 4. Limits as $K\to\infty$, $N\to\infty$, $\varepsilon\to 0$. Now observe that $B(\mu)-B(\sigma)=\Phi(\mu)-\Phi(\sigma)$, $\quad B_N-b_{\mu,N}=\Phi_N-\Phi_{\mu,N}$, and

$$|L\varphi_K|=|\widetilde{L}\varphi_K|+|\langle\Phi_N-\Phi_{\mu,N},\nabla\varphi_K\rangle|\leqslant|\widetilde{L}\varphi_K|+C\cdot K^{-1}|\Phi_N-\Phi_{\mu,N}|.$$

Hence, (14) and (15) imply that

$$\int \psi\, d(\mu_t - \sigma_t) \leqslant C_1 \cdot \int_0^t \int |\Phi(\mu) - \Phi(\sigma)|\, d\sigma_s\, ds + R_{\rm co} + R_{\rm op} + R_{\rm appr} + J_0(K,N), \tag{16}$$

where

$$R_{\rm op} := C_2(\psi) \int_0^t \int |\widetilde{L}\varphi_K|\, d(\mu_s + \sigma_s)\, ds,$$

$$R_{\rm appr} := \big(C_1 + C \cdot K^{-1}\big) \cdot \int_0^t \int |\Phi_N(\mu) - \Phi_{\mu,N}|\, d(\mu_s + \sigma_s)\, ds,$$

$$R_{\rm co} := 2C_1 \int_0^t \int |A\nabla\varphi_K| d(\mu_s + \sigma_s) ds,$$

$$J_0(K,N) = \int \varphi_K f_N\, d\mu_0 - \int \varphi_K f_N\, d\sigma_0.$$

Let us show that the R.-terms are small. By (F1) we have

$$|R_{\rm appr}| \leqslant (C_1 + C)\Big(\int_0^t \int |\Phi_N(\mu) - \Phi_{\mu,N}|\, d(\mu_s + \sigma_s)\, ds\Big) \leqslant C_3\varepsilon.$$

Let us estimate $R_{\rm op}$. Note that

$$\widetilde{L}\varphi_K = K^{-1}\varphi'(|x|/K)\langle b_{\mu,N}, \frac{x}{|x|}\rangle + K^{-2}\varphi''(|x|/K)|{\rm tr}A_N.$$

This expression is nonzero only on the set $\gamma_K := \{x\colon K \leqslant |x| \leqslant 2K\}$. Hence, for a fixed N we have

$$|\widetilde{L}\varphi_K| \leqslant C_2 I_K\big(\frac{|b_{\mu,N}|}{1+|x|} + \frac{{\rm tr}A_N}{(1+|x|)^2}\big) \overset{(F2)}{\leqslant} (C_2 \cdot Cl_N(\mu) + C_2 \cdot {\rm tr}A_N) I_K,$$

where I_K is the indicator function of γ_K. Thus, we have $R_{\rm op} \leqslant C_4(\psi, N) \cdot (\mu+\sigma)\{\gamma_K \times [0,t]\}$. Similarly, $R_{\rm co} \leqslant C_5(N)\cdot(\mu+\sigma)\{\gamma_K \times [0,t]\}$. Obviously, $(\mu+\sigma)\{\gamma_K \times [0,t]\}$ tends to zero as $K \to \infty$. By (15) we have $f_N(x,0) \in$

Lip_{C_1}; hence the Lebesgue dominated convergence theorem yields that

$$J_0(K,N) \to J_{0,N} = \int f_N\, d\mu_0 - \int f_N\, d\sigma_0 \quad \text{as} \quad K \to \infty.$$

Using this and (12), for any fixed ε, ψ, and N, we pass to the limit in (16) as $K \to \infty$ and find that

$$\int \psi\, d(\mu_t - \sigma_t) \leqslant C_1 \int\limits_0^t \int V(x) \cdot G(W_1(\mu_s,\sigma_s)) d\sigma_s\, ds + C_3\varepsilon + J_{0,N}.$$

Now we pass to the limit as $N \to \infty$. The maximum principle and the Arzelà–Ascoli theorem (which is applicable due to the fact that $f_N \in \mathrm{Lip}_{C_1}$) imply that the sequence $\{f_N(x,0)\}$ contains a subsequence converging on all compact sets. In particular, $f_N(x,0) \to \widetilde{f}(x) \in \mathrm{Lip}_{C_1}$ pointwise as $N \to \infty$. Therefore,

$$\int \psi\, d(\mu_t - \sigma_t) \leqslant C_1 \int\limits_0^t \int V(x) \cdot G(W_1(\mu_s,\sigma_s)) d\sigma_s\, ds + C_3\varepsilon + \int \widetilde{f}\, d(\mu_0 - \sigma_0). \tag{17}$$

