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COMBINATORIAL ASPECTS OF CORRELATION FUNCTIONS OF THE XXZ HEISENBERG CHAIN IN LIMITING CASES

Abstract. We discuss the connection between quantum integrable models and some aspects of enumerative combinatorics and the theory of partitions. As a basic example, we consider the spin XXZ Heisenberg chain in the limiting cases of zero and infinite anisotropy. The representation of the Bethe wave functions via Schur functions allows us to apply the theory of symmetric functions to calculating the thermal correlation functions as well as the form factors in the determinantal form. We provide a combinatorial interpretation of the correlation functions in terms of nests of self-avoiding lattice paths. The suggested interpretation is in turn related to the enumeration of boxed plane partitions. The asymptotic behavior of the thermal correlation functions is studied in the limit of small temperature provided that the characteristic parameters of the system are large enough. The leading asymptotics of the correlators are found to be proportional to the squared numbers of boxed plane partitions.

§1. Introduction

The exactly solvable Heisenberg XXZ model is a prominent model describing the interaction of spins $\frac{1}{2}$ on a chain. The integrability of the model via the algebraic Bethe ansatz has led to important results, going from the spin dynamics up to exact expressions for the correlation functions [5, 14, 17].

The correlation functions of the XXZ Heisenberg chain are of considerable interest [5, 17]. This article is a continuation of a series of works dedicated to the study of the combinatorial implications of the correlation functions of the XXZ chain for two limits of the anisotropy parameter: zero and/or infinite anisotropy [6–10]. Here we focus on the case of the $XX0$ chain, which is the zero anisotropy limit of the XXZ model. This

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limit, also known as the free fermion case of the XXZ model, was intensively studied in physics and mathematics. It is related, in particular, to low-energy quantum chromodynamics [26], to the theory of symmetric functions [20], to a third-order phase transition [22], and to the theory of plane partitions [1, 13, 20]. Plane partitions (three-dimensional Young diagrams) appear in probability theory [11, 12, 29], enumerative combinatorics [27], the theory of faceted crystals [24], topological string theory [25], and the theory of random walks on lattices [3, 4, 13, 18].

We study the asymptotic behavior of the thermal correlation functions of the $XX0$ model in the limit of low temperature provided that the chain is long enough while the number of flipped spins is moderate. Namely, in this limit the thermal correlation functions are related to random matrix models [3].

We will consider the periodic $XX0$ Heisenberg chain. The representation of the Bethe wave functions via Schur functions [20] allows one to apply the well-developed theory of symmetric functions to calculating the thermal correlation functions as well as the form factors. In the present paper, we are interested in the correlation function of the creation operator of n excitations on consecutive sites of the chain, which will be called the *survival probability of the domain wall*. Special attention will be paid to the combinatorial objects appearing in the calculations (the generating functions of plane partitions and random walks, the q-binomial determinants) and to the combinatorial interpretation of the obtained results. We will calculate the leading terms of the asymptotics of the correlation function under consideration, provided that the characteristic parameters of the system are large enough, including the critical exponents of these correlation functions in the low temperature limit, and the related amplitudes. These amplitudes are found to be proportional to the squared numbers of boxed plane partitions.

§2. An outline of the XXZ Heisenberg chain

The quantum XXZ Heisenberg chain consisting of $M+1$ sites is described in the absence of a magnetic field by the Hamiltonian (see [14])

$$\widehat{H}_{XXZ} = -\frac{1}{2}\sum_{k=0}^{M}(\sigma_{k+1}^{-}\sigma_{k}^{+} + \sigma_{k+1}^{+}\sigma_{k}^{-} + \frac{\Delta}{2}\,(\sigma_{k+1}^{z}\sigma_{k}^{z} - 1)), \tag{1}$$

where $\Delta \in \mathbb{R}$ is the anisotropy parameter. The local spin operators $\sigma_k^{\pm} = \frac{1}{2}(\sigma_k^x \pm i\sigma_k^y)$ and σ_k^z, depending on the lattice argument k, are defined as $(M+1)$-fold tensor products:

$$\sigma_k^{\#} = \sigma^0 \otimes \cdots \otimes \sigma^0 \otimes \underbrace{\sigma^{\#}}_{k} \otimes \sigma^0 \otimes \cdots \otimes \sigma^0, \tag{2}$$

where σ^0 is the 2×2 identity matrix and $\sigma^{\#}$ at the kth site is a Pauli matrix, $\sigma^{\#} \in \mathfrak{su}(2)$ (here $\#$ is either x, y, z or $\pm$). The commutation rules are given by the relations

$$[\,\sigma_k^+, \sigma_l^-\,] = \delta_{kl}\,\sigma_l^z, \quad [\,\sigma_k^z, \sigma_l^{\pm}\,] = \pm 2\,\delta_{kl}\,\sigma_l^{\pm},$$

where δ_{kl} is the Kronecker symbol.

The spin operators act in the state space $\mathfrak{H}_{M+1} = \bigotimes_{k=0}^{M} \mathfrak{h}_k$ which is the product of $M+1$ copies of the linear space $\mathfrak{h}_k \equiv \mathbb{C}^2$. The state space $\mathfrak{H}_{M+1}$ is spanned by the state vectors $\bigotimes_{k=0}^{M} |s\rangle_k$, where $s = \uparrow, \downarrow$ corresponds to either the "spin-up" state or the "spin-down" state, $|\uparrow\rangle \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ or $|\downarrow\rangle \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. The action of the operators (2) is defined by the following rules:

$$\sigma^- \,|\uparrow\rangle = |\downarrow\rangle, \qquad \sigma^- \,|\downarrow\rangle = 0, \qquad \sigma^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

$$\sigma^+ \,|\uparrow\rangle = 0, \qquad \sigma^+ \,|\downarrow\rangle = |\uparrow\rangle, \qquad \sigma^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

We consider the XXZ Heisenberg model on a periodic chain with the boundary conditions $\sigma_{k+(M+1)}^{\#} = \sigma_k^{\#}$.

Let the sites with "spin-down" states be labeled by decreasing coordinates $M \geqslant \mu_1 > \mu_2 > \ldots > \mu_N \geqslant 0$, which constitute a strict partition $\boldsymbol{\mu} \equiv (\mu_1, \mu_2, \ldots, \mu_N)$. Another important partition $\boldsymbol{\lambda} \equiv (\lambda_1, \lambda_2, \ldots, \lambda_N)$ is a weakly decreasing sequence of nonnegative integers $L \geqslant \lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_N \geqslant 0$. The relation $\lambda_j = \mu_j - N + j$, where $1 \leqslant j \leqslant N$, connects the parts of $\boldsymbol{\lambda}$ to those of $\boldsymbol{\mu}$. Therefore, we can write $\boldsymbol{\lambda} = \boldsymbol{\mu} - \boldsymbol{\delta}_N$, where $\boldsymbol{\delta}_N$ is the partition $(N-1, N-2, \ldots, 1, 0)$.

