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QUANTUM GROUPS: FROM KULISH–RESHETIKHIN DISCOVERY TO CLASSIFICATION

Abstract. The aim of this paper is to provide an overview of the results about classification of quantum groups that were obtained in [10, 11].

**Dedicated to P. P. Kulish
on the occasion of his 70th birthday**

§1. Introduction

The first example of a quantum group was found by Kulish and Reshetikhin in [13]. They discovered what was later named $U_q(\mathfrak{sl}_2)$ in relation to the study of the inverse quantum scattering method. Later, Drinfeld [3] and Jimbo [9] independently developed a general notion of quantum group. Today there are many different approaches to what quantum group is and the term has no clear meaning. Informally speaking, the quantum group is a deformation of a universal enveloping algebra of some Lie algebra $\mathfrak{g}$. Of course, the precise meaning should be given to the term deformation. We will be using the following definition.

Definition 1.1. A quantum group is a topologically free cocommutative mod $\hbar$ Hopf algebra over $\mathbb{C}[[\hbar]]$ such that $H/\hbar H$ is a universal enveloping algebra of some Lie algebra $\mathfrak{g}$ over $\mathbb{C}$.

It is well known that many problems about Lie groups become simpler when they are written on the language of Lie algebras. In general the existence of almost one-to-one correspondence between Lie groups and Lie algebras is one of the central parts of Lie theory. Therefore, it is desirable to obtain a notion of quantum algebra that will help to simplify problems about quantum groups gradually. The first natural attempt was to look at the linear part of comultiplication of a quantum group H. Indeed, one can

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define a co-Poisson structure $\delta : U(\mathfrak{g}) \to U(\mathfrak{g}) \otimes U(\mathfrak{g})$ by the formula

$$\delta(x) = \frac{\Delta(a) - \Delta^{21}(a)}{\hbar} \mod \hbar,$$

where $x \equiv a \mod \hbar$. Furthermore, from a co-Poisson structure on $U(\mathfrak{g})$ one gets a Lie bialgebra structure on $\mathfrak{g}$ and the co-Poisson structure is uniquely determined by this Lie bialgebra structure. The process of recovering the (non-unique) quantum group structure from the Lie bialgebra structure is known as quantization.

The following problem naturally arises.

Conjecture 1.2 (Drinfeld's quantization conjecture)**.** Any Lie bialgebra can be quantized.

The conjecture was solved by Etingof and Kazhdan in [5, 6].

Kazhdan and Etingof not only proved Drinfeld's quantization conjecture but found a correct notion of quantum algebra. It was very important because it was not difficult to see that there might be many different quantizations of a given Lie bialgebra over $\mathbb{C}$. They constructed a canonical co-Poisson structure on $U(\mathfrak{g}) \otimes \mathbb{C}[[\hbar]]$. This structure is much finer then the co-Poisson structure discussed above. The Lie groups – Lie algebras correspondence has an analogy in the quantum world.

Theorem 1.3. *Let* Qgroup *be the category of quantum groups in the sense of Definition 1.1. Let* LieBialg *be the category of topologically free Lie bialgebras over* $\mathbb{C}[[\hbar]]$ *with* $\delta \equiv 0 \mod \hbar$. *Then there exists a dequantization functor* $deQuant$: Qgroup $\to$ LieBialg *that is an equivalence of categories.*

In their solution of Drinfeld's quantization conjecture, Etingof and Kazhdan constructed a functor $Quant$: LieBialg $\to$ Qgroup, which informally can be called *universal quantization formula* or *quantum Baker–Campbell–Hausdorff formula*. They proved that if one starts with a Lie bialgebra $L[[\hbar]]$ and applies the functor $Quant$ to it and then $deQuant$, the resulting Lie bialgebra will be isomorphic to $L[[\hbar]]$. The same is true if one starts with a quantum group H: $Quant(deQuant(H))$ will be isomorphic to H.

One of the uses of Lie groups – Lie algebras correspondence is the classification of semisimple Lie groups, because the classification of semisimple Lie algebras is a much easier problem. In the same way one can use Theorem 1.3 as an approach to the classification of quantum groups over semisimple Lie algebras. This was done in the works [10, 11]. The rest of the paper is dedicated to the exposition of the main results of that works.