Step 5. The final estimate. Observe that

$$\int \widetilde{f}(x) d(\mu_0 - \sigma_0) \leqslant C_1 W_1(\mu_0,\sigma_0),$$

and, since ε is an arbitrary positive number, (17) gives the inequality

$$\int \psi\, d(\mu_t - \sigma_t) \leqslant C_1 \int\limits_0^t G(W_1(\mu_s,\sigma_s)) \Bigl(\int V(x) d\sigma_s\Bigr)\, ds + C_1 W_1(\mu_0,\sigma_0). \tag{18}$$

Passing to the supremum over $\psi \in \mathcal{FC}_0^\infty(H)$ with $|\nabla\psi(x)| \leqslant 1$, we find that

$$W_1(\mu_t,\sigma_t) \leqslant C_1 W_1(\mu_0,\sigma_0) + C_1 \int\limits_0^t G(W_1(\mu_s,\sigma_s)) \Bigl(\int V(x) d\sigma_s\Bigr)\, ds.$$

Since $\sigma \in M_T(V)$, this yields that

$$W_1(\mu_t,\sigma_t) \leqslant C_1 W_1(\mu_0,\sigma_0) + C \int\limits_0^t G(W_1(\mu_s,\sigma_s)) ds.$$

If $W_1(\mu_0,\sigma_0)=0$, then the integration gives $W_1(\mu_t,\sigma_t)\equiv 0$. In the general case we obtain

$$W_1(\mu_t,\sigma_t)\leqslant F^{-1}\Big(F(C_1W_1(\mu_0,\sigma_0))-Ct\Big),$$

where $F(v)=\int\limits_v^1 G(u)^{-1}du$ and F^{-1} is the inverse function to F. Finally, we recall that $C_1=2e^{2\theta T_0}$. □

We now proceed to the cylindrical case. Assume that

(T0) $\beta^j\geqslant\beta_0>0$.

We consider only solutions to (7) from the class $\mathcal{K}_2:=\mathcal{P}_2(H)\cap M_{T_0}(V)$. Let us list our assumptions on the drift term:

(T1) For every solution $\mu\in\mathcal{K}_2$, we have $\Phi(\mu,x,t)\in l^2$ for $\mu\times[0,T_0]$-a.e. (x,t) and $\|\Phi\|_{l^2}\in L^2(\mu)$.

(T2) There exists a continuous increasing function G on $[0,+\infty)$ such that $G(0)=0$ and

$$|\Phi(\mu,x,t)-\Phi(\sigma,x,t)|\leqslant\sqrt{V(x)}\cdot G(\|\mu_t-\sigma_t\|_{TV}) \qquad (19)$$

for all $(x,t)\in H\times[0,T]$ and all solutions $\mu,\sigma\in\mathcal{K}_2$.

Theorem 2.2. *Assume that conditions* (T0), (T1), (T2) *hold. If the initial data is such that* $\mu_0\in\mathcal{P}_2(H)$, $\sqrt{V}\in L^1(\mu_0)$, *and*

$$\int\limits_{0+}\frac{du}{G^2(\sqrt{u})}=+\infty,$$

then a solution to the Cauchy problem (7) *in* $\mathcal{K}_2$ *is unique, provided it exists. Moreover, for every two solutions* $\mu=(\mu_t)_{t\in[0,T_0]}$ *and* $\sigma=(\sigma_t)_{t\in[0,T_0]}$ *from* $\mathcal{K}_2$ *one has*

$$\|\mu_t-\sigma_t\|_{TV}\leqslant F^{-1}\Big(F(\|\mu_0-\sigma_0\|_{TV}-Ct\Big),$$

where F^{-1} *is the inverse function to* $F(v)=\int\limits_v^1 G(\sqrt{u})^{-2}du$. *In particular, if* $G(u)=u$, *then we have*

$$\|\mu_t-\sigma_t\|_{TV}\leqslant\|\mu_0-\sigma_0\|_{TV}e^{Ct}.$$

Example 2.2. Consider

$$b^i=-\lambda_i x_i+f^i(x)\int\varphi(y)d\mu_t(y),\quad \beta^j\geqslant\beta_0>0.$$

Assume that f has at most linear growth and φ is globally bounded. Moreover, assume that $\mu_0 \in \mathcal{P}_2(H)$. Then there is at most one solution to (7) in $M_{T_0}(|x|^2)$.

