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R. Vidunas, A. V. Kitaev

COMPUTATION OF *RS*-PULLBACK TRANSFORMATIONS FOR ALGEBRAIC PAINLEVÉ VI SOLUTIONS

Abstract. Algebraic solutions of the sixth Painlevé equation can be constructed with the help of *RS*-transformations of the hypergeometric equations. Construction of these transformations includes specially ramified rational coverings of the Riemann sphere and corresponding Schlesinger transformations (*S*-transformations). Some algebraic solutions can be constructed from rational coverings alone, without obtaining the corresponding pullbacked isomonodromy Fuchsian system, i.e., without *S* part of the *RS* transformations. At the same time one and the same covering can be used to pullback different hypergeometric equations, resulting in different algebraic Painlevé VI solutions. In case of high degree coverings construction of *S* parts of the *RS*-transformations may represent some computational difficulties. This paper presents computations of explicit *RS*-pullback transformations, and derivation of algebraic Painlevé VI solutions from them. As an example, we present computation of all seed solutions for pull-backs of hyperbolic hypergeometric equations.

To Professor Peter Kulish in honor of his 70th birthday

§1. Introduction

The sixth Painlevé equation is, canonically,

$$\frac{d^2y}{dt^2} = \frac{1}{2}\left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-t}\right)\left(\frac{dy}{dt}\right)^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{y-t}\right)\frac{dy}{dt}$$

Key words and phrases: The sixth Painlevé equation, Schlesinger transformations, isomonodromy deformations, algebraic function, rational covering, hypergeometric equation.

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$$+\frac{y(y-1)(y-t)}{t^2(t-1)^2}\left(\alpha+\beta\frac{t}{y^2}+\gamma\frac{t-1}{(y-1)^2}+\delta\frac{t(t-1)}{(y-t)^2}\right), \qquad (1.1)$$

where α, β, γ, $\delta \in \mathbb{C}$ are parameters. As well-known [11], its solutions define isomonodromic deformations (with respect to t) of the 2×2 matrix Fuchsian equation with 4 singular points ($\lambda = 0, 1, t$, and ∞):

$$\frac{d}{dz}\Psi = \left(\frac{A_0}{z}+\frac{A_1}{z-1}+\frac{A_t}{z-t}\right)\Psi, \qquad \frac{d}{dz}A_k = 0 \quad \text{for } k\in\{0,1,t\}. \qquad (1.2)$$

The standard correspondence is due to Jimbo and Miwa [11]. We choose the traceless normalization of (1.2), so we assume that the eigenvalues of A_0, A_1, A_t are, respectively, $\pm\theta_0/2$, $\pm\theta_1/2$, $\pm\theta_t/2$, and that the matrix $A_\infty := -A_1 - A_2 - A_3$ is diagonal with the diagonal entries $\pm\theta_\infty/2$. Then the corresponding Painlevé equation has the parameters

$$\alpha = \frac{(\theta_\infty-1)^2}{2}, \quad \beta = -\frac{\theta_0^2}{2}, \quad \gamma = \frac{\theta_1^2}{2}, \quad \delta = \frac{1-\theta_t^2}{2}. \qquad (1.3)$$

We refer to the numbers θ_0, θ_1, θ_t and θ_∞ as *local monodromy differences*.

For any numbers $\nu_1, \nu_2, \nu_t, \nu_\infty$, let us denote by $P_{VI}(\nu_0, \nu_1, \nu_t, \nu_\infty; t)$ the Painlevé VI equation for the local monodromy differences $\theta_i = \nu_i$ for $i \in \{0, 1, t, \infty\}$, via (1.3). Note that changing the sign of ν_0, ν_1, ν_t or $1-\nu_\infty$ does not change the Painlevé equation. Fractional-linear transformations for the Painlevé VI equation permute the 4 singular points and the numbers $\nu_0, \nu_1, \nu_t, 1-\nu_\infty$.

Similarly, for any numbers $\nu_1, \nu_2, \nu_t, \nu_\infty$ and a solution $y(t)$ of $P_{VI}(\nu_0, \nu_1, \nu_t, \nu_\infty; t)$, let us denote by $E(\nu_0, \nu_1, \nu_t, \nu_\infty; y(t); z)$ a Fuchsian equation (1.2) corresponding to $y(t)$ by the Jimbo-Miwa correspondence. The Fuchsian equation is determined uniquely up to conjugation of A_0, A_1, A_t by a diagonal matrix (dependent on t only). In particular, $y(t) = t$ can be considered as a solution of $P_{VI}(e_0, e_1, 0, e_\infty; t)$. The equation $E(e_0, e_1, 0, e_\infty; t; z)$ is a Fuchsian equation with 3 singular points, actually without the parameter t. Its solutions can be expressed in terms of Gauss hypergeometric function see [11] or the Appendix in [23]. We refer to $E(e_0, e_1, 0, e_\infty; t; z)$ as a *matrix hypergeometric equation*, and see it as a matrix form of Euler's ordinary hypergeometric equation.

We consider pullback transformations of 2×2 Fuchsian systems $d\Psi(z)/dz = M(z)\Psi(z)$. They have the following general form:

$$z \mapsto R(x), \qquad \Psi(z) \mapsto S(x)\,\Psi(R(x)), \qquad (1.4)$$

where $R(x)$ is a rational function of x, and $S(x)$ is a Schlesinger transformation, usually designed to remove apparent singularities. For transformations to parametric isomonodromic equations, $R(x)$ and $S(x)$ may depend algebraically on parameter(s) as well. The transformed equation is

$$\frac{d\Psi(x)}{dx} = \left(\frac{dR(x)}{dx} S^{-1}(x) M(R(x)) S(x) - S^{-1}(x) \frac{dS(x)}{dx} \right) \Psi(x). \qquad (1.5)$$

In [12, 13], these pullback transformations are called *RS-transformations*, meaning that they are compositions of a rational change of the independent variable $z \mapsto R(x)$ and the Schlesinger transformation $S(x)$. The Schlesinger transformation $S(x)$ is analogous here to *projective equivalence* transformations $y(x) \to \theta(x)y(x)$ of ordinary differential equations. To merge terminology, we refer to these pullback transformations as *RS-pullbacks*, or *RS-pullback transformations*. If $S(x)$ is the identity transformation, we have a *direct pullback* of a Fuchsian equation.

The subject of this article is construction of *RS*-pullback transformations of matrix hypergeometric equations to isomonodromic Fuchsian systems with 4 singular points. To have so few singular points of the transformed equation, we usually have to start with a matrix hypergeometric equation with restricted local monodromy differences, and the R-part $R(x)$ must define a specially ramified covering of $\mathbb{P}^1$. In particular, the covering usually may ramify only above the 3 singular points of the hypergeometric equation, except that there is one additional simple (i.e., order 2) ramification point is allowed. Coverings ramified over 4 points of $\mathbb{P}^1$ in this way are called here *almost Belyi coverings*. Recall that a *Belyi function* is a rational function on an algebraic curve with at most 3 critical values; the respective covering of $\mathbb{P}^1$ by the algebraic curve is ramified above a set of 3 points only.

Suitable starting hypergeometric equations and ramification patterns of almost Belyi coverings can be classified rather easily [7, 13]. This is similar to classification of algebraic transformations of Gauss hypergeometric functions [24, 25], where Belyi functions typically occur. The computationally hard problem is construction of almost Belyi coverings from a priori suitable ramification patterns. This leads us towards Grothendieck's theory of dessins d'enfant. In particular, Hurwitz spaces for almost Belyi coverings with a fixed ramification pattern define isomonodromy parameters for the pullbacked Fuchsian equations. Effective computations of high degree

almost Belyi coverings are presented in [22]. In this paper, we use three coverings computed in [13].

Computation of S-parts of suitable RS-transformations does not look hard in principle. However, this problem is not as straightforward as finding suitable projective equivalence transformations for scalar differential equations. General Schlesinger transformations can be constructed by composing several simple Schlesinger transformations (each shifting just two local monodromy differences), as was done in [1,2,11]. More effectively, the method in [23] constructs Schlesinger transformations in one go, avoiding factorization of high degree polynomials when shifting local monodromy differences at all conjugate roots by the same integer. In the context of isomonodromy problems, this approach is adopted in [9] as well.