The N-excitation state vectors $|\,\Psi_N(\mathbf{u})\rangle$, i.e., the vectors with N "down" spins, are given by the formula

$$|\,\Psi_N(\mathbf{u})\rangle = \sum_{\boldsymbol{\mu}} \chi_{\boldsymbol{\mu}}^{XXZ}(\mathbf{u}) \left(\prod_{k=1}^{N} \sigma_{\mu_k}^{-} \right) |\Uparrow\rangle, \tag{3}$$

where the summation is over all admissible partitions $\boldsymbol{\mu}$. The parameters $\mathbf{u}$ (or $\mathbf{u}_N$), $\mathbf{u} \equiv (u_1, u_2, \ldots, u_N)$ are sets of arbitrary complex numbers. For instance, $\mathbf{u}^2 \equiv (u_1^2, u_2^2, \ldots, u_N^2)$. The state $|\Uparrow\rangle$ corresponds to all spins "up": $|\Uparrow\rangle \equiv \bigotimes_{n=0}^{M} |\uparrow\rangle_n$. Besides,

$$\chi_{\boldsymbol{\mu}}^{XXZ}(\mathbf{u}) = \sum_{S_{p_1,p_2,\ldots,p_N}} \mathcal{A}_S(u_1, u_2, \ldots, u_N)\, u_{p_1}^{2\mu_1} u_{p_2}^{2\mu_2} \ldots u_{p_N}^{2\mu_N}, \tag{4}$$

where the summation runs over all permutations

$$S_{p_1,p_2,\ldots,p_N} \equiv S\begin{pmatrix} 1, & 2, & \ldots, & N \\ p_1, & p_2, & \ldots, & p_N \end{pmatrix}.$$

The amplitude $\mathcal{A}_S$ is

$$\mathcal{A}_S(u_1, u_2, \ldots, u_N) \equiv \prod_{1\leqslant j<i\leqslant N} \frac{1 - 2\Delta u_{p_i}^2 + u_{p_i}^2 u_{p_j}^2}{u_{p_i}^2 - u_{p_j}^2}.$$

The state vectors (3) are eigenstates of the Hamiltonian (1),

$$\widehat{H}_{XXZ}\, |\,\Psi_N(\mathbf{u}_N)\rangle = E_N\, |\,\Psi_N(\mathbf{u}_N)\rangle,$$

with the eigenvalues $E_N \equiv E_N(u_1, \ldots, u_N) = -\frac{1}{2}\sum_{i=1}^{N}(u_i^2 + u_i^{-2} - 2\Delta)$, if and only if u_l $(1 \leqslant l \leqslant N)$ satisfy the *Bethe equations*:

$$u_l^{2(M+1)} = (-1)^{N-1} \prod_{k=1}^{N} \frac{1 - 2\Delta u_l^2 + u_l^2 u_k^2}{1 - 2\Delta u_k^2 + u_l^2 u_k^2}. \tag{5}$$

The XXZ model was considered in [7–9] for two limits of the anisotropy parameter: $\Delta = 0$ and $\Delta \to \infty$.

• *The free fermion limit*, $\Delta \to 0$. As $\Delta \to 0$, the Hamiltonian (1) takes the form

$$\widehat{H}_{XX} \equiv -\frac{1}{2}\sum_{k=0}^{M}(\sigma_{k+1}^{-}\sigma_k^{+} + \sigma_{k+1}^{+}\sigma_k^{-}).$$

Up to an irrelevant prefactor, the amplitude (4) looks as follows:

$$\chi_{\boldsymbol{\mu}}^{XX}(\mathbf{u}) = \det(u_j^{2\mu_k})_{1\leqslant j,k\leqslant N} \prod_{1\leqslant n<l\leqslant N} (u_l^2 - u_n^2)^{-1}.$$

The reduced Bethe equations (5) are now exactly solvable:

$$u_j^{2(M+1)} = (-1)^{N-1}, \qquad u_j^2 = e^{i\frac{2\pi}{M+1}I_j}, \qquad 1 \leqslant j \leqslant N, \tag{6}$$

where I_j are integers or half-integers depending on whether N is odd or even, $M \geqslant I_1 > I_2 > \cdots > I_N \geqslant 0$. The eigenenergies are equal to

$$E_N^{XX}(I_1, I_2, \ldots, I_N) \;=\; -\sum_{l=1}^{N} \cos\Big(\frac{2\pi I_l}{M+1}\Big).$$

• *The strong anisotropy limit*, $\Delta \to -\infty$. In this limit, the system is described by the effective Hamiltonian (see [8])

$$\widehat{H}_{\mathrm{SA}} = -\,\frac{1}{2}\sum_{k=0}^{M} \mathcal{P}(\sigma_{k+1}^{-}\sigma_k^{+} + \sigma_{k+1}^{+}\sigma_k^{-})\mathcal{P},$$

where $\mathcal{P} \equiv \prod\limits_{k=0}^{M}(1 - \widehat{q}_{k+1}\widehat{q}_k)$ and the projections $\check{q}_k$, $\widehat{q}_k$ onto the "spin-up" and "spin-down" states are defined as

$$\check{q}_k \;\equiv\; \frac{1}{2}\,(\sigma_k^0 + \sigma_k^z) = \begin{pmatrix} 1\,0 \\ 0\,0 \end{pmatrix}_k, \quad \widehat{q}_k \;\equiv\; \frac{1}{2}\,(\sigma_k^0 - \sigma_k^z) = \begin{pmatrix} 0\,0 \\ 0\,1 \end{pmatrix}_k.$$

The projection $\mathcal{P}$ cuts out states having "down" spins at a pair of neighboring sites, since $\widehat{q}_k\ |\!\uparrow\rangle_k = 0$, $\widehat{q}_k\ |\!\downarrow\rangle_k = |\!\downarrow\rangle_k$ (analogously, $\check{q}_k\ |\!\downarrow\rangle_k = 0$, $\check{q}_k\ |\!\uparrow\rangle_k = |\!\uparrow\rangle_k$).

The wave function (4) takes the form

$$\chi_{\boldsymbol{\mu}}^{\mathrm{SA}}(\mathbf{u}) = \det(u_j^{2(\mu_k - N + k)})_{1\leqslant j,k\leqslant N} \prod_{1\leqslant n<l\leqslant N} (u_l^2 - u_n^2)^{-1},$$

where $\boldsymbol{\mu}$ is the strict decreasing partition, $M \geqslant \mu_1 > \mu_2 > \ldots > \mu_N \geqslant 0$, of the "down-spin" states and the parts of $\boldsymbol{\mu}$ satisfy the *exclusion condition*: $\mu_i > \mu_{i+1} + 1$. The Bethe equations (5) in this limit are also exactly solvable:

$$u_k^{2(M+1-N)} = (-1)^{N-1}\prod_{j=1}^{N} u_j^{-2}, \qquad u_k^2 = e^{i\frac{2\pi I_k - P}{M+1-N}}, \qquad 1 \leqslant k \leqslant N,$$

where I_j are integers or half-integers depending on whether N is odd or even, and $P \equiv \frac{2\pi}{M+1} \sum_{j=1}^{N} I_j$. The eigenenergies have the form

$$E_N(I_1, I_2, \dots, I_N) = - \sum_{l=1}^{N} \cos\Big(\frac{2\pi I_k - P}{M+1-N} \Big),$$

where $M - N \geqslant I_1 > I_2 > \cdots > I_N \geqslant 0$.

The crucial fact in the study of these two limiting models is that their state vectors are expressed through *Schur functions* (see [20]):

$$\begin{aligned} S_{\boldsymbol{\lambda}}(x_1, x_2, \dots, x_N) \quad &\equiv \quad \frac{\det(x_j^{\lambda_k+N-k})_{1\leqslant j,k\leqslant N}}{\det(x_j^{N-k})_{1\leqslant j,k\leqslant N}} \\ &= \quad \det(x_j^{\lambda_k+N-k})_{1\leqslant j,k\leqslant N} \prod_{1\leqslant n<l\leqslant N} (x_l - x_n)^{-1}, \end{aligned} \tag{7}$$

where $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \dots, \lambda_N)$ is an N-tuple of nonincreasing nonnegative integers, $L \geqslant \lambda_1 \geqslant \lambda_2 \geqslant \dots \geqslant \lambda_N \geqslant 0$.

With the help of (7), the state vectors (3) can be written in the limiting cases as follows.

- **The free fermion case**:

$$|\Psi_N(\mathbf{u})\rangle = \sum_{\boldsymbol{\lambda}\subseteq\{\mathcal{M}^N\}} S_{\boldsymbol{\lambda}}(\mathbf{u}^2) \left(\prod_{k=1}^{N} \sigma_{\mu_k}^{-} \right) |\Uparrow\rangle, \tag{8}$$

where $\boldsymbol{\lambda} = \boldsymbol{\mu} - \boldsymbol{\delta}_N$, $\mathcal{M} \equiv M + 1 - N \geqslant \lambda_1 \geqslant \lambda_2 \geqslant \dots \geqslant \lambda_N \geqslant 0$.