Indeed,

$$\|\Phi\|_{l^2}^2 \leqslant \sum_{j=1}^{\infty} |f^i|^2 = \|f\|_{L^2(\mu_t+\sigma_t)}^2 \leqslant C(1+|x|^2) < +\infty,$$

i.e., (T1) holds, and

$$\Big|\int \varphi(y)d\mu_t(y) - \int \varphi(y)d\sigma_t(y)\Big| \leqslant C_1\|\mu_t - \sigma_t\|_{TV},$$

so assumption (T2) is also fulfilled.

Proof. Consider solutions $\mu = (\mu_t)_{t\in[0,T_0]}$ and $\sigma = (\sigma_t)_{t\in[0,T_0]}$ to (7) from the class $\mathcal{K}_2$ with initial conditions μ_0 and σ_0 from $\mathcal{P}_2(H)$, respectively. Assume that $\sqrt{V} \in L^1(\mu_0+\sigma_0)$. Fix a function $\psi \in \mathcal{FC}_0^\infty(H)$ such that $|\psi| \leqslant 1$. Fix d such that $\psi(x) = \psi_0(P_d x)$ (which exists by the definition of $\mathcal{FC}_0^\infty(H)$).

Fix a cut-off function $\varphi \in C_0^\infty(\mathbb{R}^1)$ such that $0 \leqslant \varphi \leqslant 1$, $\varphi(x) = 1$ if $|x| < 1$, and $\varphi(x) = 0$ if $|x| > 2$. Moreover, assume that for some $C > 0$ and all $x \in \mathbb{R}$, one has $|\varphi''(x)|^2 + |\varphi'(x)|^2 \leqslant C\varphi(x)$. For each $K \geqslant 1$ set $\varphi_K(t,x) := \varphi(t/K)\cdot\varphi(|x|/K)$.

We split the proof again into several steps.

Step 1. Finite-dimensional smooth approximation of the drift. Given Φ satisfying (T1), for every $\varepsilon > 0$ we can find $b_{\mu,N} \in C^\infty(\mathbb{R}^N)$, $N \geqslant d$, such that

$$\int_0^{T_0}\int_H |B_N(x,t,\mu) - b_{\mu,N}(P_N x,t)|^2(\mu_t+\sigma_t)(dx)dt < \varepsilon^2$$

for all solutions μ, $\sigma \in \mathcal{K}_2$. To prove this, we observe that for each N there is a smooth bounded function $\Phi_{\mu,N} \in C_b^\infty(\mathbb{R}^N \times [0,T_0])$, $N \geqslant d$, such that

$$\int_0^{T_0}\int_H |\Phi_N(x,t,\mu) - \Phi_{N,\mu}(P_N x,t)|^2(\mu_t+\sigma_t)(dx)dt < \varepsilon^2$$

for all solutions μ, $\sigma \in \mathcal{K}_2$. Indeed, one can pick $M \in \mathbb{N}$ such that

$$\sum_{k=M+1}^{\infty} \int_0^{T_0} \int |\Phi^k|^2 (\mu_t + \sigma_t)(dx)dt \leqslant \varepsilon^2/2.$$

Next, for each Φ^k there exists a smooth bounded function $\bar{\Phi}_k$ depending only on the first n_k space variables and t such that

$$\int_0^{T_0} \int |\Phi^k - \bar{\Phi}_k|^2 (\mu_t + \sigma_t)(dx)dt < \varepsilon^2 (2M)^{-1}.$$

For $k > M$, set $\bar{\Phi}_k \equiv 0$. Set $N = \max\{M, n_1, \dots, n_M\}$ and $\Phi_{N,\mu} = (\bar{\Phi}_1, \dots \bar{\Phi}_N, 0, \dots)$. Then

$$\begin{aligned}\int_0^{T_0} \int |\Phi_N(x,t,\mu) - \Phi_{N,\mu}(P_N x, t)|^2 (\mu_t + \sigma_t)(dx)dt \\ \leqslant \sum_{k=1}^{M} \int_0^{T_0} \int |\Phi^k - \Phi_{N,\mu}|^2 (\mu_t + \sigma_t)(dx)dt \\ + \sum_{k=M+1}^{\infty} \int_0^{T_0} \int |\Phi^k|^2 (\mu_t + \sigma_t)(dx)dt < \varepsilon^2.\end{aligned}$$

Hence $b_{\mu,N}(\mu, x, t) := (-\lambda_1 x_1, \dots, -\lambda_N x_N) + \Phi_{N,\mu}(\mu, x, t)$ is the desired approximating sequence.

