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ORTHOGONAL PAIRS AND MUTUALLY UNBIASED BASES

Abstract. The goal of our article is a study of related mathematical and physical objects: orthogonal pairs in $\mathrm{sl}(n)$ and mutually unbiased bases in $\mathbb{C}^n$. An *orthogonal pair* in a simple Lie algebra is a pair of Cartan subalgebras that are orthogonal with respect to the Killing form. The description of orthogonal pairs in a given Lie algebra is an important step in the classification of *orthogonal decompositions*, i.e., decompositions of the Lie algebra into a direct sum of Cartan subalgebras pairwise orthogonal with respect to the Killing form. One of the important notions of quantum mechanics, quantum information theory, and quantum teleportation is the notion of mutually unbiased bases in the Hilbert space $\mathbb{C}^n$. Two orthonormal bases $\{e_i\}_{i=1}^n, \{f_j\}_{j=1}^n$ are *mutually unbiased* if and only if $|\langle e_i|f_j\rangle|^2 = \frac{1}{n}$ for any $i, j = 1, \dots, n$. The notions of mutually unbiased bases in $\mathbb{C}^n$ and orthogonal pairs in $\mathrm{sl}(n)$ are closely related. The problem of classification of orthogonal pairs in $\mathrm{sl}(n)$ and the closely related problem of classification of mutually unbiased bases in $\mathbb{C}^n$ are still open even for the case $n = 6$. In this article, we give a sketch of our proof that there is a complex four-dimensional family of orthogonal pairs in $\mathrm{sl}(6)$. This proof requires a lot of algebraic geometry and representation theory. Further, we give an application of the result on the algebraic geometric family to the study of mutually unbiased bases. We show the existence of a real four-dimensional family of mutually unbiased bases in $\mathbb{C}^6$, thus solving a long-standing problem.

§1. Introduction

An *orthogonal pair* in a semisimple Lie algebra is a pair of Cartan subalgebras that are orthogonal with respect to the Killing form. The description of orthogonal pairs in a given Lie algebra is an important step in the classification of *orthogonal decompositions*, i.e., decompositions of the Lie algebra into a sum of Cartan subalgebras pairwise orthogonal with respect to the Killing form.

Key words and phrases: orthogonal pairs, mutually unbiased bases (MUB), complex Hadamard matrices, generalized Hadamard matrices.

Orthogonal decompositions first came up in the theory of integer lattices in the paper by Thompson [23]. Then the theory of such decompositions was substantially developed [16]. The classification problem of orthogonal pairs in $\mathrm{sl}(n, \mathbb{C})$ is closely related to the classification of complex Hadamard $n \times n$ matrices [16, 4].

Independently, the study in quantum theory brought into light the notion of mutually unbiased bases, objects of constant use in quantum information theory, quantum tomography, etc. [8, 21]. It was revealed that mutually unbiased bases are a unitary version of orthogonal pairs [4]. This links the subject to various vibrant problems in mathematical physics.

One of the reasons why mutually unbiased bases are important in practice is that they provide a crucial mathematical tool that allows one to transfer quantum information with minimal loss in the channel. Reliable protocols in quantum channels are based on a choice of the maximum number of mutually unbiased bases in a relevant vector space of quantum states of transmitted particles. For instance, the protocol BB84, which utilizes three such bases in a two-dimensional vector space, enables one to significantly extend the distance between the source and the receiver of quantum information. Constructing the maximum number of mutually unbiased bases in vector spaces of higher dimension is important for producing reliable protocols in quantum channels.

Also, one of the important problems of quantum teleportation is to check the purity of the result of teleportation by means of quantum tomography. This is used in real experiments on the teleportation of entangled particles (cf. [17]). The quantum tomography with minimal error bar is again based on mutually unbiased bases [5, 9].

Despite a simple definition, the classification of orthogonal pairs is a very hard problem of algebraic geometric flavor. We will consider pairs in the Lie algebra $\mathrm{sl}(n, \mathbb{C})$. According to the famous Winnie-the-Pooh conjecture [14], orthogonal decompositions are possible in this algebra when n is a power of a prime number only. This suggests the idea that the behavior of the objects under study strongly depends on the arithmetic properties of the number n. For $n = 1, 2, 3$, there is a unique, up to natural symmetries, orthogonal pair. For $n = 5$, there are three of them [15, 19], while for $n = 4$ (the first nonprime integer), there is a one-dimensional family of pairs parameterized by a rational curve.

The first positive integer that is not a power of a prime is $n = 6$. The Winnie-the-Pooh conjecture is open even for this case. Researchers in quantum information theory have independently come to the unitary version of the Winnie-the-Pooh conjecture, which claims the nonexistence of $n+1$ mutually unbiased bases in the n-dimensional complex space [14] when n is not a power of a prime. The case $n = 6$ is the subject of problem number 13 in the popular list of problems in quantum information theory [20].

In this paper, we outline the proof of the existence of a four-dimensional family of orthogonal pairs in the Lie algebra $\mathrm{sl}(6, \mathbb{C})$. The existence of such a family was conjectured by the authors (unpublished). Independently, mathematical physicists came to the conjecture on the existence of a four-dimensional family of pairs of mutually unbiased bases in $\mathbb{C}^6$, see [22, 18]. Despite many efforts, the proof of the existence of the family was not available until recently [3]. Our proof is quite involved and requires a lot of algebraic geometry. In this paper, we give a relatively short survey of the main steps of the proof and describe explicit constructions that lead to the existence of the family.

Then, we give an application of the result on the algebraic geometric family of pairs to the study of mutually unbiased bases. We show the existence of a real four-dimensional manifold parameterizing the pairs of such bases in $\mathbb{C}^6$, thus confirming the conjecture of physicists. The proof is based on a construction of a principal homogeneous bundle over the locus $\mathcal{M}_{\mathbb{R}}$ parameterizing the pairs of mutually unbiased bases.

In [1], we interpreted orthogonal pairs and decompositions as representations of the algebra $B(\Gamma)$ for a suitable choice of a graph Γ (see Sec. 2.2). This algebra is a so-called *homotope* over the path algebra of the graph Γ regarded as a topological space. In its turn, the path algebra of a graph is Morita equivalent to the group algebra of the fundamental group of the graph. This is useful for calculating the moduli space of representations of $B(\Gamma)$. Orthogonal pairs in $\mathrm{sl}(n)$ correspond to representations of the algebra $B(\Gamma)$ where Γ is the complete bipartite graph $\Gamma_{n,n}$.

One of the key points of our proof is a hidden geometry of an elliptic fibration of the moduli space X of six-dimensional representations of $B(\Gamma_{3,3})$, where $\Gamma_{3,3}$ is the full bipartite graph of length $(3, 3)$. We define three functions on X which determine a map $X \to \mathcal{U}$, where $\mathcal{U}$ is a three-dimensional affine space. After the factorization of X by the permutation group $S_3 \times S_3$, the fibre is actually isomorphic to (an open affine subset in) two disjoint copies of an elliptic curve. The advantage of this map is that the original problem of describing orthogonal pairs in $\mathrm{sl}(6, \mathbb{C})$ can be interpreted in terms of "gluing" four copies of X in such a way that all constructions are basically implemented relatively over $\mathcal{U}$. The geometry of the elliptic fibration is a powerful tool which eventually allowed us to show the existence of the four-dimensional family. In particular, we study the interplay between relevant involutions acting on the elliptic fibers. This part is based on the heavy use of algebraic geometry. Let us mention important formula (12), which probably needs a more conceptual explanation than just a verification.

If we think about the main steps of the proof in terms of the 6×6 matrix A that conjugates one Cartan subalgebra in the orthogonal pair to the other one (*suitable*, or *generalized Hadamard* matrix), then we first present this matrix in two blocks of 3×6 matrices and then decompose each of these 3×6 blocks into two 3×3 blocks.

Equivalently, the first decomposition is about decomposing the set of vertices in one of the rows of the full bipartite graph $\Gamma_{6,6}$ into two disjoint subsets with 3 elements in each. This has a geometric interpretation presented in the statement of Theorem 12 that the higher dimensional components of the moduli space $X(6,6)$ of six-dimensional representations of the algebra $B_{6,6}$, a quotient of the algebra $B(\Gamma_{6,6})$, are birationally identified with a fiber product of two copies of the representation moduli spaces $X(3,6)$ for the algebra $B_{3,6}$, which is a quotient of $B(\Gamma_{3,6})$.

