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REPRESENTATIONS OF QUANTUM CONJUGACY CLASSES OF ORTHOSYMPLECTIC GROUPS

ABSTRACT. Let G be the complex symplectic or special orthogonal group and $\mathfrak{g}$ its Lie algebra. With every point x of the maximal torus $T \subset G$ we associate a highest weight module M_x over the Drinfeld-Jimbo quantum group $U_q(\mathfrak{g})$ and a quantization of the conjugacy class of x by operators in $\mathrm{End}(M_x)$. These quantizations are isomorphic for x lying on the same orbit of the Weyl group, and M_x support different representations of the same quantum conjugacy class.

Dedicated to P. P. Kulish
on the occasion of his 70th birthday

§1. INTRODUCTION

This paper is a sequel of a series of works on quantization of semisimple conjugacy classes of a non-exceptional simple Poisson group G, [1–5]. It is done in the spirit of [6] devoted to $G = SL(n)$ and can be viewed as a uniform approach to quantization that includes the results of [1–5] as a special case. The earlier constructed quantum conjugacy classes were realized by operators on certain modules of the quantized universal enveloping algebra $U_q(\mathfrak{g})$ of the Lie algebra $\mathfrak{g}$ of the group G. For a large number of examples, this theory is parallel to the $U(\mathfrak{g})$-equivariant quantization of semisimple adjoint orbit in $\mathfrak{g} \simeq \mathfrak{g}^*$, [1, 7, 8]. In both cases, G and $\mathfrak{g}$, the quantized algebra of polynomial functions is represented on parabolic Verma modules, respectively, over $U_q(\mathfrak{g})$ and $U(\mathfrak{g})$. However, adjoint orbits in G are in a greater supply than in $\mathfrak{g}$. Quantization of some of them requires more general modules, which cannot be obtained by induction from a character of the parabolic extension of the stabilizer, [3, 4]. Moreover, the latter itself disappears as a natural subalgebra in $U_q(\mathfrak{g})$. This observation makes us take a more general look at already constructed quantum homogeneous

spaces and conclude that they were obtained through a very special choice of the initial point. Such points are distinguished by their isotropy subgroups, whose triangular decomposition perfectly matches the triangular decomposition of G. All they are of Levi type, as for semisimple orbits in $\mathfrak{g}$, and their basis of simple positive roots of is a part of the basis of the total group. That is violated for stabilizers of non-Levi type appearing among conjugacy classes in G. At the same time, one can apply a generic Weyl group transformation to the initial point in $\mathfrak{g}$ and break the nice inclusion of root bases even in the Levi case. In this respect, a generic initial point whose stabilizer is isomorphic to a Levi subgroup has much similarity with essentially non-Levi one. It makes sense therefore to extend the original approach to quantization and consider all points on the maximal torus (the Cartan subalgebra) for initial. They belong to the same conjugacy class if and only if they lie on the same orbit of the Weyl group. We associate a module of highest weight with every such point and realize the quantization of its cojugacy class by linear operators on that module. Points on the same Weyl group orbit give rise to isomorphic quantizations, which can be regarded as different representations of the same quantum homogeneous space. They can also be thought of as different polarizations of the same algebra.

There are other interesting problems related to quantum homogeneous spaces, such as quantization of associated vector bundles, star product formulation etc. That is well understood for classes with Levi isotropy subgroups, through the mechanism of parabolic induction, [10–14]. At the same time, the difference between Levi and non-Levi conjugacy classes is qualitative, and alternative representations of Levi classes could be a bridge between the two cases. A uniform approach to quantization may help to understand the non-Levi case too.

1.1. Preliminaries. Let G be the complex orthogonal or symplectic connected algebraic group of $N \times N$-matrices preserving a non-degenerate skew-diagonal symmetric or, respectively, symplectic form on $\mathbb{C}^N$. Given a basis $\{w_i\}_{i=1}^N \in \mathbb{C}^N$, we fix the triangular decomposition $\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{h} \oplus \mathfrak{g}_+$ so that the Cartan subalgebra is represented by diagonal matrices, while the nilpotent subalgebras $\mathfrak{g}_\pm$ by strictly upper ($+$) and lower ($-$) triangular matrices. The basis elements w_i carry weights $\varepsilon_i \in \mathfrak{h}^*$ satisfying $\varepsilon_{i'} = -\varepsilon_i$, where $i' = N + 1 - i$. Fix the inner product $(.,.)$ on $\mathfrak{h}^*$ so that the weights with $i \leqslant \frac{N}{2}$ form an orthogonal basis. Let n designate the rank of $\mathfrak{g}$. We choose a basis Π^+ of simple roots in $\mathfrak{h}^*$ as $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$, $i < n$, and

$\alpha_i = \varepsilon_n$, $\alpha_i = 2\varepsilon_n$, $\alpha_i = \varepsilon_{n-1} + \varepsilon_n$ for, respectively, $\mathfrak{g} = \mathfrak{so}(2n+1)$, $\mathfrak{g} = \mathfrak{sp}(2n)$, and $\mathfrak{g} = \mathfrak{so}(2n)$. Denote by R and R^+ the sets of all and positive roots of $\mathfrak{g}$. When we need to distinguish the roots systems of a subgroup, we mark it with the corresponding subscript.