Step 2. "The adjoint problem." Similarly to Step 1 of the proof of Theorem 2.1, we extend $b^i_{\mu,N}$ to the whole space $\mathbb{R}^{N+1}$ as follows: $b^i_{\mu,N}(x,t) = b^i_{\mu,N}(x,T_0)$ if $t > T_0$ and $b^i_{\mu,N}(x,t) = b^i_{\mu,N}(x,0)$ if $t < 0$. Let $f = f_N$ be the $C^{2,1}(\mathbb{R}^N \times [0,t])$-solution to the finite-dimensional Cauchy problem in $\mathbb{R}^N$ (see [14, Theorems 3.2.4, 3.2.6])

$$\partial_s f + \widetilde{L} f = 0, \quad f|_{s=t} = \psi, \quad \widetilde{L} f := \sum_{j=1}^{N} \beta^j \partial^2_{x_j x_j} f + b^j_{\mu,N} \partial_{x_j} f.$$

The maximum principle implies that $\sup |f| \leqslant \max |\psi| \leqslant 1$.

Step 3. Similarly to the proof of Theorem 2.1, we plug the test functions $\varphi_K \cdot f$ into (6) for the solutions μ and σ and subtract one from another:

$$\int \varphi_K(t,x)\psi(x)\,(\mu_t-\sigma_t)(dx)$$
$$\leqslant \int_0^t \int \varphi_K |B(\mu)-b_{\mu,N}|\cdot|\nabla f|\,d(\sigma_s+\mu_s)\,ds + \|\mu_0-\sigma_0\|_{TV}$$
$$+\,2\int_0^t \int \Big[|A\nabla\varphi_K|\cdot|\nabla f| + |f|\cdot|L\varphi_K|\Big] d(\mu_s+\sigma_s)ds$$
$$+\int_0^t \int \varphi_K |B(\sigma)-B(\mu)||\nabla f|\,d\sigma_s\,ds.$$

Set $I_f := \Big(\int_0^t \int \varphi|\nabla f|^2 d\sigma_s ds\Big)^{1/2}$. By the Cauchy inequality,

$$\int \varphi_K\psi\,d(\mu_t-\sigma_t)$$
$$\leqslant \int_0^t\int \varphi\cdot|B(\mu)-B(\sigma)|\cdot|\nabla f|\,d\sigma_s\,ds + \|\mu_0-\sigma_0\|_{TV} + R_{\mathrm{co}} + R_{\mathrm{op}} + R_{\mathrm{appr}}, \tag{20}$$

where

$$R_{\mathrm{op}} := \int_0^t \int |\widetilde{L}\varphi_K|\,d(\mu_s+\sigma_s)\,ds,$$
$$R_{\mathrm{co}} := 2I_f\Big(\int_0^t \int |A\nabla\varphi_K|^2 d(\mu_s+\sigma_s)ds\Big)^{1/2},$$
$$R_{\mathrm{appr}} := \Big(C\cdot K^{-1} + I_f\Big)\Big(\int_0^t \int |B(\mu)-b_{\mu,N}|^2\,d(\mu_s+\sigma_s)\,ds\Big)^{1/2}.$$

Step 4. A bound for I_f. In order to find a bound for $\int_0^t\int \varphi|\nabla f|^2 d\sigma_s ds$, we plug the test function $f^2\cdot\varphi_K$ into (6) for σ:

$$\int \psi^2\varphi d\sigma_t - \int f^2(x,0)\varphi d\sigma_0 = \int_0^t\int_{\mathbb{R}^d}(\partial_s + L_\sigma)(f^2\varphi)d\sigma_s ds$$

$$= \int_0^t\int_{\mathbb{R}^d}\Big[2\varphi|\sqrt{A}\nabla f|^2 + 2\varphi f\langle b_{\mu,N} - B(\sigma),\nabla f\rangle + f^2L_\sigma\varphi + 2f(A\nabla f,\nabla\varphi)\Big]d\sigma_s ds.$$