Further, the vertices in the row of length 6 in the full bipartite graph $\Gamma_{3,6}$ are decomposed into two disjoint subsets with 3 elements in each. This boils down to the decomposition of the unique four-dimensional component of the moduli space $X(3,6)$ of representations for $B_{3,6}$ into a fiber product of two copies of the moduli space $X = X_{3,3}$ for representations of the algebra $B(\Gamma_{3,3})$, as in Theorem 9. In the text, we do this in the reverse order: first decompose $X = X_{3,6}$ and then $X = X_{6,6}$.

The fiber products are taken over the moduli spaces of representations for the algebras $A(n)$, $n = 3, 6$ (see Sec. 2.5). We construct the Morita equivalence of the algebra $A(n)$ with the deformed preprojective algebra, for arbitrary n. Deformed preprojective algebras are intensively studied by many authors (cf. [10, 6]). For our purposes, this Morita equivalence is important because we can use a result of Crawley-Boewey [7] to infer the irreducibility of the representation moduli space $Y(n)$ for $A(n)$. The symplectic geometry of $Y(n)$ is a part of the symplectic approach to the study of pairs of mutually unbiased bases discussed in [2], where its relation via mirror symmetry to the Birkhoff–von Neumann polytope of doubly stochastic matrices was discovered.

We construct an involution on the quotient space $X(3,6)/S_3$. The crucial step in our argument is to show that this involution agrees with a map $X(3,6)/S_3 \to Y(6)$ and an involution σ' on $Y(6)$. The proof of this fact (Proposition 16) uses the property of automorphisms on varieties of general type to be of finite order. We use the point $x_0 \in X(6,6)$ corresponding to the standard pair of Cartan subalgebras, which has a regular behavior with respect to our constructions, to prove the existence of a four-dimensional component that contains this point.

Then, we shift our attention to mutually unbiased bases. We compare the space $\mathcal{M}_{\mathbb{R}}$ parameterizing the pairs of mutually unbiased bases with the space $\mathcal{M}^{\theta}$ parameterizing the stable points of an antiholomorphic involution θ acting on the moduli space of orthogonal pairs. We show that $\mathcal{M}_{\mathbb{R}}$ is open in $\mathcal{M}^{\theta}$. The proof is based on considering a principal homogeneous bundle over $\mathcal{M}^{\theta}$ and characterizing its restriction to $\mathcal{M}_{\mathbb{R}}$ by means of the Sylvester theorem characterizing positive Hermitian matrices. This describes, in principle, the strict polynomial inequalities that define $\mathcal{M}_{\mathbb{R}}$ inside $\mathcal{M}^{\theta}$. Since the point x_0 is in $\mathcal{M}_{\mathbb{R}}$ and the real dimension of $\mathcal{M}_{\mathbb{R}}$ equals the complex dimension of the corresponding component in $X(6,6)$, we infer the existence of a real four-dimensional family of pairs of mutually unbiased bases.

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§2. Algebraic preliminaries

2.1. Orthogonal Cartan subalgebras. Consider a simple Lie algebra L over an algebraically closed field of characteristic zero. Let K be the Killing form on L. In 1960, J. G. Thompson, in the course of construction of integer quadratic lattices with interesting properties, introduced the following definitions.

Definition. *Two Cartan subalgebras H_1 and H_2 in L are said to be orthogonal if $K(h_1, h_2) = 0$ for all $h_1 \in H_1, h_2 \in H_2$.*

Definition. *A decomposition of L into a direct sum of Cartan subalgebras $L = \oplus_{i=1}^{h+1} H_i$ is said to be orthogonal if H_i is orthogonal to H_j for all $i \neq j$.*

An intensive study of orthogonal decompositions has been undertaken since then (see the book [16] and references therein). For the Lie algebra $\mathrm{sl}(n)$, A. I. Kostrikin et al. arrived at the following conjecture, called the *Winnie-the-Pooh conjecture* (cf. *ibid.*, where, in particular, the name of the conjecture is explained by a wordplay in the Russian translation of Milne's book).

Conjecture 1. *The Lie algebra $\mathrm{sl}(n)$ has an orthogonal decomposition if and only if $n = p^m$ for a prime number p.*

The conjecture has proved to be notoriously difficult. Even the nonexistence of an orthogonal decomposition for $\mathrm{sl}(6)$, when n is 6, i.e., the first number that is not a prime power, is still open. Also, it is important to find the maximum number of pairwise orthogonal Cartan subalgebras in $\mathrm{sl}(n)$ for any given n, as well as to classify them up to obvious symmetries.

We recall an interpretation of the problem in terms of systems of minimal projectors and its relation to the representation theory of the Temperley–Lieb algebras.

Let $\mathrm{sl}(V)$ be the Lie algebra of traceless operators in V. The Killing form is given by the trace of the product of operators. A Cartan subalgebra H in V defines a unique maximal set of minimal orthogonal projectors in V. Indeed, H can be extended to the Cartan subalgebra H' in $\mathrm{gl}(V)$ spanned by H and the identity operator E. The rank 1 projectors in H' are pairwise orthogonal and comprise the required set. We say that these projectors are *associated* to H.

If p is a minimal projector in H', then the trace of p is 1, hence, $p - \frac{1}{n}E$ is in H. If projectors p and q are associated to orthogonal Cartan subalgebras, then

$$\mathrm{Tr}(p - \frac{1}{n}E)(q - \frac{1}{n}E) = 0,$$

which is equivalent to

$$\mathrm{Tr}\, pq = \frac{1}{n}. \tag{1}$$

We say that a pair of minimal projectors is *algebraically unbiased* if it satisfies this equation.

Therefore, an orthogonal pair of Cartan subalgebras is in a one-to-one correspondence with two maximal sets of minimal orthogonal projectors such that every pair of projectors from different sets is algebraically unbiased. Similarly, orthogonal decompositions of $\mathrm{sl}(n)$ correspond to $n+1$ pairwise algebraically unbiased sets of minimal orthogonal projectors. In the analysis of the problem, it is worthwhile not only to consider maximal sets of orthogonal projectors, but also to study pairwise unbiasedness for various subsets of maximal sets. This suggests to consider the representation theory of reduced Temperley–Lieb algebras of arbitrary graphs without loops, which we describe in the next section.

2.2. Reduced Temperley–Lieb algebras. Let Γ be a connected simply laced graph without loops (i.e., without edges with coinciding ends). Denote by $V(\Gamma)$ and $E(\Gamma)$ the sets of vertices and edges of the graph. Let F be a field of characteristic zero.

Fix $r \in F^*$. We define the reduced Temperley–Lieb algebra $B_r(\Gamma)$ as the unital algebra over F with generators x_i numbered by the vertices $i \in V(\Gamma)$ subject to the following relations:

- $x_i^2 = x_i$ for every i in $V(\Gamma)$,
- $x_i x_j x_i = r x_i$, $x_j x_i x_j = r x_j$ if there is an edge (i,j) in Γ,
- $x_i x_j = x_j x_i = 0$ if there is no edge (i,j) in Γ.

If we replace the last relation by $x_i x_j = x_j x_i$ (under the same condition on (i,j)), we get the standard Temperley–Lieb algebra $TL_r(\Gamma)$. It follows that $B_r(\Gamma)$ is a quotient of the Temperley–Lieb algebra $TL_r(\Gamma)$ of the graph Γ. In its turn, the Temperley–Lieb algebra is a quotient of the Hecke algebra of the graph, hence the algebra $B_r(\Gamma)$ is a special quotient of the Hecke algebra (see [1]). Thus the representation theory of $B_r(\Gamma)$ is a part of the representation theory of Hecke algebras of graphs. Note that the representation theory of $B_r(\Gamma)$ is difficult, and the measure of difficulty is the rank of the first homology of the graph as a topological space. Clearly, any automorphism of the graph Γ induces an automorphism of the algebra $B_r(\Gamma)$.

