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DEGENERATELY INTEGRABLE SYSTEMS

Abstract. This is a short survey of degenerate integrability is Hamiltonian mechanics. The first section contains a short description of degenerately integrable systems. It is followed by a number of examples which include spin Calogero model, Casimir models, integrable models on symplectic leaves of Poisson Lie groups and some others. Bibliography: 27 titles.

Dedicated to P.P. Kulish
on the occasion of his 75 birthday

Introduction

Degenerately integrable systems are also known as superintegrable systems and as noncommutative integrable systems. We will use the term "degenerate integrability" to avoid possible confusion with supermanifolds, Lie superalgebras and supergeometry.

Degenerate integrability generalizes well known Liouville integrability of Hamiltonian systems on a $2n$-dimensional symplectic manifold to the case when the dimension of invariant tori is $k < n$. When $k = n$ we have to the usual Liouville integrability. This notion in its modern form, and the term, were first introduced in [18]. Then a series of examples related to Lie groups was found in [10]. First examples were known much earlier, see for example [8, 11, 21, 22].

The first section is a short introduction to degenerate integrability. Then we give few examples of degenerately integrable systems. First example is the Kepler system, which is also the classical counterpart of the hydrogen atom. Its degenerate integrability can be traced back to [8, 21, 22]. The next example is Casimir integrable systems. They can be regarded as degenerations of Gaudin models. These examples are important for understanding semiclassical asymptotic of q-6j symbols for simple Lie algebras. In section 3 spin Calogero–Moser systems, rational spin Ruijsenaars system and the duality between these systems is described. This section is a

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concise version of [24]. Spin generalization of Calogero–Moser system was first found in [13]. The better title of this section would be spin Calogero–Moser–Sutherland–Olshanetsky–Perelomov systems [3, 17, 20, 26]. For Liouville integrability of spin Calogero–Moser systems see [15, 16]. For the duality for non-spin case see [4, 7, 18]. For further discussion of duality in the spin case see [2]. Further generalization of Calogero–Moser systems suggested in [5, 6]. Section 5 contains the proof of degenerate integrability of relativistic spin Calogero–Moser systems, of the relativistic spin Ruijsenaars system and the duality between them. Results from this sections seem to be new. The last sections describes the degenerate integrability of Toda type systems on symplectic leaves of simple Poisson Lie groups with standard Poisson Lie structure. It is based on [24]. The proof of degenerate integrability in the linearized case was done in [12].

The degenerate integrability of Calogero–Moser system which is discussed in [27] has somewhat different nature.

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§1. Degenerate Integrable systems

1.1. Degenerate Integrable systems. An integrable system on a $2n$ dimensional symplectic manifold is called *degenerate* if all the invariant submanifolds have dimension $k < n$. The nondegenerate case $k = n$ corresponds to the usual Liouville integrability (non-degenerate case). Abusing the language we will assume $k \leqslant n$ and will treat $k = n$ as a particular case of degenerate integrable systems.

Definition 1. *A degenerate integrable system on a symplectic manifold $(\mathcal{M}_{2n}, \omega)$ consists of a Poisson subalgebra $C_J(\mathcal{M}_{2n})$ in $C(\mathcal{M}_{2n})$ of rank $2n - k$ which has a Poisson center $C_I(\mathcal{M}_{2n})$ of rank k.*

A Hamiltonian dynamics generated by the function $H \in C(\mathcal{M})$ is said to be degenerately integrable if $H \in C_I(M)$. If $J_1, \dots, J_{2n-k}$ are independent functions from $C_J(\mathcal{M})$, we have

$$\{H, J_i\} = 0, \quad i = 1, \dots, 2n - k.$$

In other words, functions J_i are integrals of motion for H. One can say that Hamiltonian fields generated by J_i describe the symmetry of the Hamiltonian flow generated by H. In this sense, functions from $C_I(\mathcal{M}_{2n})$ are natural to call (Poisson commuting) Hamiltonians, while functions $C_J(\mathcal{M}_{2n})$ with be called integrals of motion for Hamiltonians.

The level surface $\mathcal{M}(c_1, \dots, c_{2n-k}) = \{x \in \mathcal{M} | J_i(x) = c_i\}$ of functions J_i is called generic relative to $C_I(\mathcal{M}_{2n})$ if for n independent functions $I_1, \dots, I_k \in C_I(\mathcal{M}_{2n})$ the form $dI_1 \wedge \dots \wedge dI_k$ does not vanish identically on it. Then the following holds [18]:

Theorem 1. (1) *Flow lines of any $H \in C_I(\mathcal{M}_{2n})$ are parallel to level surfaces of J_i.*
(2) *Each connected component of a generic level surface has canonical affine structure generated by the flow lines of $I_1, \dots, I_k$.*
(3) *The flow lines of H are linear in this affine structure.*

When $k = n$ this theorem reduces to the Liouville integrability. As a consequence, each generic level surface is isomorphic to $\mathbb{R}^l \times (S^1)^{k-l}$ for some $0 \leqslant l \leqslant k$.

The notion of degenerate integrability has a simple semiclassical meaning. In the Liouville integrable systems when there are n Poisson commuting integrals on a $2n$ dimensional symplectic manifold the semiclassical spectrum of quantum integrals is either non-degenerate or has stable degeneracy which is determined by the number of connected components of fibers in the Lagrangian fibration given by Hamiltonians.

In degenerate integrable systems the semiclassical spectrum of quantized commuting integrals I_i is expected to be degenerate with the multiplicity $h^{n-k} vol(p^{-1}(b))(1 + O(h))$. Quantization of the Poisson algebra generated by J_i gives the associative algebra, which describes the symmetry of the joint spectrum of quantum integrals.

Geometrically, a degenerate integrable system consists of two Poisson projections

$$\mathcal{M}_{2n} \xrightarrow{\pi} P_{2n-k} \xrightarrow{p} B_k \tag{1}$$

where P_{2n-k} and B_k are Poisson manifolds and B_k has trivial Poisson structure. In the algebraic setting P_{2n-k} is the spectrum (of primitive ideals) of $C_J(\mathcal{M})$ and B_k is the spectrum of $C_I(\mathcal{M})$. Fibers of p are (possibly disjoint unions of) symplectic leaves of P.

One should emphasize that degenerate integrability is a special structure which is stronger then Liouville integrability: invariant tori now have dimension $k < n$. In the extreme case $k = 1$ all trajectories are periodic. A degenerately integrable system may also be Liouville integrable, but degenerate integrability carries more information.

The projection $p \circ \pi : \mathcal{M} \to B_k$ defines the mapping of tangent bundles $d(p \circ \pi) : T\mathcal{M} \to TB_k$. This gives the distribution

$$D_B = \omega^{-1}(\ker(d(p \circ \pi))^{\perp}) \subset T\mathcal{M}$$

where the symplectic form ω is regarded as an isomorphism $T\mathcal{M} \simeq T^*\mathcal{M}$ and of $\ker(d(p \circ \pi))^{\perp} \subset T^*\mathcal{M}$ is the subbundle orthogonal to $\ker(d(p \circ \pi)) \subset T\mathcal{M}$.