Denote by T the maximal torus of G exponentiating the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$. Given a point $x \in T$, denote by $K \subset G$ its centralizer subgroup with the Lie algebra $\mathfrak{k}$, which is a reductive subalgebra of maximal rank in $\mathfrak{g}$. The triangular decomposition of $\mathfrak{g}$ induces a triangular decomposition $\mathfrak{k} = \mathfrak{k}_+ \oplus \mathfrak{h} \oplus \mathfrak{k}_-$. There are inclusions $\mathrm{R}_\mathfrak{k} \subset \mathrm{R}_\mathfrak{g}$ and $\mathrm{R}_\mathfrak{k}^+ \subset \mathrm{R}_\mathfrak{g}^+$, but not $\Pi_\mathfrak{k}^+ \subset \Pi_\mathfrak{g}^+$ in general. If the latter holds, K is said to be a regular Levi subgroup of G. If K is not isomorphic to a Levi subgroup, we call it pseudo-Levi. We call it regular if a maximal Levi subgroup among those contained in K is regular. Similar terminology is used for its Lie algebra $\mathfrak{k}$. Collectively we call K and $\mathfrak{k}$ generalized Levi subgroups and subalgebras.

The canonical inner product $(.,.)$ on the dual vector space $\mathfrak{h}^*$ identifies it with $\mathfrak{h}$. Let $h_\lambda \in \mathfrak{h}$ denote the image of $\lambda \in \mathfrak{h}^*$ under this isomorphism. Fix a generalized Levi subalgebra $\mathfrak{k} \subset \mathfrak{g}$. By $\mathfrak{c}_\mathfrak{k}^*$ we denote the set of weights $\lambda \in \mathfrak{h}^*$ such that $(\lambda, \alpha) = 0$ for all $\alpha \in \mathrm{R}_\mathfrak{k}$ and by $\mathfrak{c}_{\mathfrak{k},\mathrm{reg}}^* \subset \mathfrak{c}_\mathfrak{k}^*$ the set of weights such that $(\lambda, \alpha) = 0 \Leftrightarrow \alpha \in \mathrm{R}_\mathfrak{k}$. For each $\lambda \in \mathfrak{c}_\mathfrak{k}^*$ the element $e^{2h_\lambda} \in G$ commutes with K, and $\mathfrak{k}$ is exactly the centralizer Lie algebra of $x = e^{2h_\lambda}$ once $\lambda \in \mathfrak{c}_{\mathfrak{k},\mathrm{reg}}^*$.

Denote by O_x the conjugacy class of x. The coordinate ring $\mathbb{C}[O_x]$ is a quotient of $\mathbb{C}[G]$ by a certain G-invariant ideal. To describe this ideal, observe that x determines a 1-dimensional representation χ_x of the subalgebra of invariants in $\mathbb{C}[G]$ (under the conjugation action). Apart from $SO(2n)$, it is generated by traces of the matrix powers of (X_{ij}), where X_{ij} are the coordinate functions on G. In the special case of $SO(2n)$ one has to add one more invariant that is sensible to the flip of the Dynkin diagram, in order to separate two $SO(2n)$-classes within a $O(2n)$-class whose eigenvalues are all distinct from ± 1. Furthermore, the matrix X, when restricted to O_x, satisfies an equation $p(X) = 0$ with a polynomial p in one variable. The entries of the matrix $p(X)$ are polynomial functions in X_{ij}. The defining ideal of O_x is generated by the entries of $p(X)$ over the kernel of χ_x, provided p is the minimal polynomial for x.

A pseudo-Levi subgroup K contains a Cartesian product of two blocks of the same type as G. They correspond to the eigenvalues ± 1 of the matrix x, which are simultaneously present in its spectrum. For the symplectic group, it is $SP(2m) \times SP(2p)$, where $m, p \geqslant 1$. For the odd orthogonal group, it

is $SO(2m) \times SO(2p+1)$, where $m \geqslant 2$, $p \geqslant 0$. For the even orthogonal group, one has $SO(2m) \times SO(2p)$, where $m, p \geqslant 2$. The lower bounds on m, p come from the isomorphism $SO(2) \simeq GL(1)$: if the multiplicities of ± 1 are small, then the isotropy subgroup stays within the Levi type. We distinguished such conjugacy classes as borderline Levi because they share some properties of both types, [5].