§2. First steps of the classification

Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$. We have seen that the classification of quantum groups over $\mathfrak{g}$ is equivalent to the classification of Lie bialgebra structures on $\mathfrak{g}[[\hbar]] := \mathfrak{g} \otimes \mathbb{C}[[\hbar]]$. It is easy to see that any Lie bialgebra structure on $\mathfrak{g}[[\hbar]]$ gives rise to a Lie bialgebra structure on $\mathfrak{g}((\hbar)) := \mathfrak{g} \otimes \mathbb{C}((\hbar))$ and any Lie bialgebra structure on $\mathfrak{g}((\hbar))$ becomes a Lie biagebra structure on $\mathfrak{g}[[\hbar]]$ after a multiplication by an appropriate power of $\hbar$. Therefore, it is enough to classify Lie bialgebra structures on $\mathfrak{g}((\hbar))$.

Let us first look at the classification of Lie bialgebra structures on semisimple Lie algebras over an algebraically closed field $\mathbb{F}$ of characteristic zero. This classification was obtained by Belavin and Drinfeld [1]. We will now give a brief outline of their results. Let δ be a Lie bialgebra structure on $\mathfrak{g}$. First, one notices that the "compatibility condition" for δ is equivalent to the fact that δ is a cocycle. From the triviality of cohomology of simple Lie algebras we see that there exists $r \in \mathfrak{g} \otimes \mathfrak{g}$ such that $\delta = dr$. The condition that δ is a Lie bialgebra structure can be rewritten in terms of r : it turns out that after an appropriate scaling r should satisfy the classical Yang-Baxter equation. There are two quite different cases, r skewsymmetric or non-skewsymmetric. In the first case there is no hope to obtain a meaningful classification. However, there is a lot of structure associated to a skewsymmetric r-matrix, these objects are intimately related to quasi-Frobenius Lie algebras [1]. In the second case Belavin and Drinfeld found the explicit formulas for r-matrices up to conjugation.

Theorem 2.1. *Let $\mathfrak{g}$ be a simple Lie algebra over an algebraically closed field of characteristic zero. Then any Lie bialgebra structure on $\mathfrak{g}$ is coboundary. Let r be a corresponding r-matrix. If r is not skewsymmetric then for some root decomposition we have*

$$r = r_0 + \sum_{\alpha>0} e_{-\alpha} \otimes e_\alpha + \sum_{\alpha \in \mathrm{Span}(\Gamma_1)^+} \sum_{k \in \mathbb{N}} e_{-\alpha} \wedge e_{\tau^k(\alpha)}.$$

Here Γ_1, Γ_2 are the subsets of the set of simple roots, $\tau : \Gamma_1 \to \Gamma_2$ is isometric bijection, and for every $\alpha \in \Gamma_1$ there exists $k \in \mathbb{N}$ such that $\tau^k(\alpha) \in \Gamma_2 \setminus \Gamma_1$. The triple $(\Gamma_1, \Gamma_2, \tau)$ is called admissible. *The tensor $r_0 \in \mathfrak{h} \otimes \mathfrak{h}$ must satisfy the following two conditions:*

(1) *$r_0 + r_0^{21} = \sum t_k \otimes t_k$, where t_k is an orthonormal basis of $\mathfrak{h}$,*

(2) *for any $\alpha \in \Gamma_1$ we have $(\tau(\alpha) \otimes \mathrm{id} + \mathrm{id} \otimes \alpha) r_0 = 0$.*

It is worth noticing that there is an equivalent way to distinguish skewsymmetric and non-skewsymmetric r-matrices: in the first case the Drinfeld double $D(\mathfrak{g})$ is isomorphic to $\mathfrak{g}\otimes\mathbb{F}[\varepsilon]$, $\varepsilon^2=0$, in the second case $D(\mathfrak{g})\simeq\mathfrak{g}\oplus\mathfrak{g}$, see [17].