Proposition 1. *Leaf of D_B through $x \in \mathcal{M}$ coincides with $\pi^{-1}(\pi(x))$.*

We will say that two degenerate integrable systems $(\mathcal{M}, P, B)$ and $(\mathcal{M}', P', B')$ are *spectrally equivalent* if there is a collection of mappings

- $\phi : \mathcal{M} \to \mathcal{M}'$, a mapping of Poisson manifolds,
- $\phi_1 : P \to P'$, a mapping of Poisson manifolds,
- $\phi_2 : B \simeq B'$, a diffeomorphism.

such that the following diagram is commutative

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{\phi} & \mathcal{M}' \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi'} \\ P & \xrightarrow{\phi_1} & P \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p'} \\ B & \xrightarrow{\phi_2} & B' \end{array}$$

Note that the mappings ϕ and ϕ_1 may not be diffeomorphisms. If they are diffeomorphisms then the systems are called equivalent or diffeomorphic degenerately integrable systems.

1.2. Action-angle variables. Degenerate integrable systems admit action-angle variables, see [18].

For a generic point $c \in P_{2n-k}$ the level surface $\pi^{-1}(c)$ admits angles coordinates φ_i. This is an affine coordinate system generated by by flow lines of Hamiltonian vector fields of integrals $I_1, \dots I_k$ [18]. In a tubular neighborhood of $p^{-1}(c)$ the symplectic form ω on $\mathcal{M}$ can be written as

$$\omega = \omega_c + \sum_{i=1}^{k} d\varphi_i \wedge dI_i,$$

where ω_c is the symplectic leave through c in P_{2n-k}.

1.3. Kepler system. In this case the phase space is $M = \mathbb{R}^6$ with coordinates, $p_i, q_i, i = 1, 2, 3$ and with symplectic form

$$\omega = \sum_{i=1}^{3} dp_i \wedge dq^i$$

The Hamiltonian is

$$H = \frac{1}{2}p^2 - \frac{\gamma}{|q|}$$

The non-commutative Poisson algebra of integrals is generated by momenta M_i and components of the Lenz vector A_i:

$$M_1 = p_2q^3 - p_3q^2, \quad M_2 = p_3q^1 - p_1q^3, \quad M_3 = p_1q^2 - p_2q^1$$

$$A_1 = p_2M_3 - p_3M_2 + \gamma\frac{q^1}{|q|}, \quad A_2 = p_3M_1 - p_1M_3 + \gamma\frac{q^2}{|q|},$$

$$A_3 = p_1M_2 - p_2M_1 + \gamma\frac{q^3}{|q|}$$

In vector notations $M = p \times q$ and $A = p \times M + \gamma\frac{q}{|q|}$. Components of M and A have the following Poisson brackets:

$$\{M_i, M_j\} = \varepsilon_{ijk}M_k, \quad \{M_i, A_j\} = \varepsilon_{ijk}A_k, \quad \{A_i, A_j\} = -2H\varepsilon_{ijk}M_k \quad (2)$$

$$\{H, M_i\} = \{H, A_i\} = 0$$

The momentum vector M and the Lenz vector A satisfy extra relations

$$(M, A) = 0, \quad (A, A) = \gamma^2 + 2(M, M)H \quad (3)$$

Denote by P_5 the 5-dimensional Poisson manifold which is a real affine algebraic submanifold in $\mathbb{R}^7$ with coordinates M_i, A_i, H defined by relations (3) and with Poisson brackets (2).

Formulae for M, A, and H in terms of p and q coordinates describe the Poisson projection $\mathbb{R}^6 \to P_5$. The following describes level surfaces of H is P_5.

The level surface $H = E < 0$ is the coadjoint orbit $O_{-E} \subset so(4)^*$. This orbit is isomorphic to $S^2 \times S^2$ where each S^2 has radius $\gamma/\sqrt{2|E|}$ and $S^2 \times S^2$ is naturally embedded into $so(3)^* \times so(3)^* \simeq \mathbb{R}^3 \times \mathbb{R}^3$. We used the natural isomorphism $so(4)^* \simeq so(3)^* \times so(3)^*$ where left and right $so(3)^*$ components are given by $L_i = M_i - \frac{A_i}{\sqrt{2|E|}}$ and $R_i = M_i - \frac{A_i}{\sqrt{2|E|}}$.

The level surface $H = 0$ is coadjoint orbit in $e(3)^*$ which is isomorphic to TS^2 and the sphere has radius γ, $(A, A) = \gamma^2$.

The level surface $H = E > 0$ is the hyperboloid O_E which is the coadjoint orbit in $so(3,1)^*$ with natural coordinates M and $B = \frac{A}{\sqrt{2E}}$ and with Casimir functions $(M,B) = 0$ and $(B,B) - (M,M) = \gamma^2$.

All of these level surfaces are symplectic manifolds and we just described symplectic leaves of the Poisson manifold P_5.

This structure correspond to the following sequence of Poisson maps:

$$\mathbb{R}^6 \to P_5 \to \mathbb{R}$$

where

$$P_5 \simeq \sqcup_{E<0} S^2 \times S^2 \sqcup_{E=0} TS^2 \sqcup_{E>0} O_E \tag{4}$$

The first projection is the map $(p,q) \to (M(p,q), A(p,q), H(p,q)$ and second one projects P_5 to the E-axis.

§2. Casimir integrable systems

2.0.1. *Casimir integrable systems.* Here we describe complex algebraic version the Casimir system. In this section G is a complex algebraic group and $\mathfrak{g}$ is it Lie algebra. The phase space of the Casimir system is the Hamiltonian reduction of the product of coadjoint orbits $\mathcal{O}_1 \times \cdots \times \mathcal{O}_n$

$$\mathcal{M}_{\mathcal{O}_1,\ldots,\mathcal{O}_n} = \{(x_1,\ldots,x_n) \in \mathcal{O}_1 \times \cdots \times \mathcal{O}_n | x_1 + \cdots + x_n = 0\}/G$$

Here we assume that each orbit is regular (passes through a regular element of $\mathfrak{h}^*$).

