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MULTIVARIATE JACOBI POLYNOMIALS AND THE SELBERG INTEGRAL. II

Abstract. The problem of harmonic analysis for infinite-dimensional classical groups and symmetric spaces leads to a family of probability measures with infinite-dimensional support. In the present paper, we construct these measures in a different way, which makes it possible to substantially extend the range of the parameters. The measures that we obtain can be interpreted as the result of formal analytic continuation of the N-dimensional beta distributions which appear in the Selberg integral. Our procedure of analytic continuation, based on Carlson's theorem, turns N into a complex parameter.

§1. Introduction

The multivariate Jacobi polynomials we are dealing with are the Heckman–Opdam polynomials (see Heckman's lecture notes [7]) related to the BC_N root system with formal root multiplicities, $N = 1, 2, \dots$. These are symmetric polynomials in N variables, orthogonal with respect to a weight function on the cube $[-1, 1]^N$ depending on a triple of parameters (a, b, θ), and indexed by the set $\mathcal{V}_N$ of Young diagrams with at most N rows. For some special values of the parameters (in particular, θ has to take one of the values $\frac{1}{2}, 1, 2$), the Jacobi polynomials have a representation-theoretic interpretation: they are the irreducible characters of compact orthogonal or symplectic groups, or else indecomposable spherical functions on certain compact symmetric spaces (see, e.g., Okounkov–Olshanski [15]).

In what follows, the parameters (a, b, θ) are assumed to be fixed. We define, for every $N = 2, 3, \dots$, a stochastic matrix Λ^N_{N-1} of format $\mathcal{V}_N \times \mathcal{V}_{N-1}$, which describes the "branching rule" of the Jacobi polynomials. Next, we define, for every $N = 1, 2, \dots$, a family $\{M_N^{(z,z')}\}$ of probability

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measures on the discrete set $\mathcal{V}_N$ depending on two additional continuous parameters (z, z'). These measures are called the *z-measures* (see Definition 3.1; note that starting from Sec. 2 we slightly change the above notation).

The goal of the paper is to prove Theorem 4.1, which says that for fixed parameters (z, z'), the z-measures form a *coherent family* in the sense that they satisfy the *coherency relation*

$$\sum_{\lambda \in \mathcal{V}_N} M_N^{(z,z')}(\lambda) \Lambda_{N-1}^N(\lambda, \nu) = M_\nu^{(z,z')}, \qquad N = 2, 3 \dots, \quad \nu \in \mathcal{V}_{N-1}. \quad (1)$$

By a general theory (see Olshanski [16]), Theorem 4.1 implies the following result (Theorem 4.2 below): the coherent family

$$\{M_N^{(z,z')} : N = 1, 2, \dots\}$$

determines a probability measure $M_\infty^{(z,z')}$ on an infinite-dimensional space Ω. (That space serves as the *boundary* for the chain of the spaces $\mathcal{V}_N$ linked by the stochastic matrices Λ_{N-1}^N, see Okounkov–Olshanski [15] and Borodin–Olshanski [3].)

One can show that for the special values of the parameters (a, b, θ) that are related to classical groups and symmetric spaces, the measures $M_\infty^{(z,z')}$ have a representation-theoretic meaning: they govern the spectral decomposition of certain unitary representations of infinite-dimensional classical groups. This fact is one of the motivations for the present work.

For those special values of the parameters (a, b, θ), the coherency relation (1) can be obtained, case-by-case, from a representation-theoretic construction developed in Pickrell [19], Neretin [14], and Olshanski [16]. However, in order to work with general parameters, one has to invent other methods.

For the first time, an example of a coherency relation of type (1), with general parameter θ, appeared in Kerov's paper [9]. Then further results were obtained in Borodin–Olshanski [2], Olshanski [17], Olshanski–Osinenko [18], and also (in a somewhat different direction) in the recent papers Gorin–Olshanski [5, 6].

The present paper is an extension of our previous paper [18], where we worked out the particular case $\theta = 1$. This case is exceptional, because then the multivariate Jacobi polynomials can be explicitly expressed, in a simple way, through the classical univariate Jacobi polynomials. For $\theta \neq 1$, such an expression does not exist, and the computational work becomes more cumbersome. Nevertheless, the structure of the proof remains the same, and we tried to closely follow the exposition of [18] in the hope that a comparison with [18] will help the reader to digest tedious computations.

As in [18], our approach relies on the observation that when one of the parameters z, z' is an integer, a degeneration occurs: the support of the measure $M_\infty^{(z,z')}$ becomes a finite-dimensional subset and the measure itself can be identified with the multidimensional beta distribution which serves as the integrand in the Selberg integral. In our proof of the coherency relation (1), we first verify it in the degenerate case with the help of a generalized Selberg integral. Then, to handle the general case, we use a procedure of analytic continuation from integer points to a complex domain via Carlson's theorem.

§2. Preliminaries

The Jacobi polynomials. Throughout the paper, we fix a triple (a, b, θ) of real parameters such that $\theta > 0$ and $a \geqslant b \geqslant -\frac{1}{2}$. (Perhaps, the above condition on (a, b) is redundant and it would be possible to impose weaker constraints, but here we follow [15, Proposition 1.1].)

Let H_N denote the Hilbert space of all functions on the cube $[-1, 1]^N$ that are invariant under the permutations of coordinates and are square integrable with respect to the measure $m_N^{a,b,\theta}$ on $[-1, 1]^N$ which has the density

$$\prod_{i=1}^{N} (1 - x_i)^a (1 + x_i)^b V^{2\theta}(x)$$

relative to the Lebesgue measure, where $V(x)$ stands for the Vandermonde determinant,

$$V(x) = \prod_{1 \leqslant i < j \leqslant N} (x_i - x_j),$$

and $V^{2\theta}(x) := \left((V(x))^2\right)^\theta$.

Let $\mathcal{V}_N$ denote the set of N-tuples of nonincreasing nonnegative integers. The N-variate Jacobi polynomials with parameters (a, b, θ) are symmetric polynomials $\pi_\lambda^{a,b,\theta}(x_1, \ldots, x_N)$ indexed by arbitrary N-tuples $\lambda \in \mathcal{V}_N$; these polynomials are uniquely determined by the property of orthogonality with respect to the inner product in H_N and the triangularity condition

$$\pi_\lambda^{a,b,\theta}(x_1, \ldots, x_N) = x_1^{\lambda_1} \ldots x_N^{\lambda_N} + \ldots,$$

where the dots stand for the sum of lower terms with respect to the lexicographic order on monomials. See, e.g., [7, 10–12].