We want to obtain a version of Belavin–Drinfeld classification over the non-closed field $\mathbb{C}((\hbar))$. Let again $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$. First notice that we have a natural notion of equivalence for Lie bialgebras on $\mathfrak{g}((\hbar))$: $\delta_1\sim\delta_2$ if and only if there exists $\lambda\in\mathbb{C}((\hbar))$ and $X\in G(\mathbb{C}((\hbar)))$ such that $\delta_1=\lambda\mathrm{Ad}_X\delta_2$. Here G is an algebraic group associated to $\mathfrak{g}$.

Any Lie bialgebra structure on $\mathfrak{g}((\hbar))$ can be lifted to $\mathfrak{g}\otimes\overline{\mathbb{C}((\hbar))}$. Over the algebraically closed field $\overline{\mathbb{C}((\hbar))}$ we have the Belavin–Drinfeld classification. Therefore, any Lie bialgebra structure on $\mathfrak{g}((\hbar))$ is given by an r-matrix of the form $\lambda\mathrm{Ad}_X r$, where r is an r-matrix from the Belavin–Drinfeld list or a skewsymmetric r-matrix. One can prove that for a non-skew matrix up to equivalence λ is either 1 or $\sqrt{\hbar}$. Therefore, for any non-skew matrix from the Belavin–Drinfeld list there are two sets $H^1_{BD}(r_{BD})$ and $\overline{H}^1_{BD}(r_{BD})$ of equivalence classes of r-matrices. $H^1_{BD}(r_{BD})$ parametrizes the equivalence classes of r-matrices of the form $\mathrm{Ad}_X r_{BD}$ that define a Lie bialgebra structure on $\mathfrak{g}((\hbar))$ and, respectively, $\overline{H}^1_{BD}(r_{BD})$ parametrizes equivalence classes of matrices of the form $\sqrt{\hbar}\mathrm{Ad}_X r_{BD}$. We call $\overline{H}^1_{BD}(r_{BD})$ and $H^1_{BD}(r_{BD})$ the set of, respectively, twisted and non-twisted Belavin–Drinfeld cohomologies. Analogously for a skewsymmetric r-matrix r we define the Frobenius cohomology set $H^1_F(r)$.

There is an alternative way to see the difference between twisted and non-twisted Lie bialgebra structures. Let us look at the structure of the Drinfeld double $D(\mathfrak{g})$. It easily follows from methods developed in [15] that there are three possible cases: $D(\mathfrak{g}((\hbar)))$ can be isomorphic to $\mathfrak{g}((\hbar))\oplus\mathfrak{g}((\hbar))$, $\mathfrak{g}((\hbar))[\sqrt{\hbar}]$ or to $\mathfrak{g}((\hbar))[\varepsilon]$, where $\varepsilon^2=0$. These possibilities precisely correspond to the non-twisted, twisted and skew cases respectively.

We have shown that all Lie bialgebra structures on $\mathfrak{g}$ fall into one of the three types: non-twisted, twisted or skew. In what follows we will examine each case in more detail.

§3. Non-twisted case

We have defined $H^1_{BD}(r_{BD})$ as a set of equivalence classes of Lie bialgebra structures. However, there is an equivalent definition that appeals only to the inner structure of $\mathfrak{g}((\hbar))$. In what follows G is an algebraic group that corresponds to $\mathfrak{g}$.

Definition 3.1. The *centralizer* $C(r)$ of an r-matrix r is the set of all $X \in G(\overline{\mathbb{C}((\hbar))})$ such that $\mathrm{Ad}_X r = r$.

Definition 3.2. $X \in G(\overline{\mathbb{C}((\hbar))})$ is called a *non-twisted Belavin–Drinfeld cocycle* for r_{BD} if for any $\sigma \in \mathrm{Gal}(\overline{\mathbb{C}((\hbar))}/\mathbb{C}((\hbar)))$ we have $X^{-1}\sigma(X) \in C(r_{BD})$. The set of non-twisted cocycles will be denoted by $Z(r_{BD}) = Z(G, r_{BD})$.

Definition 3.3. Two cocycles $X_1, X_2 \in Z(r_{BD})$ are called *equivalent* if there exist $Q \in G(\mathbb{C}((\hbar)))$ and $C \in C(r_{BD})$ such that $X_1 = QX_2C$.