The coadjoint action of the Lie group G on $\mathfrak{g}^*$ is Hamiltonian. The moment map $\mathcal{O}_1 \times \cdots \times \mathcal{O}_n \to \mathfrak{g}^*$ for the diagonal action of G on $\mathcal{O}_1 \times \cdots \times \mathcal{O}_n$ acts is

$$(x_1,\ldots,x_n) \mapsto x_1 + \cdots + x_n$$

It is G-invariant, therefore we have a natural map of Poisson manifolds

$$\mu : \widetilde{\mathcal{M}}_{\mathcal{O}_1\times\cdots\times\mathcal{O}_n} = (\mathcal{O}_1 \times \cdots \times \mathcal{O}_n)/G \to \mathfrak{g}^*/\mathrm{Ad}^*_G$$

Here the quotient space is the GIT quotient. The Hamiltonian reduction gives symplectic leaves of Poisson manifold $\widetilde{\mathcal{M}}_{\mathcal{O}_1\times\cdots\times\mathcal{O}_n}$:

$$\mathcal{M}_{\mathcal{O}_1\times\cdots\times\mathcal{O}_n|\mathcal{O}_{n+1}} = \mu^{-1}(\mathcal{O}_{n+1})$$

We have natural symplectomorphisms:

$$\mathcal{M}_{\mathcal{O}_1\times\cdots\times\mathcal{O}_n|\mathcal{O}_{n+1}} \simeq \mathcal{M}_{\mathcal{O}_1\times\cdots\times\mathcal{O}_n,-\mathcal{O}_{n+1}}$$

and $\mathcal{M}_{\mathcal{O}_1,\ldots,\mathcal{O}_n} = \mathcal{M}_{\mathcal{O}_1,\ldots,\mathcal{O}_n|\{0\}}$.

Define the Poisson manifold $\mathcal{P}_{IJ}$ as the fibered product[1].

$$\mathcal{P}_{I,J} = \widetilde{\mathcal{M}}_{\mathcal{O}_{i_1},\dots,\mathcal{O}_{i_k}} \widetilde{\times}_{\mathfrak{g}^*/G} \widetilde{\mathcal{M}}_{\mathcal{O}_{i_1},\dots,\mathcal{O}_{i_k}}$$

where (I, J) is a partition of $(1, \dots, n)$ as above and the twist is $x \mapsto -x$. The following Poisson maps define the Casimir integrable system in the complex algebraic setting:

$$\mathcal{M}_{\mathcal{O}_1,\dots,\mathcal{O}_n} \to \mathcal{P}_{I,J} \to \mathcal{B}_{I,J}\mathfrak{g}^*/\operatorname{Ad}^*_G$$

where $B_{I,J}$ is the image of the last map and the maps are

$$\operatorname{Ad}^*_G(x_1, \dots, x_n) \mapsto (\operatorname{Ad}^*_G(x_{i_1}, \dots, x_{i_k}), \operatorname{Ad}^*_G(x_{j_1}, \dots, x_{j_{n-k}})) \mapsto \\ \operatorname{Ad}^*_G(x_{i_1} + \dots + x_{i_k}) = \operatorname{Ad}^*_G(x_{j_1} - \dots - x_{j_{n-k}})$$

The variety $B_{I,J}$ has dimension r but it is, "generically" smaller then $\mathfrak{g}^*/\operatorname{Ad}^*_G$.

2.1. "Relativistic" Casimir systems. We will keep the same data as in the previous sections. Let $\mathcal{C}_i \subset G$ be conjugation orbits, $i = 1, \dots, n$. The moduli space of flat G-connections on a sphere with n punctures is a Poisson manifold with the Atiyah–Bott Poisson structure. Assigning conjugacy classes to punctures fixes a symplectic leaf of this Poisson manifold:

$$\mathcal{M}_{\mathcal{C}_1,\dots,\mathcal{C}_n} = \{(g_1, \dots, g_n) \in \mathcal{C}_1 \times \dots \times \mathcal{C}_n | g_1 \dots g_n = 1\}/G$$

where G acts on the Cartesian product by diagonal conjugations. The Poisson structure on the moduli space itself, i.e. on $\mathcal{M} = \{(g_1, \dots, g_n) \in G \times \dots \times G | g_1 \dots g_n = 1\}/G$ can be described using classical factorizable r-matrices as in [9].

The group G acts on the product $\mathcal{C}_1 \times \mathcal{C}_n$ by diagonal conjugations. This action is Poisson and the mapping

$$\mathcal{C}_1 \times \mathcal{C}_n \to G, \quad (g_1, \dots, g_n) \to g_1 \dots g_n$$

[1]Recall that given two projections $\pi_{1,2} : M_{1,2} \to N$, the fibered product of M_1 and M_2 over N is

$$M_1 \times_N M_2 = \{(x_1, x_2) \in M_1 \times M_2 | \pi_1(x_1) = \pi_2(x_2)\}$$

If $\sigma : M_2 \to M_2$ is a diffeomorphism, the fibered product twisted by σ is

$$M_1 \widetilde{\times}_N M_2 = \{(x_1, x_2) \in M_1 \times M_2 | \pi_1(x_1) = \pi_2(\sigma(x_2))\}$$

is the group valued moment map for this action [1]. It commutes with the conjugation action of G and gives the Poisson map

$$\widetilde{\mathcal{M}}_{\mathcal{C}_1,\dots,\mathcal{C}_n} \to G/\operatorname{Ad}_G$$

where

$$\widetilde{\mathcal{M}}_{\mathcal{C}_1,\dots,\mathcal{C}_n} = \{(g_1,\dots,g_n) \in \mathcal{C}_1 \times \cdots \times \mathcal{C}_n\}/G$$

As in the previous section, define the Poisson varieties

$$\mathcal{P}_{I,J}(\mathcal{C}_1,\dots,\mathcal{C}_n) = \widetilde{\mathcal{M}}_{\mathcal{C}_{i_1},\dots,\mathcal{C}_{i_k}} \times_{G/\operatorname{Ad}_G} \widetilde{\mathcal{M}}_{\mathcal{C}_{j_1},\dots,\mathcal{C}_{j_{n-k}}}$$

where I, J is a partition $(1,\dots,n) = I \sqcup J$. Where the twisted fibered product is defined in the previous section. The twist is given by $\sigma : g \mapsto g^{-1}$.

Relativistic Casimir integrable system is described by the following sequence of Poisson maps

$$\mathcal{M}_{\mathcal{C}_1,\dots,\mathcal{C}_n} \to \mathcal{P}_{I,J}(\mathcal{C}_1,\dots,\mathcal{C}_n) \to \mathcal{B}_{I,J}(\mathcal{C}_1,\dots,\mathcal{C}_n) \subset G/\operatorname{Ad}_G$$

acting as

$$\begin{aligned} &\operatorname{Ad}_G(g_1,\dots,g_n) \mapsto (\operatorname{Ad}_G(g_{i_1}\dots g_{i_k}), \\ &\operatorname{Ad}_G(g_{j_1}\dot{s}g_{j_{n-k}})) \mapsto [g_{i_1}\dots g_{i_k}] = [(g_{j_1}\dots g_{j_{n-k}})^{-1}] \in G/\operatorname{Ad}_G \end{aligned}$$

Here $\mathcal{B}_{I,J}$ is the image of the last map, which has dimension r but is, generally, smaller then $\mathfrak{g}^*/\operatorname{Ad}_G^*$.