- **The strong anisotropy case**:

$$|\,\Psi_N(\mathbf{u})\rangle = \sum_{\widetilde{\boldsymbol{\lambda}}\subseteq\{(M-2(N-1))^N\}} S_{\widetilde{\boldsymbol{\lambda}}}(\mathbf{u}^2) \left(\prod_{k=1}^{N} \sigma_{\widetilde{\mu}_k}^{-} \right) |\Uparrow\rangle, \tag{9}$$

where $\widetilde{\boldsymbol{\lambda}} = \widetilde{\boldsymbol{\mu}} - 2\boldsymbol{\delta}_N$, $M + 2(1-N) \geqslant \widetilde{\lambda}_1 \geqslant \widetilde{\lambda}_2 \geqslant \dots \geqslant \widetilde{\lambda}_N \geqslant 0$; the parts of $\widetilde{\boldsymbol{\mu}}$ satisfy the inequalities $\widetilde{\mu}_i > \widetilde{\mu}_{i+1} + 1$.[1]

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[1]As shown in [28], in the case of the integrable q-boson model, a representation for the N-particle states arises that is analogous to (8) and (9).

The scalar products of state vectors in both limits are calculated by means of the Binet–Cauchy formula:

$$\begin{aligned}\mathcal{P}_{L/n}(\mathbf{y},\mathbf{x}) &\equiv \sum_{\boldsymbol{\lambda}\subseteq\{(L/n)^N\}} S_{\boldsymbol{\lambda}}(\mathbf{y})S_{\boldsymbol{\lambda}}(\mathbf{x}) \\ &= \left(\prod_{l=1}^{N} y_l^n x_l^n\right)\frac{\det(T_{kj})_{1\leqslant k,j\leqslant N}}{\mathcal{V}_N(\mathbf{y})\mathcal{V}_N(\mathbf{x})},\end{aligned} \tag{10}$$

where $\mathcal{V}_N(\mathbf{y}) \equiv \prod_{1\leqslant k<j\leqslant N}(y_j - y_k)$ is the Vandermonde determinant. The summation $\sum_{\boldsymbol{\lambda}\subseteq\{(L/n)^N\}}$ runs over the weakly decreasing sequences of integers satisfying $L \geqslant \lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_N \geqslant n$, $n \geqslant 0$. The entries T_{jk} take the form

$$T_{kj} = \frac{1-(x_k y_j)^{N+L-n}}{1-x_k y_j}. \tag{11}$$

§3. Correlation functions

We will consider the calculation of the *survival probability of the domain wall*:

$$\mathcal{F}(\boldsymbol{\theta}^{\mathrm{g}}_{N-n}, n, \beta) \equiv \frac{\langle \Psi_{N-n}(\boldsymbol{\theta}^{\mathrm{g}}) \mid \bar{\mathsf{F}}_n\, e^{-\beta\mathcal{H}}\, \bar{\mathsf{F}}_n \mid \Psi_{N-n}(\boldsymbol{\theta}^{\mathrm{g}})\rangle}{\langle \Psi_{N-n}(\boldsymbol{\theta}^{\mathrm{g}}) \mid e^{-\beta\mathcal{H}} \mid \Psi_{N-n}(\boldsymbol{\theta}^{\mathrm{g}})\rangle}, \tag{12}$$

where $\beta\in\mathbb{C}$ and $\bar{\mathsf{F}}_n \equiv \prod_{j=0}^{n-1}\sigma_j^-$ is the operator of creating n consecutive "spin-down" states on sites of the chain, i.e., of creating a *domain wall*. The shorthand notation $|\Psi_{N-n}(\boldsymbol{\theta}^{\mathrm{g}})\rangle \equiv |\Psi_{N-n}(e^{i\boldsymbol{\theta}^{\mathrm{g}}/2})\rangle$ implies that the eigenstate is calculated for an $(N-n)$-particle Bethe solution taken in the exponential form $\mathbf{u}^2_{N-n} = e^{i\boldsymbol{\theta}_{N-n}}$. The notation $\boldsymbol{\theta}^{\mathrm{g}}_{N-n} = (\theta^{\mathrm{g}}_1, \theta^{\mathrm{g}}_2, \ldots, \theta^{\mathrm{g}}_{N-n})$ corresponds to an $(N-n)$-particle solution of the Bethe equation for the ground state:

$$\theta^{\mathrm{g}}_j = \frac{2\pi}{M+1}\left(\frac{N-n+1}{2} - j\right), \quad 1\leqslant j\leqslant N-n. \tag{13}$$

Besides, $\mathcal{H}$ means either $\widehat{H}_{XX}$ or $\widehat{H}_{\mathrm{SA}}$, and $\bar{\mathsf{F}}_0$ is the identity operator $\mathbf{I}$, i.e., $\mathcal{F}(\boldsymbol{\theta}^{\mathrm{g}}_{N-n}, 0, \beta) = 1$.

In order to calculate the form factor of the operator $\bar{\mathsf{F}}_n$,

$$\langle \Psi_N(\mathbf{v}) \mid \bar{\mathsf{F}}_n \mid \Psi_{N-n}(\mathbf{u})\rangle,$$

we define an auxiliary operator $\mathsf{D}^n(\mathbf{u})$ which acts on the expectation $\langle\cdot\rangle_{\mathbf{u}}$ regarded as a function of $\mathbf{u}$ as follows:

$$\mathsf{D}^n(\mathbf{u})\,\langle\cdot\rangle_{\mathbf{u}} \;\equiv\; \mathsf{D}_{u_{N-n+1},u_{N-n+2},\dots,u_N}\left(\frac{\mathcal{V}_N(\mathbf{u}_N^2)}{\mathcal{V}_{N-n}(\mathbf{u}_{N-n}^2)}\times\langle\cdot\rangle_{\mathbf{u}}\right),$$

where

$$\mathsf{D}_{u_{N-n+1},u_{N-n+2},\dots,u_N} \;\equiv\; \mathsf{D}^{n-1}_{u_{N-n+1}}\circ\mathsf{D}^{n-2}_{u_{N-n+2}}\circ\ldots\circ\mathsf{D}^{0}_{u_N},$$

$$\mathsf{D}^{j}_{u_{N-j}} \;\equiv\; \lim_{u^2_{N-j}\to 0}\frac{1}{j\,!}\,\frac{d^j}{d(u^2_{N-j})^j},\qquad 0\leqslant j\leqslant n-1.$$

Now we are ready to formulate the following theorem.

Theorem 1 ([9]). *The action of the operator* $\mathsf{D}^n(\mathbf{u})$ *on the scalar product* $\langle\Psi(\mathbf{v}_N)\mid\Psi(\mathbf{u}_N)\rangle$ *gives the form factor of the domain wall creation operator* $\bar{\mathsf{F}}_n$:

$$\langle\Psi(\mathbf{v}_N)\mid\bar{\mathsf{F}}_n\mid\Psi(\mathbf{u}_{N-n})\rangle \;=\; \mathsf{D}^n(\mathbf{u})\langle\Psi(\mathbf{v}_N)\mid\Psi(\mathbf{u}_N)\rangle. \tag{14}$$

Proof. We evaluate the left-hand side of (14) using both the definition (8) and the properties of the Schur functions (7):

$$\langle\Psi_N(\mathbf{v})\mid\bar{\mathsf{F}}_n\mid\Psi_{N-n}(\mathbf{u})\rangle \;=\; \left(\prod_{l=1}^{N-n}u_l^{2n}\right)\sum_{\boldsymbol{\lambda}\subseteq\{\mathcal{M}^{N-n}\}}S_{\widehat{\boldsymbol{\lambda}}}(\mathbf{v}_N^{-2})S_{\boldsymbol{\lambda}}(\mathbf{u}^2_{N-n}), \tag{15}$$

where $\mathcal{M}\equiv M-N+1\geqslant\lambda_1\geqslant\lambda_2\geqslant\ldots\geqslant\lambda_{N-n}\geqslant 0$. Besides, $\widehat{\boldsymbol{\lambda}}$ is of length N, since $\widehat{\lambda}_p=\lambda_p$ for $1\leqslant p\leqslant N-n$ and $\widehat{\lambda}_{N-n+1}=\widehat{\lambda}_{N-n+2}=\dots=\widehat{\lambda}_N=0$.