Due to the maximum principle,

$$2\beta_0\int_0^t\int_{\mathbb{R}^d}\varphi|\nabla f|^2 d\sigma_s ds \leqslant 2\int_0^t\int_{\mathbb{R}^d}\varphi|\sqrt{A}\nabla f|^2 d\sigma_s ds$$

$$\leqslant 2 + 2\int_0^t\int_{\mathbb{R}^d}|b_{\mu,N} - B(\sigma)|\cdot|\nabla f| d\sigma_s ds + 2R_c,$$

where

$$R_c := \int_0^t\int_{\mathbb{R}^d}|L_\sigma\varphi|/2 + C_1(\psi)|A\nabla\varphi| d\sigma_s ds.$$

Using the inequality $ab \leqslant 2^{-1}\gamma a^2 + (2\gamma)^{-1}b^2$ with $\gamma = \beta_0^{-1}$, we obtain

$$2\beta_0\int_0^t\int_{\mathbb{R}^d}|\nabla f|^2 d\sigma_s ds \leqslant 2 + \frac{1}{\beta_0}\int_0^t\int_{\mathbb{R}^d}|b_{\mu,N} - B(\sigma)|^2 d\sigma_s ds$$

$$+ \beta_0\int_0^t\int|\nabla f|^2 d\sigma_s ds + 2R_c,$$

i.e.,

$$\beta_0\int_0^t\int_{\mathbb{R}^d}|\nabla f|^2 d\sigma_s ds \leqslant 2 + \frac{1}{\beta_0}\int_0^t\int_{\mathbb{R}^d}|b_{\mu,N} - B(\sigma)|^2 d\sigma_s ds + 2R_c.$$

Plugging this bound into (20) and using the Cauchy inequality, we obtain

$$\int \varphi_K \psi \, d(\mu_t - \sigma_t) \leqslant \sqrt{\int_0^t \int |B(\mu) - B(\sigma)|^2 d\sigma_s \, ds}$$

$$\times \sqrt{2\beta_0^{-1} + \beta_0^{-2} \int_0^t \int_{\mathbb{R}^d} |b_{\mu,N} - B(\sigma)|^2 d\sigma_s ds + 2R_c}$$

$$+ R_{\rm co} + R_{\rm op} + R_{\rm appr}.$$

Step 5. The limits. The final estimate. Arguing similarly to Step 4 of the proof of Theorem 2.1 and taking into account (19), we pass to the limits as $K \to \infty$, $N \to \infty$, and $\varepsilon \to 0$ and find that

$$\int \psi \, d(\mu_t - \sigma_t) \leqslant \|\mu_0 - \sigma_0\|_{TV}$$

$$+ \sqrt{C \int_0^t G^2(\|\mu_s - \sigma_s\|_{TV}) ds} \cdot \sqrt{2\beta_0^{-1} + \beta_0^{-2} C \int_0^t G^2(\|\mu_s - \sigma_s\|_{TV}) ds}.$$

Passing to the supremum over $\psi \in \mathcal{FC}_0^\infty(H)$ with $|\psi| \leqslant 1$ and observing that $\|\mu_s - \sigma_s\|_{TV} \leqslant 2$ and $t \leqslant T_0$, we obtain

$$\|\mu_t - \sigma_t\|_{TV} \leqslant \|\mu_0 - \sigma_0\|_{TV}$$

$$+ \sqrt{C \int_0^t G^2(\|\mu_s - \sigma_s\|_{TV}) ds} \cdot \sqrt{2\beta_0^{-1} + \beta_0^{-2} C \cdot T_0 \cdot G^2(2) ds}$$

$$\equiv \|\mu_0 - \sigma_0\|_{TV} + C_{\rm sum} \sqrt{\int_0^t G^2(\|\mu_s - \sigma_s\|_{TV}) ds}.$$

If $\|\mu_0 - \sigma_0\|_{TV} = 0$, then the integration yields that $\|\mu_t - \sigma_t\|_{TV} \equiv 0$. In the general case we obtain

$$\|\mu_t - \sigma_t\|_{TV} \leqslant F^{-1}\Big(F(\|\mu_0 - \sigma_0\|_{TV} - Ct\Big),$$

where $F(v) = \int_v^1 G(\sqrt{u})^{-2}$ and F^{-1} is the inverse function to F. □

§3. The existence of probability solutions

The question of existence of a probability solution to the Cauchy problem for a nonlinear equation is in a sense easier. One can establish the existence of solutions for equations of a more general form:

$$\partial_t \mu_t = \partial^2_{e_i e_j}(a^{ij}(x,t,\mu)\mu_t) - \partial_{e_i}(b^i(x,t,\mu)\mu_t), \quad \mu_0 = \nu, \tag{21}$$

where μ_t, ν are Borel probability measures on H. A solution is constructed as the limit of solutions for finite-dimensional equations. The essential part of the proof is justifying this limit. The finite-dimensional existence is ensured by [10].