The quantized polynomial algebra $\mathbb{C}_\hbar[O_x]$, $\hbar = \log q$, is described as follows. The algebra $\mathbb{C}[G]$ is replaced with $\mathbb{C}_\hbar[G]$, which is an equivariant quantization of a special Poisson bracket on G, [9]. This bracket makes G a Poisson–Lie homogeneous space over the Poisson group G equipped with the Drinfeld–Sklyanin bracket [15], with respect to the conjugation action. The algebra $\mathbb{C}_\hbar[G]$ admits an equivariant embedding into the corresponding quantum group $U_\hbar(\mathfrak{g}) \supset U_q(\mathfrak{g})$. As a subalgebra in $U_\hbar(\mathfrak{g})$, it is generated by the entries of the matrix $\mathcal{Q} = (\pi \otimes \mathrm{id})(\mathcal{R}_{21}\mathcal{R})$, where $\mathcal{R}$ is the universal R-matrix of $U_\hbar(\mathfrak{g})$ and π stands for the representation homomorphism $U_\hbar(\mathfrak{g}) \to \mathrm{End}(\mathbb{C}^N)$. The factor $\mathcal{R}_{21}$ is obtained by flip of the tensor legs of $\mathcal{R}$. This embedding makes a $U_\hbar(\mathfrak{g})$-module into a $\mathbb{C}_\hbar[G]$-module and the representation homomorphism of $\mathbb{C}_\hbar[G]$ automatically $U_\hbar(\mathfrak{g})$-equivariant.

The subalgebra of invariants in $\mathbb{C}_\hbar[G]$ coincides with its centre, which is generated by q-traces of the matrix powers of $\mathcal{Q}$ (apart from the special case of $SO(2n)$, as mentioned above). The "quantum initial points" can be described as follows. Let $\rho_\mathfrak{k} = \frac{1}{2} \sum_{\alpha \in \mathcal{R}_\mathfrak{k}^+} \alpha$ be the Weyl vector of the isotropy subalgebra $\mathfrak{k}$. Let $\mathfrak{c}_\mathfrak{k}^*$ be the orthogonal complement to $\mathbb{C}\Pi_\mathfrak{k}^+$. Denote $\mathfrak{C}_{\mathfrak{k},\mathrm{reg}}^* = \frac{1}{\hbar}\mathfrak{c}_{\mathfrak{k},\mathrm{reg}}^* + \mathfrak{c}_\mathfrak{k}^* + \rho_\mathfrak{k} - \rho$ and $\mathfrak{C}_\mathfrak{k}^* = \frac{1}{\hbar}\mathfrak{c}_\mathfrak{k}^* + \mathfrak{c}_\mathfrak{k}^* + \rho_\mathfrak{k} - \rho$. By construction, all $\lambda \in \mathfrak{C}_\mathfrak{k}^* \subset \frac{1}{\hbar}\mathfrak{h}^* \oplus \mathfrak{h}^*$ satisfy $q^{2(\lambda+\rho,\alpha)} = q^{(\alpha,\alpha)}$ for all q if $\alpha \in \Pi_\mathfrak{k}^+$ while $\lambda \in \mathfrak{C}_{\mathfrak{k},\mathrm{reg}}^* \subset \mathfrak{C}_\mathfrak{k}^*$ satisfies this condition only if $\alpha \in \Pi_\mathfrak{k}^+$.

With $\lambda \in \mathfrak{C}_{\mathfrak{k},\mathrm{reg}}^*$ we associate a module M_λ of highest weight λ, so that the image of $\mathbb{C}_\hbar[G]$ in $\mathrm{End}(M_\lambda)$ is a quantization of $\mathbb{C}_\hbar[O_x]$. It is a parabolic Verma module if and only if $\mathfrak{k}$ is a regular Levi subalgebra. Irregular Levi subgroups also appear as stabilizers of initial points in $\mathfrak{g}$, so our approach is as well applicable to the $U(\mathfrak{g})$-equivariant quantization of adjoint orbits in $\mathfrak{g}$.