Definition 3.4. The set of equivalence classes of non-twisted cocycles is denoted by $H^1_{BD}(r_{BD}) = H^1_{BD}(G, r_{BD})$ and is called the *non-twisted Belavin–Drinfeld cohomology.*

We were able to compute H^1_{BD} for the algebras of $A-D$ series. First let us make a small remark about A_n case. In this case $\mathfrak{g}((\hbar))$ is naturally acted upon by the group $GL(n)$ and we can compute the cohomology with respect to conjugation by $GL(n)$ or $SL(n)$. To distinguish between these cases we write $H^1_{BD}(GL(n), r_{BD})$ and $H^1_{BD}(SL(n), r_{BD})$.

If $(\Gamma_1, \Gamma_2, \tau)$ is an admissible triple then the set $\alpha, \tau(\alpha), \ldots, \tau^k(\alpha)$, where $\alpha \in \Gamma_1 \setminus \Gamma_2$ and $\tau^k(\alpha) \in \Gamma_2 \setminus \Gamma_1$ will be called a string of τ. The following table describes H^1_{BD} for algebras of type $A-D$. The cohomology is called trivial if $|H^1_{BD}(r_{BD})| = 1$.

Algebra	Triple type	H^1_{BD} for an arbitrary field	H^1_{BD} for $\mathbb{C}((\hbar))$
A_n		trivial ($GL(n)$ case)	
B_n		trivial	
C_n		trivial	
D_n	there exists a string of τ that contains α_{n-1} and α_n	$F^*/(F^*)^2$	2 elements
	α_{n-1} and α_n do not belong to the same string of τ	trivial	

Remark 3.5. In this paper α_n, α_{n-1} are the branchendpoints in the Dynkin diagram for D_n

Remark 3.6. One can similarly define Belavin–Drinfeld cohomologies over an arbitrary field F as a tool to understand Lie bialgebra structures on $\mathfrak{g}(F)$.

The result for $H^1_{BD}(SL(n), r_{BD})$ is more interesting. Let $\alpha_{i_1}, \dots, \alpha_{i_k}$ be a string of τ, $\tau(\alpha_{i_p}) = \alpha_{i_{p+1}}$. If $\tau(\alpha_{i_p})$ is not defined, then anyway we define the corresponding string, which consists of one element $\{\alpha_{i_p}\}$ only.

For any string $S = \{\alpha_{i_1}, \dots, \alpha_{i_k}\}$ of τ, we define the weight of S by $w_S = \sum_p i_p$. Moreover, for any Belavin–Drinfeld triple we will also formally consider the string $\{\alpha_n\}$ with weight n. Let $t_1, \dots, t_n$ be the ends of the strings with weights $w_1, \dots, w_n$. We note that some indices in $w_1, \dots, w_n$ are missing unless Γ_1 is an empty set and $w_n = n$ is always present. Let $N = GCD(w_1, \dots, w_n)$.

Theorem 3.7. *The number of elements of $H^1_{BD}(SL(n), r)$ is N. Each cohomology class contains a diagonal matrix $D = A_1 A_2$, where $A_2 \in C(GL(n), r)$ and $A_1 \in \mathrm{diag}(n, \mathbb{C}((\hbar)))$. Two such diagonal matrices $D_1 = A_1 A_2$ and $D_2 = B_1 B_2$ are contained in the same class of $H^1_{BD}(SL(n), r)$ if and only if $\det(A_1) = \det(B_1)$ in $\mathbb{C}((\hbar))^*/(\mathbb{C}((\hbar))^*)^N$.*

§4. Twisted case

As in non-twisted case there is a way to define $\overline{H}^1_{BD}$ without mentioning Lie bialgebra structures.