§3. Calogero–Moser systems

3.1. Degenerate integrability. Spin Calogero–Moser systems are parameterized by pairs $(\mathfrak{g}, \mathcal{O})$ where $\mathfrak{g}$ is a simple Lie group and $\mathcal{O}$ is a co-adjoint orbit in $\mathfrak{g}$. Calogero and Moser discovered such systems for Lie algebras of type A and coadjont orbit of rank 1. Sutherland generalized them to trigonometric and hyperbolic potentials. Olshanetsky and Perelomov generalized them to all simple Lie algebras and to elliptic potentials. Here we will focus on trigonometric potentials.

The degenerate integrability of spin Calogero–Moser systems is given by the following collection of Possin projections.

$$\begin{array}{ccccc} T^*G & \longrightarrow & \mathfrak{g}^* \times_{\mathfrak{h}^*/W} \mathfrak{g}^* & \underset{R}{\overset{L}{\rightrightarrows}} & \mathfrak{g}^* \\ \downarrow & & \downarrow & & \downarrow \\ T^*G/\operatorname{Ad}_G & \longrightarrow & (\mathfrak{g}^* \times_{\mathfrak{h}^*/W} \mathfrak{g}^*)/G & \xrightarrow{p} & \mathfrak{h}^*/W \simeq \mathfrak{g}^*/\operatorname{Ad}_G^* \end{array}$$

Here $\mathfrak{g}^* \times_{\mathfrak{h}^*/W} \mathfrak{g}^*$ is the fibred product of two copies of $\mathfrak{g}^*$ over $\mathfrak{h}^*$. The maps in the upper row of the diagram act as $(x, g) \mapsto (x, -\operatorname{Ad}^*_g(x))$, $L(x, y) = x$, and $R(x, y) = y$. Here and below we assume that the co-adjoint bundle T^*G is trivialized by left translations $T^*G \simeq \mathfrak{g}^* \times G$ and has a standard symplectic structure of a cotangent bundle. The lower horizontal sequence of Poisson maps is at heart of degenerate integrability of spin Calogero–Moser systems [24].

Recall that classical spin Calogero–Moser systems are parameterized by co-adjoint orbits $\mathcal{O} \subset \mathfrak{g}^*$. If $\mathcal{O}$ is passing through $t \in \mathfrak{h}^*$, then it is also passing through each $w(t)$ where $w \in W$ is an element of the Weyl group. We will denote such orbit passing through t by $\mathcal{O}_{[t]}$ where $[t] \in \mathfrak{h}^*/W$ is the orbit of t with respect to the Weyl group action. When $\mathfrak{g}$ is a real compact form of a simple Lie algebra, we can identify $\mathfrak{h}^*/W$ with $h^* \geqslant 0 = \{\sum_{i=1}^{r} x_i \omega_i | x_i \in \mathbb{R}_{\geqslant 0}\}$, where are fundamental weights of $\mathfrak{g}$ and r is the rank of $\mathfrak{g}$.

For a generic co-adjoint orbit $\mathcal{O}_{[t]}$ the phase space of the corresponding spin Calogero–Moser system is the symplectic leaf $S_{[t]} = \mu^{-1}(\mathcal{O}_{[t]})/G$ where $\mu : T^*G \to \mathfrak{h}^*$ is the moment map for the adjoint action of G:

$$\mu(x, g) = x - \operatorname{Ad}^*_g(x) \in \mathfrak{g}^*$$

Here $x \in \mathfrak{g}^*, g \in G$.

The sequence of projections from the diagram above produces the sequence of Poisson projections

$$S_{[t]} \to \sqcup_{[s] \in \mathfrak{h}^*/W} \mathcal{M}_{[s], -[s] | [t]} \to \mathcal{B}_{[t]} \subset \mathfrak{h}^*/W \qquad (5)$$

Here the moduli space $\mathcal{M}_{[s_1],[s_2]|[t]}$ is defined as

$$\mathcal{M}_{[s_1],[s_2]|[t]} = \{(x_1, x_2) \in \mathcal{O}_{[s_1]} \times \mathcal{O}_{[s_2]} | x_1 + x_2 \in \mathcal{O}_{[t]}\}/G$$

and $\mathcal{B}_{[t]} = \{[s] \in \mathfrak{h}^*/W | \mathcal{M}_{[s],-[s]|[t]} \neq \varnothing\}$. Note that $\mathcal{B}_{[t]}$ is unbounded but if $t \neq 0$ it does not contain the vicinity of zero. Its dimension is $r = \operatorname{rank}(G)$. The series of projections (5) describes the degenerate integrability of classical spin Calogero–Moser model. The Hamiltonian of the classical spin Calogero–Moser system is the pull-back of the quadratic Casimir function on $\mathfrak{h}^*/W$ to $S_{[t]}$. Taking into account the isomorphism $S_{[t]} \simeq (T^*\mathfrak{h} \times \mathcal{O}_{[t]}//H)/W$ (assuming, as above, that t is generic), were $\mathcal{O}_{[t]}//H$ is the Hamiltonian reduction of $\mathcal{O}_{[t]}$ with respect to the coadjoint action of H, the Hamiltonian of classical spin Calogero–Moser system can be written as

$$H_{sCM} = \langle p, p\rangle + \sum_{\alpha\in\Delta_+} \frac{\mu_\alpha \mu_{-\alpha}}{(h_{\alpha/2} - h_{-\alpha/2})^2}$$

where p, h_α are coordinate functions on $T^*\mathfrak{h}$ and $\mu_\alpha\mu_{-\alpha}$ is a function on $\mathcal{O}_{[t]}//H$ (the Hamiltonian reduction of $\mathcal{O}_{[t]}$ with respect to the action of the Cartan subgroup) see [24] for details. One can check that the Poisson algebra $C(S_{[t]})$ is isomorphic to the subalgebra of W-invariant functions from $Pol(p, h_\alpha^{\pm 1}) \otimes C(\mathcal{O}_t//H)$ with the Poisson structure

$$\{p_i, p_j\} = 0, \quad \{p_i, h_\alpha\} = \alpha_i h_\alpha, \quad \{h_\alpha, h_\beta\} = 0$$

Poisson algebra $C(\mathcal{O}_t//H)$ of functions on the Hamiltonian reduction of $\mathcal{O}_t$ with respect to the Hamiltonian action of $H$is the quotient of the Poisson algebra of H-invariant functions on $\mathcal{O}_t$ with respect to the Poisson ideal generated by Cartan components of μ_i.

Note that the evolution with respect to a central function F on $\mathfrak{g}^*$ is quite simple:

$$(X, g) \mapsto (X, e^{t\nabla F(X)} g)$$

where ∇F is the gradient (with respect to the Killing from on $\mathfrak{g}$ of F. This formula becomes somewhat complicated after the projection $T^*G \to T^*G/G$.