RS-pullback transformations to isomonodromic Fuchsian systems with 4 singular points gives solutions of the sixth Painlevé equations that are algebraic, because those solutions are determined algebraically by matrix entries of pullbacked equations (1.5) while those entries are algebraic functions in x and the isomonodromy parameter. The second author conjectured in [13] that all algebraic solutions of the sixth Painlevé equation can be obtained by RS-pullback transformations of matrix hypergeometric equations, up to Okamoto transformations [19]. This conjecture is certainly true if the monodromy group of the Fuchsian systems is finite, due to celebrated Klein's theorem [15]. Richard Fuchs [10] soon considered extension of Klein's theorem to algebraic solutions of Painlevé equations. Recently, Ohyama and Okumura [18] showed that algebraic solutions of Painlevé equations from the first to the fifth do arise from pull-back transformations of confluent hypergeometric equations, affirming the formulation of R. Fuchs.

The pullback method for computation of algebraic Painlevé VI solutions was previously suggested in [2, 7, 12, 13]. This method is substantially different from the representation-theoretic approach of Dubrovin–Mazzocco [8] and Boalch [4, 5]. Recently, Lisovyy and Tykhyy [16] used the representation-theoretic method to complete classification of algebraic Painlevé VI solutions. The mentioned conjecture in [13] is still interesting as a generalization of Klein's theorem. There is a similar situation with classification of algebraic solutions of the Lamé equation, where representation-theoretic methods (as in [3]) compete with Klein's pullback method (as in [17]).

One important observation is that the same rational covering $R(x)$ can be used in several *RS*-pullback transformations. For example, here we apply the same degree 10 covering to pullback three different matrix hypergeometric equations $E(1/7, 1/2, 0, 1/3; t; z)$, $E(2/7, 1/2, 0, 1/3; t; z)$ and $E(3/7, 1/2, 0, 1/3; t; z)$. We obtain Painlevé solutions of, respectively,

$$P_{VI}(1/7, 1/7, 1/7, 2/3; t),$$
$$P_{VI}(2/7, 2/7, 2/7, 1/3; t)$$

and

$$P_{VI}(3/7, 3/7, 3/7, 2/3; t),$$

unrelated by fractional-linear or Okamoto transformations. The first Painlevé solution is a fractional-linear version of solution [14, (3.16)–(3.17)]. The second Painlevé solution is the same as in [6, page 106]. The third Painlevé solution is new.

The article is organized as follows. Section 2 presents the covering of degree 10 for our exmaples; it was previously used in [14]. There we also mention how some Painlevé VI solutions can be computed from the rational coverings alone, without computation of full *RS*-transformations. This kind of possibility is noticed in [7, 13], and is summarized in Theorem 3.1 below. In Section 4 a more general Theorem 4.1 from [23] is cited. Thereby a direct formula for algebraic Painlevé VI solutions is given, with minimum information from full *RS*-transformations. In Section 5, representative *RS*-pullback transformations of "hyperbolic" hypergeometric equations $E(1/2, 1/3, 0, 1/7; t; z)$ and $E(1/2, 1/3, 0, 1/8; t; z)$ to isomonodromic Fuchsian systems with 4 singularities are sumarized, and the corresponding Painlevé VI solutions are presented (hereby complementing [14]). The Appendix presents a formula for composition of two quadratic transformations of Painlevé VI solutions; a general degree formula for the almost Belyi coverings relevant to algebraic Painlevé VI solutions; and geometric interpretation of the latter formula.

The authors prepared Maple 9.5 worksheets supplementing this article and [22, 23], with the formulas in Maple input format, and demonstration of key computations. To access the worksheet, readers may contact the authors, or search a current website of the first author on the internet.

§2. The working covering and RS-transformations

First we introduce notation for ramification patterns, and later for RS-transformations. A ramification pattern for an almost Belyi covering of degree n is denoted by $R_4(P_1|\,P_2|\,P_3)$, where P_1, P_2, P_3 are three partitions of n specifying the ramification orders above three points. The ramification pattern above the fourth ramification locus is assumed to be $2+1+1+\ldots+1$. By *the extra ramification point* we refer to the simple ramification point in the fourth fiber. The Hurwitz space for such a ramification pattern is generally one-dimensional [26, Proposition 3.1].

We use only genus 0 almost Belyi coverings, and write them as $\mathbb{P}^1_x \to \mathbb{P}^1_z$, meaning that the projective line with the projective coordinate x is mapped to the projective line with the coordinate z. Then the total number of parts in P_1, P_2, P_3 must be equal to $n+3$, according to [13, Proposition 2.1]; this is a consequence of Riemann-Hurwitz formula.

The similar notation for a ramification pattern for a Belyi function is $R_3(P_1|\,P_2|\,P_3)$, as in [1], [14]. The total number of parts in P_1, P_2, P_3 must be equal to $n+2$, as stated in [24, Lemma 2.4] or [13, Proposition 2.1].

Our working almost Belyi covering has the following ramification pattern:

$$R_4\big(7+1+1+1\mid 2+2+2+2+2\mid 3+3+3+1\big). \tag{2.6}$$

The covering has degree 10. The three specified fibers with ramified points can be brought to any three distinct locations by a fractional-linear transformation of $\mathbb{P}^1_z$. We assign the first partition to $z=0$, and the next two partitions — to $z=1$ and $z=\infty$ respectively. Similarly, by a fractional-linear transformation of $\mathbb{P}^1_x$ we may choose any three x-points[1] as $x=0$, $x=1$, $x=\infty$.

All coverings with ramification pattern (2.6) can be computed on modern computers either using the most straightforward method, or an improved method [22] that uses differentiation. Up to fractional-linear transformations and reparametrization, there is one general such covering given

[1]Strictly speaking, the x-points in our settings are curves, or branches, parametrized by an isomonodromy parameter t or other parameter, since the Hurwitz spaces for almost Belyi maps are one-dimensional. For simplicity, we ignore the dimensions introduced by such parameters, and consider a one-dimensional Hurwitz space as a generic point.

by

$$\varphi_{10}(x)=\frac{x^7\,F_{10}}{4\,G_{10}^3},\qquad\text{or}\qquad \varphi_{10}(x)-1=\frac{P_{10}^2}{4\,G_{10}^3},\tag{2.7}$$

where

$$\begin{aligned}F_{10}&=9s^2x^3-2(2s^3+6s^2+15s-16)x^2+3(8s^2+8s-13)x-36(s-1),\\ G_{10}&=2(s+1)x^3-(s^2+4s+10)x^2+6(s+2)x-9,\\ P_{10}&=3sx^5-3(2s^2+6s+7)x^4+2(s^3+6s^2+30s+35)x^3\\ &\quad-18(s^2+4s+7)x^2+54(s+2)x-54.\end{aligned}\tag{2.8}$$

The extra ramification point is $x=7(s-1)/s(s+1)$.

For direct applications to the Painlevé VI equation, it is required to normalize the point above $z=\infty$ with the deviating ramification order 1 and the three nonramified points above $\{0,1,\infty\}\subset\mathbb{P}^1_z$ as $x=0$, $x=1$, $x=\infty$, $x=t$. We refer to explicit almost Belyi coverings normalized this way as *properly normalized*. A properly normalized covering with ramification pattern (2.6) was first computed in [13]. To get a properly normalized expression, we reparametrize

$$s=-\frac{(u+2)(u^2-u+2)}{2(u-1)},\tag{2.9}$$

and make the fractional-linear transformation

$$\begin{aligned}x\mapsto{}&\frac{(u-1)^2(u^2+3)w}{9(u^2-u+2)^2}(2x-1)\\ &-\frac{(u-1)(u^5+u^4-2u^3+18u^2-9u+27)}{9(u^2-u+2)^2},\end{aligned}\tag{2.10}$$

where $w=\sqrt{(u-1)(u+5)(u^2+3)}$. The obtained expression is

$$\begin{aligned}\widehat{\varphi}_{10}(x)={}&-\frac{(u-1)^2(u+2)^2w^3}{8(u+5)(u^3+u^2-2u+6)^3}\\ &\times\frac{x\,(x-1)\,(x-t_{10})\,(x-t_{10}^*)^7}{\big((x^2-x)\,(x-\frac12-L_3)-L_4(x-\frac12)+L_5\big)^3},\end{aligned}\tag{2.11}$$

where

$$t_{10} = \frac{1}{2} + \frac{u^9+3u^8-3u^7+7u^6-21u^5+21u^4-161u^3-27u^2-144u-108}{2(u-1)^3(u+2)^2(u^2+3)\sqrt{(u-1)(u+5)(u^2+3)}},$$

$$t^*_{10} = \frac{1}{2} + \frac{u^5+u^4-2u^3+18u^2-9u+27}{2(u-1)(u^2+3)\sqrt{(u-1)(u+5)(u^2+3)}}, \tag{2.12}$$

and

$$L_3 = \frac{(u^5+4u^4+u^3+18u^2+24u+36)(u^7+14u^4-21u^3+42u^2+36)}{8(u-1)^2(u+5)(u^2+3)^3(u^3+u^2-2u+6)},$$

$$L_4 = \frac{3(u^{10}-6u^8+28u^7-99u^6+252u^5-668u^4+1008u^3-1212u^2+672u-408)}{8w(u-1)^3(u^2+3)(u^3+u^2-2u+6)},$$

$$L_5 = \frac{u^{15}+5u^{14}+28u^{12}+98u^{11}-126u^{10}+616u^9-184u^8+333u^7+1785u^6-1512u^5+3276u^4+6048u^2+3888u+1296}{16w(u-1)^3(u+5)(u^2+3)^3(u^3+u^2-2u+6)}.$$