The condition (1) on two minimal projectors to be algebraically unbiased can be reformulated as algebraic relations:

$$pqp = \frac{1}{n}p, \quad qpq = \frac{1}{n}q.$$

It follows from Sec. 2.1 that a pair of orthogonal Cartan subalgebras in the Lie algebra $\mathrm{sl}(n)$ defines a representation of $B_{\frac{1}{n}}(\Gamma_{n,n})$, where $\Gamma_{n,n}$ is the full bipartite graph with n vertices in both rows, and every generator x_i is represented by a rank 1 projector. The generators in one row correspond to the system of orthogonal projectors related to one Cartan subalgebra. Since the sum of all minimal projectors in one system is the identity matrix, the representation descends to a representation of the algebra

$$B_{n,n} = B_{\frac{1}{n}}(\Gamma_{n,n})/(\sum p_i - 1, \sum q_j - 1),$$

where p_i's are the idempotents corresponding to one row, and q_j's, to the other row. The representations of $B_{n,n}$ where every generating idempotent is presented by a minimal projector are in a one-to-one correspondence with the orthogonal pairs of Cartan subalgebras in $\mathrm{sl}(n)$. The moduli space of six-dimensional representations for $B_{6,6}$ is the central object of this paper.

It is instructive to think about $B_r(\Gamma)$ as a *homotope* of the path algebra of a quiver (see below).

2.3. The path algebra of a graph. Let again Γ be a simply laced graph without loops. Consider it as a topological space. Let $\mathcal{P}(\Gamma)$ be the Poincaré groupoid of the graph Γ, i.e., the category whose objects are vertices of the graph and morphisms are homotopic classes of paths. The composition of morphisms is given by the concatenation of paths.

Denote by $F\Gamma$ the algebra over F with the free F-basis numbered by the morphisms in $\mathcal{P}(\Gamma)$ and multiplication induced by the concatenation of paths (when it makes sense; when it does not, the product is zero). Let e_i be the element of $F\Gamma$ that is the constant path at the vertex i. Any oriented edge (ij) can be interpreted as a morphism in $\mathcal{P}(\Gamma)$, hence it gives an element l_{ij} in $F\Gamma$. These are the generators. The defining relations are

- $e_i e_j = \delta_{ij} e_i, \quad e_i l_{jk} = \delta_{ij} l_{ik}, \quad l_{jk} e_i = \delta_{ki} l_{jk};$
- $l_{ij} l_{ji} = e_i, \quad l_{ji} l_{ij} = e_j, \quad l_{ij} l_{km} = 0$ if $j \neq k$.

We regard $F\Gamma$ as an algebra with unit:

$$1 = \sum_{i \in V(\Gamma)} e_i.$$

Let Γ be, in addition, a connected graph. Then the categories of representations for $F\Gamma$ and for the fundamental group of the graph are equivalent. To see this, fix $t \in V(\Gamma)$. Denote by $F[\pi(\Gamma, t)]$ the group algebra of the fundamental group $\pi(\Gamma, t)$. Consider the projective $F\Gamma$-module $P_t = F\Gamma e_t$. Clearly, P_t is a $F\Gamma$-$F[\pi(\Gamma, t)]$-bimodule. Note that P_t are isomorphic as left

$F\Gamma$-modules for all choices of t. Indeed, the right multiplication by an element corresponding to a path starting at t_1 and ending at t_2 gives an isomorphism $P_{t_1} \simeq P_{t_2}$.

The bimodule P_t induces a Morita equivalence of $F\Gamma$ with $F[\pi(\Gamma, t)]$. Thus, the categories $F\Gamma$-mod and $F[\pi(\Gamma, t)]$-mod are equivalent. Moreover, the algebra $F\Gamma$ is isomorphic to the matrix algebra over $F[\pi(\Gamma, t)]$, with the size of (square) matrices equal to $|V(\Gamma)|$.

Mutually inverse functors that induce an equivalence between the categories $F\Gamma$-mod and $F[\pi(\Gamma, t)]$-mod are

$$V \mapsto P_t \otimes_{F[\pi(\Gamma,t)]} V, \quad W \mapsto \mathrm{Hom}_{F\Gamma}(P_t, W). \tag{2}$$

In order to define an isomorphism $F\Gamma \to \mathrm{Mat}_n(F[\pi(\Gamma, t)])$, fix a system of paths $\{\gamma_i\}$ connecting the vertex t with every vertex i. For any element $\pi \in F[\pi(\Gamma, t)]$ consider the element $\gamma_i^{-1}\pi\gamma_j$ in $F\Gamma$. The homomorphism is defined by the assignment

$$\gamma_i^{-1}\pi\gamma_j \mapsto \pi \cdot E_{ij},$$

where E_{ij} stands for the elementary matrix with the only nontrivial entry 1 at the (ij)th position. This is clearly a well-defined ring isomorphism.

The fundamental group $\pi(\Gamma, t)$ is free, with the number of generators equal to the rank of the first homology of the graph regarded as a topological space.

2.4. Homotopes and reduced Temperley–Lieb algebras. Recall the definition of a homotope. Given a unital algebra A and an element $\Delta \in A$, one can define a new algebra structure on A by the multiplication

$$a \circ b = a\Delta b.$$

The new algebra might not have a unit. For this reason, we adjoin a unit to it and denote the new algebra by B:

$$B = F \cdot 1_B \oplus B^+,$$

where B^+ is the two-sided ideal in B that coincides with A as a vector space with the new multiplicaion. We say that B is the *homotope* over A with respect to Δ.

Algebraic properties of homotopes and their general representation theory is available in [1].

Consider again a simply laced graph Γ without loops. Fix $r \in F^*$. The *(generalized) Laplace operator of the graph* Γ is the element Δ in the algebra $F\Gamma$ of the Poincaré groupoid of the graph given by the formula

$$\Delta = 1 + \sqrt{r} \sum l_{ij}, \tag{3}$$

where the sum is taken over all oriented edges.

Consider the algebra $F\Gamma_\Delta = F \cdot 1 \oplus F\Gamma_\Delta^+$, the unital homotope over $F\Gamma$ with respect to the element Δ. Note that the algebra is independent of the choice of a square root of r. Denote by x_i's the elements in $F\Gamma_\Delta^+$ that correspond to e_i's in $F\Gamma$. The following theorem realizes $B_r(\Gamma)$ as a unital homotope over the Poincaré groupoid $F\Gamma$.

Theorem 2 ([1])**.** *There is a unique isomorphism of algebras and maximal ideals in them*

$$B_r(\Gamma) \cong F\Gamma_\Delta, \quad B_r^+(\Gamma) \cong F\Gamma_\Delta^+ \tag{4}$$

that takes x_i to e_i.

This theorem allows us to relate the moduli spaces of representations of $B_r(\Gamma)$ with the moduli spaces of the path algebra of the graph. Since the latter algebra is Morita equivalent to the fundamental group of the graph, a link to the representation theory of the free group is implied.

2.5. The algebra $A(n)$ and Morita equivalence. Let us define the deformed preprojective algebra $\Pi_{\vec{\lambda}}(Q)$ of a loop-free quiver Q. Denote by Q_0 and Q_1 the sets of vertices and arrows of Q, respectively. Let us construct the *double* quiver Q^d by adding to each arrow $a \in Q_1$ the opposite arrow $a^* \in Q_1^d$. Define the commutator c as the element $\sum_{a\in Q_1}[a, a^*] \in FQ^d$. For a vector $\vec{\lambda} = (\lambda_1, \ldots, \lambda_m) \in F^m$, $m = |Q_0|$, we define the deformed preprojective algebra as follows:

$$\Pi_{\vec{\lambda}}(Q) = FQ^d / \langle c - \sum_{i=1}^{k} \lambda_i e_i \rangle. \tag{5}$$

Fix $r_i \in F^*$, $i = 1, \ldots, n$. Consider the star quiver $\mathcal{Q}$ with one central vertex and n vertices at the boundary. The central vertex is connected with every vertex on the boundary by one outbound arrow. Let $\vec{\lambda} = (-r_1, \ldots, -r_n, 1)$, $\sum_{i=1}^n r_i = k$, where $k \in \mathbb{N}$ and $-r_i$, $i = 1, \ldots, n$, correspond to the vertices on the boundary, while 1 corresponds to the central vertex.

