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A VARIANT OF THE LEVINE–MOREL MOVING LEMMA

Abstract. We consider a version of the lemma proved by Levine–Morel in their book "Algebraic cobordisms". Being reformulated in the Chow group context the lemma turns out to be valid in any characteristic and its proof is substantially shortened.

Dedicated to A.V.Yakovlev on his 75th birthday

In this note we present a rather short essentially self-contained proof of a Chow version of Proposition [2, Prop. 3.3.1]. The validity of the result in any positive characteristic was used in the paper [4] in order to extend the main result of [3] to odd characteristics.

Theorem 1. *Suppose that k is an infinite field of any characteristic. Let W be a k-smooth scheme and $\mathrm{Ch}(W)$ be its Chow group, let $i : Z \hookrightarrow W$ be a k-smooth closed subscheme. Let Y be a k-smooth irreducible variety and $f : Y \to W$ be a projective morphism. Then there are irreducible k-smooth varieties $Y_1, Y_2, \ldots, Y_n$, projective k-morphisms $f_j : Y_j \to W$ $(j = 1, 2, \ldots, n)$ and integers $r_1, r_2, \ldots, r_n$ such that*

- *for each j morphisms f_i and $i : Z \hookrightarrow W$ are transverse;*
- *one has an equality $\sum r_j f_{j,*}([Y_j]) = f_*([Y])$ for pushforward morphims landing in $\mathrm{Ch}(W)$.*

The original lemma [2, Prop. 3.3.1] referred to the algebraic cobordisms context, therefore it was explicitly stated for characteristic zero case. The Chow variant of this lemma is valid in any characteristic. The outline of the proof is the same as before. But the proof of our variant of the moving lemma seems to be much more transparent and short due to essential simplifications, since we prove less.

Choose a closed subset $C \subset Y$ such that the morphisms $Y - C \to W$ and $Z \hookrightarrow W$ are transverse and try to reduce the dimension of C. Thus,

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the proof will proceed by induction of $\dim C$. The Claim 2 below describes the inductive step. Theorem 1 follows immediately from Claim 2.

Claim 2. *Under the hypotheses of Theorem* 1 *let* $C \subset Y$ *be a closed subscheme such that the morphism* $f|_{Y-C} : Y - C \to W$ *is transverse to the embedding* $Z \hookrightarrow W$. *Then there exist* k*-smooth varieties* Y_1, Y_2, *projective morphisms* $f_j : Y_j \to W$ *and open embeddings* $j_i : Y_i - C_i \subset Y_i$ $(j = 1, 2)$ *such that*

- *for each* i *morphisms* $f_i \circ j_i : Y_i - C_i \to W$ *and inclusion* $Z \hookrightarrow W$ *are transverse;*
- $\dim C_i < \dim C$ *for each* i*;*
- *one has* $f_*([Y]) = f_{1,*}([Y_1]) - f_{2,*}([Y_2])$ *in* $\mathrm{Ch}(W)$.

Proof. We subdivide the proof of the Claim into three steps.

Step 1. We may assume that the variety Y is a closed subvariety of W due to the following transversality lemma, which is a straightforward consequence of [4, Lemma 2.2].

Lemma 3. *Under conditions of the above Claim take projection* $p : W \times \mathbf{P}^n \to W$ *and a closed embedding* $g : Y \hookrightarrow W \times \mathbf{P}^n$ *such that* $f = p \circ g$. *Then* $Y - C$ *and* $Z \times \mathbf{P}^n$ *are transversal subvarieties of* $W \times \mathbf{P}^n$. *If a closed embedding* $g : Y \hookrightarrow W \times \mathbf{P}^n$ *of smooth varieties is such that* $Y - C$ *is transverse to the embedding* $Z \times \mathbf{P}^n \hookrightarrow W \times \mathbf{P}^n$ *then the map* $f = p \circ g : Y \to W$ *is such that* $f|_{Y-C} : Y - C \to W$ *is transverse to the embedding* $Z \hookrightarrow W$.

Replacing W by $W \times \mathbf{P}^n$ and Z by $Z \times \mathbf{P}^n$ we may now assume that Y is a smooth closed subvariety of W. In this case the set C is contained in $Y \cap Z$.

Step 2. In this step we form a commutative diagram of smooth irreducible projective varieties

$$\begin{array}{ccc} \widetilde{Y} & \xrightarrow{\ i\ } & T \\ {\scriptstyle \pi_Y}\downarrow & & \downarrow{\scriptstyle p} \\ Y & \xrightarrow{\ j\ } & W \end{array} \qquad (1)$$

with projective morphisms π_Y and p such that $(\pi_Y)_*[\widetilde{Y}] = [Y] \in \mathrm{Ch}(W)$, $\widetilde{Y} \subset T$ is a smooth divisor, the morphism $p : T - (\pi_Y)^{-1}(C) \to W$ transverse and finally $\dim(\pi_Y)^{-1}(C) = \dim C$.

Take a very ample divisor $D \subset W$. Denote by $\pi : \widetilde{W} \to W$ the blow up of W in Y and denote by $E \subset \widetilde{W}$ the exceptional divisor. Then for a large m the divisor $mD - E$ is very ample on $\widetilde{W}$. Take the intersection T of $\dim W - \dim Y - 1$ hyperplane sections in sufficiently general position in the complete linear system $mD - E$. By the Bertini type theorem [1, Thm.2.1] we can choose T such that:

(1) T is a smooth irreducible subvariety of $\widetilde{W}$;
(2) T and E are transverse to each other and the variety $T \cap E$ is irreducible;
(3) $\dim \pi^{-1}(C) \cap T = \dim C$ (in fact, $\dim(\pi^{-1}(C)) = \dim C + (\dim W - \dim Y - 1)$);

To explain one more condition recall that $Z - C \subset W$ and $Y \subset W$ are transverse in W. It follows that the inverse image $\pi^{-1}(Z - C)$ coincides with the strict transform of $Z - C$ under the blowing up $\pi : \widetilde{W - C} \to W - C$ of $W - C$ in the smooth center $Y - C$. Note that $\pi^{-1}(Z - C)$ is smooth being isomorphic to the blowing up $\widetilde{Z - C}$ in smooth center $(Y \cap Z) - C$. The fourth condition on T is this:

(4) $T \subset \widetilde{W}$ *must be transverse* to the smooth locally closed subscheme $\widetilde{Z - C} \subset \widetilde{W}$.