Theorem 4.1. *$a\mathrm{Ad}_X r_{BD}$ defines a Lie bialgebra structure on $\mathfrak{g}(\mathbb{C}((\hbar)))$ if and only if X is a non-twisted cocycle for the field $\mathbb{C}((\hbar))[\sqrt{\hbar}]$ and $\mathrm{Ad}_{X^{-1}\sigma_0(X)} r_{BD} = r^{21}_{BD}$. Here σ_0 is the non-trivial element of the group $\mathrm{Gal}(\mathbb{C}((\hbar))[\sqrt{\hbar}]/\mathbb{C}((\hbar)))$.*

To deal with the condition $\mathrm{Ad}_{X^{-1}\sigma_0(X)} r_{BD} = r^{21}_{BD}$ we classified all triples $(\Gamma_1, \Gamma_2, \tau)$ such that r^{21}_{BD} and r_{BD} are conjugate. In each case we found a suitable $S \in G(F)$ such that $r^{21}_{BD} = \mathrm{Ad}_S r_{BD}$. Then we can define Belavin–Drinfeld cocycles and cohomologies similar to the non-twisted case. In all cases $S^2 = \pm 1$.

Definition 4.2. $X \in G(\overline{\mathbb{C}((\hbar))})$ is called a *Belavin–Drinfeld twisted cocycle* if for any $\sigma \in \mathrm{Gal}(\overline{\mathbb{C}((\hbar))}/\mathbb{C}((\hbar))[\sqrt{\hbar}])$ we have $X^{-1}\sigma(X) \in C(r_{BD})$

and $SX^{-1}\sigma_0(X) \in C(r_{BD})$. The set of Belavin–Drinfeld twisted cocycles is denoted by $\overline{Z}(r_{BD}) = \overline{Z}(G, r_{BD})$.

Definition 4.3. Two twisted cocycles X_1, X_2 are called *equivalent* if there exist $Q \in G(\mathbb{C}((\hbar)))$ and $C \in C(r_{BD})$ such that $X_1 = QX_2C$. The set of equivalence classes of twisted cocycles is called the *twisted Belavin–Drinfeld cohomology* and is denoted by $\overline{H}^1_{BD}(r_{BD}) = \overline{H}^1_{BD}(G, r_{BD})$.

<table>
<tr><th>Algebra</th><th colspan="2">Triple type</th><th>$\overline{H}^1_{BD}$ for $\mathbb{C}((\hbar))$</th></tr>
<tr><td rowspan="2">A_n</td><td colspan="2">$s\tau = \tau^{-1}s$, where s is the non-trivial automorphism of the Dynkin diagram</td><td>one element</td></tr>
<tr><td colspan="2">other</td><td>empty</td></tr>
<tr><td rowspan="2">B_n</td><td colspan="2">Drinfeld-Jimbo</td><td>one element</td></tr>
<tr><td colspan="2">not DJ</td><td>empty</td></tr>
<tr><td rowspan="2">C_n</td><td colspan="2">Drinfeld-Jimbo</td><td>one element</td></tr>
<tr><td colspan="2">not DJ</td><td>empty</td></tr>
<tr><td rowspan="5">D_n</td><td rowspan="2">even n</td><td>Drinfeld-Jimbo</td><td>one element</td></tr>
<tr><td>not DJ</td><td>empty</td></tr>
<tr><td rowspan="3">odd n</td><td>$\Gamma_1 = \{\alpha_{n-1}\}$
$\tau(\alpha_{n-1}) = \alpha_n$;
$\Gamma_1 = \{\alpha_n\}$
$\tau(\alpha_n) = \alpha_{n-1}$;
$\Gamma_1 = (\alpha_{n-1}, \alpha_k)$, $k \neq n$
$\tau(\alpha_{n-1}) = \alpha_k, \tau(\alpha(k)) = \alpha_n$;
$\Gamma_1 = (\alpha_n, \alpha_k)$, $k \neq n-1$
$\tau(\alpha_n = \alpha_k), \tau(\alpha_k) = \alpha_{n-1}$</td><td>two elements</td></tr>
<tr><td>Drinfeld-Jimbo</td><td>one element</td></tr>
<tr><td>not DJ</td><td>empty</td></tr>
</table>

Here the cohomology for $\mathfrak{sl}_n$ is considered with respect to the group $GL(n)$. For the results for A_n over arbitrary field see [16].