Consider the algebra $A(n)$ with generators $P, q_1, \ldots, q_n$ and relations

$$P^2 = P, \quad q_i^2 = q_i, \quad q_i P q_i = r_i q_i, \quad \sum_{i=1}^{n} q_i = 1. \tag{6}$$

Proposition 3. *The algebra $A(n)$ is Morita equivalent to the deformed preprojective algebra $\Pi_{\vec{\lambda}}(\mathcal{Q})$.*

Denote by $Y(n)$ the GIT moduli space of n-dimensional $A(n)$-representations where P is represented by a projector of rank k and the idempotents q_j are represented by projectors of rank 1. The above proposition allows us to apply the results of Crawley–Boewey [6, 7]. By checking his assumptions for the star quiver, we get that the variety $Y(n)$ is irreducible and has dimension $2(n-k-1)(k-1)$.

2.6. Coproducts of algebras and moduli of representations. Consider the quotient algebra

$$B_{k,n} = B_{\frac{1}{n}}(\Gamma_{k,n})/(\sum q_j - 1),$$

where q_j's are the idempotents corresponding to the vertices of the row of length n in the bipartite graph $\Gamma_{k,n}$. A decomposition of the set of vertices in one row of the graph $\Gamma_{n,n}$ into two disjoint subsets with k and $n-k$ elements defines two subalgebras $B_{k,n}$ and $B_{n-k,n}$ in the algebra $B_{n,n}$. The intersection of these two subalgebras in $B_{n,n}$ is identified with the algebra $A(n)$. The importance of the algebra $A(n)$ for us is explained by the following proposition.

Proposition 4. *The algebra $B_{n,n}$ is a fiber coproduct of $B_{k,n}$ and $B_{n-k,n}$ over $A(n)$.*

For an algebra A, denote by $\mathrm{Rep}_n A$ the affine variety parameterizing the n-dimensional representations of A. The above proposition implies the following corollary.

Corollary 5. *For every positive l, we have the fiber product decomposition*

$$\mathrm{Rep}_l B_{n,n} = \mathrm{Rep}_l B_{k,n} \times_{\mathrm{Rep}_l A(n)} \mathrm{Rep}_l B_{n-k,n}.$$

Denote by $\mathcal{M}_n A = \mathrm{Rep}_n A/\mathrm{GL}(n)$ the GIT moduli space of A-representations. Unfortunately, fiber coproduct decompositions for algebras do not imply fiber product decompositions for the moduli spaces of representations, primarily due to the presence of nontrivial automorphisms of representations.

Denote $X(k,n) = \mathcal{M}_n B_{k,n}$ and $Y(n) = \mathcal{M}_n A(n)$. Consider the open subset $Y(n)_o$ in $Y(n)$ of points corresponding to irreducible representations. Let $X(k,n)_o$ be the open subset in $X(k,n)$ of points corresponding to $B_{k,n}$-representations that restrict to irreducible $A(n)$-representations.

Proposition 6. *We have*

$$X(n,n)_o = X(k,n)_o \times_{Y(n)_o} X(n-k,n)_o$$

§3. Moduli spaces of representations for subgraphs of the graph $\Gamma_{6,6}$

3.1. The representation moduli spaces X, Y, and S. Let us consider the full bipartite graph $\Gamma_{3,3}$ with 3 vertices in both rows. Denote by $p_i, i = 1,2,3$ (respectively, $q_j, j = 1,2,3$), the idempotents in $B_{\frac{1}{6}}(\Gamma_{3,3})$ corresponding to the vertices in the first (respectively, second) row of the graph. Let $X = X_{3,3}$ be the GIT moduli space of six-dimensional representations for the algebra $B_{\frac{1}{6}}(\Gamma_{3,3})$ where all idempotents p_i and q_j are represented by projectors of rank 1.

One can check that $X \simeq (F^*)^4$. To this end, one can interpret the algebra $B_{\frac{1}{6}}(\Gamma_{3,3})$ as a *homotope* of the path algebra of the graph (see Sec. 2 and [1]). A homotope B over an algebra A has a canonical maximal two-sided ideal B^+, which is endowed with the left module structure of A that commutes with the right action of B (see Sec. 2.4). This allows us to consider the functor $\mathrm{Hom}_B(B^+, -) : \mathrm{modB} \to \mathrm{mod}A$.

Applying this general theory to $B_{\frac{1}{6}}(\Gamma_{3,3})$ as a homotope of the path algebra $F\Gamma_{3,3}$ of the graph, and taking into account the fact that $F\Gamma_{3,3}$ is Morita equivalent to the group algebra of the fundamental group of the graph, which is the free group with four generators, imply that the above functor has an interpretation as a functor that takes $B_{\frac{1}{6}}(\Gamma_{3,3})$-modules to representations of the fundamental group. Moreover, the representations that are parameterized by X are taken to representations of dimension 1. The moduli space of the latter is $(F^*)^4$, hence we have a map $X \to (F^*)^4$. One can see that the map is one-to-one on closed points, due to the interpretation of closed points as equivalence classes of representations. Thus the map is a birational morphism. Since $(F^*)^4$ is smooth, in particular, normal, it follows that the map is an isomorphism.

The algebra A_3 has generators P and q_j, $j = 1,2,3$, satisfying the relations $P^2 = P$, $q_j^2 = q_j$, and $q_j P q_j = \frac{1}{2} q_j$. This algebra is endowed with an *involution* σ, which is of particular importance for us. It is given by $\sigma : P \mapsto 1 - P$. Let Y be the GIT moduli space of six-dimensional representations of A_3 in which P is represented by a projector of rank 3 and q_j's, by projectors of rank 1. This is a four-dimensional variety.

The algebra homomorphism $A_3 \to B_{\frac{1}{6}}(\Gamma_{3,3})$ defined on the generators by $P \mapsto \sum p_i$ and $q_j \mapsto q_j$ defines a map $f : X_{3,3} \to Y$, which is in fact a quasi-finite map of degree 12.

We will also consider the algebra C with generators P and Q and relations $P^2 = P$, $Q^2 = Q$. The moduli space of six-dimensional representations of this algebra where both P and Q are represented by projectors of rank 3 and $\mathrm{Tr}PQ = \frac{3}{2}$ is denoted by S. It has dimension 2. We have a morphism $g : Y \to S$ defined by the algebra homomorphism $C \to A_3$ that takes $P \mapsto P$ and $Q \mapsto \sum q_j$.

We will consider another copy of A_3 with generators denoted by Q and p_i, $i = 1, 2, 3$, which play the roles of P and $q_j, j = 1, 2, 3$, respectively, in the first copy. Then we have the following commutative square of algebras, where we write algebras together with their generators:

$$\begin{array}{ccccc} & & B_{\frac{1}{6}}(\Gamma_{3,3})(p_1,p_2,p_3;q_1,q_2,q_3) & & \\ & \nearrow & & \nwarrow & \\ A_3(Q;p_1,p_2,p_3) & & & & A_3(P;q_1,q_2,q_3). \\ & \nwarrow & & \nearrow & \\ & & C(P;Q) & & \end{array} \qquad (7)$$

In the north-west pointed arrows of this diagram, P is taken to $\sum p_i$, and in the north-east pointed arrows, Q goes to $\sum q_j$. We also have the induced commutative square of moduli spaces:

$$\begin{array}{ccccc} & & X & & \\ & \swarrow^{f\circ\tau} & & \searrow^{f} & \\ Y & & & & Y, \\ & \searrow & & \swarrow & \\ & & S & & \end{array} \qquad (8)$$

where τ is the involution on X that comes from the involution on the algebra $B_{\frac{1}{6}}(\Gamma_{3,3})$ defined by exchanging p_i with q_i, for $i = 1, 2, 3$.

Let us introduce functions u_1, u_2 on Y:

$$u_1 = 6^2(\mathrm{Tr}Pq_1Pq_2 + \mathrm{Tr}Pq_1Pq_3 + \mathrm{Tr}Pq_2Pq_3), \qquad (9)$$

$$u_2 = 6^3(\mathrm{Tr}Pq_1Pq_2Pq_3 + \mathrm{Tr}Pq_1Pq_3Pq_2). \quad (10)$$

One can easily check that u_1 is $\mathrm{Tr}PQPQ$ up to a constant multiplier, while u_2 can be expressed as a linear combination of $\mathrm{Tr}PQPQPQ$, $\mathrm{Tr}PQPQ$, and the unit. It follows that u_1 and u_2 are well-defined regular functions on S; moreover, they generate the algebra of functions $F[S]$.