Suppose that some positive continuous function V on H is fixed. Given a positive function $\alpha \in C([0,T_0])$ and $\tau \in (0,T_0]$, consider the class $M_{\tau,\alpha}(V)$ of all nonnegative finite Borel measures $\mu = (\mu_t)_{t\in[0,\tau]}$ such that for all $t \in [0,\tau]$ we have

$$\int V(x)\,d\mu_t \leqslant \alpha(t).$$

We shall say that a sequence $\mu^n = (\mu^n_t)_{t\in[0,\tau]}$ from $M_{\tau,\alpha}$ is V-convergent to $\mu = (\mu_t)_{t\in[0,\tau]} \in M_{\tau,\alpha}$ if for all $t \in [0,\tau]$

$$\lim_{n\to\infty} \int F(x) d\mu^n_t = \int F(x) d\mu_t \tag{22}$$

for every continuous function F on H such that

$$\lim_{R\to\infty} \sup_{x\in H\setminus B_R} F(x)\cdot V^{-1}(x) = 0.$$

Let us introduce our assumptions on the coefficients.

(H1) There exists a function V on H such that

$$V(x) > 0, \qquad \lim_{\|x\|\to+\infty} V(x) = +\infty,$$

and two mappings Λ_1 and Λ_2 of the space $C^+([0,T_0])$ into $C^+([0,T_0])$ such that for all $\tau \in (0,T_0]$ and all $\alpha \in C^+([0,T_0])$ the functions a^{ij} and b^i are defined on $M_{\tau,\alpha} = M_{\tau,\alpha}(V)$ and for all $\mu \in M_{\tau,\alpha}$ and $(x,t) \in H\times[0,\tau]$ one has

$$L_\mu V(x,t) \leqslant \Lambda_1[\alpha](t) + \Lambda_2[\alpha](t)V(x).$$

We shall call such a function V a Lyapunov function for the operator L_μ.

(H2) For all $\tau \in (0, T_0]$, $\alpha \in C^+([0, T_0])$, $\sigma \in M_{\tau,\alpha}$, and $x \in H$, the mappings

$$t \mapsto a^{ij}(x, t, \sigma) \quad \text{and} \quad t \mapsto b^i(x, t, \sigma)$$

are Borel measurable on $[0, \tau]$, and for every cylinder $K \subset H$ with a compact finite-dimensional base, the mappings

$$x \mapsto b^i(x, t, \sigma) \quad \text{and} \quad x \mapsto a^{ij}(x, t, \sigma)$$

are bounded on K uniformly in $\sigma \in M_{\tau,\alpha}$ and $t \in [0, \tau]$ and continuous on K uniformly in $\sigma \in M_{\tau,\alpha}$ and $t \in [0, \tau]$. Moreover, if a sequence $\mu^n \in M_{\tau,\alpha}$ is V-convergent to $\mu \in M_{\tau,\alpha}$, then for all $(x, t) \in H \times [0, \tau]$

$$\lim_{n\to\infty} a^{ij}(x, t, \mu^n) = a^{ij}(x, t, \mu), \quad \lim_{n\to\infty} b^i(x, t, \mu^n) = b^i(x, t, \mu).$$

(H3) For every $d \in \mathbb{N}$, $\tau \in (0, T_0]$, $\alpha \in C^+([0, T_0])$ and $\sigma \in M_{\tau,\alpha}$, the matrix $A_d(x, t, \sigma) = (a^{ij}(x, t, \sigma))_{1\leqslant i,j\leqslant d}$ is symmetric and nonnegative definite.

Theorem 3.1. *Assume that conditions* (H1)–(H3) *hold and* $V \in L^1(\nu)$. *Then*

(i) *there is* $\tau \in (0, T_0]$ *such that the Cauchy problem* (21) *has a probability solution* $\mu = (\mu_t)_{t\in[0,\tau]}$ *on* $[0, \tau]$*; moreover, a choice of* τ *depends only on* Λ_1 *and* Λ_2*;*

(ii) *if* Λ_1 *and* Λ_2 *are constant, then the Cauchy problem* (21) *has a solution on the whole interval* $[0, T_0]$.