Since $C \subset Y$ one has $\pi^{-1}(C) \cap T = \pi_Y^{-1}(C)$. Thus $\dim \pi_Y^{-1}(C) = \dim C$.

Lemma 4. *If T satisfy the first and the fourth conditions, then $Z \hookrightarrow W$ is transverse to the composition $T - \pi_Y^{-1}(C) \hookrightarrow \widetilde{W} \xrightarrow{\pi} W$.*

Proof. Consider a commutative diagram:

$$\begin{array}{ccccccc}
(\widetilde{Z - C}) \cap T & \longrightarrow & \widetilde{Z - C} & \longrightarrow & Z - C & \longrightarrow & Z \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
T - \pi_Y^{-1}(C) & \longrightarrow & \widetilde{W} - \pi^{-1}(C) & \longrightarrow & W - C & \longrightarrow & W
\end{array}$$

Clearly, $\widetilde{W - C} = \widetilde{W} - \pi^{-1}(C)$. The left-hand side square is transverse due to the fourth condition on T. The middle square is transverse due to transversality of $Z - C$ and Y in W. Transversality of a right-hand square is obvious. Therefore the ambient rectangle is transverse as well by Lemma [4, Lemma 2.2]. The Lemma follows. □

Set $\widetilde{Y} := T \cap E \subset T$. By the second condition (2) the variety $\widetilde{Y}$ is a smooth irreducible divisor on smooth T. The restriction of π to $\widetilde{Y}$ defines a morphism $\pi_Y : \widetilde{Y} \to Y$.

Lemma 5. *The morphism $\pi_Y : \widetilde{Y} \to Y$ is dominant of degree* 1. *In particular, one has $(\pi_Y)_*[\widetilde{Y}] = [Y] \in \mathrm{Ch}(W)$.*

Proof. The restriction of the $\pi_Y : \widetilde{Y} \to Y$ is dominant by dimensional reasons. For each closed point $y \in Y$ the fibre $\pi^{-1}(y)$ coincides with the projective space $\mathbf{P}^r_{k(y)}$, where $r = \dim W - \dim Y - 1$. The restriction of the sheaf $\mathcal{O}(mD - E)$ to $\pi^{-1}(y)$ coincides with $\mathcal{O}_{\mathbf{P}^r_{k(y)}}(1)$ under the identification of $\pi^{-1}(y)$ with $\mathbf{P}^r_{k(y)}$. The variety T is an intersection of r divisors of the linear system $mD - E$. Thus $T \cap \pi^{-1}(y) \neq \varnothing$. It follows that the morphism π_Y is dominant. By the condition (2) on T dimensions of $\widetilde{Y}$ and Y are the same. Thus the function field extension $k(\widetilde{Y})/k(Y)$ is finite. It follows that there exists a non-empty open $U \subset Y$ such that for each closed point $y \in U$ the scheme intersection $T \cap \pi^{-1}(y)$ is a finite k-scheme. The variety T is an intersection of r divisors of the linear system $mD - E$ and the restriction of the sheaf $\mathcal{O}(mD - E)$ to $\pi^{-1}(y)$ coincides with $\mathcal{O}_{\mathbf{P}^r_{k(y)}}(1)$ under the identification of $\pi^{-1}(y)$ with $\mathbf{P}^r_{k(y)}$. Thus, for each closed point $y \in U$ the scheme $T \cap \pi^{-1}(y)$ coincides with $\mathrm{Spec}(k(y))$. It follows that the degree $[k(\widetilde{Y}) : k(Y)] = 1$. Whence the lemma. □

Let $i : \widetilde{Y} \hookrightarrow T$, $j : Y \hookrightarrow W$ be the inclusions, p be the composition $T \hookrightarrow \widetilde{W} \xrightarrow{\pi} W$ and π_Y be as above. Clearly, we get a commutative diagram of the form (1), which satisfies the properties mentioned just below the diagram (1).

Step 3. Represent the class $[\widetilde{Y}]$ in $\mathrm{Ch}(T)$ as the difference $[Y_1] - [Y_2]$ of two very ample divisor classes. We may choose effective representatives Y_i $(i = 1, 2)$ such that:

- Y_i is smooth;
- $Y_i \subset T$ is transverse to the smooth subscheme $(\widetilde{Z - C}) \cap T \subset T$;
- $\dim(Y_i \cap \widetilde{C}) < \dim \widetilde{C}$.

Let $f_i : Y_i \to W$ be the composition $Y_i \hookrightarrow T \xrightarrow{p} W$. Denote $Y_i \cap \widetilde{C}$ by C_i. Then $f_i \circ j_i : Y_i - C_i \to W$ is transverse to the embedding $Z \hookrightarrow W$ by Lemma 4, the second condition on Y_i and [4, Lemma 2.2]. The second condition required by the Claim 2 is provided by the third condition on Y_i.

The third requirement from the Claim 2 is satisfied by Lemma 5, since $[\widetilde{Y}] = [Y_1] - [Y_2]$ in $\mathrm{Ch}(T)$ and the diagram (1) commutes. □

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