The stochastic matrices Λ^N_{N-1}. We introduce also the normalized Jacobi polynomials:

$$\Phi_\lambda^{a,b,\theta}(x_1,\dots,x_N)=\frac{\pi_\lambda^{a,b,\theta}(x_1,\dots,x_N)}{\pi_\lambda^{a,b,\theta}(1,\dots,1)}.$$

Set $x_N = 1$ in $\Phi_\lambda^{a,b,\theta}(x_1,\dots,x_N)$ and expand the resulting symmetric polynomial in the basis $\{\Phi_\nu^{a,b,\theta}(x_1,\dots,x_{N-1}),\ \nu\in\mathcal{V}_{N-1}\}$:

$$\Phi_\lambda^{a,b,\theta}(x_1,\dots,x_{N-1},1)=\sum_{\nu\in\mathcal{V}_{N-1}}\Lambda^N_{N-1}(\lambda,\nu)\Phi_\nu^{a,b,\theta}(x_1,\dots,x_{N-1}),$$

where the coefficients $\Lambda^N_{N-1}(\lambda,\nu)$ depend on the parameters (a,b,θ) but we omit them to simplify the notation.

Lemma 2.1. *For every $N = 2, 3, \dots$, the $\mathcal{V}_N\times\mathcal{V}_{N-1}$ matrix Λ^N_{N-1} built from the coefficients $\Lambda^N_{N-1}(\lambda,\nu)$ of the above expansion is stochastic.*

Proof. Setting $x_1=\dots=x_N=1$, we obtain that $\sum\Lambda^N_{N-1}(\lambda,\nu)=1$ for any λ. The nonnegativity of the coefficients $\Lambda^N_{N-1}(\lambda,\nu)$ is a nontrivial fact, which can be derived from the two-step branching rule for the Koornwinder polynomials established in [20, Sec. 5]. For seven special values of parameters corresponding to groups and symmetric spaces (see [15, Sec. 6, Table II]), this fact is obvious. □

Lemma 2.2. *The coefficient $\Lambda^N_{N-1}(\lambda,\nu)$ is nonzero if and only if there exists $\mu\in\mathcal{V}_{N-1}$ such that $\mu\prec\lambda$ and $\nu\prec\mu\cup\{0\}$, that is,*

$$\lambda_1\geqslant\mu_1\geqslant\lambda_2\geqslant\dots\geqslant\lambda_{N-1}\geqslant\mu_{N-1}\geqslant\lambda_N,$$
$$\mu_1\geqslant\nu_1\geqslant\mu_2\geqslant\dots\geqslant\mu_{N-1}\geqslant\nu_{N-1}\geqslant0.$$

Proof. This also can be derived from the results of [20, Sec. 5]. □

The space Ω and the Markov kernels Λ^∞_N. The sequence of sets $\{\mathcal{V}_N\}$ linked by the stochastic matrices Λ^N_{N-1} is a projective chain in the sense of [3, Sec. 2.2], and Theorem 2.4 below describes its boundary (also in the sense of [3, Sec. 2.2]).

To describe the boundary, we need some additional notation. Let $\mathbb{R}_+$ be the set of nonnegative reals, $\mathbb{R}_+^\infty$ be the direct product of countably many copies of $\mathbb{R}_+$, and $\Omega \subset \mathbb{R}_+^\infty \times \mathbb{R}_+^\infty \times \mathbb{R}_+$ be the subset of triples $\omega = (\alpha, \beta, \delta)$ such that

$$\alpha = (\alpha_1 \geqslant \alpha_2 \geqslant \ldots \geqslant 0), \quad \beta = (\beta_1 \geqslant \beta_2 \geqslant \ldots \geqslant 0),$$
$$\beta_1 \leqslant 1, \quad \delta \geqslant 0, \quad \sum \alpha_i + \sum \beta_i \leqslant \delta.$$

Due to the last inequality, Ω is a locally compact space with respect to the product topology of $\mathbb{R}_+^\infty \times \mathbb{R}_+^\infty \times \mathbb{R}_+$.

The following definition is taken from [15]. For $\omega = (\alpha, \beta, \delta) \in \Omega$ we set

$$\gamma = \delta - \sum \alpha_i - \sum \beta_i \geqslant 0$$

and

$$\Psi(x; \omega) := e^{\gamma(x-1)} \prod_{i=1}^{\infty} \frac{1 + \frac{\beta_i(2-\beta_i)}{2}(x-1)}{\left(1 - \frac{\alpha_i(2\theta+\alpha_i)}{2\theta^2}(x-1)\right)^\theta}, \qquad x \in [-1, 1]. \tag{2}$$

Given $N = 1, 2, \ldots$ and $\omega \in \Omega$, the product $\Psi(x_1; \omega) \ldots \Psi(x_N; \omega)$ is a continuous symmetric function on $[-1, 1]^N$, and therefore it can be expanded in the normalized Jacobi polynomials, which form an orthogonal basis in the space H_N. We denote the corresponding coefficients by $\Lambda_N^\infty(\omega, \lambda)$:

$$\Psi(x_1; \omega) \ldots \Psi(x_N; \omega) = \sum_{\lambda \in \mathcal{V}_N} \Lambda_N^\infty(\omega, \lambda) \Phi_\lambda^{a,b,\theta}(x_1, \ldots, x_N). \tag{3}$$

(Note that the coefficients $\Lambda_N^\infty(\omega, \lambda)$ depend on (a, b, θ).)

Taking $x_1 = \ldots x_N = 1$ in (3), we get

$$\sum_{\lambda \in \mathcal{V}_N} \Lambda_N^\infty(\omega, \lambda) = 1, \quad \text{for every} \quad \omega \in \Omega.$$

Next, it follows from the results of [15] that $\Lambda_N^\infty(\omega, \lambda) \geqslant 0$. Thus, for every fixed N the coefficients $\Lambda_N^\infty(\omega, \lambda)$ form a Markov kernel. We denote it by Λ_N^∞.