In both cases

$$\sup_{t\in[0,\tau]} \int V(x) d\mu_t < \infty.$$

Proof. Let us introduce an auxiliary class of measures: for any $\alpha(t) \in C^+([0, T_0])$ and $\tau > 0$, let $N_{\tau,\alpha}$ denote the class of nonnegative measures $\mu = (\mu_t) \in M_{\tau,\alpha}$ such that

$$\Big|\int \varphi d\mu_t - \int \varphi d\mu_s\Big| \leqslant \Lambda(\tau, \alpha, \varphi)|t - s|$$

for all functions $\varphi \in \mathcal{FC}_0^\infty(H)$, where

$$\Lambda(\tau, \alpha, \varphi) := \sup\Big\{|L_\mu\varphi(x)| \colon\ x \in X,\, \mu \in M_{\tau,\alpha}\Big\}$$

does not depend on $\mu \in M_{\tau,\alpha}$. Due to (H2), this supremum is finite. Observe that the weak convergence of μ_t^n for each fixed t obviously follows from the V-convergence of μ^n. The set $N_{\tau,\alpha}$ is a convex compact set in the space of finite Borel measures. Moreover, the V-convergence of measures from $N_{\tau,\alpha}$ is equivalent to the weak convergence in the following sense: every sequence $\{\mu^n\} = \{\mu^n_t(dx)dt\} \in N_{\tau,\alpha}$ contains a subsequence $\{\mu^{n_l}\}$ such that it converges weakly to μ on $H \times [0,\tau]$ and $\mu^{n_l}_t$ converges weakly to μ_t on H for each fixed $t \in [0,\tau]$. Next, if a sequence $\{\mu^n_t\} \in N_{\tau,\alpha}$ is weakly convergent, then it is V-convergent. These assertions are easy generalizations of analogous finite-dimensional results (see [10, Lemmas 1, 2]).

We construct a solution to (21) as a certain limit of solutions to finite-dimensional problems. For each $d \in \mathbb{N}$ consider

$$A_d\colon (x,t,\mu) \mapsto (a^{ij}(P_d x,t,\mu))_{1\leqslant i,j\leqslant d}, \quad b_d\colon (x,t,\mu) \mapsto (b^i(P_d x,t,\mu))_{1\leqslant i\leqslant d}.$$

Set $L^d_\mu = a^{ij}_d \partial^2_{x_i x_j} + b^i_d \partial_{x_i}$, $1 \leqslant i,j \leqslant d$. Then the problem

$$\partial_t \mu_t = \partial^2_{x_i x_j}(a^{ij}_d(x,t,\mu)\mu_t) - \partial_{x_i}(b^i_d(x,t,\mu)\mu_t), \quad \mu_0 = \nu^d \tag{23}$$

with $\nu^d = \nu \circ P_d^{-1}$ has a probability solution $\mu^d = (\mu^d_t)_{t\in[0,\tau_d]}$ with some $\tau_d > 0$ (see [10, Theorem 1]). This follows from the fact that $V_d = P_d \circ V$ is a Lyapunov function for this finite-dimensional problem and

$$L^d_\mu V_d \leqslant \Lambda_1[\alpha] + \Lambda_2[\alpha] V_d$$

with the same coefficients Λ_1 and Λ_2. Furthermore, a choice of τ_d is determined only by Λ_1 and Λ_2 ([10, Remark 3]); hence $\tau_d \equiv \tau$ can be taken independent of d. If $\Lambda_j \equiv \mathrm{const}$, then $\tau = T_0$ ([10, Corollary 4]). We consider solutions $(\mu^d_t)_{t\in[0,\tau]}$ as measures on H, setting $\mu^d_t(B \times U) = 0$ for every $B \subset \mathbb{R}^d$ and nonempty $U \subset H \setminus \mathbb{R}^d$.

Fix a function $\varphi(x) = \varphi_0(x_1,\dots,x_m) \in \mathcal{FC}^\infty_0(H)$, and let $S \subset \mathbb{R}^m$ denote its compact support. For every $d \geqslant m$ we have

$$\int\limits_S \varphi d\mu^d_t - \int\limits_S \varphi d\nu^d = \int\limits_0^t \int\limits_S L^d_\mu \varphi d\mu^d_s ds. \tag{24}$$