Applying the orthogonality relation

$$\Big\langle\Uparrow\!\Big|\prod_{k=1}^{N}\sigma^+_{\mu_k}\prod_{l=1}^{N}\sigma^-_{\nu_l}\Big|\!\Uparrow\Big\rangle=\prod_{n=1}^{N}\delta_{\mu_n\nu_n},$$

we calculate the scalar product of the state vectors (8) by means of the Binet–Cauchy formula (10):

$$\langle\Psi(\mathbf{v}_N)\mid\Psi(\mathbf{u}_N)\rangle \;=\; \sum_{\boldsymbol{\lambda}\subseteq\{\mathcal{M}^N\}}S_{\boldsymbol{\lambda}}(\mathbf{v}_N^{-2})S_{\boldsymbol{\lambda}}(\mathbf{u}_N^2) \;=\; \frac{\det(T^{\mathrm{o}}_{kj})_{1\leqslant k,j\leqslant N}}{\mathcal{V}(\mathbf{u}_N^2)\mathcal{V}(\mathbf{v}_N^{-2})}, \tag{16}$$

where the summation is over all partitions $\boldsymbol{\lambda}$ with at most N parts each of which is less than or equal to $\mathcal{M}=M-N+1$. The entries T^{o}_{kj} in (16) are given by formula (11) taken at $n=0$:

$$T^{\mathrm{o}}_{kj} = \frac{1-(u_k^2/v_j^2)^{M+1}}{1-u_k^2/v_j^2}. \tag{17}$$

For $\mathbf{v}_N = \mathbf{u}_N$, the scalar product (16) gives the squared "length," $\mathcal{N}^2(\mathbf{u}_N) \equiv \langle \Psi(\mathbf{u}_N) \mid \Psi(\mathbf{u}_N)\rangle$, of the states (8).

A direct evaluation of the right-hand side of (14) by means of $\mathsf{D}^n(\mathbf{u})$ also leads to the right-hand side of (15) provided that the scalar product is expressed in terms of Schur functions according to (16). $\square$

Theorem 1 enables us to obtain two summation rules for products of Schur functions which are crucial in establishing combinatorial results for the correlation functions in question.

Theorem 2 ([9]). *The following formulas for sums of products of Schur functions hold*:

$$\sum_{\boldsymbol{\lambda}\subseteq\{\mathcal{M}^{N-n}\}} S_{\widehat{\boldsymbol{\lambda}}}(\mathbf{v}_N^{-2}) S_{\boldsymbol{\lambda}}(\mathbf{u}^2_{N-n}) \;=\; \left(\prod_{l=1}^{N-n} u_l^{-2n}\right) \frac{\det(\bar{T}_{kj})_{1\leqslant k,j\leqslant N}}{\mathcal{V}(\mathbf{u}^2_{N-n})\mathcal{V}(\mathbf{v}_N^{-2})}, \tag{18}$$

$$\sum_{\boldsymbol{\lambda}\subseteq\{\mathcal{M}^{N-n}\}} S_{\boldsymbol{\lambda}}(\mathbf{v}_{N-n}^{-2}) S_{\widehat{\boldsymbol{\lambda}}}(\mathbf{u}^2_{N}) \;=\; \left(\prod_{l=1}^{N-n} v_l^{2n}\right) \frac{\det(\tilde{T}_{kj})_{1\leqslant k,j\leqslant N}}{\mathcal{V}(\mathbf{v}^{-2}_{N-n})\mathcal{V}(\mathbf{u}_N^{2})}, \tag{19}$$

where the entries of the matrices $(\bar{T}_{kj})_{1\leqslant k,j\leqslant N}$ *and* $(\tilde{T}_{kj})_{1\leqslant k,j\leqslant N}$ *are*

$$\begin{aligned} &\bar{T}_{kj} = T^{\mathrm{o}}_{kj}, && 1\leqslant k\leqslant N-n, && 1\leqslant j\leqslant N,\\ &\bar{T}_{kj} = v_j^{-2(N-k)}, && N-n+1\leqslant k\leqslant N, && 1\leqslant j\leqslant N, \end{aligned} \tag{20}$$

and

$$\begin{aligned} &\tilde{T}_{kj} = T^{\mathrm{o}}_{kj}, && 1\leqslant k\leqslant N, && 1\leqslant j\leqslant N-n,\\ &\tilde{T}_{kj} = u_j^{2(N-k)}, && 1\leqslant k\leqslant N, && N-n+1\leqslant j\leqslant N. \end{aligned}$$

Here the entries T^{o}_{kj} given by (17) *are used.*

Proof. We calculate the right-hand side of (14) using the determinantal form of the scalar product given by (16):

$$\langle \Psi_N(\mathbf{v}) \mid \bar{\mathsf{F}}_n \mid \Psi_{N-n}(\mathbf{u})\rangle \;=\; \mathsf{D}^n(\mathbf{u}) \left(\frac{\det(T^{\mathrm{o}}_{kj})_{1\leqslant k,j\leqslant N}}{\mathcal{V}_N(\mathbf{u}^2)\mathcal{V}_N(\mathbf{v}^{-2})}\right). \tag{21}$$

Further, using (17), we obtain from (21) that

$$\langle \Psi_N(\mathbf{v}) \mid \bar{\mathsf{F}}_n \mid \Psi_{N-n}(\mathbf{u})\rangle \;=\; \frac{\det(\bar{T}_{kj})_{1\leqslant k,j\leqslant N}}{\mathcal{V}_{N-n}(\mathbf{u}^2)\mathcal{V}_N(\mathbf{v}^{-2})}, \tag{22}$$

where the entries of $\bar{T}$ are given by (20). Since the right-hand sides of (15) and (22) coincide, the relation (18) for Schur functions does indeed take place. By the same arguments, the validity of (19) can be established. $\square$

To calculate the survival probability of the domain wall, we insert a resolution of the identity operator into the numerator of (12) taken, however, at an arbitrary parametrization, see [9]:

$$\langle \Psi(\mathbf{v}_{N-n}) \mid \bar{\mathsf{F}}_n^+ \, e^{-\beta\mathcal{H}} \, \bar{\mathsf{F}}_n \mid \Psi(\mathbf{u}_{N-n}) \rangle = \sum_{\{\boldsymbol{\theta}_N\}} \langle \Psi(\mathbf{v}_{N-n}) \mid \bar{\mathsf{F}}_n^+ \mid \Psi(e^{i\boldsymbol{\theta}_N/2}) \rangle \tag{23}$$

$$\times \, \langle \Psi(e^{i\boldsymbol{\theta}_N/2}) | \, \bar{\mathsf{F}}_n \mid \Psi(\mathbf{u}_{N-n}) \rangle \, \frac{e^{-\beta E_N(\boldsymbol{\theta}_N)}}{\mathcal{N}^2(\boldsymbol{\theta}_N)} \tag{24}$$

$$= \, \mathsf{D}^n(\mathbf{u}) \, \mathsf{D}^n(\mathbf{v}^{-1}) \langle \Psi(\mathbf{v}_N) \mid e^{-\beta\mathcal{H}} \mid \Psi(\mathbf{u}_N) \rangle. \tag{25}$$

The decomposition (24) turns into (25) if (14) is used for each of the form factors in (24).