Coherent families of measures. From the very construction of the kernels Λ_N^∞ it follows that they are consistent with the stochastic matrices Λ_{N-1}^N:

$$\Lambda_N^\infty \Lambda_{N-1}^N = \Lambda_{N-1}^\infty. \tag{4}$$

Definition 2.3. *Let* $\{M_N : N = 1, 2, \dots\}$ *be a family of complex measures with finite variation on the sets* $\mathcal{V}_N$. *We say that the family* $\{M_N\}$ *is* coherent *if* $M_N \Lambda^N_{N-1} = M_{N-1}$ *for every* $N \geqslant 2$. *That is,*

$$\sum_{\lambda \in \mathcal{V}_N} M_N(\lambda)\Lambda^N_{N-1}(\lambda, \nu) = M_{N-1}(\nu), \quad \textit{for every} \quad \nu \in \mathcal{V}_{N-1}. \tag{5}$$

The following theorem is a special case of the results obtained in [15] and [16].

Theorem 2.4. *There exists a one-to-one correspondence* $M_\infty \leftrightarrow \{M_N\}$ *between the probability measures on* Ω *and the coherent families of probability measures, given by the following formula:*

$$M_N(\lambda) = \int_\Omega M_\infty(d\omega)\Lambda^\infty_N(\omega, \lambda), \quad N = 1, 2, \dots, \quad \lambda \in \mathcal{V}_N. \tag{6}$$

We say that M_∞ is the *boundary measure* of the coherent family $\{M_N\}$.

The degenerate case. Given $K = 1, 2, \dots$, we set

$$\mathcal{V}_N(K) = \{\lambda \in \mathcal{V}_N : \lambda_1 \leqslant K\} \subset \mathcal{V}_N.$$

The set $\mathcal{V}_N(K)$ is finite, and its elements can be identified with Young diagrams contained in the rectangular diagram $\square(N, K)$ with N rows and K columns.

Next, we denote by $\Omega(K)$ the subset of Ω formed by those triples (α, β, δ) for which

$$\alpha \equiv 0, \qquad \beta_{K+1} = \beta_{K+2} = \dots = 0, \qquad \delta = \beta_1 + \dots + \beta_K.$$

We identify $\Omega(K)$ with the closed subset of $[0, 1]^N$ consisting of the vectors with real nonincreasing coordinates:

$$\Omega(K) = \{(\beta_1 \geqslant \ldots \geqslant \beta_K) \in [0, 1]^N\}.$$

The proof of the following proposition is exactly the same as in [18].

Proposition 2.5. *If* $\omega \in \Omega(K)$, *then* $\Lambda^\infty_N(\omega, \lambda) = 0$ *for all* $\lambda \in \mathcal{V}_N \setminus \mathcal{V}_N(K)$.

Corollary 2.6. *If* M_∞ *is a probability measure concentrated on* $\Omega(K) \subset \Omega$, *then the corresponding measures* $M_N = M_\infty \Lambda^\infty_N$ *are concentrated on the subsets* $\mathcal{V}_N(K) \subset \mathcal{V}_N$.

Conversely, one may prove that if $\{M_N\}$ is a coherent family of probability measures concentrated on the subsets $\mathcal{V}_N(K)$ for a certain fixed K, then the corresponding boundary measure M_∞ is concentrated on $\Omega(K)$. Such coherent families are called *degenerate*.

§3. z-Measures

We now define the main object of the paper. First, we set

$$D := \{z \in \mathbb{C} : \operatorname{Re} z > -(1+b)/2\}. \tag{7}$$

For $z \in D$, the function $(1+x)^z$ is square integrable with the Jacobi weight $(1-x)^a(1+x)^b$ on $[-1,1]$. Therefore, the function

$$f_{z|N}(x_1,\dots,x_N) := \prod_{i=1}^N (1+x_i)^z, \quad z \in D,$$

lies in H_N.

Definition 3.1. *The* z-measure *on $\mathcal{V}_N$ with parameters $(z,z') \in D \times D$ is the complex measure given by*

$$M_N(\lambda \mid z,z',a,b,\theta) = \frac{(f_{z|N}, \pi_\lambda^{a,b,\theta})(\pi_\lambda^{a,b,\theta}, f_{\bar{z}'|N})}{(f_{z|N}, f_{\bar{z}'|N})\|\pi_\lambda^{a,b,\theta}\|^2}, \qquad \lambda \in \mathcal{V}_N, \tag{8}$$

where $(\cdot,\cdot)$ stands for the inner product in H_N and $\|\cdot\|$ is the corresponding norm.

Note that $(f_{z|N}, f_{\bar{z}'|N}) \neq 0$ (see Lemma 3.8 below). An explicit formula for $M_N(\lambda \mid z,z',a,b,\theta)$ is given below in Proposition 3.9.

From formula (8) it is easy to see that

$$\sum M_N(\lambda \mid z,z',a,b,\theta) = 1.$$

Thus, if the parameters $z,z' \in D$ are such that $M_N(\lambda \mid z,z',a,b,\theta)$ is real and nonnegative for all λ, then $M_N(\lambda \mid z,z',a,b,\theta)$ defines a probability measure. For instance, this is so in the case when z and z' are conjugate to each other.

In what follows, we use the notation

$$\varepsilon = \frac{a+b+1}{2}$$

and

$$1^N = \underbrace{1,\dots,1}_{N}.$$

In the next two lemmas we compute the quantities $\pi_\lambda^{a,b,\theta}(1^N)$ and $\|\pi_\lambda^{a,b,\theta}\|$.

Lemma 3.2. *For $\lambda \in \mathcal{V}_N$ we have*

$$\pi_\lambda^{a,b,\theta}(1^N) = \frac{1}{2^{|\lambda|}} \frac{\Gamma(\lambda_i - \lambda_j + \theta(j-i+1))}{\Gamma(\lambda_i - \lambda_j + \theta(j-i))} \frac{\Gamma(\theta(j-i))}{\Gamma(\theta(j-i+1))}$$
$$\times \frac{\Gamma(\lambda_i + \lambda_j + \theta(2N-j-i+1) + 2\varepsilon)}{\Gamma(\lambda_i + \lambda_j + \theta(2N-j-i) + 2\varepsilon)} \frac{\Gamma(\theta(2N-j-i) + 2\varepsilon)}{\Gamma(\theta(2N-j-i+1) + 2\varepsilon)}$$
$$\times \prod_{i=1}^{N} \frac{\Gamma(2\lambda_i + 2\theta(N-i) + 2a + 1)}{\Gamma(2\lambda_i + 2\theta(N-i) + 2\varepsilon)} \frac{\Gamma(2\theta(N-i) + 2\varepsilon)}{\Gamma(2\theta(N-i) + 2a + 1)}$$
$$\times \prod_{i=1}^{N} \frac{\Gamma(\lambda_i + \theta(N-i) + 2\varepsilon)}{\Gamma(\lambda_i + 2\theta(N-i) + a + \frac{1}{2})} \frac{\Gamma(\theta(N-i) + a + \frac{1}{2})}{\Gamma(\theta(N-i) + 2\varepsilon)}.$$