Obviously, $\mu^d \in N_{\tau,\alpha}$. Hence there exists a subsequence of indices n_k such that μ^{n_k} is V-converging to μ on the strip $H \times [0,\tau]$ as $k \to \infty$. Moreover, the sequence $\mu^{n_k}_t$ converges weakly to μ_t for all $t \in [0,\tau]$. Next, ν^d converges weakly to ν as $d \to \infty$. Assumption (H2) ensures the pointwise convergence of the sequences $a^{ij}(x,t,\mu^{n_k})$ and $b^i(x,t,\mu^{n_k})$ and their equicontinuity. By the Arzelà–Ascoli theorem (after relabeling indices), the sequences $a^{ij}(x,t,\mu^{n_k})$ and $b^i(x,t,\mu^{n_k})$ uniformly converge to $a^{ij}(x,t,\mu)$

and $b^i(x,t,\mu)$ on compact sets in $H\times[0,\tau]$, respectively. Clearly,

$$
\begin{aligned}
&\Big|\int_0^t\int L_{\mu^{n_k}}\varphi d\mu_s^{n_k}ds-\int_0^t\int L_\mu\varphi d\mu_s ds\Big|\\
&\qquad\leqslant\Big|\int_0^t\int_S L_{\mu^{n_k}}\varphi d\mu_s^{n_k}ds-\int_0^t\int_S L_\mu\varphi d\mu_s^{n_k}ds\Big|\\
&\qquad\qquad+\Big|\int_0^t\int_S L_\mu\varphi d\mu_s^{n_k}ds-\int_0^t\int_S L_\mu\varphi d\mu_s ds\Big|.
\end{aligned}
$$

The second summand on the right-hand side tends to zero as $k\to\infty$ due to the weak convergence of the measures $\mu_t^{n_k}(dx)dt$, the first summand on the right-hand side tends to zero by the uniform convergence of the coefficients. One can pass to the limit in (24) as $k\to\infty$ and obtain

$$\int\varphi d\mu_t-\int\varphi d\nu=\int_0^t\int L_\mu\varphi d\mu_s ds.$$

Here we have used the fact that

$$\int\varphi d\mu_t^{n_k}\to\int\varphi d\mu_t\quad\text{and}\quad\int\varphi d\nu^{n_k}\to\int\varphi d\nu\quad\text{as}\quad k\to\infty.$$

By definition, this means that $(\mu_t)_{t\in[0,\tau]}$ is a solution to the Cauchy problem (21). □

Remark. As it was mentioned in the proof, V-convergence is equivalent to weak convergence on the set $N_{\tau,\alpha}$. It is introduced mainly for technical purposes: assumption (H2) for unbounded drifts is easier to verify in terms of V-convergence. For instance, if the drift term has the form

$$b(\mu,x,t)=\int K(x,y)d\mu_t(y)$$

for some continuous vector kernel K, and for some function V and continuous functions $C_1(x)$, $C_2(x)$ we have

$$|K(x,y)|\leqslant C_1(x)+C_2(x)V^{1-\gamma}(y),\quad \gamma\in(0,1),$$

then (H2) is fulfilled.

Finally, we formulate sufficient conditions for the existence and uniqueness of a probability solution for the Cauchy problem (7) with

$$b(\mu, x, t) = Rx + \int K(x, y)d\mu_t(y), \qquad b^j = \langle b, e_j\rangle,$$

where R is a nonpositive self-adjoint operator with eigenbasis $\{e_j\}$, $j \in \mathbb{N}$, and eigenvalues $r = \{-r_j\}$, $j \in \mathbb{N}$. The following theorem is an immediate corollary of Theorem 2.1 and Theorem 3.1 with $V(x) = 1 + |x|^2$.

Theorem 3.2. *Let $K(\cdot,\cdot)\colon\ H \times H \to H$ be a continuous kernel, and let $\sum\limits_{j=1}^{\infty} \beta^j < +\infty$. Assume that for some $C_0 > 0$*

$$|K(x, y) - K(x, z)| \leqslant C_0 \cdot (1 + |x|^2) \cdot |y - z|.$$

Assume also that there exists a sequence of smooth bounded mappings K_n such that for all $(x, y) \in H \times H$ we have $K_n(x, y) \to K(x, y)$ as $n \to \infty$,

$$\langle K_n(x, y) - K_n(z, y), x - z\rangle \leqslant \theta|x - z|^2 + \|x - z\|_r^2,$$

and $|K_n(x, y)| \leqslant C_4(1 + |x|)(1 + |y|^{2-\delta})$ for some $\delta > 0$. Then for any $\mu_0 \in \mathcal{P}_2(H)$, the Cauchy problem (7) has a unique probability solution in $\mathcal{P}_2(H)$.

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