3.2. Rank 1 orbits for SL_n. In this case

$$\mu_{ij} = \phi_i \psi_j - \delta_{ij}\kappa,$$

where $\kappa = \frac{1}{n}\sum_{i=1}^{n} \phi_i\psi_i$. The Hamiltonian reduction with respect to the action of the Cartan subgroup introduces constraint $\mu_{ii} = 0$ which implies $\phi_i\psi_i = \kappa$. In this case

$$\mu_{ij}\mu_{ji} = \phi_i\psi_i\phi_j\psi_j = \kappa^2$$

which means, in particular that the Hamiltonian reduction of a rank 1 orbit is a point. The spin Calogero–Moser system for such orbits becomes Calogero–Moser system with the Hamiltonian. Which is equal to

$$H_{CM} = \langle p, p\rangle + \sum_{i<j} \frac{\kappa^2}{4\sin(\frac{q_i - q_j}{2})^2}$$

for the compact real form of G.

§4. Rational spin Ruijsenaars systems

4.1. Degenerate integrability. Let us denote by (T^*G,p) the Poisson manifold which is T^*G as a manifold, which we assume trivialized $T^*G \simeq \mathfrak{g}\times G$ by left translations with the Poisson algebra structure on $C^\infty(T^*G)$ defined uniquely by the following properties:

- The subalgebras $C^\infty(\mathfrak{g}^*)$ and $C^\infty(G)$ are Poisson subalgebras with the standard and trivial Poisson structures respectively.
- Poisson bracket between a linear function $X\in\mathfrak{g}$ on $\mathfrak{g}^*$ and $f\in C^\infty(G)$ is
$$\{X,f\}=(L_X-R_X)f$$
where L_X and R_X are the left and right invariant vector fields on G generated by X.

Note that this Poisson structure differs from the standard symplectic structure on the cotangent bundle to a manifold. Symplectic leaves of (T^*G,p) are products $\mathcal{O}\times\mathcal{C}$ where $\mathcal{O}\subset\mathfrak{g}^*$ is a co-adjoint orbit and $\mathcal{C}\subset G$ is a conjugacy class.

The adjoint action of the group G (the extension of the adjoint action from G to T^*G) on (T^*G,p) is Poisson, thus $(T^*G,p)/G$ has a natural Poisson structure. The symplectic leaves of the quotient space are $(\mathcal{O}\times\mathcal{C})/G$ where G acts diagonally on the product.

It is easy to check that the map $T^*G\to(T^*G,p)$ acting as $(x,g)\to(x-\mathrm{Ad}_g^*(x),g)$ is Poisson and it is clear that it commutes with the adjoint G-actions. It induces Poisson map
$$\phi: T^*G/\mathrm{Ad}_G\to(T^*G,p)/\mathrm{Ad}_G\ .$$
We also have a natural projection
$$\pi:(T^*G,p)/\mathrm{Ad}_G\to G/\mathrm{Ad}_G\ .$$
acting as $\mathrm{Ad}_G(x,g)\mapsto\mathrm{Ad}_G\,g$. This projection is Poisson with the trivial Poisson structure on the base.

Restricting $\widetilde{\psi}$ to a symplectic leaf of T^*G/Ad_G[2], we have the sequence of Poisson maps describing degenerate integrability of rational spin Ruijenaars (sR) systems
$$S(\mathcal{O})\to P(\mathcal{O})\to B(\mathcal{O})\subset G/\mathrm{Ad}_G\ .$$
Here $S(\mathcal{O})$ is the symplectic leaf of T^*G/Ad_G corresponding to the coadjoint orbit $\mathcal{O}\in\mathfrak{g}^*$, $P(\mathcal{O})=\phi(S(\mathcal{O}))=(\mathcal{O}\times\mathcal{O})/G\subset(T^*G,p)/G$, and

[2]Symplectic leaves of T^*G/Ad_G are described earlier.

$B(\mathcal{O}) = \pi(P(\mathcal{O}))$. Note that π is surjective and fibers of ϕ has dimension $\dim(B(\mathcal{O})$, which is, generically, r. Recall that symplectic leaves of T^*G/Ad_G are perimages of coadjoint orbits with respect to the moment map, $S(\mathcal{O}) = \{(x,g)|x - \operatorname{Ad}^*_g(x) \in \mathcal{O}\}/G$. The fiber of the last projection is the symplectic leaf of $P(\mathcal{O})$:

$$P(\mathcal{O},\mathcal{C}) = \{(x - \operatorname{Ad}^*_g(x), g)|x \in \mathcal{O}, g \in \mathcal{C}\}/G$$

As in the case of the spin Calogero–Moser, the dimension of $B(\mathcal{O})$ is r for generic $\mathcal{O}$ but for degenerate orbits it is less and for maximally degenerate non-trivial orbits it is 1-dimensional.

4.2. Hamiltonians for SL_n rank 1 orbits. Here we assume $G = SL_n$. In this case we can this of both $\mathfrak{g}$ and $\mathfrak{g}^*$ and traceless $n\times n$ matrices. We also assume that $\mathcal{O} \subset \mathfrak{g}^*$ is an orbit through a semisimple element and that $\mu = x - gxg^{-1} \in \mathcal{O}$. If we choose the cross-section of the adjoint G-action on T^G, where $x_{ij} = \delta_{ij}h_i$, the symplectic leaf $S(\mathcal{O}) \in T^*G/G$ (its open dense subset) has coordinates h_i, $\mu_{ij}\mu_{ji}$ g_{ii}. The Hamiltonian reduction imposes the constraint $\mu_{ii} = 0$. Elements g_{ij} satisfy the equation

$$(h_i - h_j)g_{ij} = \sum_{k=1}^{n} \mu_{ik}g_{kj}\ . \tag{6}$$

We will not try to solve this equations here, in order to find Hamiltonians for rank > 1 orbits. In the next section we will do it for rank 1 case (this computation can also be found in many other paper, see for example [18] [4].