The Hurwitz space parametrising this properly normalized almost Belyi covering has still genus 0. To get the rational covering $\lambda_1(\lambda)$ in [14], one has to consider $t_{10}/\widehat{\varphi}_{10}(x)$, and substitute $x \mapsto t_{10}/x$, $u \mapsto 2/s - 1$.

In [13], the following symbol is introduced to denote RS-pullback transformations of $E(e_0, e_1, 0, e_\infty; t; z)$ with respect to a covering with ramification pattern $R_4(P_0|P_1|P_\infty)$:

$$RS_4^2 \left(\begin{array}{c|c|c} e_0 & e_1 & e_\infty \\ P_0 & P_1 & P_\infty \end{array} \right), \tag{2.13}$$

where the subscripts 2 and 4 indicate a second order Fuchsian system with 4 singular points after the pullback. We assume the same assignment of the fibers $z = 0$, $z = 1$, $z = \infty$ as for the R_4-notation. Location of the x-branches $0, 1, t, \infty$ does not have to be normalized. As was noticed in [13] and [7], some algebraic Painlevé VI solutions determined by RS-pullback transformations $RS_4^2 \left(\begin{smallmatrix} 1/k_0 & 1/k_1 & 1/k_\infty \\ P_0 & P_1 & P_\infty \end{smallmatrix} \right)$, with $k_0, k_1, k_\infty \in \mathbb{Z}$, can be calculated from the rational covering alone, without actual computation of the full RS-pullbacks. We discuss this possibility in Section 3. Our covering $\widehat{\varphi}_{10}(x)$ immediately gives a solution of $P_{VI}(1/7, 1/7, 1/7, 2/3; t)$. In Section 4 we formulate a direct way to obtain algebraic Painlevé VI solutions via computation of suitable syzygies between x^2 (or x^3), P_{10}, G_{10}. We obtain algebraic solutions of

$$P_{VI}(2/7, 2/7, 2/7, 1/3; t) \quad \text{and} \quad P_{VI}(3/7, 3/7, 3/7, 2/3; t)$$

by implicitly using RS-pullback transformations

$$RS_4^2\left(\begin{array}{c|c|c} 2/7 & 1/2 & 1/3 \\ 7+1+1+1 & 2+2+2+2+2 & 3+3+3+1 \end{array}\right)$$

and

$$RS_4^2\left(\begin{array}{c|c|c} 3/7 & 1/2 & 1/3 \\ 7+1+1+1 & 2+2+2+2+2 & 3+3+3+1 \end{array}\right),$$

respectively.

§3. Pullback coverings and algebraic Painlevé VI solutions

As noticed in [13] and [7], some algebraic Painlevé VI solutions can be computed knowing just a pullback covering, without computation of pullbacked Fuchsian equations of full RS-transformations removing all apparent singularities of a direct pullback. Here we formulate the most interesting general situation.

Theorem 3.1. *Let k_0, k_1, k_∞ denote three integers, all ≥ 2. Let $\varphi : \mathbb{P}^1_x \to \mathbb{P}^1_z$ denote an almost Belyi map, dependent on a parameter t. Suppose that the following conditions are satisfied:*

(i) *The covering $z = \varphi(x)$ is ramified above the points $z = 0$, $z = 1$, $z = \infty$; there is one simply ramified point $x = y$ above $\mathbb{P}^1_z \setminus \{0, 1, \infty\}$; and there are no other ramified points.*

(ii) *The points $x = 0$, $x = 1$, $x = \infty$, $x = t$ lie above the set $\{0, 1, \infty\} \subset \mathbb{P}^1_z$.*

(iii) *The points in $\varphi^{-1}(0) \setminus \{0, 1, t, \infty\}$ are all ramified with the order k_0. The points in $\varphi^{-1}(1) \setminus \{0, 1, t, \infty\}$ are all ramified with the order k_1. The points in $\varphi^{-1}(\infty) \setminus \{0, 1, t, \infty\}$ are all ramified with the order k_∞.*

Let a_0, a_1, a_t, a_∞ denote the ramification orders at $x = 0, 1, t, \infty$, respectively. Then the point $x = y$, as a function of $x = t$, is an algebraic solution of

$$P_{VI}\left(\frac{a_0}{k_{\varphi(0)}}, \frac{a_1}{k_{\varphi(1)}}, \frac{a_t}{k_{\varphi(t)}}, 1 - \frac{a_\infty}{k_{\varphi(\infty)}}; t\right). \tag{3.14}$$

Proof. Let $R_4(P_0|P_1|P_\infty)$ denote the ramification pattern of the covering $z = \varphi(x)$. We aim for an RS-pullback transformation

$$RS_4^2\left(\begin{array}{c|c|c} 1/k_0 & 1/k_1 & 1+1/k_\infty \\ P_0 & P_1 & P_\infty \end{array}\right)$$

with respect to $\varphi(x)$. Let d denote the degree of $\varphi(x)$. For time being, we assume that the point $x=\infty$ lies above $z=\infty$.

The direct pullback of the hypergeometric equation $E(1/k_0, 1/k_1, 0, 1+1/k_\infty; t; z)$ with respect to $\varphi(x)$ has apparent singularities at the points mentioned in part *(iii)* above. Nonapparent singularities are possibly $x=0$, $x=1$, $x=t$ and $x=\infty$. The lower-left entry of the direct pullback is equal, up to a factor independent of x, to $\varphi'/\varphi(1-\varphi)$, which is the logarithmic derivative of $\varphi/(\varphi-1)$. The poles of this rational function are simple, and they are precisely the points above $z=0$ and $z=1$. The zeroes of the rational function are the following: the extra ramification point of φ (a simple zero); and the points above $z=\infty$, with multiplicities one less than the respective ramification orders.

Notice that if we apply a Schlesinger transformation of the upper triangular form $S=\frac{1}{\sqrt{(x-\alpha_1)(x-\alpha_2)}}\left(\begin{smallmatrix} x-\alpha_1 & \alpha_3 \\ 0 & x-\alpha_2 \end{smallmatrix}\right)$, where α_1, α_2, α_3 are independent of x, then the lower-left entry of the matrix differential equation changes by the factor $(x-\alpha_1)/(x-\alpha_2)$ and a factor independent of x. If the point $x=\alpha_2$ is above $z=\infty$, this Schlesinger transformation (with appropriate α_3) decreases the local monodromy differences at $x=\alpha_1$ and $x=\alpha_2$ by 1. Similarly, the Schlesinger transformation $S=\frac{1}{\sqrt{x-\alpha_1}}\left(\begin{smallmatrix} x-\alpha_1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ changes the local monodromy differences at $x=\alpha_1$ and $x=\infty$ by 1, and it multiplies the lower-left entry by the factor $x-\alpha_1$ (and a factor independent of x).