3.2. The space $\mathcal{U}$. Now we consider a new function on Y:

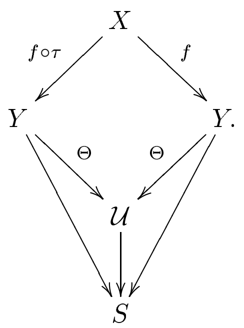
$$u_3 = (6^2\,\mathrm{Tr}\,Pq_1Pq_2 - 1)(6^2\,\mathrm{Tr}\,Pq_2Pq_3 - 1)(6^2\,\mathrm{Tr}\,Pq_3Pq_1 - 1). \quad (11)$$

We have the three-dimensional affine space $\mathcal{U} = \mathrm{Spec}F[u_1, u_2, u_3]$. It is endowed with natural surjective maps $\mathcal{U} \to S$ and $\Theta : Y \to \mathcal{U}$. The variety $\mathcal{U}$ is important for us because many calculations that we perform are done relatively over $\mathcal{U}$. It would be interesting to find a representation-theoretic meaning for $\mathcal{U}$.

Proposition 7. *Consider two systems (p_1, p_2, p_3) and (q_1, q_2, q_3) of orthogonal projectors of rank 1 in a vector space satisfying the condition $\mathrm{Tr}\,p_iq_j = \frac{1}{6}$. Let $P = p_1 + p_2 + p_3$ and $Q = q_1 + q_2 + q_3$. Then the following identity holds:*

$$\prod_{(i,j)\in\{1,2,3\}} (6^2\,\mathrm{Tr}(Pq_iPq_j) - 1) = \prod_{(i,j)\in\{1,2,3\}} (6^2\,\mathrm{Tr}(Qp_iQp_j) - 1). \quad (12)$$

This proposition, together with the above remarks on u_1 and u_2, allows us to extend the diagram (8) to the following commutative diagram:



(13)

The induced map $X \to Y \times_{\mathcal{U}} Y$ is an embedding. The variety $Y \times_{\mathcal{U}} Y$ is a divisor in $Y \times_S Y$, $\dim Y \times_{\mathcal{U}} Y = 5$, $\dim Y \times_S Y = 6$.

Let S_3 be the group of permutations of three elements. We consider the variety $X' = X/(S_3 \times S_3)$ where the action of $S_3 \times S_3$ on X is induced by the action on $B_{\frac{1}{6}}(\Gamma_{3,3})$ by independent permutations of p_i's and q_j's.

Similarly, $Y' = Y/S_3$ where S_3 acts on A_3 by permuting q_j's, which gives an action on Y. We have the induced maps $X' \to Y' \to \mathcal{U}$.

Proposition 8 ([3]). *The fiber of the composite map $X' \to \mathcal{U}$ over a generic closed point $u \in \mathcal{U}$ is a disjoint union of two isomorphic elliptic curves, while the fiber of $Y' \to \mathcal{U}$ is just one elliptic curve. The map $X' \to Y'$ maps two components of the fiber of X' over u isomorphically to the fiber of Y' over u.*

3.3. The representation moduli space $X(3,6)$**.** Let us consider the full bipartite graph $\Gamma_{3,6}$ with 3 vertices in the first row and 6 vertices in the second row. Denote by p_i, $i = 1,2,3$ (respectively, q_j, $j = 1,\dots,6$), the idempotents in $B_{\frac{1}{6}}(\Gamma_{3,6})$ corresponding to the vertices in the first (respectively, second) row of the graph. Consider the algebra $B_{3,6}$, the quotient of $B_{\frac{1}{6}}(\Gamma_{3,6})$ by the two-sided ideal generated by $\sum q_j - 1$. Let $X(3,6)$ be the GIT moduli space of six-dimensional representations of the algebra $B_{3,6}$ where all idempotents p_i and q_j are represented by projectors of rank 1.

Consider the map $X(3,6) \to X$ induced by the algebra homomorphism $B_{\frac{1}{6}}(\Gamma_{3,3}) \to B_{3,6}$ defined by $p_i \mapsto p_i$ and $q_j \mapsto q_j$. We will also consider a second copy of $B_{\frac{1}{6}}(\Gamma_{3,3})$, with generators p_i, $i = 1,2,3$, and q_j, $j = 4,5,6$, and a second map $X_{3,6} \to X$, induced by the similar algebra homomorphism $B_{\frac{1}{6}}(\Gamma_{3,3}) \to B_{3,6}$ defined by $p_i \mapsto p_i$ and $q_j \mapsto q_j$. Combining it with two maps $f, \sigma \circ f : X \to Y$, we obtain the commutative diagram

$$\begin{array}{ccccc} & & X(3,6) & & \\ & \swarrow^{p_2} & & \searrow^{p_1} & \\ X & & & & X. \\ & \searrow_{\sigma\circ f} & & \swarrow_{f} & \\ & & Y & & \end{array} \tag{14}$$

Theorem 9. *The variety $X(3,6)$ is irreducible of dimension 4. The variety $X \times_Y X$ has only one irreducible component of dimension 4, all the other components being of lower dimension. The map $h : X(3,6) \to X \times_Y X$ induced by the above diagram establishes a birational isomorphism of $X(3,6)$ with the four-dimensional irreducible component of $X \times_Y X$.*

Note that it is quite plausible that $X \times_Y X$ is in fact also irreducible, which would mean that the map h is birational.

3.4. The representation moduli spaces $Y(6)$ and $X(6,6)$. Consider the algebra $A(6)$ with generators P and q_j, $j=1,\dots,6$, and relations

$$P^2 = P, \quad q_j^2 = q_j, \quad q_j P q_j = \frac{1}{2} q_j, \quad \sum q_j = 1.$$

The algebra $A(6)$ is endowed with the involution given by $P \mapsto 1-P$ and $q_j \mapsto q_j$. Denote by $Y(6)$ the GIT moduli space of six-dimensional representations of the algebra $A(6)$ where P is represented by a projector of rank 3 and the idempotents q_j are represented by projectors of rank 1. The involution on $A(6)$ induces an involution $\sigma' : Y(6) \to Y(6)$.

The algebra $A(n)$ is Morita equivalent to the deformed preprojective algebra of the star graph Q with one central vertex and n vertices on the boundary, the central vertex being connected with every boundary vertex by one edge (see Sec. 2.5). According to a result of Crawley–Boewey [6, 7], this implies that the variety $Y(6)$ is irreducible and has dimension 8.

There is an algebra homomorphism $A(6) \to B_{3,6}$ that takes P to $\sum p_i$. It defines a map $g : X(3,6) \to Y(6)$. Consider the action of the group S_3 on the algebra $B_{3,6}$ that permutes the generators p_1, p_2, p_3. Clearly, g is an S_3-invariant map. Recall that, according to Theorem 9, the variety $X(3,6)/S_3$ is irreducible.

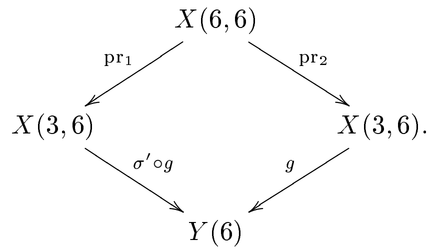
Theorem 10. *The morphism $g : X(3,6)/S_3 \to Y(6)$ maps $X(3,6)/S_3$ birationally onto its image in $Y(6)$.*

The proof of this theorem heavily uses the fact established in Proposition 8 that the fiber of $X/S_3 \times S_3$ over a generic point $\mathcal{U}$ is a disjoint union of two copies of an elliptic curve. This allows us to use the geometry of elliptic curves and elliptic fibrations.

Consider a second copy of the algebra $B_{3,6}$, whose generators we denote by (p_4, p_5, p_6) and $(q_j,\ j = 1,\dots,6)$. The corresponding moduli space of representations of this algebra is again identified with $X(3,6)$.