In this case

$$\mu_{ij} = \phi_i\psi_j - \delta_{ij}\kappa$$

where $\kappa =< \phi,\psi > /n$ as in the rank 1 case of Calogero Moser. The equation (6) implies

$$(h_i - h_j)g_{ij} = \phi_i \sum_k \psi_k g_{kj} - \kappa g_{ij}$$

From here we have

$$g_{ij} = \frac{1}{h_i - h_j + \kappa}\phi_i \sum_k \psi_k g_{kj} \tag{7}$$

This gives the system of equations for $\psi_i\phi_i$

$$\sum_{i=1}^{n} \frac{\phi_i\psi_i}{h_i - h_j + \kappa} = 1 \tag{8}$$

and the identity

$$g_{ii} = \frac{\phi_i}{\kappa} \sum_{k=1}^{n} \psi_k g_{ki} \tag{9}$$

The equation (8) can be solved explicitly:

$$\phi_i \psi_i = \prod_{j \neq i} \frac{h_i - h_j + \kappa}{h_i - h_j}$$

Equations (9) and (7) give the formula for g_{ij}

$$g_{ij} \ = \ \frac{\phi_i \phi_j^{-1} \kappa g_{jj}}{h_i - h_j + \kappa} \ .$$

Reduced Poisson brackets are log-linear in coordinates h_i, u_i[3]

$$\{h_i, h_j\} = 0, \ \ \{h_i, u_j\} = \delta_{ij}, \ \ \{u_i, u_j\} = 0$$

where u_i is related to g_{ii} as

$$g_{ii} = u_i \prod_{j \neq i}^{n} \frac{h_i - h_j + \kappa}{h_i - h_j}$$

The first two elementary G-invariant functions of g are

$$\begin{aligned} \mathrm{tr}(g) \ &= \ \sum_{i=1}^{n} g_{ii} \ , \\ \mathrm{tr}(g^2) \ &= \ \kappa^2 \sum_{ij} g_{ii} g_{jj} \ \frac{1}{(h_i - h_j + \kappa)(h_j - h_i + \kappa)} \ . \end{aligned}$$

The second function gives the Hamiltonian of the rational Ruijsenaars system.

$$H^{rR} = \chi_{\omega_2}(g) = \frac{1}{2}(tr(g^2) - tr(g)^2) = -\sum_{i<j} u_i u_j \prod_{a \in \{ij\}, b \in \{ij\}^\vee} \frac{h_a - h_b + \kappa}{h_a - h_b}$$

Here $\{i, j\} \subset \{1, \ldots, n\}$ and $\{i, j\}^\vee$ is its complimentary subset. Characters of fundamental representations $\chi_{\omega_i}(g)$ evaluated on elements g described above are classical analogs of rational Macdonald operators.

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[3]To be more precise the algebra of functions on $S(\mathcal{O})$ is isomorphic to the algebra of symmetric polynomials in $p_i, u^{\pm 1}$.

4.3. Duality. A duality relation between spin Calogero–Moser systems and systems which we will call rational spin Ruijsenaars systems was observed in [19] [9] (see also references therein). This is a duality between two Liouville integrable systems which maps angle variables of one system to the action variable of the other system. The duality between spin Calogero–Moser and rational spin Ruijsenaars systems (as the duality of degenerately integrable systems) was found in [24]. Here we will recall this property.

Let $F(G(x,\gamma))$ be the fiber of the projection $\psi : T^*G/G \to (\mathfrak{g}^* \times_{\mathfrak{g}^*/G} \mathfrak{g}^*)/G$ containing $G(x,\gamma)$. Recall that $\psi(G(x,\gamma)) = G(x, -\operatorname{Ad}^*_\gamma(x))$. It is clear that

$$F(G(x,\gamma)) = G(x, Z_x\gamma)$$

where $Z_x = \{g \in G | \operatorname{Ad}^*_g(x) = x\}$. This fiber is the Liouville torus of the spin C–M system passing through the point $G(x,\gamma)$. It projects to $\operatorname{Ad}_G(x) \in \mathfrak{g}^*/G$ on the base of the last projection in (5). Hamiltonian flows of functions on $\mathfrak{g}^*/G$ generate angle variable for spin C–M system, i.e. an affine coordinate on $F(G(x,\gamma))$. Generic fiber $F(G(x,\gamma))$ has dimension $r = \operatorname{rank}(G)$.

Define $\widetilde{F}(G(x\gamma))$ as a fiber of the map $\widetilde{\psi} : T^*G/G \to (T^*G, p)/G$ which contains $G(x,\gamma)$. Recall that $\widetilde{\psi}(x,\gamma) = (x - \operatorname{Ad}^*_\gamma(x), \gamma)$. It is easy to see that

$$\widetilde{F}(G(x,\gamma)) = G(x + C_\gamma, \gamma)$$

Here $C_\gamma = \{x \in \mathfrak{g}^* | \operatorname{Ad}^*_\gamma(x) = x\}$. This fiber is the Liouville torus of the rational spin Rujsenaars system passing through $G(x,\gamma)$. Hamiltonian flows of functions on G/Ad_G generate an affine coordinate system on it which is the collection of angle variables for the rational spin Ruijenaars system.

Theorem 2. *The fibers $F(G(x,\gamma))$ and $\widetilde{F}(G(x,\gamma))$ are dual in a sense that*

$$F(G(x,\gamma)) \cap \widetilde{F}(G(x,\gamma)) = G(x,\gamma)$$

For rank 1 orbits, when both systems are Liouville integrable, this duality reduces to the one from [4, 19].

§5. Relativistic spin Calogero–Moser and spin Ruijsenaars systems

5.1. Relativistic spin Calogero–Moser system.

5.1.1. *Hamiltonian structure and degenerate integrability of relativistic spin Caloero–Moser and Ruijsenaars models.* The underlying Poisson manifold for relativistic spin Calogery–Moser system is a "nonlinear" version of T^*G which is known as a Heisenberg double $H(G)$ of G with the standard Poisson Lie structure. Equivalently $H(G)/G$ where G acts by diagonal conjugations can be regarded as the moduli space of flat connections on a punctures torus (see [9]).

As a manifold the Heisenberg double is $H(G) = G \times G$. A point (x, y) should be regarded as a pair of monodromies o the local system on a punctures torus around two fundamental cycles. The monodromy around the puncture is $xyx^{-1}y^{-1}$. The Poisson structure on $H(G)$ can be described in terms of r-matrices for standard Poisson Lie structure on G. Poisson brackets between coordinate functions can be written as [7]:

$$\begin{aligned}
\{x_1, x_2\} &= r_{12}x_1x_2 - x_1x_2r_{21} + x_1r_{21}x_2 - x_2r_{12}x_1 \\
\{x_1, y_2\} &= -r_{21}x_1y_2 - x_1y_2r_{21} + x_1r_{21}y_2 - y_2r_{12}x_1 \\
\{y_1, y_2\} &= r_{12}y_1y_2 - y_1y_2r_{21} + y_1r_{21}y_2 - y_2r_{12}y_1
\end{aligned} \tag{10}$$

Here x and y are matrix elements of $x \in G$ in some finite dimensional representation with some basis (these matrix elements from a basis in the space of regular functions on G). The matrix r_{12} is the result of evaluation of the universal classical r-matrix from section 6.1 in the tensor product of two finite dimensional represenations of G.

The phase space of relativistic Calogero–Moser system is the symplectic leaf of the moduli space $H(G)/G$ corresponding to fixing the conjugacy class of the monodromy $xyx^{-1}y^{-1}$ around the puncture. In terms of Poisson geometry, this symplectic leaf can be described as follows.