Let h denote the number of distinct apparent singularities above $z=\infty$. There are in total $(d+3)-4-h$ apparent singularities above $z=0$ and $z=1$. We can construct $d-1-h$ simple Schlesinger transformations of the forms presented just above, so that α_1 runs through the set of apparent singularities above $z=0$ and $z=1$, and each point $x=\alpha_2$ or $x=\infty$ above $z=\infty$ is chosen n_x times, where

$$n_x=\begin{cases} \text{the ramification order at } x, \text{ minus 1}, & \text{if } x=\infty \text{ or} \\ & \text{an apparent singularity;} \\ \text{the ramification order at } x, & \text{otherwise.} \end{cases}$$

The composite effect of these $d-1-h$ transformations is removal of all apparent singularities above $z=0$, $z=1$, $z=\infty$; and reducing the local monodromy difference at $x=\infty$ from $a_\infty+a_\infty/k_\infty$ to $1+a_\infty/k_\infty$. The local monodromy differences at the other singularities are $a_0/k_{\varphi(0)}$, $a_1/k_{\varphi(1)}$,

$a_t/k_{\varphi(t)}$ after the composite transformation. Hence the transformed equation has (at most) four singularities. The transformed equation is

$$E\left(\frac{a_0}{k_{\varphi(0)}}, \frac{a_1}{k_{\varphi(1)}}, \frac{a_t}{k_{\varphi(t)}}, 1+\frac{a_\infty}{k_\infty}; \widetilde{y}(t); x\right), \tag{3.15}$$

where the Painlevé VI solution $\widetilde{y}(t)$ is determined by lower-left entry of the transformed equation. The lower-left entry is changed from $\varphi'/\varphi(1-\varphi)$ to a rational function whose numerator has only one root. The single root must be the extra ramification point of $\varphi(x)$. Hence $\widetilde{y}(t)$ can be identified with the branch $x = y$. It is a solution of

$$P_{VI}\big(a_0/k_{\varphi(0)}, a_1/k_{\varphi(1)}, a_t/k_{\varphi(t)}, 1+a_\infty/k_\infty; t\big)$$

which is the same equation as (3.14).

If the point $x = \infty$ does not lie above $z = \infty$, we can move the point $z = \infty$ by the fractional-linear transformations. That would only permute the three fibers, and change the rational function φ to $1/\varphi$, $1/(1-\varphi)$, $1-1/\varphi$ or $\varphi/(\varphi-1)$. Action of fractional-linear transformations on local monodromy differences is compatible with the form (3.14). □

The above theorem is a special case of [13, Theorem 2.1], with all k^i_j's equal to 1, and with correct parameters in (3.14). Theorem 4.5 in [7] is a more general statement, but without identification of transformed local monodromy differences.

In [13], it is regularly implied that the Painlevé VI solutions obtained with Theorem 3.1 arise from *RS*-pullback transformations of the type $RS^2_4\left(\begin{smallmatrix}1/k_0\\P_0\end{smallmatrix}\middle|\begin{smallmatrix}1/k_1\\P_1\end{smallmatrix}\middle|\begin{smallmatrix}1/k_\infty\\P_\infty\end{smallmatrix}\right)$. However, the above proof actually uses transformation $RS^2_4\left(\begin{smallmatrix}1/k_0\\P_0\end{smallmatrix}\middle|\begin{smallmatrix}1/k_1\\P_1\end{smallmatrix}\middle|\begin{smallmatrix}1+1/k_\infty\\P_\infty\end{smallmatrix}\right)$. On the other hand, it is apparent from classification [14] of rational coverings for RS^2_4-pullback transformations relevant to the sixth Painlevé equation that either $k_0 = 2$ or $k_1 = 2$ or $k_\infty = 2$. Once we assume $k_\infty = 2$, the transformations types $RS^2_4\left(\begin{smallmatrix}1/k_0\\P_0\end{smallmatrix}\middle|\begin{smallmatrix}1/k_1\\P_1\end{smallmatrix}\middle|\begin{smallmatrix}1/k_\infty\\P_\infty\end{smallmatrix}\right)$ and $RS^2_4\left(\begin{smallmatrix}1/k_0\\P_0\end{smallmatrix}\middle|\begin{smallmatrix}1/k_1\\P_1\end{smallmatrix}\middle|\begin{smallmatrix}1+1/k_\infty\\P_\infty\end{smallmatrix}\right)$ are the same or related by extra Schlesinger transformations. If $k_0 = 2$ or $k_1 = 2$, we still can relate the two transformation types via Schlesinger transformations. Hence, the *RS*-pullback transformation implied in Theorem 3.1 can be realized as $RS^2_4\left(\begin{smallmatrix}1/k_0\\P_0\end{smallmatrix}\middle|\begin{smallmatrix}1/k_1\\P_1\end{smallmatrix}\middle|\begin{smallmatrix}1/k_\infty\\P_\infty\end{smallmatrix}\right)$ as well.

Application of Theorem 3.1 to $\widehat{\varphi}_{10}(x)$ gives this solution of

$$P_{VI}(1/7, 1/7, 1/7, 2/3; t_{10}) :$$

$$y_{71} = \frac{1}{2} + \frac{(u+5)(u^6-u^5+3u^4-13u^3+4u^2-18u-12)}{2(u-1)(u+2)(u^3+u^2-2u+6)\sqrt{(u-1)(u+5)(u^2+3)}}. \tag{3.16}$$

A parametrization of t_{10} is given in (2.12). To get the solution of

$$P_{VI}(1/3, 1/7, 1/7, 6/7; t_{10})$$

in [14, (3.6)–(3.7)], one has to consider the function t_{10}/y_{71} and substitute $u \mapsto 2/s - 1$. Our implied RS-transformation is

$$RS_4^2 \left(\begin{array}{c|c|c} 1/7 & 1/2 & 1/3 \\ 7+1+1+1 & 2+2+2+2+2 & 3+3+3+1 \end{array} \right).$$

§4. Painlevé solutions from more general RS-pullback transformations

By the Jimbo-Miwa correspondence, a Painlevé VI solution is determined by the lower-left entry of a pullbacked Fuchsian system. By the results in [23, Section 4], that lower-left entry is determined by a syzygy (U_2, V_2, W_2) between F, G, H; that is, a polynomial solution of $FU_2 + GV_2 + HW_2 = 0$. If the shift δ of local monodromy differences at $x = \infty$ is small, that syzygy is determined by degree bounds of its components. The following theorem summarizes the situation.

Theorem 4.1. *Let $z = \varphi(x)$ denote a rational covering, and let $F(x)$, $G(x)$, $H(x)$ denote polynomials in x. Let k denote the order of the pole of $\varphi(x)$ at $x = \infty$. Suppose that the direct pullback of $E(e_0, e_1, 0, e_\infty; t; z)$ with respect to $\varphi(x)$ is a Fuchsian equation with the following singularities:*

- *Four singularities are $x = 0$, $x = 1$, $x = \infty$ and $x = t$, with the local monodromy differences d_0, d_1, d_t, d_∞, respectively. The point $x = \infty$ lies above $z = \infty$.*
- *All other singularities in $\mathbb{P}^1_x \setminus \{0, 1, t, \infty\}$ are apparent singularities. The apparent singularities above $z = 0$ (respectively, above $z = 1$, $z = \infty$) are the roots of $F(x) = 0$ (respectively, of $G(x) = 0$, $H(x) = 0$). Their local monodromy differences are equal to the multiplicities of those roots.*

Let us denote $\Delta = \deg F + \deg G + \deg H$, and let $\delta \le \max(2, k)$ denote a non-negative integer such that $\Delta + \delta$ is even. Suppose that (U_2, V_2, W_2) is

a syzygy between the three polynomials F, G, H, satisfying, if $\delta = 0$,

$$\deg U_2 = \frac{\Delta}{2} - \deg F, \quad \deg V_2 = \frac{\Delta}{2} - \deg G, \quad \deg W_2 < \frac{\Delta}{2} - \deg H, \tag{4.17}$$

or, if $\delta > 0$,

$$\deg U_2 < \tfrac{\Delta+\delta}{2} - \deg F, \quad \deg V_2 < \tfrac{\Delta+\delta}{2} - \deg G, \quad \deg W_2 = \tfrac{\Delta-\delta}{2} - \deg H. \tag{4.18}$$

Then the numerator of the (simplified) rational function

$$\begin{aligned} &\frac{U_2W_2}{G}\left(\frac{(e_0-e_1+e_\infty)}{2}\frac{\varphi'}{\varphi} - \frac{(FU_2)'}{FU_2} + \frac{(HW_2)'}{HW_2}\right) \\ &+ \frac{(e_0-e_1-e_\infty)}{2}\frac{V_2W_2}{F}\frac{\varphi'}{\varphi-1} + \frac{(e_0+e_1-e_\infty)}{2}\frac{U_2V_2}{H}\frac{\varphi'}{\varphi(\varphi-1)}, \end{aligned} \tag{4.19}$$

has degree 1 in x, and the x-root of it is an algebraic solution of

$$P_{VI}(d_0, d_1, d_t, d_\infty + \delta; t).$$

Proof. See Theorem 5.1 in [23]. □

Alternative forms of expression (4.19) are given in formulas (5.17)–(5.22) in [23]. For greater δ, formula (4.19) is still valid for a suitable syzygy (U_2, V_2, W_2), but that syzygy depends on initial coefficients of local solutions at $z = 0$ of the original hypergeometric equation. Taking only small shifts $\delta < \max(2, k)$ at $x = \infty$ seems to be enough to generate interesting "seed" solutions of the sixth Painlevé equation.