Proof. This is a particular case of a result due to Opdam, see [7, Theorem 3.6.6] and [15, (2.12)]. □

Lemma 3.3. *For $\lambda \in \mathcal{V}_N$ we have*

$$\|\pi_\lambda^{a,b,\theta}\|^2 = \prod_{1 \leqslant i < j \leqslant N} \frac{\Gamma(\lambda_i - \lambda_j + \theta(j-i-1) + 1)}{\Gamma(\lambda_i - \lambda_j + \theta(j-i) + 1)} \frac{\Gamma(\lambda_i - \lambda_j + \theta(j-i+1))}{\Gamma(\lambda_i - \lambda_j + \theta(j-i))}$$
$$\times \prod_{i=1}^{N} 2^{2\lambda_i + (2N-2i)\theta + 2\varepsilon} \prod_{1 \leqslant i < j \leqslant N} \frac{\Gamma(\lambda_i + \lambda_j + \theta(2N-j-i-1) + 2\varepsilon + 1)}{\Gamma(\lambda_i + \lambda_j + \theta(2N-j-i) + 2\varepsilon + 1)}$$
$$\times \prod_{i=1}^{N} \Gamma(\lambda_i + (N-i)\theta + b + 1) \prod_{1 \leqslant i < j \leqslant N} \frac{\Gamma(\lambda_i + \lambda_j + \theta(2N-j-i+1) + 2\varepsilon)}{\Gamma(\lambda_i + \lambda_j + \theta(2N-j-i) + 2\varepsilon)}$$
$$\times N! \prod_{i=1}^{N} \frac{\Gamma(\lambda_i + (N-i)\theta + 1)\Gamma(\lambda_i + (N-i)\theta + 2\varepsilon + 1)\Gamma(\lambda_i + (N-i)\theta + a + 1)}{\Gamma(2\lambda_i + (2N-2i)\theta + 2\varepsilon + 1)\Gamma(2\lambda_i + (2N-2i)\theta + 2\varepsilon)}.$$

Proof. See [7, Sec. 3.5]. □

Proposition 3.4. *For $\lambda \in \mathcal{V}_N$ and $z \in D$ we have*

$$(f_z, \pi_\lambda^{a,b,\theta}) = N! 2^{|\lambda|+N(z+(N-1)\theta+2\varepsilon)} \prod_{i=1}^{N} \frac{\Gamma(z+1+(i-1)\theta)}{\Gamma(z+1+(i-1)\theta-\lambda_i)}$$
$$\times \prod_{i=1}^{N} \frac{\Gamma(z+(N-i)\theta+b+1)\Gamma(\lambda_i+(N-i)\theta+2\varepsilon)\Gamma(\lambda_i+(N-i)\theta+a+1)}{\Gamma(2\lambda_i+2(N-i)\theta+2\varepsilon)\Gamma(\lambda_i+(2N-1-i)\theta+2\varepsilon+z+1)}$$
$$\times \prod_{1\leqslant i<j\leqslant N} \frac{\Gamma(\lambda_i-\lambda_j+\theta(j-i+1))}{\Gamma(\lambda_i-\lambda_j+\theta(j-i))} \frac{\Gamma(\lambda_i+\lambda_j+\theta(2N-j-i+1)+2\varepsilon)}{\Gamma(\lambda_i+\lambda_j+\theta(2N-j-i)+2\varepsilon)}.$$

Proof. *Step* 1. Let us prove this formula for $z = K = 1, 2, \dots$. We shall use the *dual Cauchy identity* for multivariate Jacobi polynomials, which can be found in [13]. With every $\lambda \in \mathcal{V}_N(K)$ we associate

$$\mu = (N-\lambda'_K, \dots, N-\lambda'_1) \in \mathcal{V}_K(N),$$

where λ' is the diagram conjugate to λ. The dual Cauchy identity says that

$$\prod_{i=1}^{N}\prod_{j=1}^{K}(x_i+y_j) = \sum_{\lambda\subset K^N} \pi_\lambda^{a,b,\theta}(x)\pi_\mu^{\tilde b,\tilde a,\tilde\theta}(y), \tag{9}$$

where

$$\tilde a + 1 = \frac{a+1}{\theta}, \quad \tilde b + 1 = \frac{b+1}{\theta}, \quad \tilde\theta = \frac{1}{\theta}.$$

Taking the inner product with $\pi_\lambda^{a,b,\theta}$ and setting $y = 1^K$, we obtain

$$(f_{K|N}, \pi_\lambda^{a,b,\theta}) = ||\pi_\lambda^{a,b,\theta}||^2 \pi_\mu^{\tilde b,\tilde a,\tilde\theta}(1^K). \tag{10}$$

The explicit expression for $\pi_\mu^{\tilde b,\tilde a,\tilde\theta}(1^K)$ is given by Lemma 3.2, but we need to rewrite it in terms of λ. We shall use the following three lemmas whose proofs will be given a bit later.

Lemma 3.5. *We have*

$$\prod_{j=1}^{K} \frac{\Gamma(2\mu_j+2\tilde\theta(K-j)+2\tilde b+1)}{\Gamma(2\tilde\theta(K-j)+2\tilde b+1)} \frac{\Gamma(\tilde\theta(K-j)+\tilde b+\frac{1}{2})}{\Gamma(\mu_j+\tilde\theta(K-j)+\tilde b+\frac{1}{2})}$$
$$= \frac{2^{2|\mu|}}{\theta^{|\mu|}} \prod_{i=1}^{N} \frac{\Gamma(K+(N-i)\theta+b+1)}{\Gamma(\lambda_i+(N-i)\theta+b+1)}. \tag{11}$$

Lemma 3.6. *We have*

$$\prod_{1\leqslant i<j\leqslant K}\frac{\Gamma(\mu_i-\mu_j+\tilde\theta(j-i+1))}{\Gamma(\mu_i-\mu_j+\tilde\theta(j-i))}\frac{\Gamma(\tilde\theta(j-i))}{\Gamma(\tilde\theta(j-i+1))}$$
$$=\prod_{i=1}^{N}\frac{\Gamma(K+1+(i-1)\theta)}{\Gamma(K+1+(i-1)\theta-\lambda_i)\Gamma(\lambda_i+1+(N-i)\theta)}$$
$$\times\prod_{1\leqslant i<j\leqslant N}\frac{\Gamma(\lambda_i-\lambda_j+1+(j-i)\theta)}{\Gamma(\lambda_i-\lambda_j+1+(j-i-1)\theta)}.$$