The map $H(G) \to G$, $(x, y) \mapsto x$ is the G-valued moment map for the left action of the group on $H(G)$ (regarded as non-linear version of the cotangent bundle on G trivialized by left translations). The map $H(G) \to G$, $(x, y) \to yxy^{-1}$ is the group valued moment map for the corresponding right action of G. The map $\mu : H(G) \to G$, $\mu : (x, y) \mapsto xyx^{-1}y^{-1}$ is the group valued map corresponding to the conjugation action. For details on group valued moment maps see [1]. Thus,

$$\mathcal{M}(\mathcal{C}) = \mu^{-1}(\mathcal{C}) \subset H(G)/G$$

is the symplectic leaf corresponding to the conjugacy class $\mathcal{C}$ of the monodromy around the puncture. Here G acts by conjugation.

Hamiltonians of the relativistic spin Calogero–Moser system corresponding to the conjugacy class $\mathcal{C}$ are conjugation invariant functions on G, i.e. functions on G/G. The Hamiltonian corresponding to $f \in C^G(G)$ is $H_f(x,y) = f(x)$.

The degenerate integrability of the relativistic spin Calogero–Moser system is described by restricting the following sequences of Poisson maps:

$$(G \times G)/G \to (G \tilde{\times}_{G/\operatorname{Ad}_G} G)/G \to G/\operatorname{Ad}_G \tag{11}$$

to the symplectic leaf $\mathcal{M}(\mathcal{C})$. Here the fibered product is twisted as in the relativistic Casimir system by $g \mapsto g^{-1}$ and $[(x,y)] \mapsto [(x, yx^{-1}y^{-1})] \mapsto [x]$. For compact simple Lie group G this gives the degenerate integrability of relativistic spin Calogero systems:

$$\mathcal{M}(\mathcal{C}) \to \{G(g_1,g_2)|G(g_1) = G(g_2), \quad g_1g_2 \in \mathcal{C}\} \to \mathcal{B}(\mathcal{C}) \subset G/G$$

where $\mathcal{B}(\mathcal{C}) = \{\mathcal{C}' \in G/G | \mathcal{M}(\mathcal{C}', \mathcal{C}'^{-1}, \mathcal{C}) \neq \varnothing\}$.

Hamiltonians of relativistic spin Ruijsenaars system are $H_f(x,y) = f(y)$ where $f \in C^G(G)$ is a function on G, invariant with respect to conjugations.

The degenerate integrability of relativistic spin Ruijsenaars system is given by restricting maps

$$(G \times G)/G \to (G \times G)/G \to G/G$$

where $G(x,y) \mapsto G(xyx^{-1}y^{-1}, y) \mapsto Gy$, to a symplectic leaf of $(G \times G)/G$.

Two systems are related as follows.

Proposition 2. *The mapping $G \times G \to G \times G$, $(x,y) \mapsto (y, x^{-1})$ is a Poisson map and it induces the symplectomorphism $\mathcal{M}(\mathcal{C}) \mapsto \mathcal{M}(\mathcal{C})$ which maps relativistic spin Calogero–Moser system to relativistic spin Rujsenaars system.*

The proof is straightforward.

5.1.2. *Duality.* Let us prove that the two systems are dual in a sense of intersection property of Lioville tori.

Let π_1 be the projection

$$(G \times G)/G \to (G \tilde{\times}_{G/G} G)/G, \quad G(x,y) \mapsto G(x, yx^{-1}y^{-1})$$

and π_2 be the projection

$$(G \times G)/G \to (G \times G)/G, \quad G(x,y) \mapsto G(xyx^{-1}y^{-1}, y),$$

Denote fibers of these projections through the point $G(x,y) \in (G \times G)/G$ by $F_1(G(x,y))$ and $F_2(G(x,y))$ respectively. The following is easy to prove.

Proposition 3. *For generic* $(x, y$ *we have:*

1) $F_1(G(x,y)) = \{G(x,yz)|z \in Z_x\}$ *where* Z_x *is the centralizer of* x *in* G.
2) $F_2(G(x,y)) = \{G(xz,y)|z \in Z_y\}$
3) $F_1(G(x,y)) \cap F_2(G(x,y)) = G(x,y)$

5.1.3. *Hamiltonians for rank 1 conjugacy classes in* SL_n. Assume that $z = xyx^{-1}y^{-1} \in SL_n$ belongs to the rank 1 conjugacy class. For generic rank 1 conjugacy class this means

$$z = u \operatorname{diag}(q^{n-1}, q, \ldots, q)u^{-1}$$

for some $u \in SL_n$ and $q \in \mathbb{C}^*$. Equivalently, we can write

$$z_{ij} = \phi_i \psi_j + q^{-1}\delta_{ij}$$

where $(\phi, \psi) = \sum_{i=1}^{n} \psi_i \phi_i = q^{n-1} - q^{-1}$.

Hamiltonians of relativistic Calogero–Moser and relativistic Ruijenaars system are

$$H_k^{rCM} = \chi_{\omega_k}(x), \quad H_k^{sR} = \chi_{\omega_k}(y)$$

Let us compute them in appropriate coordinates.

First, assume x is semisimple and bring it to the diagonal form with eigenvalues $x_1, \ldots, x_n$. From the definition of z we have

$$y_{ij}x_j = \sum_{k=1}^{n} x_k y_{kj} = \phi_i \sum_{k=1}^{n} \psi_k x_k y_{kj} - q^{-1}x_i y_{ij} \tag{12}$$

From here we have:

$$y_{ij} = \frac{\phi_i \sum_{k=1}^{n} \psi_k x_k y_{kj}}{x_j - q^{-1}x_i}$$

Multiplying by $\phi_i\psi_i$ and taking sum over i gives the following equation for $\psi_i\phi_i$:

$$\sum_{i=1}^{n} \frac{\psi_i \phi_i x_i}{x_j - q^{-1}x_i} = 1$$

Solving this equation we have

$$\psi_i\phi_i = (1-q^{-1})x_i^{-1}\prod_{j\neq i}^{n}\frac{1-qx_jx_i^{-1}}{1-x_jx_i^{-1}}$$

When $i=j$, (12) implies

$$y_{ii} = \frac{\phi_i}{x_i(1-q^{-1})}\sum_k \psi_k x_k y_{ki}$$

Solving this for $\sum_k \psi_k x_k y_{ki}$ we have

$$y_{ij} = \frac{\phi_i\phi_j^{-1}(1-q^{-1})y_{jj}}{1-q^{-1}x_ix_j^{-1}}$$

Now we can compute Hamiltonians of rsR model in terms of y_{ii} and x_i. For the first two we have

$$tr(y) = \sum_{j=1}^{n} y_{jj}$$

$$tr(y^2) = \sum_{ij}^{n}\frac{(1-q^{-1})^2 y_{ii}y_{jj}}{(1-q^{-1}x_ix_j^{-1})(1-q^{-1}x_jx_i^{-1})}$$