§5. Skewsymmetric case

Following the pattern in [1], it can be easily proved that the classification of Lie bialgebra structures related to skew (triangular) r-matrices on $\mathfrak{g}((\hbar))$ is equivalent to the classification of quasi-Frobenius Lie subalgebras of $\mathfrak{g}((\hbar))$. This can be used to prove that if r is skewsymmetric then r has to be defined over $\mathbb{C}((\hbar))$. However, different r-matrices defined over $\mathbb{C}((\hbar))$ can be conjugate over $\overline{\mathbb{C}((\hbar))}$. We can define Frobenius cohomology similarly to Belavin–Drinfeld cohomology. We call two r-matrices equivalent if there exists $a \in \mathbb{C}((\hbar))$, $X \in G(\mathbb{C}((\hbar)))$ such that $r_1 = a\mathrm{Ad}_X r_2$. If r defines a Lie bialgebra structure on $\mathfrak{g}((\hbar))$ then we define the Frobenius cohomology set $H^1_F(r)$ to be the set of equivalence classes of r-matrices that are conjugate to r over $\overline{\mathbb{C}((\hbar))}$. We do not have a classification of skew r-matrices even over an algebraically closed field, but this cohomology can be computed in a way similar to Belavin–Drinfeld case.

Definition 5.1. The *centralizer* $C(r)$ of an r-matrix r is the set of all $X \in G(\overline{\mathbb{C}((\hbar))})$ such that $\mathrm{Ad}_X r = r$.

Definition 5.2. $X \in G(\overline{\mathbb{C}((\hbar))})$ is called a *non-twisted Frobenius cocycle* for r if for any $\sigma \in \mathrm{Gal}(\overline{\mathbb{C}((\hbar))}/\mathbb{C}((\hbar)))$ we have $X^{-1}\sigma(X) \in C(r)$. The set of non-twisted cocycles will be denoted by $Z_F(r) = Z_F(G, r)$.

Definition 5.3. Two cocycles $X_1, X_2 \in Z_F(r)$ are called *equivalent* if there exists $Q \in G(\mathbb{C}((\hbar)))$ and $C \in C(r)$ such that $X_1 = QX_2C$.

Definition 5.4. The set of equivalence classes of Frobenius cocycles is denoted by $H^1_F(r) = H^1_F(G, r)$ and is called the *Frobenius cohomology.*

Example 5.5. Let r_J be the Jordan r-matrix, i.e. $r_J = E \wedge H$. Then $H^1_F(r_J)$ is trivial. Here $\{E, F, H\}$ is the standard basis in $\mathfrak{sl}_2$

§6. Historical Remarks

Quantum groups (as in Definition 1.1) were defined by Drinfeld in his talk at the International Congress of Mathematicians in Berkeley, 1986. Relations between quantum groups and quantum algebras (quantization and dequantization functors, quantum Baker-Campbell-Hausdorff formula) were obtained by Etingof and Kazhdan in a series of papers [5, 6].

The first example of a quantum group of the non-twisted type is due to Kulish and Reshetikhin [13]. Generalizations for all simple Lie algebras were obtained by Drinfeld and Jimbo [3, 9], where they found quantum groups which quantize Lie bialgebra structures on $\mathfrak{g}$ defined by $\Gamma_1 = \Gamma_2 = \varnothing$.

Further classes of Lie bialgebra structures on $\mathfrak{g}$, related to certain triples $(\Gamma_1, \Gamma_2, \tau)$, were quantized by Kulish and Mudrov in [12].

Finally, Etingof, Schiffman, and Schedler quantized all Lie bialgebra structures defined by all admissible triples $(\Gamma_1, \Gamma_2, \tau)$ [7].

There are no explicit formulas for quantum groups related to the twisted Belavin–Drinfeld cohomologies.

Construction of quantum groups of skewsymmetric type appeared in the work of Drinfeld [4] by means of a certain twisting element F. The first explicit formula for F is due to Coll, Gerstenhaber, and Giaquinto [2]. This formula was used by Kulish and Stolin to explicitly quantize a certain nonstandard Lie bialgebra structure on the polynomial Lie algebra $\mathfrak{sl}_2[u]$.

This paper is dedicated to Petr P. Kulish on the occasion of his 70-years jubilee. The authors are thankful for valuable remarks to G. Rozenblum who joins the congratulations.

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