Lemma 3.7. *Set* $\tilde\varepsilon=(\tilde b+\tilde a+1)/2$. *We have*

$$\prod_{1\leqslant i<j\leqslant K}\frac{\Gamma(\mu_i+\mu_j+\tilde\theta(2K-j-i+1)+2\tilde\varepsilon)}{\Gamma(\mu_i+\mu_j+\tilde\theta(2K-j-i)+2\tilde\varepsilon)}\frac{\Gamma(\tilde\theta(2K-j-i+1)+2\tilde\varepsilon)}{\Gamma(\tilde\theta(2K-j-i)+2\tilde\varepsilon)}$$
$$\times\prod_{j=1}^{K}\frac{\Gamma(2\tilde\theta(K-j)+2\tilde\varepsilon)}{\Gamma(2\mu_j+2\tilde\theta(K-j)+2\tilde\varepsilon)}\frac{\Gamma(\mu_j+\tilde\theta(K-j)+2\tilde\varepsilon)}{\Gamma(\tilde\theta(K-j)+2\tilde\varepsilon)}$$
$$=\theta^{|\mu|}\prod_{i=1}^{N}\frac{\Gamma(2\lambda_i+(2N-2i)\theta+2\varepsilon+1)}{\Gamma(\lambda_i+(2N-1-i)\theta+2\varepsilon+K+1)}$$
$$\times\prod_{1\leqslant i<j\leqslant N}\frac{\Gamma(\lambda_i+\lambda_j+1+(2N-j-i)\theta+2\varepsilon)}{\Gamma(\lambda_i+\lambda_j+1+(2N-j-i-1)\theta+2\varepsilon)}.$$

Combining Lemmas 3.2, 3.3, 3.5, 3.6, and 3.7 with formula (10), we obtain the desired result for the positive integer values of $z=K=1,2,\dots$.

Step 2. In order to prove the proposition for general values of z, we apply Carlson's theorem, or, rather, its weaker version saying that a function that is holomorphic in a right half-plane $\operatorname{Re} z>\text{const}$, has at most polynomial growth at infinity, and vanishes at the integer points of this half-plane, is identically zero; see, e.g., [1, Theorem 2.8.1].

Let us examine the equality to be proved (see the formulation of Proposition 3.4 above). The difference between the left- and right-hand sides is a holomorphic function in the variable $z\in D$ which vanishes at the positive integer points by virtue of Step 1. It remains to check that this function has at most polynomial growth. But this follows from the the fact that

$$\prod_{i=1}^{N}\frac{\Gamma(z+1+(i-1)\theta)}{\Gamma(z+1+(i-1)\theta-\lambda_i)}\frac{\Gamma(z+(N-i)\theta+b+1)}{\Gamma(\lambda_i+(2N-1-i)\theta+2\varepsilon+z+1)}$$

has at most polynomial growth in D, which, in turn, follows from the well-known asymptotic formula

$$\frac{\Gamma(\mathrm{const}_1+x)}{\Gamma(\mathrm{const}_2+x)} = x^{\mathrm{const}_1-\mathrm{const}_2}(1+O(1/x)), \tag{12}$$

where $x \to \infty$ in a right half-plane.

This completes the proof of the proposition, but we have to return to the three lemmas stated above. □

Proof of Lemma 3.5. By the duplication formula for the gamma function

$$\Gamma(2x) = \frac{\Gamma(x)\Gamma(x+\frac{1}{2})}{\sqrt{\pi}} 2^{2x-1},$$

the left-hand side of (11) equals

$$\prod_{j=1}^{K} \frac{\Gamma(\mu_j+\tilde{\theta}(K-j)+\tilde{b}+1)}{\Gamma(\tilde{\theta}(K-j)+\tilde{b}+1)}.$$

Then (11) follows from the formula

$$\prod_{j=1}^{K} \frac{\Gamma(\mu_j+\tilde{\theta}(K-j)+x)}{\Gamma(\tilde{\theta}(K-j)+x)} = \frac{1}{\theta^{|\mu|}} \prod_{i=1}^{N} \frac{\Gamma(K+(N-i)\theta+\theta x)}{\Gamma(\lambda_i+(N-i)\theta+\theta x)}, \tag{13}$$

which can be proved as follows.

Let us use the Pochhammer symbol

$$(u)_n = \Gamma(u+n)/\Gamma(u), \qquad n=0,1,2,\dots.$$

The left-hand side of (13) is equal to

$$\prod_{j=1}^{K} (\tilde{\theta}(K-j)+x)_{\mu_j}, \tag{14}$$

while the right-hand side is equal to

$$\frac{1}{\theta^{|\mu|}} \prod_{i=1}^{N} (\lambda_i+(N-i)\theta+\theta x)_{K-\lambda_i}. \tag{15}$$

It is easy to see that both (14) and (15) are equal to

$$\prod_{(j,i)\in\mu} (\tilde{\theta}(K-j)+x+i-1) \tag{16}$$

and, therefore, they are equal to each other. Here, in (16), (j,i) means the box in the jth row and ith column of the diagram μ and the product is taken over all boxes of μ. □

Lemma 3.6 can be easily proved by induction on K.

Proof of Lemma 3.7. Performing simple transformations, we obtain that the left-hand side of the equality stated in this lemma equals

$$\left(\prod_{i=1}^{K}\prod_{j=i+1}^{K+1}(\tilde\theta(2K-i-j+1)+2\tilde\varepsilon+1+\mu_i+\mu_j)_{\mu_{j-1}-\mu_j}\right)^{-1}, \tag{17}$$

while the right-hand side equals

$$\left(\prod_{i=1}^{N}\prod_{j=i}^{N}(\theta(2N-i-j)+2\varepsilon+1+\lambda_i+\lambda_j)_{\lambda_{j-1}-\lambda_j}\right)^{-1}. \tag{18}$$

Now, the claim follows from the observation that both (17) and (18) are equal to

$$\prod_{(j,i)\in\mu}((\mu_i+j-1)+(2K-2i-l(i,j))\tilde\theta+2\tilde\varepsilon)^{-1},$$

where $l(i,j)=\mu'_j-i+1$. □

Lemma 3.8. *For $z,z'\in D$ we have*

$$(f_{z|N},f_{\bar z'|N})=N!2^{N(2\varepsilon+z'+z+(N-1)\theta)}$$
$$\times\prod_{i=1}^{N}\frac{\Gamma(1+z+z'+b+(i-1)\theta)\Gamma(a+1+(i-1)\theta)\Gamma(i\theta)}{\Gamma(2\varepsilon+1+z+z'+(N+i-2)\theta)\Gamma(\theta)}.$$

In particular, $(f_{z|N},f_{\bar z'|N})\neq 0$.