We can apply this theorem to obtain algebraic solutions of

$$P_{VI}(1/7, 1/7, 1/7, 2/3; t), \quad P_{VI}(2/7, 2/7, 2/7, 1/3; t)$$

and

$$P_{VI}(3/7, 3/7, 3/7, 2/3; t).$$

Implicitly, we apply pullback transformations

$$RS_4^2\left(\begin{matrix} 1/7 \\ 7+1+1+1 \end{matrix}\middle|\begin{matrix} 1/2 \\ 2+2+2+2+2 \end{matrix}\middle|\begin{matrix} 1/3 \\ 3+3+3+1 \end{matrix}\right),$$

$$RS_4^2\left(\begin{matrix} 2/7 \\ 7+1+1+1 \end{matrix}\middle|\begin{matrix} 1/2 \\ 2+2+2+2+2 \end{matrix}\middle|\begin{matrix} 1/3 \\ 3+3+3+1 \end{matrix}\right)$$

and

$$RS_4^2\left(\begin{matrix} 3/7 \\ 7+1+1+1 \end{matrix}\middle|\begin{matrix} 1/2 \\ 2+2+2+2+2 \end{matrix}\middle|\begin{matrix} 1/3 \\ 3+3+3+1 \end{matrix}\right),$$

respectively. Like in Section 3, we work with the covering $z = \varphi_{10}(x)$ rather than with the normalized covering $z = \widehat{\varphi}_{10}(x)$ while computing syzygies, and apply reparametrization (2.9) and normalizing fractional-linear transformation (2.10) at the latest stage. We have $k = 1$. Therefore recall the definition of F_{10}, G_{10} and P_{10} in (2.8). We take $\delta = 0$ for the second RS-transformation, or $\delta = 1$ for the other two. We have to compute syzygies between $F = x$ (or, respectively, $F = x^2$, or $F = x^3$) and $G = P_{10}$, $H = G_{10}$.

The syzygy for a solution of $P_{VI}(1/7, 1/7, 1/7, 2/3; t)$ is $(G_{10}, 0, -x)$, up to a scalar multiple. With this trivial syzygy, the solution is the same $\varphi_{10}(t_{10})$ as in (3.16). In fact, Theorem 4.1 reduces to Theorem 3.1 whenever one of syzygy components is zero; see [23, Remark 5.2].

The full RS-pullback $RS_4^2\left(\begin{smallmatrix}1/7\\7+1+1+1\end{smallmatrix}\middle|\begin{smallmatrix}1/2\\2+2+2+2+2\end{smallmatrix}\middle|\begin{smallmatrix}1/3\\3+3+3+1\end{smallmatrix}\right)$ would give a solution $\widetilde{y}_{71}(t_{70})$ of $P_{VI}(1/7, 1/7, 1/7, -2/3; t_{10})$ as well. The equation $P_{VI}(1/7, 1/7, 1/7, 8/3; t_{10})$ is identical. It turns out that the same Painlevé solution can be obtained by applying Theorem 4.1 with $\delta = 3$. (Have a look at the second part of [23, Remark 5.3].) However, since $\delta = 3 > \max(2, 1)$ we are not given restrictions on the syzygy (U_2, V_2, W_2), and additional knowledge of the normalized solutions of $E(1/7, 1/2, 0, 1/3; t; z)$ at $z = \infty$ is needed. The syzygy can be eventually computed to be

$$\Big(-63s^2x^4 + (74s^3 + 222s^2 + 285s - 52)x^3 \\ -2(8s^4 + 48s^3 + 257s^2 + 297s - 130)x^2 + 6(16s^3 + 64s^2 + 101s - 52)x \\ -144s^2 - 288s + 234,\ 21s,\ 26(s+1)^2x - 126s\Big).$$

The numerator of simplified expression (4.19) is then indeed linear in x. The solution $\widetilde{y}_{71}(t_{10})$ is rather stupendous:

$$\frac{1}{2}+\frac{(u+5)(65u^{18}+195u^{17}-195u^{16}+325u^{15}-1104u^{14}+\ldots-248931u^2-299835u+222534)}{10(u+2)\sqrt{(u-1)^3(u+5)(u^2+3)}(13u^{15}+65u^{14}+42u^{11}-1050u^{10}+\ldots-37611u^2+63927u-783)}.$$

On the other hand, this solution can be obtained by applying a series of Okamoto transformations to $y_{71}(t_{10})$.

To get a solution of $P_{VI}(2/7, 2/7, 2/7, 1/3; t)$ we apply Theorem 4.1 with $(F, G, H) = \big(x^2, P_{10}, G_{10}\big)$. With $\delta = 0$, the degree specifications in (4.17) are

$$\deg U_2 = 3, \qquad \deg V_2 = 0, \qquad \deg W_2 < 2. \tag{4.20}$$

As expected, there is one syzygy satisfying these bounds, up to a constant multiple:

$$\big(3sx^3-(2s^2+6s+13)x^2+6(2s+3)x-18,-1,-2(s+2)x+6\big)\,. \tag{4.21}$$

With this syzygy, expression (4.19) is equal to

$$\frac{4\,\big(s(2s^2+4s-19)x-3(2s^2-12s+7)\big)}{7\,F_{10}}. \tag{4.22}$$

The form is as expected: the numerator has degree 1 in x, while the denominator is a cubic polynomial in x. After reparametrization (2.9) and normalizing fractional-linear transformation (2.10) the denominator polynomial surely factors as $x(x-1)(x-t_{10})$, with t_{10} given in (2.12). The x-root of the transformed numerator gives the following solution $y_{72}(t_{10})$ of $P_{VI}(2/7,2/7,2/7,1/3;t_{10})$:

$$\begin{aligned} y_{72} &= \frac{1}{2} \\ &+ \frac{(u+5)(u^8+u^7+u^6-u^5+8u^4-82u^3-54u^2-90u-108)}{2(u+2)(u^6+2u^5-3u^4+8u^3-26u^2+60u-6)\sqrt{(u-1)(u+5)(u^2+3)}}. \end{aligned} \tag{4.23}$$

To relate to Boalch's parametrization in [6, page 106] for the same solution, we have to substitute $u\mapsto(s+5)/(s-1)$ into the expressions for y_{72} and t_{10}.

A solution $\widetilde{y}_{72}(t_{10})$ of $P_{VI}(2/7,2/7,2/7,-1/3;t_{10})$ can be computed without extra knowledge of the normalized solutions at $z=\infty$. The identical Painlevé equation is $P_{VI}(2/7,2/7,2/7,7/3;t_{10})$, and Theorem 4.1 can be applied with $\delta=2$. The following syzygy fits into formula (4.19):

$$\begin{aligned} \Big(&-69s(s+1)x^3+(32s^3+128s^2+325s-65)x^2 \\ &-6(32s^2+59s-15)x+288s-90, \\ &-5s-5,42sx^2-10(s+1)(s+2)x+30+30s\Big). \end{aligned}$$

Application of Theorem 4.1 with $(F,G,H)=\big(x^3,P_{10},G_{10}\big)$ and $\delta=1$ gives a solution of $P_{VI}(3/7,3/7,3/7,2/3;t)$. The degree bounds are $\deg U_2<3$, $\deg V_2<1$, $\deg W_2=2$. An appropriate syzygy is

$$\big(-(s+4)x^2+(2s+7)x-6,-1,2x^2-2(s+2)x+6\big) \tag{4.24}$$

Simplified expression (4.19) has the unique x-root $x = -(2s-5)(4s-7)/s(10s-11)$. After reparametrization (2.9) and normalizing fractional-linear transformation (2.10) we derive the following solution $y_{73}(t_{10})$ of $P_{VI}(3/7, 3/7, 3/7, 2/3; t_{10})$:

$$y_{73} = \frac{1}{2} + \frac{(u+5)(5u^7-10u^6+5u^5-20u^4+13u^3-68u^2-3u-30)}{2(u-1)^2(u+2)(5u^3+5u^2+11u+9)\sqrt{(u-1)(u+5)(u^2+3)}}. \tag{4.25}$$

This solution cannot be obtained by Okamoto, fractional-linear and quadratic transformations from previously know solutions.