Now consider the algebra $B_{6,6}$ that is the quotient of the algebra $B_{\frac{1}{6}}(\Gamma_{6,6})$ with generators p_i, $i=1,\dots,6$, and q_j, $j=1,\dots,6$, by the two-sided ideal generated by the elements $\sum p_i - 1$ and $\sum q_j - 1$. Let $X_{6,6}$ be the GIT moduli space of six-dimensional representations of the algebra $B_{6,6}$ where all idempotents p_i and q_j are represented by projectors of rank 1.

Note that the above two copies of the algebra $B_{3,6}$ are mapped into the algebra $B_{6,6}$ by sending the generators p_i to p_i and q_j to q_j. We have chosen the indices of the generators in the two copies in such a way that they agree with the indices of the generators in the algebra $B_{6,6}$. These two maps induce two maps $X(6,6) \to X(3,6)$. All the above maps can be combined into a commutative diagram:



(15)

Lemma 11. *There exists a point x_0 in $X(6,6)$ such that the tangent space T_{x_0} at x_0 has dimension 4, the differentials at x_0 of the maps pr_1 and pr_2 are isomorphisms of T_{x_0} with the tangent spaces at the images of x_0, and the differential of the map $s : X(6,6) \to Y(6)$ induces an embedding of T_{x_0} into the tangent space to $Y(6)$ at $s(x_0)$. The point $s(x_0) \in Y(6)$ corresponds to an irreducible representation of $A(6)$.*

Proof. Recall that the *standard pair* (see [14]) of Cartan subalgebras in $\mathrm{sl}(n,\mathbb{C})$ consists of the diagonal Cartan subalgebra H_0 in a fixed basis $\{e_i\}$ and the subalgebra H_1 that is linearly spanned by $(P, \ldots, P^{n-1})$ where P is the operator of the cyclic permutation of the basis vectors $e_i \mapsto e_{i+1/\mathrm{mod}\, n}$.

The transition matrix A from the basis $\{e_i\}$ to the basis $\{f_j\}$ related to the second Cartan subalgebra has the following coefficients:

$$A = \{a_{ij} = \frac{1}{\sqrt{n}} \epsilon^{(i-1)(j-1)}\}, \quad i,j = 1, \ldots, n, \tag{16}$$

where ϵ is a primitive root, $\epsilon^n = 1$.

One can calculate the tangent space to $X_{6,6}$ at the point corresponding to the standard pair and check that it has dimension 4 (cf. [24]).

Let us exchange the 3rd and the 4th columns of the matrix A. This corresponds to reordering the projectors p_i, thus changing the projections $X(6,6) \to X(3,6)$. It is a direct check to show that all claims of the lemma are satisfied for this choice of x_0 and the projections. ☐

Theorem 12. *The induced morphism $X(6,6) \to X(3,6) \times_{Y(6)} X(3,6)$ establishes a one-to-one correspondence between the set of irreducible components of $X(6,6)$ and $X(3,6) \times_{Y(6)} X(3,6)$ of dimension greater than or equal to 4 and the birational isomorphisms between the corresponding components.*

The proof in [3] is based on the calculation of the locus of points in $X(3,6)\times_{Y(6)}X(3,6)$ for which the fiber in $X(6,6) \to X(3,6)\times_{Y(6)}X(3,6)$ is different from just one point and showing that it has dimension less than 4.

§4. A four-dimensional component in $X(6,6)$

4.1. The invariance of the image under an involution. The main technical result that implies the existence of a four-dimensional component in $X(6,6)$ is the following statement of independent interest.

Theorem 13. *The image of $X(3,6)$ under the map $g : X(3,6) \to Y(6)$ has a nonempty Zariski open subset that is invariant under the involution σ'.*

We describe the main steps of the proof of Theorem 13.

According to Theorem 9, the variety $X(3,6)$ is irreducible, and it is mapped birationally onto the unique four-dimensional irreducible component of $X \times_Y X$. Consider the map $h : X \times_Y X \to Y \times_S Y$.

Proposition 14. *The image under h of the four-dimensional irreducible component of $X\times_Y X$ has a nonempty Zariski open subset that is invariant under the involution (σ,σ).*

The map $X(3,6) \to Y\times_S Y$ factors through the quotient map $X(3,6) \to X(3,6)/S_3$, where the action of S_3 on $X(3,6)$ is induced by the permutations of p_i, $i = 1,2,3$.

Proposition 15. *The induced morphism $X(3,6)/S_3 \to Y \times_S Y$ isomorphically maps a Zariski open subset in $X(3,6)/S_3$ into $Y \times_S Y$.*

Propositions 14 and 15 imply that the involution (σ,σ) induces an involution π on a Zariski open subset of $X(3,6)/S_3$.

The map g allows for the factorization through the quotient $X(3,6) \to X(3,6)/S_3$, thus inducing a map $g : X(3,6)/S_3 \to Y(6)$.

Proposition 16. *We have $g\pi = \sigma' g$.*

Proof. First, we prove that the involution π commutes with the action of S_6 on $X(3,6)/S_3$ that is induced by the permutations of q_j, $j = 1,\dots,6$, in the algebra $B(3,6)$. Consider the product $Y\times_S Y$ that is defined by the two maps $Y \to S$ induced by the maps $C \to A_3$ given by $Q \mapsto q_1+q_2+q_3$ and $Q \mapsto 1-q_1-q_2-q_3$.

Let us construct a morphism $Y(6) \to Y \times_S Y$. It corresponds to a decomposition of the set $(1,2,3,4,5,6)$ into a disjoint union of two three-element subsets and a choice of an ordering of elements in each subset. We can assign two algebra homomorphisms $A_3 \to A(6)$ to this combinatorial data. The first map takes the idempotents q_j's of A_3 to q_j's with indices in the first subset, ordered in the prescribed way, and similarly for the second homomorphism. Together, these homomorphisms define a morphism $Y(6) \to Y \times Y$, which is easily seen to descend to a morphism $Y(6) \to Y \times_S Y$. When composed with g, this morphism gives us a morphism $X(3,6)/S_3 \to Y \times_S Y$.

We choose two particular decompositions of the set $(1,2,3,4,5,6)$ into a disjoint union of two subsets. One is $((1,2,3),(4,5,6))$, and the other one is $((1,2,4),(3,5,6))$. As above, they define two morphisms $X(3,6)/S_3 \to Y \times_S Y$. Let us consider two functions on the variety $X(3,6)/S_3$:

$$z_1 = \mathrm{Tr} P q_1 P q_2, \quad z_2 = \mathrm{Tr} P q_5 P q_6.$$

Let $\mathcal{Z} = \mathrm{Spec} F[z_1, z_2]$. The natural morphism $X(3,6)/S_3 \to \mathcal{Z}$ factors through both morphisms $X(3,6)/S_3 \to Y \times_S Y$. Hence we get a commutative diagram:

$$\begin{array}{ccccc} & & X(3,6)/S_3 & & \\ & \swarrow & & \searrow & \\ Y \times_S Y & & & & Y \times_S Y. \\ & \searrow & & \swarrow & \\ & & \mathcal{Z} & & \end{array} \tag{17}$$

The involution (σ, σ) acts along the fibers of both morphisms $Y \times_S Y \to \mathcal{Z}$. Denote by π and π' the involutions on $X(3,6)/S_3$ where π was defined above and is attached to one of the morphisms $X(3,6)/S_3 \to Y \times_S Y$, while π' is similarly attached to the other morphism $X(3,6)/S_3 \to Y \times_S Y$. Both π and π' act along the fibers of the map $X(3,6)/S_3 \to \mathcal{Z}$. Therefore, the product $\pi\pi'$ also acts along the fibers of the same map. The fibers of the map over a generic point are compactified to a surface of general type. Thus $\pi\pi'$ is a birational automorphism of a surface of general type. The group of birational automorphisms of a variety of general type is finite (cf. [12]). Therefore, the element $\pi\pi'$ is of finite order. One can find a smooth fixed point of $\pi\pi'$ on $X(3,6)/S_3$ such that $\pi\pi'$ acts by the identity on the tangent space at this point. The point is the projection to $X(3,6)/S_3$ of the point in $X(6,6)$ corresponding to the "standard orthogonal pair" of Cartan subalgebras in $\mathrm{sl}(6,\bar{F})$. Since $\pi\pi'$ is of finite order, it follows that it is the identity on the whole $X(3,6)/S_3$. Therefore, $\pi = \pi'$.