The explicit expression (15) for the form factor allows us to express the survival probability of the domain wall in terms of Schur functions starting with relation (24):

$$\begin{aligned} &\mathcal{F}(\boldsymbol{\theta}^{\mathrm{g}}_{N-n}, n, \beta) \, = \\ &= \frac{1}{\mathcal{N}^2(\boldsymbol{\theta}^{\mathrm{g}}_{N-n})(M+1)^{N-n}} \sum_{\{\boldsymbol{\theta}_{N-n}\}} e^{-\beta(E_{N-n}(\boldsymbol{\theta}_{N-n}) - E_{N-n}(\boldsymbol{\theta}^{\mathrm{g}}_{N-n}))} \\ &\times \left| \mathcal{V}(e^{i\boldsymbol{\theta}_{N-n}}) \sum_{\boldsymbol{\lambda} \subseteq \{\mathcal{M}^{N-n}\}} S_{\widehat{\boldsymbol{\lambda}}}(e^{-i\boldsymbol{\theta}_{N-n}}) S_{\boldsymbol{\lambda}}(e^{i\boldsymbol{\theta}^{\mathrm{g}}_{N-n}}) \right|^2, \end{aligned} \tag{26}$$

where the summation is over all solutions to the Bethe equation (6) and $\boldsymbol{\theta}^{\mathrm{g}}_{N-n}$ is the ground state solution (13) of the system of $N-n$ particles.

§4. q-Binomial determinants and generating functions of plane partitions

Let us show that the correlators obtained above are related to generating functions of *boxed plane partitions* and *self-avoiding lattice walks.*

An array $(\pi_{ij})_{i,j\geqslant 1}$ of nonnegative integers that is nonincreasing as a function both of i and j is called a *boxed plane partition* $\boldsymbol{\pi}$. The entries π_{ij} are the *parts* of the plane partition, and its volume is $|\boldsymbol{\pi}| = \sum_{i,j\geqslant 1} \pi_{ij}$. Each plane partition is represented by the three-dimensional Young diagram consisting of cubes arranged into stacks so that the stack with coordinates (i,j) is of height π_{ij}. It is said that a plane partition is contained in a box $\mathcal{B}(L,N,P)$ if $i \leqslant L$, $j \leqslant N$, and $\pi_{ij} \leqslant P$ for all cubes of the Young diagram (see Fig. 1).

The generating function of plane partitions in the box $\mathcal{B}(L,N,P)$ is the formal series $Z_q(L,N,P) \equiv \sum_{\{\pi\}} q^{|\pi|}$ (the summation is over all partitions in the box), and it takes the form (see [13, 20])

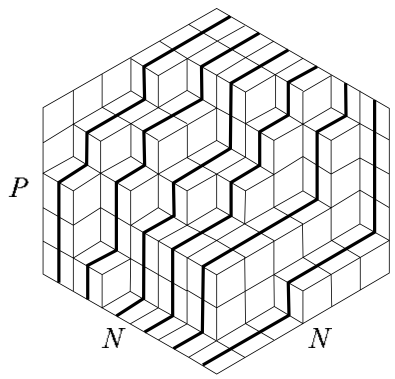

Fig. 1. A three-dimensional Young diagram.

$$Z_q(L,N,P) = \prod_{j=1}^{L}\prod_{k=1}^{N}\prod_{i=1}^{P}\frac{1-q^{i+j+k-1}}{1-q^{i+j+k-2}} = \prod_{j=1}^{L}\prod_{k=1}^{N}\frac{1-q^{P+j+k-1}}{1-q^{j+k-1}}.$$

The limit $q \to 1$ leads to the MacMahon formula [21]:

$$A(L,N,P) = \prod_{j=1}^{L}\prod_{k=1}^{N}\prod_{i=1}^{P}\frac{i+j+k-1}{i+j+k-2} = \prod_{j=1}^{L}\prod_{k=1}^{N}\frac{P+j+k-1}{j+k-1}.$$

To study the asymptotic behavior of the correlation functions, we need the determinant of $(\bar{\mathsf{T}})_{1\leqslant j,k\leqslant N}$ taken under the q-parametrization

$$\mathbf{v}_N^{-2} = \mathbf{q}_N \equiv (q, q^2, \dots, q^N), \qquad \mathbf{u}_N^2 = \mathbf{q}_N/q = (1, q, \dots, q^{N-1}). \tag{27}$$

For arbitrary P and $L \leqslant N$, these entries take the form

$$\begin{aligned} \bar{\mathsf{T}}_{kj} &= \frac{1-q^{(P+1)(j+k-1)}}{1-q^{j+k-1}}, \quad 1 \leqslant k \leqslant L, \qquad 1 \leqslant j \leqslant N, \\ \bar{\mathsf{T}}_{kj} &= q^{j(N-k)}, \qquad\qquad L+1 \leqslant k \leqslant N, \quad 1 \leqslant j \leqslant N. \end{aligned} \tag{28}$$

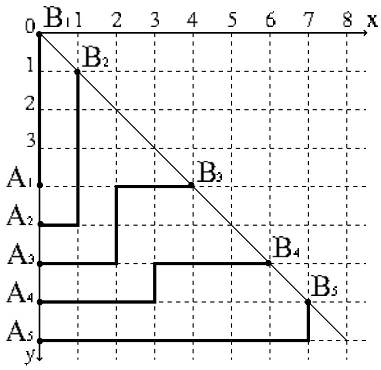

Fig. 2. An S-tuple $(w_1, w_2, \ldots, w_S)$ of self-avoiding lattice paths for $S = 5$.

The matrix $(\bar{\mathsf{T}})_{1\leqslant j,k\leqslant N}$ consists of two blocks with sizes $L \times N$ and $(N - L) \times N$. It is appropriate to call $\det \bar{\mathsf{T}}$ a *Kuperberg-type* determinant, since it is closely related to the determinant obtained in [19] for the problem of enumerating the alternating sign matrices [13, 23].

Several definitions are in order. We use the standard definition of the *q-binomial determinant* (see [27]):

$$\begin{pmatrix} \mathbf{a} \\ \mathbf{b} \end{pmatrix}_q \equiv \begin{pmatrix} a_1, & a_2, & \ldots & a_S \\ b_1, & b_2, & \ldots & b_S \end{pmatrix}_q \equiv \det \left(\begin{bmatrix} a_j \\ b_i \end{bmatrix} \right)_{1\leqslant i,j\leqslant S},$$

where $\mathbf{a}$ and $\mathbf{b}$ are ordered tuples, $0 \leqslant a_1 < a_2 < \cdots < a_S$ and $0 \leqslant b_1 < b_2 < \cdots < b_S$. The entries $\begin{bmatrix} a_j \\ b_i \end{bmatrix}$ are the *q-binomial coefficients*:

$$\begin{bmatrix} N \\ r \end{bmatrix} \equiv \frac{(1 - q^N)(1 - q^{N-1}) \ldots (1 - q^{N-r+1})}{(1 - q)(1 - q^2) \ldots (1 - q^r)}, \qquad q \in \mathbb{R}.$$

As $q \to 1$, the q-binomial coefficients are replaced by the binomial coefficients and the q-binomial determinant is transformed into the *binomial determinant*:

$$\begin{pmatrix} \mathbf{a} \\ \mathbf{b} \end{pmatrix} \equiv \begin{pmatrix} a_1, & a_2, & \ldots & a_S \\ b_1, & b_2, & \ldots & b_S \end{pmatrix} = \det \left(\begin{pmatrix} a_j \\ b_i \end{pmatrix} \right)_{1\leqslant i,j\leqslant S}. \tag{29}$$

The binomial determinant (29) gives the number of self-avoiding walks across a two-dimensional lattice [16]. A nest of self-avoiding lattice paths is shown in Fig. 2, where each path w_i belonging to a tuple $(w_1, w_2, \dots, w_S)$ goes from $A_i = (0, a_i)$ to $B_i = (b_i, b_i)$, $1 \leqslant i \leqslant S$ (only northward and eastward steps are allowed).