Poisson algebra $C(\mathcal{M}(\mathcal{C})$ is isomorphic to the algebra of symmetric Laurant polynomials in y_{ii} and x_i (with respect to the diagonal action of the symmetric group) with following Poisson brackets between x and y:

$$\{x_i,x_j\}=0,\quad \{x_i,u_j\}=\delta_{ij}x_iu_j,\quad \{u_i,u_j\}=0$$

where

$$y_{ii} = u_i\prod_{j\neq i}\frac{1-q^{-1}x_jx_i^{-1}}{1-x_jx_i^{-1}}$$

The Hamiltonians $\chi_{\omega_i}(y)$ are classical analogs of Macdonald operators. The Hamiltonian of the relativistic Ruijsenaars model is

$$H_2 = \chi_{\omega_2}(y) = -q^{-1}\sum_{i<j}u_iu_j\prod_{a\in\{ij\},b\in\{ij\}^\vee}\frac{1-q^{-1}x_ax_b^{-1}}{1-x_ax_b^{-1}}$$

The mapping $(x,y)\mapsto(y,x^{-1})$ intertwine the relativistic Calogero–Moser system and Relativistic Ruijenaars system. So, the Hamiltonian of relativistic Calogero–Moser model is given by essentially the same formulae.

§6. Characteristic systems on simple Poisson Lie groups with standard Poisson Lie structure

6.1. Symplectic leaves and degenerate integrability of characteristic system. Standard Poisson Lie structure on a simple Lie group requires a choice of a Borel subgroup in G. This fixes a Catran subalgebra $\mathfrak{h}$, the root system and positive roots. Assuming that the tangent bundle TG is trivialized by left translations $TG \simeq \mathfrak{g} \times G$, the Poison bivector field corresponding to the standard structure is

$$\eta(x) = \mathrm{Ad}_x(r) - r, \quad r = \frac{1}{2}\sum_{i=1}^{r} h^i \otimes h_i + \sum_{\alpha>0} E_\alpha \otimes F_\alpha$$

Here α and positive roots of $\mathfrak{g}$, E_α, F_α are corresponding elements of the basis in $\mathfrak{g}$, r is the rank of $\mathfrak{g}$, h_i is a basis in the Cartan subalgebra $\mathfrak{h}$ and h^i is the dual basis with respect to the Killing form. We assume that $\mathfrak{g} \wedge \mathfrak{g} \subset \mathfrak{g} \otimes \mathfrak{g}$.

Symplectic leaves of any Poisson Lie group are orbits of the dressing action of the dual Poisson Lie group. For a simple Lie group G with the standard Poisson Lie structure are known to be fibers of the fibration of double Bruhat cells over tori inside of the Cartan subgroup H of G. Recall that a double Bruhat cell in G is the intersection of a Bruhat cell for B and a Bruhat cell for B^-:

$$G^{u,v} = BuB \cap B^- vB^-$$

where BuB is defined as $B\overline{u}B \subset G$, where $u \in W$ and $\overline{u} \in N(H) \subset G$ is its representative in the normalizer of H, and B^-vB^- is defined similarly.

Generalized minors give a natural fibration

$$\begin{array}{ccc} G^{u,v} & \longleftarrow & S^{u,v} \\ \downarrow & & \\ T^{u,v} & & \end{array}$$

For the explicit description of it see, for example [23] and references therein.

Hamiltonians of the characteristic integrable system are central functions on G. There are only r independent central functions which can be chosen as characters of fundamental representations. Their restriction to a generic symplectic leave of G generate a degenerately integrable system [23]. Poisson projections describing degenerate integrability can be described as follows:

$$S_{u,v} \to P^{u,v} \to \mathrm{Ad}_G S_{u,v}. \tag{13}$$

Here $P^{u,v} = (S^{u,v} \times S^{u,v})/\mathrm{Ad}_{G^*}$ where $S^{u,v} \times S^{u,v} \subset G \times G$ and the dual Poisson Lie group G^* is embedded in $G \times G$ as usual $G^* = \{(b^+, b^-) \in B \times B^- \subset G \times G | [b^+]_0 = [b^{-^{-1}}]_0\}$, where $[b]_0$ is the Cartan component of $b \in B$. The first map is the diagonal embedding, the second map is the projection to $(G \times G)/\mathrm{Ad}_{G\times G}$ followed by the projection to any of the factors in the Cartesian product.

In other words, characteristic Hamiltonian systems are integrable and their Liouville tori are intersections of adjoint orbits of G and of orbits of the dressing action of G^* (which are symplectic leaves of G).

6.1.1. *Hamiltonian flows as the factorization dynamics.* Let G be a factorizable Poisson–Lie group. Standard Poisson Lie group structure on a simple Lie group is an example of a factorizable Poisson Lie group. Let $I(G) \subset C^\infty(G)$ be the subspace of Ad_G-invariant functions on G.

Let G^* be the dual Poisson Lie group to G. It has a natural embedding to $G \times G$ described above. The multiplication in G, together with this embedding gives the mapping $G^* \to G$, $(b_+, b_-) \mapsto b_+ b_-^{-1}$. When the inverse exists $g \mapsto (g_+, g_-)$ (in a vicinity of the unit element in G it is unique when it exists), it is called the factorization map. Note that at the level of Lie algebras there is always a linear isomorphism $\mathfrak{g} \to \mathfrak{g}^*$, such that $x = x_+ + x_0 + x_- \mapsto (x_+ + \frac{x_0}{2}, -x_- - \frac{x_0}{2})$. It is called the factorization isomorphism.

The dynamics of characteristic systems can be described explicitly by the following theorem [25]:

Theorem 3. *Assume the factorization map is defined and unique on an open dense subset of G, then*

i) *$I(G)$ is a commutative Poisson algebra in $C^\infty(G)$.*

ii) *In a neighborhood of $t = 0$ the flow lines of the Hamiltonian flow induced by $H \in I(G)$ passing through $x \in G$ at $t = 0$ have the form*

$$x(t) = g_\pm(t)^{-1} x g_\pm(t),$$

where the mappings $g_\pm(t)$ are determined by

$$g_+(t) g_-(t)^{-1} = \exp\left(tI\left(d_l H(x)\right)\right),$$

and $I : \mathfrak{g}^* \longrightarrow \mathfrak{g}$ is the inverse to the factorization isomorphism. Here $d_l H(x) \in \mathfrak{g}^*$ is the left differential of $H(x)$. For $X \in \mathfrak{g}$, *assuming the left trivialization of* TG *we have* $< d_l H(x), X >= \frac{d}{dt} H(e^{tX}x)|_{t=0}$ *where* $< .,. >: \mathfrak{g}^* \times \mathfrak{g} \to \mathbb{C}$ *is the natural pairing (assuming we are over* $\mathbb{C}$*).*

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