This implies that π commutes with the transposition $(34) \in S_6$. Clearly, π commutes with all elements in S_6 that permute elements inside the subsets $(1,2,3)$ and $(4,5,6)$. Together with the transposition (34), they generate the whole group S_6. Thus π commutes with this group.

Now we consider the product of as many copies of $Y \times_S Y$ as there exist decompositions of the set $(1,2,3,4,5,6)$ into a disjoint union of two three-element subsets and choices of an ordering of elements in each subset. Taking the product of the above maps for each individual copy of $Y \times_S Y$ defines a morphism $\psi : Y(6) \to \prod(Y \times_S Y)$. One can check that this map is birationally an embedding.

The variety $\prod(Y \times_S Y)$ has the involution σ'' defined by the action of (σ, σ) on every component $Y \times_S Y$. It is obvious from the definition that $\sigma''\psi = \psi\sigma'$. Denote $\phi = \psi g : X(3,6)/S_3 \to \prod(Y \times_S Y)$. Since π commutes with the action of S_6, it follows that $\sigma''\phi = \phi\pi$.

Since g and ϕ are both birationally embeddings, it follows that $g\pi = \sigma' g$. □

It would be nice to have a more conceptual proof for this result.

Clearly, Proposition 16 implies a proof of Theorem 13.

4.2. The main algebraic geometric result.

Theorem 17. *There exists a four-dimensional irreducible component of $X(6,6)$ that contains the point x_0 constructed in Lemma* 11.

Proof. Proposition 16 implies that the variety $\bar{T}$ which is the locus of the points $(\bar{x}, \pi\bar{x})$, where $\bar{x}$ runs over the set of points $X(3,6)/S_3$ such that $\pi\bar{x}$ is well defined, is a subvariety in $X(3,6)/S_3 \times_{Y(6)} X(3,6)/S_3$. Let T be its preimage in $X(3,6) \times_{Y(6)} X(3,6)$. Consider the open subset $T_o \subset T$ of points that lie over the locus Y_o of irreducible representations for the algebra $A(6)$. According to Proposition 6, the open subset $X(6,6)_o$ is isomorphic to $X(3,6)_o \times_{Y(6)_o} X(3,6)_o$. Thus T_o is a subvariety in $X(6,6)_o$. Note that $\bar{T}$ is irreducible by construction, and T might have several components. By construction, $\bar{T}$ and all components of T have dimension 4.

Now consider the point $x_0 \in X(6,6)$ constructed in Lemma 11. By the lemma, x_0 lies over Y_o, i.e., it corresponds to a point in T_o under the isomorphism from Proposition 6. Since the dimension of the tangent space to $X(6,6)$ at this point is 4 and T_o is of dimension 4, it follows that x_0 is a smooth point on T_o. Hence the irreducible component of T_o that contains x_0 is an irreducible component of $X(6,6)$. □

Since $X(6,6)$ can be interpreted as the moduli space of orthogonal pairs in sl(6), as it was explained in Sec. 2.2, we have the following result.

Corollary 18. *There exists a four-dimensional family of orthogonal pairs in* sl(6)*, which contains the standard pair.*

It might be instructive to reformulate Proposition 16 in terms of elementary linear algebra.

Proposition 19. *Let $\mathcal{W}$ be the irreducible variety parameterizing the* 6×6 *matrices P of rank* 3 *with* $\frac{1}{2}$ *on the diagonal that satisfy $P^2 = P$ and admit a decomposition into three matrices p_i of rank* 1 *with* $\frac{1}{6}$ *on the diagonal (which implies $p_i^2 = p_i$):*

$$P = p_1 + p_2 + p_3.$$

Then for almost all $P \in \mathcal{W}$, the matrix $1 - P$ is also in $\mathcal{W}$.

Chances are that this statement is true for all $P \in \mathcal{W}$.

§5. Mutually unbiased bases

5.1. Mutually unbiased bases and a system of projectors. The terminology of unbiased bases first appeared in physics.

Let V be an n-dimensional complex vector space with a fixed Hermitian metric $\langle\ ,\ \rangle$. Two orthonormal Hermitian bases $\{e_i\}$ and $\{f_j\}$ in V are *mutually unbiased* if for all (i,j)

$$|\langle e_i, f_j\rangle|^2 = \frac{1}{n}. \tag{18}$$

There are two types of obvious transformations acting on the set of mutually unbiased bases. First, one can independently change the phase of all vectors in both bases:

$$e_j \mapsto \exp(\sqrt{-1}\alpha_j)e_j,$$
$$f_j \mapsto \exp(\sqrt{-1}\beta_j)f_j.$$

Second, one can transform all bases by a simultaneous linear transformation from $\mathrm{GL}(n,\mathbb{C})$.

Let $\{p_i\}$ be the orthogonal (i.e., $p_i p_j = 0$ for $i \neq j$) system of minimal projectors in V related to the base $\{e_i\}$, and $\{q_j\}$ be the system of minimal projectors related to the base $\{f_j\}$. Since both bases are orthonormal, all projectors are Hermitian, i.e., satisfy $p_j^\dagger = p_j$ and $q_j^\dagger = q_j$. Moreover, the condition that the bases are mutually unbiased is equivalent to the condition

$$\operatorname{Tr} p_i q_j = \frac{1}{n}$$

for all (i, j). The converse is also true: two orthogonal systems of Hermitian projectors satisfying the above equation uniquely define a mutually unbiased pair of bases up to the first type of transformations, i.e., up to changing the phases of basic vectors.

It follows from Sec. 2.1 that a pair of mutually unbiased bases defines a pair of orthogonal Cartan subalgebras in the Lie algebra $\mathrm{sl}(n, \mathbb{C})$. The requirement that the projectors are Hermitian means that the pair of Cartan subalgebras is special. We will see in the next subsection that they parameterize a real submanifold in the moduli space of all pairs of Cartan subalgebras.

5.2. The moduli of mutually unbiased bases as a "positive" real form of the moduli of orthogonal pairs. Let $\bar{\mathcal{X}}$ be the (singular) algebraic variety over $\mathbb{C}$ that parameterizes all pairs of orthogonal Cartan subalgebras in the Lie algebra $\mathrm{sl}(V)$ where $V \simeq \mathbb{C}^n$. Since it is identified with the variety $\mathrm{Rep}_n B_{n,n}$, it is an affine variety. The group $\mathrm{GL}(V)$ acts on $\bar{\mathcal{X}}$, and the GIT quotient $\bar{\mathcal{M}} = \bar{\mathcal{X}}/\mathrm{GL}(V)$ is the moduli space of orthogonal pairs in V. Since it is a GIT factor of an affine variety, it is affine too.

As we know, an orthogonal pair is uniquely defined by a pair of orthogonal systems of minimal projectors, where any pair of projectors from different systems are algebraically unbiased. For brevity, we will call such a pair of systems of projectors a *configuration*. A configuration is defined by an n-dimensional representation of the algebra $B_{n,n}$, which is known to be always irreducible (cf. [13]).

We reduce $\bar{\mathcal{X}}$ to its open subvariety $\mathcal{X}$ of smooth points, and we denote $\mathcal{M} = \mathcal{X}/\mathrm{GL}(V)$. Let us consider the real subvariety $\mathcal{X}_{\mathbb{R}}$ in $\mathcal{X}$ that is the locus of points that correspond to algebraically unbiased pairs of orthogonal systems of Hermitian projectors. The unitary group $U(n)$ acts on $\mathcal{X}_{\mathbb{R}}$, and the quotient $\mathcal{M}_{\mathbb{R}} = \mathcal{X}_{\mathbb{R}}/U(n)$ is the moduli of mutually unbiased bases.

Consider the involution that acts on $\mathcal{X}$ by Hermitian conjugation of all projectors:

$$p \mapsto p^\dagger.$$

Clearly, the involution is antiholomorphic, and $\mathcal{X}_\mathbb{R}$ is the locus of stable points of the involution. It is easy to check that the involution descends to an involution θ on $\mathcal{M}$ and that $\mathcal{M}_\mathbb{R}$ is embedded into the stable locus $\mathcal{M}^\theta$ of the involution on $\mathcal{M}$. We will show that $\mathcal{M}_\mathbb{R}$ is an open subset in $\mathcal{M}^\theta$.