Now we are ready to formulate the following theorem.

Theorem 3 ([9]). *Let $(\bar{\mathsf{T}})_{1\leqslant j,k\leqslant N}$ be the matrix with the entries* (28) *where $\frac{P}{2} < N < P$. The determinant of $(\bar{\mathsf{T}})_{1\leqslant j,k\leqslant N}$ is given by*

$$q^{-\frac{L}{2}(L-1)(N-L)} \,\frac{\det(\bar{\mathsf{T}})_{1\leqslant j,k\leqslant N}}{\mathcal{V}(\mathbf{q}_N)\mathcal{V}(\mathbf{q}_L/q)}$$

$$= q^{-\frac{N}{2}(\mathcal{P}-1)\mathcal{P}} \begin{pmatrix} L+N, & L+N+1, & \dots & L+N+\mathcal{P}-1 \\ L, & L+1, & \dots & L+\mathcal{P}-1 \end{pmatrix}_q \tag{30}$$

$$= \prod_{k=1}^{\mathcal{P}} \prod_{j=1}^{L} \frac{1-q^{j+k+N-1}}{1-q^{j+k-1}} \;=\; Z_q(L,N,\mathcal{P}), \tag{31}$$

where $\mathcal{P} \equiv P - N + 1$ and $Z_q(L,N,\mathcal{P})$ is the generating function of plane partitions.

Proof. The proof is based on symmetric functions. Before going further, we define the elementary symmetric functions $e_r = e_r(\mathbf{x})$ depending on N variables $\mathbf{x} = (x_1, x_2, \dots, x_N)$:

$$e_r \;\equiv \sum_{i_1<i_2<\dots<i_r} x_{i_1}x_{i_2}\dots x_{i_r}.$$

The value of e_r at $\mathbf{x} = \mathbf{q} \equiv (q, q^2, \dots, q^N)$ is $e_r = q^{r(r+1)/2}\begin{bmatrix} N \\ r \end{bmatrix}$.

Let us turn to the Schur functions (7) labeled by nonstrict partitions $\boldsymbol{\lambda}$. Consider the *conjugate partitions* $\bar{\boldsymbol{\lambda}}$. The Young diagram of $\bar{\boldsymbol{\lambda}}$ is obtained by transposing the Young diagram of $\boldsymbol{\lambda}$. The parts of a nonstrict partition $\bar{\boldsymbol{\lambda}}$ satisfy the conditions $N \geqslant \bar{\lambda}_1 \geqslant \bar{\lambda}_2 \geqslant \dots \geqslant \bar{\lambda}_{\mathcal{P}} \geqslant 0$, $\mathcal{P} \equiv P + 1 - N$. It is known that

$$S_{\boldsymbol{\lambda}}(\mathbf{x}) \;=\; \det\bigl(e_{\bar{\lambda}_i - i + j}(\mathbf{x})\bigr)_{1\leqslant i,j\leqslant \mathcal{P}}.$$

In order to express $\det \bar{\mathsf{T}}$ as a q-binomial determinant, we use Theorem 2 under the q-parametrization:

$$\det \bar{\mathsf{T}} \;=\; q^{\frac{L}{2}(L-1)(N-L)}\mathcal{V}(\mathbf{q}_N)\mathcal{V}(\mathbf{q}_L/q) \sum_{\boldsymbol{\lambda}\subseteq\{\mathcal{P}^L\}} S_{\hat{\boldsymbol{\lambda}}}(\mathbf{q}_N) S_{\boldsymbol{\lambda}}(\mathbf{q}_L/q).$$

Denote the sum by $\boldsymbol{\Sigma}_{\mathsf{S}}$ and bring it to the form

$$\boldsymbol{\Sigma}_{\mathsf{S}} = \sum_{\bar{\boldsymbol{\lambda}} \subseteq \{L^{\mathcal{P}}\}} \det\bigl(e_{\bar{\lambda}_j - j + k}(\mathbf{q}_N)\bigr)_{1 \leqslant j,k \leqslant \mathcal{P}} \det\bigl(e_{\bar{\lambda}_p - p + l}(\mathbf{q}_L/q)\bigr)_{1 \leqslant l,p \leqslant \mathcal{P}},$$

where the summation is over the conjugate partitions $\bar{\boldsymbol{\lambda}}$. For $\boldsymbol{\Sigma}_{\mathsf{S}}$ we obtain

$$\begin{aligned} \boldsymbol{\Sigma}_{\mathsf{S}} &= q^{-\frac{N}{2}(\mathcal{P}-1)\mathcal{P}} \begin{pmatrix} L+N, & L+N+1, & \dots & L+N+\mathcal{P}-1 \\ L, & L+1, & \dots & L+\mathcal{P}-1 \end{pmatrix}_q \\ &= \prod_{j=1}^{L} \prod_{k=1}^{\mathcal{P}} \frac{1-q^{N+j+k-1}}{1-q^{j+k-1}} \;=\; Z_q(L,\mathcal{P},N) \;=\; Z_q(L,N,\mathcal{P}). \qquad \square \end{aligned}$$

Theorem 3 relates $\det \bar{\mathsf{T}}$ to the q-binomial determinant, which is transformed as $q \to 1$ into the binomial determinant, equal, in turn, to the number of $\mathcal{P}$-tuples of lattice self-avoiding paths between the *end points* $A_l = (0, N+L+l-1)$ and $B_l = (L+l-1, L+l-1)$, $1 \leqslant l \leqslant \mathcal{P}$, see [16]. Figure 3 gives an appropriate picture with *equidistant* points A_l and B_l (in the figure, $\mathcal{P} = L = 3$ and $N = 2$).

Since the lattice paths are self-avoiding, each of them has a horizontal part terminating at the abscissa axis. After a partial “amputation” of these horizontal parts, a configuration called a *watermelon* comes to play. A watermelon configuration consists of lattice paths connecting the points $C_l = (l-1, N+L+l-1)$ and $B_l = (L+l-1, L+l-1)$, $1 \leqslant l \leqslant \mathcal{P}$. For every path, the numbers of steps along the abscissa and ordinate axes coincide.

The generating function $Z_q(L,N,\mathcal{P})$ from (31) gives, as $q \to 1$, the number $A(L,N,\mathcal{P})$ of plane partitions inside $\mathcal{B}(L,N,\mathcal{P})$:

$$\begin{aligned} Z_q(L,N,\mathcal{P}) \;&=\; \prod_{j=1}^{L} \prod_{k=1}^{N} \frac{1-q^{\mathcal{P}+j+k-1}}{1-q^{j+k-1}} \\ &\underset{q \to 1}{\longrightarrow} A(L,N,\mathcal{P}) = \det\left(\begin{pmatrix} N+L+i-1 \\ L+j-1 \end{pmatrix}\right)_{1 \leqslant i,j \leqslant \mathcal{P}}. \end{aligned} \tag{32}$$

The right-hand side of (32) expresses the fact that the number $A(L,N,\mathcal{P})$ of plane partitions is equal to the number of self-avoiding lattice paths. Just the paths constituting a “watermelon” are in a bijection with the so-called *gradient lines* (Fig. 3) corresponding to a plane partition in the box $\mathcal{B}(L,N,\mathcal{P})$.

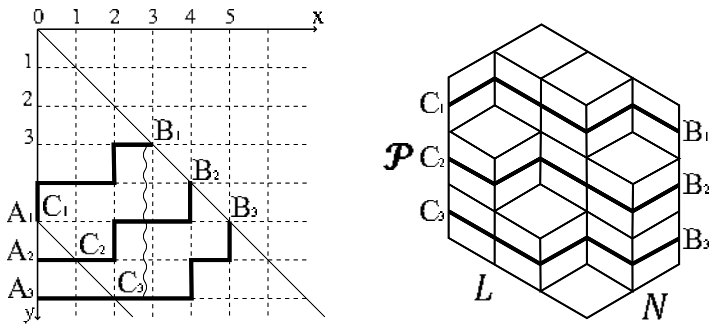

Fig. 3. Self-avoiding lattice paths constituting a *watermelon* configuration (left) and a three-dimensional Young diagram with *gradient lines*.