§5. Pull-backs of hyperbolic hypergeometric equations

Here we survey RS_4^2-pullback transformations of *hyperbolic* hypergeometric equations $E(e_0, e_1, 0, e_\infty; t; z)$; these are defined by the properties that $1/e_0, 1/e_1, 1/e_\infty$ are positive integers and $e_0+e_1+e_\infty < 1$. These pullback coverings (and corresponding Okamoto orbits of algebraic Painlevé VI solutions) are classified in [14] and [7]. The following ramification patterns are possible:

$$R_4\big(7+1+1+1 \mid 2+2+2+2+2 \mid 3+3+3+1\big), \tag{5.26}$$

$$R_4\big(3+3+3+3 \mid 2+2+2+2+2+2 \mid 7+2+1+1+1\big), \tag{5.27}$$

$$R_4\big(3+3+3+3 \mid 2+2+2+2+2+2 \mid 8+1+1+1+1\big), \tag{5.28}$$

$$R_4\big(3+3+3+3+3+3 \mid 2+2+2+2+2+2+2+2 \mid 7+7+1+1+1+1\big). \tag{5.29}$$

The coverings have degree 10, 12, 12, 18, respectively.

The generic degree 10 covering is our $\varphi_{10}(x)$, up to reparametrization. We already considered the solutions $y_{71}(t_{10})$, $y_{72}(t_{10})$, $y_{73}(t_{10})$ representing three possible Okamoto orbits.

The generic degree 12 covering with ramification (5.27) is

$$\phi_{12}(x) = \frac{4\left(x^4+2s(3s+1)x^3+2s(5s+2)x^2+4s^2x+s^2\right)^3}{27s(s+1)^3x^7\left(4x^3+4s(8s+5)x^2+s(13s+1)x+4s^2\right)}. \tag{5.30}$$

It can be normalized with the substitutions

$$s \mapsto -\frac{(u+1)^2(u-1)^2}{(u^2+7)(u^2+u+2)(u^2-u+2)}, \tag{5.31}$$

$$x \mapsto \frac{(u+1)(u-1)^2}{2(u^2-u+2)^2} - \frac{u^3(u+1)(u-1)(u^2+3)\,x}{(u^2+u+2)^2(u^2-u+2)^2}. \tag{5.32}$$

A normalized expression for $1/\phi_{12}(x)$ is presented in [13], reparametrized with $u \mapsto 1/s$. Similarly as with $\varphi_{10}(x)$, we can pullback

$$E(1/3, 1/2, 0, 1/7; t; z), \quad E(1/3, 1/2, 0, 2/7; t; z)$$

and

$$E(1/3, 1/2, 0, 3/7; t; z)$$

with respect to a properly normalized $\phi_{12}(x)$ and derive[2] algebraic solutions of, respectively, $P_{VI}(1/7, 1/7, 1/7, 5/7; t)$, $P_{VI}(2/7, 2/7, 2/7, 4/7; t)$ and $P_{VI}(3/7, 3/7, 3/7, 1/7; t)$. However, the three solutions are related by Okamoto transformations. A solution $y_{74}(t_{70})$ of $P_{VI}(1/7, 1/7, 1/7, 5/7; t)$ can be obtained using Theorem 3.1. Here is a parametrization:

$$t_{70} = \frac{(u-3)^3(u^2+u+2)^2}{2u^3(u^2+7)^2}, \quad y_{74} = \frac{(u-1)(u-3)^2(u^2+u+2)}{2u(u^2+3)(u^2+7)}. \tag{5.33}$$

It is related to the parametrization in [13] via $u \mapsto 1/s$. Solutions $y_{75}(t_{70})$, $y_{76}(t_{70})$ of, respectively, $P_{VI}(2/7, 2/7, 2/7, 4/7; t)$, $P_{VI}(3/7, 3/7, 3/7, 1/7; t)$, can be obtained using Theorem 4.1. The same solutions can be obtained

[2]The implied RS-pullback transformations are, respectively,

$$RS_4^2\left(\begin{matrix}1/3\\3+3+3+3\end{matrix}\middle|\begin{matrix}1/2\\2+2+2+2+2+2\end{matrix}\middle|\begin{matrix}1/7\\7+2+1+1+1\end{matrix}\right),$$

$$RS_4^2\left(\begin{matrix}1/3\\3+3+3+3\end{matrix}\middle|\begin{matrix}1/2\\2+2+2+2+2+2\end{matrix}\middle|\begin{matrix}2/7\\7+2+1+1+1\end{matrix}\right)$$

and

$$RS_4^2\left(\begin{matrix}1/3\\3+3+3+3\end{matrix}\middle|\begin{matrix}1/2\\2+2+2+2+2+2\end{matrix}\middle|\begin{matrix}3/7\\7+2+1+1+1\end{matrix}\right).$$

As indicated in [14], one may also consider RS-pullback transformations $RS_4^2\left(\begin{smallmatrix}1/3\\3+3+3+3\end{smallmatrix}\middle|\begin{smallmatrix}1/2\\2+2+2+2+2+2\end{smallmatrix}\middle|\begin{smallmatrix}1/2\\7+2+1+1+1\end{smallmatrix}\right)$ of $E(1/3, 1/2, 0, 1/2; t; z)$ and derive solutions of, say, $P_{VI}(1/2, 1/2, 1/2, -5/2; t)$, $P_{VI}(1/2, 1/2, 1/2, -1/2; t)$, $P_{VI}(1/2, 1/2, 1/2, 1/2; t)$. For this, other proper normalization of $\phi_{12}(x)$ has to be used, similarly as other proper normalization of $\varphi_{12}(x)$ was used in [23, Section 6] to compute solutions $y_{63}(t_{60})$, $y_{62}(t_{60})$, $y_{64}(t_{60})$ of the same equations (respectively). Incidentally, pullbacks with respect to $\phi_{12}(x)$ give exactly the same solutions $y_{63}(t_{60})$, $y_{62}(t_{60})$, $y_{64}(t_{60})$ in [23] of the same three Painlevé equations, up to reparametrization $u \mapsto (u+3)/(1-u)$.

as Okamoto transformations of $y_{74}(t_{70})$. In the notation of [21, (2.3)], we have:

$$y_{75} = K_{[-1/7,-1/7,-1/7,5/7;\,t_{70}]}\, y_{74}, \quad y_{76} = K_{[1/7,\,1/7,\,1/7,\,5/7;\,t_{70}]}\, y_{74}. \tag{5.34}$$

Here are parametrizations of $y_{75}(t_{70})$ and $y_{76}(t_{70})$:

$$y_{75} = -\frac{(u-3)^2(u^2+u+2)^2(u^2+2u+5)}{6u(u+1)(u-1)(u^2+7)}, \tag{5.35}$$

$$y_{76} = \frac{(u-1)(u-3)^2(u^2+u+2)(u^4-4u^3-6u^2-28u-11)}{2u(u^2+7)(u^6+21u^4+3u^2+39)}. \tag{5.36}$$

The solution $y_{75}(t_{70})$ is the Kleinian solution of [4], reparametrized with $u \mapsto 3s/(s-2)$.

As noticed in [14], there are two composite coverings with ramification patterns (5.28) or (5.29). They are compositions of Belyi coverings with a quadratic almost Belyi covering:

$$R_4(2 \mid 1+1 \mid 1+1) \circ R_3(\widehat{2} \mid 2 \mid 1+1) \circ R_3(\widehat{2}+1 \mid 2+\widehat{1} \mid 3), \tag{5.37}$$

$$R_4(1+1 \mid 2 \mid 1+1) \circ R_3(3+3+3 \mid 2+2+2+2+\widehat{1} \mid 7+1+1). \tag{5.38}$$

Here the compositions are from right to left, and the order 2 ramification points of a subsequent quadratic covering are indicated by the hat symbol. The algebraic Painlevé VI solutions are determined by the quadratic almost Belyi coverings. The solutions are related (via fractional-linear or Okamoto transformations) to the solution $y(t) = \sqrt{t}$ of the general equation $P_{VI}(a,b,b,1-a;t)$. We specifically have $a = b = 1/8$ or $a = b = 1/7$ if we apply Theorem 3.1 to the two composite coverings. The Belyi coverings are known from algebraic transformations of Gauss hypergeometric functions [25]. In particular, an explicit degree 9 covering is given in [25, (24)].