Proof. This is an easy consequence of the well-known Selberg integral (see [4]):

$$\int_0^1\cdots\int_0^1\left(\prod_{i=1}^{n}x_i^{\alpha-1}(1-x_i)^{\beta-1}\right)|V(x)|^{2\gamma}\,dx_1\dots dx_n$$
$$=\prod_{j=1}^{n}\frac{\Gamma(\alpha+(j-1)\gamma)\Gamma(\beta+(j-1)\gamma)\Gamma(1+j\gamma)}{\Gamma(\alpha+\beta+(n+j-2)\gamma)\Gamma(1+\gamma)}. \tag{19}$$

□

Proposition 3.9. *The right-hand side of formula* (8) *defining the z-measure is given by the following expression:*

$$\prod_{1\leqslant i<j\leqslant N} \frac{\Gamma(\lambda_i-\lambda_j+1+(j-i)\theta)}{\Gamma(\lambda_i-\lambda_j+1+(j-i-1)\theta)} \frac{\Gamma(\lambda_i+\lambda_j+1+(2N-j-i)\theta+2\varepsilon)}{\Gamma(\lambda_i+\lambda_j+1+(2N-j-i-1)\theta+2\varepsilon)}$$
$$\times \prod_{1\leqslant i<j\leqslant N} \frac{\Gamma(\lambda_i-\lambda_j+\theta(j-i+1))}{\Gamma(\lambda_i-\lambda_j+\theta(j-i))} \frac{\Gamma(\lambda_i+\lambda_j+\theta(2N-j-i+1)+2\varepsilon)}{\Gamma(\lambda_i+\lambda_j+\theta(2N-j-i)+2\varepsilon)}$$
$$\times \prod_{i=1}^{N} \frac{\Gamma(z'+(i-1)\theta+b+1)\Gamma(z'+1+(i-1)\theta)}{\Gamma(z'+1+(i-1)\theta-\lambda_i)\Gamma(\lambda_i+(2N-1-i)\theta+2\varepsilon+z'+1)}$$
$$\times \prod_{i=1}^{N} \frac{(2\lambda_i+(2N-2i)\theta+2\varepsilon)\Gamma(\lambda_i+(N-i)\theta+2\varepsilon)\Gamma(\lambda_i+(N-i)\theta+a+1)}{\Gamma(\lambda_i+(N-i)\theta+b+1)\Gamma(\lambda_i+1+(N-i)\theta)}$$
$$\times \prod_{i=1}^{N} \frac{\Gamma(2\varepsilon+1+z+z'+(N+i-2)\theta)\Gamma(\theta)}{\Gamma(1+z+z'+b+(i-1)\theta)\Gamma(a+1+(i-1)\theta)\Gamma(i\theta)}$$
$$\times \prod_{i=1}^{N} \frac{\Gamma(z+(i-1)\theta+b+1)\Gamma(z+1+(i-1)\theta)}{\Gamma(z+1+(i-1)\theta-\lambda_i)\Gamma(\lambda_i+(2N-1-i)\theta+2\varepsilon+z+1)}.$$

Proof. We combine the results of Lemmas 3.4, 3.8, and 3.3. □

Proposition 3.10. *Assume that*

$$z=K\in\{1,2,\dots\}, \quad z'=K+s,\ s>-1.$$

Then the z-measures are well defined and the whole family is degenerate: for each N, the support of the Nth measure lies in $\mathcal{V}_N(K)$. Moreover, all measures are nonnegative, and, therefore, they are probability measures.

Proof. Both parameters lie in the half-plane D, therefore the proposition readily follows from the explicit formula given in the previous proposition. □

§4. The main results. The beginning of the proof in the degenerate case

Here are the main results of the paper.

Theorem 4.1. *Assume that parameters z and z' lie in the half-plane D. Then the family of z-measures $\{M_N(\lambda \mid z,z',a,b,\theta)\}$ is coherent in the sense of Definition* 2.3.

Theorem 4.2. *Let, as above, the parameters lie in D, and assume additionally that they are such that the z-measures are nonnegative and hence are probability measures (for instance, this happens if $z' = \bar{z}$).*

Then the coherent family $\{M_N(\cdot \mid z, z', a, b, \theta) : N = 1, 2, \dots\}$ determines a probability measure $M_\infty(d\omega \mid z, z', a, b, \theta)$ on Ω – the boundary measure.

Theorem 4.2 is a direct corollary of Theorem 4.1 and the abstract Theorem 2.4. In the rest of the paper, we prove Theorem 4.1.

The idea of the proof is the following. We first examine the case when $z = K \in \{1, 2, \dots\}$ and $z' = K + s$, $s > -1$. Then, by Proposition 3.10, the corresponding z-measures are probability measures supported by the subsets $\mathcal{V}_N(K)$. By Theorem 2.4 and the argument following it, it is enough to find a probability measure M_∞ on the finite-dimensional set $\Omega(K)$ such that

$$M_N(\lambda \mid K, K+s, a, b, \theta) = \int_\Omega M_\infty(d\omega)\Lambda_N^\infty(\omega, \lambda). \tag{20}$$

By the general theory, the boundary measure M_∞ can be found as the limit as $N \to \infty$ of the images of M_N's under the map

$$\mathcal{V}_N \ni \lambda \to \left(\frac{\lambda'_1}{N}, \dots, \frac{\lambda'_K}{N}\right) \in \Omega(K),$$

but the corresponding computation is rather tedious and formally is not needed for the proof. So we only exhibit the final result:

$$M_\infty(d\omega) = \text{const}(V((1-\beta_1)^2, \dots, (1-\beta_K)^2))^{2\tilde{\theta}} \times \prod_{j=1}^K (1-(1-\beta_j)^2)^{(s+1)\tilde{\theta}-1}(1-\beta_j)^{2\tilde{b}+1} d\beta_i.$$

In this section we make equality (20) explicit, and we prove it in the next section, which concludes the proof in the degenerate case. In the last section we use Carlson's theorem to prove Theorem 4.1 for general values of the parameters z and z'.