Definition. *The generating function* $\mathsf{W}_q(L,N)$ *of "watermelon" configurations*, characterized by the total numbers of steps, L and N, along the abscissa and ordinate axes, is given by the formula

$$\mathsf{W}_q(L,N) \equiv \sum_{\{\mathbf{w}_{LN}\}} q^{|\mathbf{w}_{LN}|},$$

where $\sum_{\{\mathbf{w}_{LN}\}}$ implies the summation over all nests of paths $\mathbf{w}_{LN}$ constituting a "watermelon." Here $q^{|\mathbf{w}_{LN}|}$ is the statistical weight of a configuration of paths $\mathbf{w}_{LN}$, and $|\mathbf{w}_{LN}|$ is the volume of a configuration $\mathbf{w}_{LN}$. This volume is equal, by definition, to the sum of the volumes of the paths constituting the watermelon. The volume of a path is equal to the number of cells below the path inside the corresponding rectangle.

Due to Theorems 1 and 2, the form factor of the domain wall creation operator (15) under the q-parametrization (27) takes the form

$$\begin{aligned} &\langle \Psi(\mathbf{q}_N^{-\frac{1}{2}}) \mid \bar{\mathsf{F}}_n \mid \Psi((\mathbf{q}_{N-n}/q)^{\frac{1}{2}})\rangle \\ &= q^{\frac{n}{2}(N-n)(N-n-1)} \sum_{\boldsymbol{\lambda}\subseteq\{\mathcal{M}^{N-n}\}} S_{\widehat{\boldsymbol{\lambda}}}(\mathbf{q}_N) S_{\boldsymbol{\lambda}}(\mathbf{q}_{N-n}/q) \\ &= \frac{\det \bar{\mathsf{T}}}{\mathcal{V}(\mathbf{q}_N)\mathcal{V}(\mathbf{q}_{N-n}/q)}, \end{aligned} \tag{33}$$

where $\bar{\mathsf{T}}$ is given by (28) at $L = N - n$ and $P = M$. From (30) and (31) we obtain that the form factor $\langle \Psi(\mathbf{q}_N^{-\frac{1}{2}}) \mid \bar{\mathsf{F}}_n \mid \Psi((\mathbf{q}_{N-n}/q)^{\frac{1}{2}})\rangle$ is the generating function of plane partitions in the box $\mathcal{B}(N-n, N, \mathcal{M})$:

$$\langle \Psi(\mathbf{q}_N^{-\frac{1}{2}}) \mid \bar{\mathsf{F}}_n \mid \Psi((\mathbf{q}_{N-n}/q)^{\frac{1}{2}}) \rangle \;=\; q^{\frac{n}{2}(N-n)(N-n-1)} Z_q(N-n, N, \mathcal{M}). \tag{34}$$

As $q \to 1$, this expression becomes the MacMahon formula for the number of plane partitions in the box $\mathcal{B}(N-n, N, \mathcal{M})$:

$$\lim_{q\to 1} \langle \Psi(\mathbf{q}_N^{-\frac{1}{2}}) \mid \bar{\mathsf{F}}_n \mid \Psi((\mathbf{q}_{N-n}/q)^{\frac{1}{2}}) \rangle \;=\; A(N-n, N, \mathcal{M}). \tag{35}$$

§5. Low temperature

Assume that $M \gg 1$ and $1 \ll N, N-n \ll M$. To study the asymptotic behavior of the survival probability of the domain wall correlation function, we can replace the sums in (26) in this limit by integrals. For large β (low temperature), we approximately obtain

$$\mathcal{F}(\boldsymbol{\theta}^{\mathrm{g}}_{N-n}, n, \beta) \;\simeq\; \frac{A^2(N-n, N, M-N+1)}{\beta^{\frac{(N-n)^2}{2}}} \, \frac{\mathcal{I}_{N-n}}{\mathcal{N}^2(\boldsymbol{\theta}^{\mathrm{g}}_{N-n})},$$

where the *Mehta integral*

$$\mathcal{I}_N \;\equiv\; \frac{1}{N!} \int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} \cdots \int\limits_{-\infty}^{\infty} e^{-\frac{1}{2}\sum\limits_{l=1}^{N} x_l^2} \prod_{1\leqslant k<l\leqslant N} \bigl|x_k - x_l\bigr|^2 \frac{dx_1 dx_2 \dots dx_N}{(2\pi)^N} \tag{36}$$

is used, whose value is known:

$$\mathcal{I}_N \;=\; e^{\varphi_N}, \qquad \varphi_N \;\equiv\; \sum_{k=1}^{N} \log \frac{\Gamma(k)}{(2\pi)^{1/2}}. \tag{37}$$

Finally, we use the estimate

$$\begin{aligned}
\frac{1}{\mathcal{N}^2(\boldsymbol{\theta}^{\mathrm{g}}_{N-n})} \;&\simeq\; \frac{(2\pi)^{(N-n)(N-n-1)}}{(M+1)^{(N-n)^2}} \prod_{1\leqslant r<s\leqslant N-n} |r-s|^2 \\
&\approx\; \left(\frac{2\pi}{M+1}\right)^{N^2} e^{2\varphi_N}
\end{aligned}$$

valid for $1 \ll N, N-n \ll M$ and express the answer for the survival probability of the domain wall:

$$\mathcal{F}(\boldsymbol{\theta}^{\mathrm{g}}_{N-n}, n, \beta) \;\simeq\; A^2(N-n, N, M-N+1)\, e^{\Phi(N,M,\beta)}, \tag{38}$$

$$\Phi(N, M, \beta) \;\equiv\; N^2 \log \frac{2\pi}{M+1} - \frac{N^2}{2} \log \beta + 3\varphi_N, \tag{39}$$

where $A(N-n, N, M-N+1)$ is the number of plane partitions (35) in the box $\mathcal{B}(N-n, N, M-N+1)$ with rectangular bottom. The low temperature decay of the correlator is governed by the critical exponent $N^2/2$, while its amplitude is proportional to the squared number of plane partitions in the box $\mathcal{B}(N-n, N, M-N+1)$.

To study the asymptotic behavior, it is convenient to express φ_N through the Barnes G-function (see [2]):

$$G(z+1) = (2\pi)^{z/2} e^{\frac{-z}{2}(z+1)-\frac{\gamma}{2}z^2} \prod_{n=1}^{\infty} \Big(1+\frac{z}{n}\Big)^n e^{-z+\frac{z^2}{2n}},$$

which is an integral function satisfying the following relations: $G(1)=1$, $G(z+1)=\Gamma(z)G(z)$, and

$$G(n+1) = \frac{(n!)^n}{1^1\, 2^2 \ldots n^n} = \prod_{k=1}^{n} \Gamma(k).$$

For $\mathcal{I}_N$ (given by (36)) and φ_N (given by (37)) we obtain

$$\varphi_N = \log G(N+1) - \frac{N}{2}\log 2\pi,$$

$$\mathcal{I}_N = \frac{G(N+1)}{(2\pi)^{N/2}}.$$

The asymptotics of $\log G(z+1)$ at $z\to\infty$ is known, and it gives φ_N for $N \gg 1$:

$$\varphi_N = \frac{N^2}{2}\log N - \frac{3N^2}{4} + \mathcal{O}(\log N), \qquad N \gg 1.$$

Thus, for $\Phi(N, M, \beta)$ given by (39) we approximately obtain

$$\Phi(N, M, \beta) \simeq N^2 \log\Big(\mathsf{A}\frac{N^{3/2}}{M\beta^{1/2}}\Big),$$

where A is a constant.