Let $\mathbb{H}$ be the set of Hermitian operators in V, and $\mathbb{H}^\times$ be the open subset of invertible Hermitian operators. Define a subset $\mathcal{Y} \subset \mathbb{H}^\times \times \mathcal{X}$ by

$$\mathcal{Y} = \{(g, \{p_i, q_j\}) \in \mathbb{H}^\times \times \mathcal{X} |\ \ p_i^\dagger = g^{-1} p_i g, q_i^\dagger = g^{-1} q_i g\}.$$

Let $\mathbb{H}^\times_\pm \subset \mathbb{H}^\times$ be the open subset of invertible Hermitian matrices that are either positive or negative. Define $\mathcal{Y}_\pm \subset \mathcal{Y}$ as the open subset of those $(g, \{p_i, q_j\})$ for which $g \in \mathbb{H}^\times_\pm$.

We consider the map $\phi : \mathcal{Y} \to \mathcal{X}$ given by the projection to the second component of $\mathbb{H}^\times \times \mathcal{X}$ and the similar map $\phi_\pm : \mathcal{Y}_\pm \to \mathcal{X}$.

Denote by $\mathbb{R}^\times$ the group of nonzero real numbers. Consider the group $G = \mathbb{R}^\times \times \mathrm{PGL}(n, \mathbb{C})$ and its action on $\mathbb{H}^\times \times \mathcal{X}$ by

$$(\alpha, h)(g, \{p_i, q_j\}) = (\alpha h g h^\dagger, \{h p_i h^{-1}, h q_j h^{-1}\}).$$

It is easy to check that $\mathcal{Y}$ and $\mathcal{Y}_\pm$ are preserved by this action.

Proposition 20. *The set $\mathcal{Y}$ is a principal homogeneous G-bundle over $\mathcal{M}^\theta$. Similarly, $\mathcal{Y}_\pm$ is a principal homogeneous G-bundle over $\mathcal{M}_\mathbb{R}$.*

Proof. Let us check that the orbits of the action by $\mathbb{R}^\times$ are fibers of the map $\mathcal{Y} \to \mathcal{X}$. If $(g_1, \{p_i, q_j\})$ and $(g_2, \{p_i, q_j\})$ are in the fiber of $\mathcal{Y} \to \mathcal{X}$, then $(g_1)^{-1} g_2$ lies in the stabilizers of all projectors in the configuration. Since we consider irreducible representations of $B_{n,n}$, by Schur's lemma we have $(g_1)^{-1} g_2 = \lambda \cdot 1$. Therefore,

$$g_2 = \lambda g_1,$$

where $\lambda \neq 0$ because g_2 is invertible. Since g_1 and g_2 are Hermitian, applying the Hermitian conjugation gives

$$g_2 = \bar{\lambda} g_1.$$

Hence, $\lambda = \bar{\lambda}$, i.e., $\lambda \in \mathbb{R}^\times$.

As it was already mentioned, any configuration is given by an irreducible representation of $B_{n,n}$. Therefore, the action of $\mathrm{PGL}(n, \mathbb{C})$ on $\mathcal{X}$ is free, because the stabilizer of any configuration is a scalar matrix by Schur's lemma. It follows that the action of G on $\mathcal{Y}$ is free.

Take a point $m \in \mathcal{M}^\theta$. A point in $\mathcal{X}$ over it is presented by a configuration of projectors $\{p_i, q_j\}$. Since m is stable under the involution σ on the quotient space $\mathcal{M}$, there exists $g \in \mathrm{GL}(n, \mathbb{C})$ such that

$$p^\dagger = g^{-1} p g$$

for every projector p from the configuration. If we conjugate this equation, we get

$$p = g^\dagger p^\dagger (g^\dagger)^{-1}.$$

Together, these equations imply that $g^\dagger g^{-1}$ stabilizes all projectors p involved. It follows from Schur's lemma that $g^\dagger g^{-1} = \lambda \cdot 1$, for some nonzero multiplier $\lambda \in \mathbb{C}$. Hence

$$g^\dagger = \lambda g.$$

By taking the Hermitian dual, we have

$$\bar{\lambda} g^\dagger = g,$$

which, when combined with the previous relation, implies

$$|\lambda|^2 = 1.$$

It is easy to see that we can replace g by αg, for some $\alpha \in \mathbb{C}$, and get $g^\dagger = g$. The inverse inclusion $\phi(\mathcal{Y}) \subset \pi^{-1}(\mathcal{M}^\theta)$ is obvious. This proves that $\mathcal{Y}/G = \mathcal{M}^\theta$.

Now let us check that $\phi(\mathcal{Y}_\pm) \subset \pi^{-1}(\mathcal{M}_\mathbb{R})$. Take a point $(g, \{p_i, q_j\}) \in \mathcal{Y}_\pm$. We may assume that $g > 0$, because changing the sign of G does not change the conjugation by it. For positive nondegenerate g, it is known that there exists a decomposition

$$g = v^\dagger v,$$

for some invertible operator v. Since for all projectors p in the configuration we have

$$p^\dagger = g^{-1} p g = v^{-1} (v^\dagger)^{-1} p v^\dagger v,$$

it follows that $(v^\dagger)^{-1} p v^\dagger$ is self-adjoint. Hence, we can conjugate our configuration to a self-adjoint one.

Conversely, take a point $m \in \mathcal{M}_\mathbb{R}$. By definition, there exists a point in the π-fiber of m such that all projectors from its configuration are Hermitian. Let us take another point in the same fiber. Then every projector

p from its configuration is conjugate to the corresponding Hermitian projector r:

$$p = h^{-1} r h,$$

where $h \in \mathrm{GL}(n, \mathbb{C})$ is the same for all projectors p of the configuration. Since $r^\dagger = r$, we have

$$p^\dagger = h^\dagger r^\dagger (h^\dagger)^{-1} = h^\dagger r (h^\dagger)^{-1} = h^\dagger h p h^{-1} (h^\dagger)^{-1}.$$

Since $h^\dagger h$ is positive, we have $\pi^{-1}(\mathcal{M}_\mathbb{R}) \subset \phi(\mathcal{Y}_\pm)$. □

Corollary 21. *The subset $\mathcal{M}_\mathbb{R} \subset \mathcal{M}^\theta$ is open and is defined by a system of strict real polynomial inequalities.*

Proof. According to Sylvester's theorem, the positive Hermitian matrices are given by a system of n strict polynomial inequalities with real (even integer!) coefficients. Hence the open subset $\mathcal{Y}_\pm \subset \mathcal{Y}$ is defined by strict polynomial inequalities too. Since $\mathcal{Y}_\pm$ is invariant with respect to the free G-action, the inequalities descend to strict polynomial inequalities on $\mathcal{M}^\theta$. □

5.3. A four-dimensional family of mutually unbiased bases. Theorem 17 together with Corollary 21 imply the existence of a four-dimensional family of mutually unbiased bases in the six-dimensional complex space.

Theorem 22. *There exists a family of real dimension* 4 *of mutually unbiased bases in* $\mathbb{C}^6$.

Proof. We have an antiholomorphic involution θ on the moduli space $X(6,6)$ of six-dimensional representations of $B_{n,n}$. Let us restrict to the locus $\mathcal{M}$ of smooth points in all irreducible components of $X(6,6)$, as above. The locus of stable points of the involution on each component is a smooth real submanifold of real dimension equal to the complex dimension of the component. By Theorem 17, we have a four-dimensional irreducible component in $X(6,6)$. Hence, we need simply to check that the stable locus of θ is not empty on the smooth part of the component.

Consider the point x_0 constructed in Lemma 11. According to Theorem 17, it is a smooth point on a four-dimensional component of $X(6,6)$. Since formula (16) for the transition matrix A from the basis $\{p_i\}$ to the basis $\{q_i\}$ is a unitary matrix, the point x_0 is an element of $\mathcal{M}_\mathbb{R}$. □

Remark. Since the transformation matrix from one mutually unbiased base to another one is known to be a complex Hadamard matrix, the above theorem implies the existence of a four-dimensional family of complex Hadamard matrices of size 6×6.

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