We proceed with equality (20). It is convenient to make a change of coordinates by setting $t_j = (1-\beta_j)^2$. In these new coordinates,

$$M_\infty(d\omega) = \text{const} \prod_{j=1}^K t_j^{\tilde{b}}(1-t_j)^{(s+1)\tilde{\theta}-1} V^{2\tilde{\theta}}(t) dt_1 \dots, dt_K$$

and the normalization constant can be found from the Selberg integral (19). However, we shall not need the explicit value of this constant.

Proposition 4.3. *For* $\omega = (\beta_1, \dots, \beta_K) \in \Omega(K)$ *and* $\lambda \in \mathcal{V}_N(K)$, *the following relation holds*:

$$\Lambda_N^\infty(\omega, \lambda) = \frac{1}{2^{NK}} \prod_{j=1}^K (1 - t_j)^N \pi_\mu^{\tilde{b},\tilde{a},\tilde{\theta}}(y_1, \dots, y_K) \pi_\lambda^{a,b,\theta}(1^N),$$

where

$$y_j = \frac{1 + (1 - \beta_j)^2}{1 - (1 - \beta_j)^2} = \frac{1 + t_j}{1 - t_j}. \tag{21}$$

Proof. Recall that $\Lambda_N^\infty(\omega, \lambda)$ is the coefficient with index λ in the expansion of $\prod_{i=1}^N \Psi(x_i; \omega)$ in the normalized Jacobi polynomials. For $\omega = (\beta_1, \dots, \beta_K) \in \Omega(K)$, the expression (2) for $\Psi(x; \omega)$ simplifies significantly and we have

$$\prod_{i=1}^N \Psi(x_i; \omega) = \frac{1}{2^{NK}} \prod_{j=1}^K (\beta_j (2 - \beta_j))^N \prod_{i=1}^N \prod_{j=1}^K (x_i + y_j).$$

Thus, the result follows from the dual Cauchy identity (9). □

Now we can rewrite the desired equality (20) as

$$\frac{M_N(\lambda)}{\pi_\lambda^{a,b,\theta}(1^N)} = \frac{\text{const}}{2^{NK}} \int\limits_{[0,1]^K} \prod_{j=1}^K t_j^{\tilde{b}} (1 - t_j)^{(s+1)\tilde{\theta} - 1 + N} V^{2\tilde{\theta}}(t) \pi_\mu^{\tilde{b},\tilde{a},\tilde{\theta}}(y) \, dt.$$

Since both left- and right-hand sides of equality (20), which we have to prove, are probability measures on $\mathcal{V}_N(K)$, we may simplify the computations by ignoring any factors that do not depend on λ. In that case, we replace the equality sign by a tilde. Thus, the last equality can be rewritten as

$$\frac{M_N(\lambda)}{\pi_\lambda^{a,b,\theta}(1^N)} \sim \int\limits_{[0,1]^K} \prod_{j=1}^K t_j^{\tilde{b}} (1 - t_j)^{(s+1)\tilde{\theta} - 1 + N} V^{2\tilde{\theta}}(t) \pi_\mu^{\tilde{b},\tilde{a},\tilde{\theta}}(y) \, dt. \tag{22}$$

§5. The Kadell-type integral and the end of the proof in the degenerate case

In this section we compute the integral on the right-hand side of (22):

$$I := \int\limits_{[0,1]^K} \prod_{j=1}^{K} t_j^{\tilde b}(1-t_j)^{(s+1)\tilde\theta-1+N} V^{2\tilde\theta}(t)\pi_\mu^{\tilde b,\tilde a,\tilde\theta}(y)\,dt.$$

It is similar to the Kadell integral (see [8]) with the difference that a Jack polynomial in Kadell's integral is replaced by a Jacobi polynomial and the arguments of the polynomial are transformed.

Making a change of variables $t \to y$ and performing simple transformations, we obtain

$$I \sim \int\limits_{[1,\infty)^K} \prod_{j=1}^{K} (y_j-1)^{\tilde b}(1+y_j)^{-B} V^{2\tilde\theta}(y)\pi_\mu^{\tilde b,\tilde a,\tilde\theta}(y)\,dy,$$

where

$$B = (2K+s-1)\tilde\theta + N + \tilde b + 1.$$

Note that the integration domain changed from the cube $[0,1]^K$ to $[1,\infty)^K$. The next claim (see [18, Lemma 6.2]) allows us to return to the integration over the cube.

Lemma 5.1. *Let $F(y) = F(y_1,\dots,y_K)$ be an arbitrary polynomial. Then*

$$\int\limits_{[1,\infty)^K} F(y)\prod_{j=1}^{K}(y_j-1)^{\tilde b}(1+y_j)^{-B}dy$$
$$= \left(\frac{\sin(\pi B)}{\sin(\pi(B-b-1))}\right)^K \int\limits_{[-1,1]^K} F(y)\prod_{j=1}^{K}(y_j-1)^{\tilde b}(1+y_j)^{-B}dy. \quad (23)$$

Here the integral on the right-hand side diverges, and it should be understood in the sense of the analytical continuation with respect to B. Applying the lemma with $F(y) = V^{2\tilde\theta}(y)\pi_\mu^{\tilde b,\tilde a,\tilde\theta}(y)$, we obtain that

$$I \sim \int\limits_{[-1,1]^K} \prod_{j=1}^{K} (1-y_j)^{\tilde b}(1+y_j)^{-B} V^{2\tilde\theta}(y)\pi_\mu^{\tilde b,\tilde a,\tilde\theta}(y)\,dy.$$

Note that the integral on the right-hand side is equal to the scalar product $(f_{-B-\tilde{a}}, \pi_\mu^{\tilde{b},\tilde{a},\tilde{\theta}})$. Thus,

$$I \sim (f_{-B-\tilde{a}}, \pi_\mu^{\tilde{b},\tilde{a},\tilde{\theta}}),$$

where the explicit formula for the right-hand side is given in Proposition 3.4 and the analytic continuation with respect to the parameter B is assumed.

Now, (20) can be rewritten as

$$\frac{M_N(\lambda)}{\pi_\lambda^{a,b,\theta}(1^N)} \sim (f_{-B-\tilde{a}}, \pi_\mu^{\tilde{b},\tilde{a},\tilde{\theta}}). \tag{24}$$

The explicit formulas for all quantities in (24) are given by Lemmas 3.9, 3.2 and Proposition 3.4. However, the left-hand side is expressed in terms of λ, while the right-hand side is expressed in terms of μ. In the rest of this section we show how to overcome this difficulty.