We express the number of plane partitions in $\mathcal{B}(N-n, N, M-N+1)$ as

$$\begin{aligned} A(N-n, N, M-N+1) &= \frac{G(N+1)\, G(N-n+1)}{G(2N-n+1)} \\ &\times \frac{G(M+2-n+N)\, G(M+2-N)}{G(M+2-n)\, G(M+2)} \end{aligned}$$

and estimate it using the properties of the G-function:

$$\log A(N-n, N, M-N+1) \simeq N(N-n)\log\left(\mathsf{D}\,\frac{M-n}{2N-n}\right), \\ M-n \gg N-n, N \gg 1,$$

where D is some constant. Eventually, we obtain

$$\log \mathcal{F}(\boldsymbol{\theta}_N^{\mathrm{g}}, n, \beta) \simeq N^2 \log\left(\mathsf{A}\,\frac{N^{3/2}}{M\beta^{1/2}}\right) + \\ + 2N(N-n)\log\left(\mathsf{D}\,\frac{M-n}{2N-n}\right). \tag{40}$$

Equation (40) enables us to state that $\mathcal{F}(\boldsymbol{\theta}_N^{\mathrm{g}}, n, \beta)$ decreases as M and N increase provided that T is small enough and goes to zero, see [9].

§6. Concluding remarks

The N-particle thermal correlation functions of the domain wall creation operator $\bar{\mathsf{F}}_n$ in the $\Delta \to 0$ limit of the XXZ Heisenberg model on a cyclic chain were considered. Calculations based on the theory of symmetric functions allow us to express the answers in the determinantal form. The combinatorial aspects of the form factors and thermal correlation functions of the operator $\bar{\mathsf{F}}_n$ were studied. The representation of the form factors through q-binomial determinants stated in Theorem 3 plays an important role in establishing a connection between plane partitions and self-avoiding lattice paths. The asymptotic behavior of the correlation functions of the operator $\bar{\mathsf{F}}_n$ is estimated for sufficiently low temperatures. The low temperature approximation allows us both to extract the combinatorial pre-factor and to reduce matrix-type integrals to the partition function of the Gaussian Unitary Ensemble [15]. The correlation function demonstrates a power-law decay, and its amplitude is given by the squared number of plane partitions in a box.

Though we have focused only on the zero limit of the anisotropy parameter, the infinite anisotropy limit is studied in a similar way since the wave functions in this limit are also expressed through Schur functions.

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References

1. G. E. Andrews, *The Theory of Partitions.* Cambridge Univ. Press, Cambridge, 1998.
2. E. W. Barnes, *The theory of the G-function.* — Quart. J. Pure Appl. Math. **31** (1900), 264–314.
3. N. M. Bogoliubov, *XX Heisenberg chain and random walks.* — J. Math. Sci. **138**, No. 3 (2006), 5636–5643.
4. N. M. Bogoliubov, *The integrable models for the vicious and friendly walkers.* — J. Math. Sci. **143**, No. 1 (2007), 2729–2737.
5. N. M. Bogoliubov, A. G. Izergin, V. E. Korepin, *Correlation Functions of Integrable Systems and Quantum Inverse Scattering Method* [in Russian]. Nauka, Moscow, 1992.
6. N. M. Bogoliubov, C. Malyshev, *The correlation functions of the XX Heisenberg magnet and random walks of vicious walkers.* — Theor. Math. Phys. **159**, No. 2 (2009), 179–192.
7. N. M. Bogoliubov, C. Malyshev, *The correlation functions of the XXZ Heisenberg chain in the case of zero or infinite anisotropy, and random walks of vicious walkers.* — St.Petersburg Math. J. **22**, No. 3 (2011), 359–377.
8. N. M. Bogoliubov, C. L. Malyshev, *The Ising limit of the XXZ Heisenberg magnet and certain thermal correlation functions.* — Theor. Math. Phys. **169**, No. 2 (2011), 1517–1529.
9. N. M. Bogoliubov, C. Malyshev, *Correlation functions of $XX0$ Heisenberg chain, q-binomial determinants, and random walks.* — Nucl. Phys. B **879** (2014), 268–291.
10. N. M. Bogoliubov, C. L. Malyshev, *A combinatorial interpretation of the scalar products of state vectors of integrable models.* — Zap. Nauchn. Semin. POMI **421** (2014), 35–45.
11. A. Borodin, V. Gorin, E. M. Rains, *q-Distributions on boxed plane partitions.* — Selecta Math. (N. S.) **16**, No. 4 (2010), 731–789.
12. A. Borodin, G. Olshanski, *Infinite-dimensional diffusions as limits of random walks on partitions.* — Prob. Theory Related Fields **144**, No. 1–2 (2009), 281–318.
13. D. M. Bressoud, *Proofs and Confirmations. The Story of the Alternating Sign Matrix Conjecture.* Cambridge Univ. Press, Cambridge, 1999.
14. L. D. Faddeev, L. A. Takhtajan, *Quantum inverse scattering method and the XYZ Heisenberg model.* — Uspekhi Mat. Nauk **34**, No. 5(209) (1979), 13–63.
15. P. J. Forrester, *Log-Gases and Random Matrices.* Princeton Univ. Press, Princeton, 2010.
16. I. Gessel, G. Viennot, *Binomial determinants, paths, and hook length formulae.* — Adv. in Math. **58**, No. 3 (1985), 300–321.
17. V. E. Korepin, N. M. Bogoliubov, A. G. Izergin, *Quantum Inverse Scattering Method and Correlation Functions.* Cambridge Univ. Press, Cambridge, 1993.
18. C. Krattenthaler, A. J. Guttmann, X. G. Viennot, *Vicious walkers, friendly walkers and Young tableaux: II. With a wall.* — J. Phys. A **33**, No. 48 (2000), 8835–8866.
19. G. Kuperberg, *Another proof of the alternating-sign matix conjecture.* — Int. Math. Res. Notices **1996** (1996), 139–150.

20. I. G. Macdonald, *Symmetric Functions and Hall Polynomials.* Oxford Univ. Press, Oxford, 1995.
21. P. A. MacMahon, *Combinatory Analysis*, Vols. 1, 2. Cambridge Univ. Press, Cambridge, 1915, 1916.
22. S. N. Majumdar, G. Schehr, *Top eigenvalue of a random matrix: large deviations and third order phase transition.* — J. Stat. Mech. **2014** (2014), P01012.
23. W. H. Mills, D. P. Robbins, H. Rumsey, Jr., *Alternating sign matrices and descending plane partitions.* — J. Combin. Theory Ser. A **34**, No. 3 (1983), 340–359.
24. A. Okounkov, N. Reshetikhin, *Correlation function of Schur process with application to local geometry of a random* 3*-dimensional Young diagram.* — J. Amer. Math. Soc. **16**, No. 3 (2003), 581–603.
25. A. Okounkov, N. Reshetikhin, C. Vafa, *Quantum Calabi–Yau and classical crystals.* — In: The Unity of Mathematics (In Honor of the Ninetieth Birthday of I. M. Gelfand), P. Etingof, V. S. Retakh, I. M. Singer (eds.), Birkhäuser, Boston, 2006, pp. 597–618.
26. D. Pérez-Garcia, M. Tierz, *Mapping between the Heisenberg XX spin chain and low-energy QCD.* — Phys. Rev. X **4** (2014), 021050.
27. R. Stanley, *Enumerative Combinatorics*, Vol. 2. Cambridge Univ. Press, Cambridge, 1999.
28. N. Tsilevich, *Quantum inverse scattering method for the* q*-boson model and symmetric functions.* — Funct. Anal. Appl. **40**, No. 3 (2006), 207–217.
29. A. Vershik, *Statistical mechanics of combinatorial partitions, and their limit configurations.* — Funct. Anal. Appl., **30** No. 2 (1996), 90–105.

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