Beside the indicated coverings, there is exactly one covering (up to fractional-linear transformations) to pullback hyperbolic hypergeometric equations. It has ramification pattern (5.28):

$$\psi_{12}(x) = -\frac{4\left(9x^4+18x^3+3(2s+5)x^2-2(s-2)x+s(s-2)\right))^3}{(4s+1)^3\left(9x^4+14x^3+3(2s+3)x^2-6sx+s^2\right)}, \tag{5.39}$$

To get a proper normalization or apply Theorem 4.1, we need to choose the point $x = \infty$ appropriately; hence first a transformation

$$s \mapsto -\frac{1}{4}v^2(3v^2+8v+6), \qquad x \mapsto \frac{1}{x} - \frac{1}{2}v^2. \tag{5.40}$$

For a proper normalization, we still need to factor the remaining degree 3 factor polynomial in the denominator, and localize the points $x = 0$, $x = 1$, $x = t$ properly. This is achieved with the substitutions

$$v \mapsto \frac{(u^2-2)(u^4-4u^3+8u^2+8u+4)}{6u(u^2-2u+2)(u^2+2u+2)},$$
$$x \mapsto \frac{36u^2(u^4+4)}{(u^2+2u-2)(u^4+8u^2+4)} \times \left(\frac{8iu(u^2-2u-2)(u^2+2)^3 x}{(u^4-4u^3+8u^2+8u+4)^3} + \frac{(u^2+2i)(u^2+2iu+2)}{(u^2+2(i-1)u+2i)^3} \right).$$

Theorem 3.1 eventually gives the following solution $y_{81}(t_{80})$ of

$$P_{VI}(1/8, 1/8, 1/8, 7/8; t_{80}) :$$

$$t_{80} = \frac{i\,(u+i-1)^2\,(u-i+1)^2\left(u^2+2(i+1)u-2i\right)^3\left(u^2-2(i+1)u-2i\right)^3}{64u^2\,(u^2-2)^3\,(u^2+2)^3}, \tag{5.41}$$

$$y_{81} = -i\frac{(u+i-1)(u-i+1)(u^2+2iu+2)(u^2+2(i+1)u-2i)^2(u^2-2(i+1)u-2i)}{8u\,(u^2-2)^2\,(u^2+2)\,(u^2-2u-2)}. \tag{5.42}$$

The numerator of t_{80} can also be written as $i\left(u^2+2i\right)^2\left(u^4-12iu^2-4\right)^3$, for instance. The substitution $u \mapsto -(1+i)/u$ gives the parametrization [14, (4.12)–(4.13)].

With the same proper normalization of $\psi_{12}(x)$, one may consider *RS*-transformations

$$RS_4^2\left(\begin{matrix}1/3\\3+3+3+3\end{matrix}\middle|\begin{matrix}1/2\\2+2+2+2+2+2\end{matrix}\middle|\begin{matrix}3/8\\8+1+1+1+1\end{matrix}\right),$$

$$RS_4^2\left(\begin{matrix}1/3\\3+3+3+3\end{matrix}\middle|\begin{matrix}1/2\\2+2+2+2+2+2\end{matrix}\middle|\begin{matrix}1/4\\8+1+1+1+1\end{matrix}\right)$$

and

$$RS_4^2\left(\begin{matrix}1/3\\3+3+3+3\end{matrix}\middle|\begin{matrix}1/2\\2+2+2+2+2+2\end{matrix}\middle|\begin{matrix}1/2\\8+1+1+1+1\end{matrix}\right)$$

to derive solutions of $P_{VI}(3/8, 3/8, 3/8, 5/8; t_{80})$, $P_{VI}(1/4, 1/4, 1/4, 1/4; t_{80})$ and $P_{VI}(1/2, 1/2, 1/2, 1/2; t_{80})$, respectively. The last two equations turn out to be the same as $y_{62}(t_{60})$ and $y_{64}(t_{60})$, respectively; Theorem 4.1 gives expressions reparametrized by

$$u \to -\frac{u^4+12iu^2-4}{u^4-4iu^2-4}. \tag{5.43}$$

The solution of $P_{VI}(3/8,3/8,3/8,5/8;t_{80})$ is

$$y_{83} = \frac{i\,(u+i-1)(u-i+1)\left(u^2+2(i+1)u-2i\right)^2\left(u^2-2(i+1)u-2i\right)}{8u\,(u^2-2)^2\,(u^2+2)\,(u^6+6u^5+6u^4+16u^3-12u^2+24u-8)} \times\left(u^6-6iu^5-6u^4+16iu^3-12u^2-24iu+8\right). \tag{5.44}$$

This solution is presented in [6, pg. 102], reparametrized with $u\mapsto(i-1)s$. The same solution can be obtained by an Okamoto transformation: $y_{83}=K_{[-1/8,-1/8,-1/8,7/8;\,t_{80}]}\,y_{81}$.

As was suspected in [14], the solutions $y_{62}(t_{60})$ of

$$P_{VI}(1/4,1/4,1/4,1/4;t_{80})$$

and $y_{81}(t_{80})$ of

$$P_{VI}(1/8,1/8,1/8,7/8;t_{80})$$

are related by a sequence of two quadratic transformations. Indeed, a fractional-linear transformation of

$$K_{[-1/4,-1/4,-1/4,1/4;\,t_{60}]}\,y_{62}$$

solves $P_{VI}(0,0,1/2,1;t_{80})$, and then we can apply the following result on composition of two quadratic transformations. After substitution (5.43) the square roots are extractable; see Lemma 6.1.

§6. Appendix

Here we briefly recall or consider the following topics:

- A formula for composition of two subsequent quadratic transformations of Painlevé VI functions; see Lemma 6.1.
- A general formula for the degree of almost Belyi coverings relevant to algebraic Painlevé VI solutions; see Lemma 6.2.
- A geometric interpretation of the degree formula.

Lemma 6.1. *Suppose that $y(t)$ is a solution of $P_{VI}(0,0,a,1;t)$. Then the following expression is a solution of $P_{VI}(a/4,a/4,a/4,1-a/4;t)$:*

$$t\,\frac{\sqrt{(y-1)(t-1)}+\sqrt{y\,t}+1}{\sqrt{y\,t}+t}. \tag{6.45}$$

Proof. The result [20] of Ramani-Gramatikos-Tamizhmani states that if $Y_0(T_0)$ is a solution of $P_{VI}(0,b,c,1;T_0)$, and

$$Y_1=\frac{(\sqrt{Y_0}+1)(\sqrt{T_0}+1)}{(\sqrt{Y_0}-1)(\sqrt{T_0}-1)},\qquad T_1=\frac{(\sqrt{T_0}+1)^2}{(\sqrt{T_0}-1)^2}, \tag{6.46}$$

then $Y_1(T_1)$ is a solution of $P_{VI}(b/2, c/2, c/2, 1-b/2; T_1)$. We can transform $y(t)$ to a solution of $P_{VI}(0, a/2, a/2, 1; \ldots)$, and then apply the same transformation to get the asserted solution. (Other branches of the transformed solution can be obtained by flipping the sign of the square roots $\sqrt{(y-1)(t-1)}$ and $\sqrt{y\,t}$.) □

The following is a degree formula for pullback coverings generating algebraic Painlevé VI solutions by Theorem 3.1. In particular, it implies that the pullback covering for an icosahedral [5] solution of $P_{VI}(\nu_0, \nu_1, \nu_t, \nu_\infty; t)$ with $\nu_0, \nu_1, \nu_t, \nu_\infty \in (0,1)$ has the degree $30(\nu_0 + \nu_1 + \nu_t - \nu_\infty)$.