We rewrite the left-hand side of (24) using (8) and (10) (which can be applied since $z = K$):

$$\frac{M_N(\lambda)}{\pi_\lambda^{a,b,\theta}(1^N)} = \frac{(f_z, \pi_\lambda^{a,b,\theta})(f_{z'}, \pi_\lambda^{a,b,\theta})}{(f_z, f_{z'})||\pi_\lambda^{a,b,\theta}||^2 \pi_\lambda^{a,b,\theta}(1^N)} = \frac{\pi_\mu^{\tilde{b},\tilde{a},\tilde{\theta}}(1^K)(f_{z'}, \pi_\lambda^{a,b,\theta})}{(f_z, f_{z'})\pi_\lambda^{a,b,\theta}(1^N)}.$$

Since $(f_z, f_{z'})$ does not depend on λ, (24) is equivalent to

$$\frac{(f_{z'}, \pi_\lambda^{a,b,\theta})}{\pi_\lambda^{a,b,\theta}(1^N)} \sim \frac{(f_{-B-\tilde{a}}, \pi_\mu^{\tilde{b},\tilde{a},\tilde{\theta}})}{\pi_\mu^{\tilde{b},\tilde{a},\tilde{\theta}}(1^K)}. \tag{25}$$

Using Proposition 3.4 and Lemma 3.2 and again ignoring factors that do not depend on λ, we can rewrite (25) as

$$\prod_{i=1}^{N} \frac{1}{\Gamma(z' + 1 + (i-1)\theta - \lambda_i)\Gamma(\lambda_i + (2N - 1 - i)\theta + 2\varepsilon + z' + 1)}$$
$$\sim \prod_{j=1}^{K} \frac{1}{\Gamma(-B - \tilde{a}' + 1 + (j-1)\tilde{\theta} - \mu_j)\Gamma(\mu_j + (2K - 1 - j)\tilde{\theta} + 2\tilde{\varepsilon} - B - \tilde{a} + 1)},$$

which is easily verified by making use of (13) and Euler's reflection formula

$$\Gamma(1-z)\Gamma(z) = \frac{\pi}{\sin \pi z}.$$

This concludes the proof of Theorem 4.1 in the degenerate case.

§6. Proof in the general case

In this section we prove that Theorem 4.1 holds for arbitrary z and z' lying in the half-plane D. We do it in two steps.

Step 1. Let us fix $z = K$, N, and $\nu \in \mathcal{V}_{N-1}$. For $\lambda \in \mathcal{V}_N$, we write $\lambda \sqsupset \nu$ if λ and ν satisfy the condition of Lemma 2.2 expressing the fact that $\Lambda^N_{N-1}(\lambda, \nu) \neq 0$. Then we may write the coherency relation as

$$M_{N-1}(\nu \mid z, z', a, b, \theta) - \sum_{\lambda \in \mathcal{V}_N \colon \lambda \sqsupset \nu} M_N(\lambda \mid z, z', a, b, \theta) \Lambda^N_{N-1}(\lambda, \nu) = 0. \tag{26}$$

Since $z = K$, the measure M_N is degenerate and hence the sum on the left-hand side is finite. From the explicit formula for the z-measure given in Proposition 3.9 it is seen that $M_N(\lambda \mid z, z', a, b, \theta)$ is a rational function of the parameter z': indeed, the part of the formula that depends on z' can be represented as a product of factors of the form $\Gamma(z' + x)/\Gamma(z' + y)$, where $x - y$ is an integer.

Therefore, the left-hand side of (26) is a rational function, and it vanishes on the ray $z' = K + s$, $s > -1$, as was shown in the previous section. Hence it is identically zero.

Step 2. Conversely, let us fix an arbitrary $z' \in D$ and consider the left-hand side of (26) as a function $G(z)$ of the complex parameter z.

The function $G(z)$ is holomorphic in the half-plane D. Indeed, $z \to f_{z|N}$ is a holomorphic vector function with values in the Hilbert space H_N. This and formula (8) defining z-measures imply that the weight of any diagram is a holomorphic function in z and the series in (26) converges absolutely and uniformly on compact sets with respect to the parameter $z \in D$. Hence $G(z)$ is holomorphic.

Since $\lambda \sqsupset \nu$ and ν is fixed, all coordinates λ_i in (26) with $i > 2$ vary in a finite range, so we can assume them to be fixed. We shall prove that for $\operatorname{Re} z \gg 0$, the quantity $|M_N(\lambda \mid z, z', a, b, \theta)|$ is bounded from above by a polynomial in z multiplied by $((\lambda_1 + 1)(\lambda_2 + 1))^{-2}$, which will imply that $G(z)$ has at most polynomial growth at infinity, which, in turn, will imply that $G(z)$ is identically zero by Carlson's theorem.

Let us examine the formula for $M_N(\lambda \mid z, z', a, b, \theta)$ given in Proposition 3.9. The product of the first four lines does not depend on z and has at most polynomial growth in (λ_1, λ_2) because of the asymptotic relation (12). The product in the fifth line does not depend on λ and has at most polynomial growth in z.

Let us turn to the product in the last, sixth, line, which we split into two parts:

$$\prod_{i=3}^{N} \frac{\Gamma(z+(i-1)\theta+b+1)\Gamma(z+1+(i-1)\theta)}{\Gamma(z+1+(i-1)\theta-\lambda_i)\Gamma(\lambda_i+(2N-1-i)\theta+2\varepsilon+z+1)}$$
$$\times \prod_{i=1}^{2} \frac{\Gamma(z+(i-1)\theta+b+1)\Gamma(z+1+(i-1)\theta)}{\Gamma(z+1+(i-1)\theta-\lambda_i)\Gamma(\lambda_i+(2N-1-i)\theta+2\varepsilon+z+1)}.$$

The first product, corresponding to $i=3,\dots,N$, does not depend on (λ_1,λ_2) and has at most polynomial growth in z.

It remains to handle the product over $i=1,2$, and here we apply the same argument as in [18, Sec. 4, Step 3]. We obtain that this expression is bounded from above by a polynomial in z times $((\lambda_1+1)(\lambda_2+1))^{-p}$, where p is a positive integer which can be chosen arbitrarily large in advance. Taking p large enough, we compensate the possible polynomial growth in (λ_1,λ_2) of the expressions in the first four lines. Thus, we obtain the desired estimate.

This completes the proof of Theorem 4.1 in the general case.

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