Lemma 6.2. *In the situation of Theorem* 3.1*, we have, if* $\frac{1}{k_0}+\frac{1}{k_1}+\frac{1}{k_\infty} \neq 1$*:*

$$\deg\varphi = \left(\frac{a_0}{k_{\varphi(0)}}+\frac{a_1}{k_{\varphi(1)}}+\frac{a_t}{k_{\varphi(t)}}+\frac{a_\infty}{k_{\varphi(\infty)}}-1\right)\Big/\left(\frac{1}{k_0}+\frac{1}{k_1}+\frac{1}{k_\infty}-1\right). \tag{6.47}$$

Proof. Let d denote the degree of φ. Let b_0, b_1 respectively b_∞ denote the sums of those a_x with $x \in \{0, 1, t, \infty\}$ such that, respectively, $\varphi(x) = 0$, $\varphi(x) = 1$, $\varphi(x) = \infty$. By the Hurwitz formula, we have

$$\begin{aligned} 2d - 2(k_0-1)\frac{d-b_0}{k_0} + (k_1-1)\frac{d-b_1}{k_1} + (k_\infty-1)\frac{d-b_\infty}{k_\infty} \\ + (a_0-1) + (a_1-1) + (a_t-1) + (a_\infty-1) + 1. \end{aligned}$$

The formula follows, since $b_0 + b_1 + b_\infty = a_0 + a_1 + a_t + a_\infty$. □

Notice that this Lemma implies that it is not possible to obtain solutions like $y_{72}(t_{12})$ $y_{75}(t_{70})$ using Theorem 3.1: the degree of the covering would be negative. In other words, we cannot pullback the hyperbolic hypergeometric equation $E(1/3, 1/2, 0, 1/7; t; z)$ to the equations like $E(2/7, 2/7, 2/7, 1/3; y_{72}; z)$ or $E(2/7, 2/7, 2/7, 4/7; y_{75}; z)$. As one can see, there are just a few pullback coverings for infinitely many "hyperbolic" Painlevé VI solutions. This is in contrast to icosahedral Painlevé VI solutions (or more generally, solutions corresponding to Fuchsian systems with a finite monodromy), which can be obtained from a standard icosahedral hypergeometric equation thanks to Klein's theorem.

There is a geometric interpretation of this degree formula. If $\frac{1}{k_0} + \frac{1}{k_1} + \frac{1}{k_\infty} > 1$, then the expression $\left(\frac{1}{k_0} + \frac{1}{k_1} + \frac{1}{k_\infty} - 1\right)\pi$ is the area of the spherical triangle with the angles π/k_0, π/k_1, π/k_∞ in the standard Riemannian metric on the sphere. The spherical triangle is the image of

the upper-half plane of a Schwarz map for a hypergeometric differential equation with the local exponent differences $1/k_0$, $1/k_1$, $1/k_\infty$. The image of a Schwarz map for a scalar Fuchsian equation associated with (3.15) is a degenerate pentagon, with four angles equal to $a_0\pi/k_{\varphi(0)}$, $a_1\pi/k_{\varphi(1)}$, $a_t\pi/k_{\varphi(t)}$, $\pi - a_\infty\pi/k_{\varphi(\infty)}$, and one angles (corresponding to the extra ramification point) equal to 2π. The area of the degenerate pentagon is equal to $\left(\frac{a_0}{k_{\varphi(0)}} + \frac{a_1}{k_{\varphi(1)}} + \frac{a_t}{k_{\varphi(t)}} + \frac{a_\infty}{k_{\varphi(\infty)}} - 1\right)\pi$. If the covering $z = \varphi(x)$ can be defined over $\mathbb{R}$, then the degenerate pentagon can be triangulated into the Schwarz triangles with the angles π/k_0, π/k_1, π/k_∞, respecting analytic continuation (between the two complex half-planes) in the fiber (with respect to φ) of the degenerate pentagon. If $\frac{1}{k_0} + \frac{1}{k_1} + \frac{1}{k_\infty} < 1$ then we have hyperbolic triangles instead of spherical triangles, with the area $\left(1 - \frac{1}{k_0} - \frac{1}{k_1} - \frac{1}{k_\infty}\right)\pi$ with respect to a hyperbolic metric, but other features are the same.

Figures 1(a) and *(b)* depict Schwarz triangulations for the degree 8 map $\widehat{\varphi}_8(x)$ in [23, (2.7)]. The cut for the fifth vertex in Figure 1(b) can either include or do not reach the interior vertex. Two different figures correspond to two connected components over $\mathbb{R}$ of the Hurwitz curve $w^2 = s(s-1)(s+3)(s+8)$. The two components can be distinguished by the cut from a point above $z = 0$: in Figure 1(a) the cut goes towards a point above $z = \infty$, while in Figure 1(b) it goes towards a point above $z = 1$. One can evaluate $\widehat{\varphi}_8(x)$ at the extra ramification point:

$$\widehat{\varphi}_8(y_{26}) = -\frac{3125(u+3)(u+2)^4(2u+1)^2u^2(u-1)^3}{4(u+8)(u^3+4u^2+2u+2)^5(u-2)^2}. \tag{6.48}$$

The value $\widehat{\varphi}_8(y_{26})$ oscillates between $z = 0$ and $z = 1$ for $u \in [-3, 0]$, and the value is negative or $z = 0$, $z = \infty$ when $u \geq 1$ or $u \leq -8$. Hence, Figure 1(b) corresponds to the real component with $u \in [-3, 0]$, and Figure 1(a) corresponds to the other real component. Notice that $\widehat{\varphi}_8(y_{26})$, as a function of u, is a Belyi map.

Figure 1(d) depicts a Schwarz triangulation for the degree 12 map $\widehat{\varphi}_{12}(x)$ in [23, (2.13)]. Figure 1(c) depicts a Schwarz triangulation for a normalization of $\phi_{12}(x)$ here; this is a hyperbolic triangulation. Schwarz triangulations for our $\widehat{\varphi}_{10}(x)$ and normalized composite coverings for (5.37) are modifications of two triangulations for Belyi coverings in [25, Fig. 1]: there has to be a cut from the vertices with the angles $2\pi/7$ and $2\pi/8$. Figures 1(e), (f), (g) depict Schwarz triangulations for the degree 11, 12,

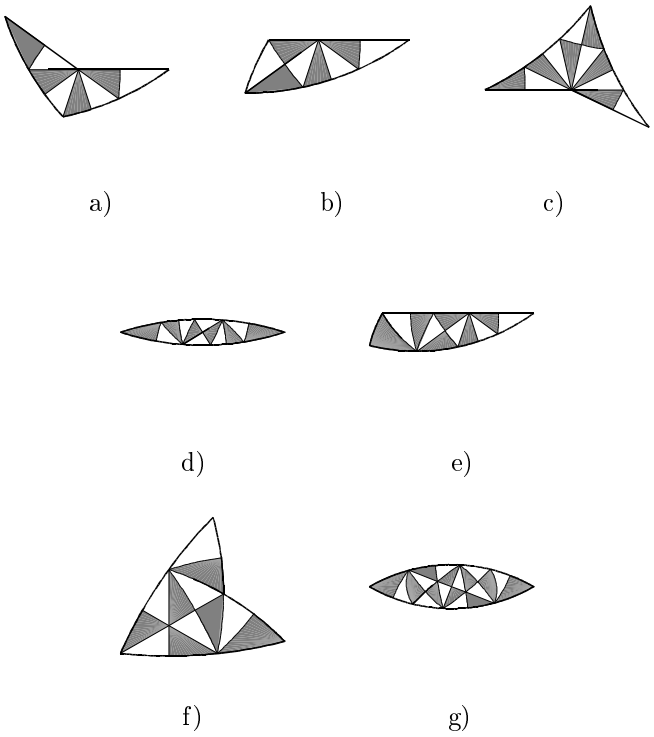

Fig. 1. Triangulations for Schwarz maps

20 maps in [22]. Note that the lens shaped figures (d) and (g) correspond precisely to Dubrovin–Mazzocco solutions.

Not all almost Belyi coverings have Schwarz tringulations. If a covering is not defined over $\mathbb{R}$, analytic continuations of Schwarz maps for the original and transformed equations do not match. For example, normalizations of $\psi_{12}(x)$ can be defined only over $\mathbb{Q}(i)$. Normalized composite coverings for (5.37), or the composite degree 20 map $\varphi_4 \circ \varphi_5(x)$ in [22, Section 5] are not defined over $\mathbb{R}$ either; nor they have Schwarz triangulations.

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Department of Mathematical Informatics,
University of Tokyo,
113-8656 Tokyo, Japan
E-mail: `rvidunas@gmail.com`

Steklov Mathematical Institute,
Fontanka 27,
St. Petersburg, 191023, Russia
E-mail: `kitaev@pdmi.ras.ru`

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