The highest weight of M_λ defines a central character of $\mathbb{C}_\hbar[G]$, whose kernel is expressed through q-traces of the matrix powers $\mathcal{Q}^k$. The matrix $\mathcal{Q}$ yields an invariant operator on $\mathbb{C}^N \otimes M_\lambda$, and its minimal polynomial is determined by module structure of the tensor product. The annihilator of M_λ is then generated by the entries of the minimal polynomial over the

kernel of the central character. The structure of $\mathbb{C}^N \otimes M_\lambda$ is the key point of this approach, and its analysis takes a great part of this exposition. Our approach makes use of some results on the Mickelsson algebras and Shapovalov inverse [16, 17] and is based on the study of the standard filtration of $\mathbb{C}^N \otimes M_\lambda$ in what follows.

1.2. Quantized universal enveloping algebra. Throughout the paper, $\mathfrak{g}$ is a complex simple Lie algebra of type B, C or D (the A-case has been considered in [6]). We assume that $q \in \mathbb{C}$ is not a root of unity. Denote by $U_q(\mathfrak{g}_\pm)$ the $\mathbb{C}$-algebra generated by $e_{\pm\alpha}$, $\alpha \in \Pi^+$, subject to the q-Serre relations, [18]. Denote by $U_q(\mathfrak{h})$ the commutative $\mathbb{C}$-algebra generated by $q^{\pm h_\alpha}$, $\alpha \in \Pi^+$. The quantum group $U_q(\mathfrak{g})$ is a $\mathbb{C}$-algebra generated by $U_q(\mathfrak{g}_\pm)$ and $U_q(\mathfrak{h})$ subject to the relations

$$q^{h_\alpha} e_{\pm\beta} q^{-h_\alpha} = q^{\pm(\alpha,\beta)} e_{\pm\beta}, \quad [e_\alpha, e_{-\beta}] = \delta_{\alpha,\beta} \frac{[h_\alpha]_q}{[\frac{(\alpha,\alpha)}{2}]_q},$$

were $[z]_q = \frac{q^z - q^{-z}}{q - q^{-1}}$. We work with the opposite comultiplication as in [18]:

$$\Delta(e_\alpha) = e_\alpha \otimes 1 + q^{h_\alpha} \otimes e_\alpha, \quad \Delta(e_{-\alpha}) = e_{-\alpha} \otimes q^{-h_\alpha} + 1 \otimes e_{-\alpha},$$
$$\Delta(q^{\pm h_\alpha}) = q^{\pm h_\alpha} \otimes q^{\pm h_\alpha},$$

for all $\alpha \in \Pi^+$. The quantized Borel subalgebras $U_q(\mathfrak{b}_\pm) \subset U_q(\mathfrak{g})$, $\mathfrak{b}_\pm = \mathfrak{g}_\pm + \mathfrak{h}$, are generated by $U_q(\mathfrak{g}_\pm)$ over $U_q(\mathfrak{h})$. The universal R-matrix is fixed to be an element of an extended tensor product of $U_q(\mathfrak{b}_-) \otimes U_q(\mathfrak{b}_+)$. Its transposed version due to the opposite comultiplication can be taken from [18], Theorem 8.3.9.

We use the notation $e_i = e_{\alpha_i}$, and $f_i = e_{-\alpha_i}$ for $\alpha_i \in \Pi^+$ in all cases apart from $i = n$, $\mathfrak{g} = \mathfrak{so}(2n+1)$, where we set $f_n = [\frac{1}{2}]_q e_{-\alpha_n}$. The corresponding commutation relation translates to $[e_n, f_n] = [h_{\alpha_n}]_q$. With this normalization of generators, the natural representation of $U_q(\mathfrak{g})$ on the vector space $\mathbb{C}^N$ is independent of q, see the next section.

## §2. Natural representation of $U_q(\mathfrak{g})$

By Γ we denote the root lattice $\Gamma = \mathbb{Z}\Pi^+$ with $\Gamma^+ = \mathbb{Z}_+\Pi^+$. Let I designate the set of integers $\{1, \ldots, N\}$. For $\beta \in \Gamma^+$ we define $P(\beta)$ to be the set of all pairs $i, j \in I$ such that $\varepsilon_i - \varepsilon_j = \beta$. Let $e_{ij} \in \text{End}(\mathbb{C}^N)$, $i, j \in I$, denote the standard matrix units. The following assignment defines

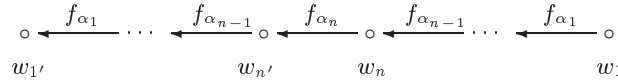a representation of $\mathfrak{g}$, which is equivalent to the natural representation:

$$\pi(e_\alpha) = \sum_{(l,r) \in P(\alpha)} e_{lr}, \quad \pi(f_\alpha) = \sum_{(l,r) \in P(\alpha)} e_{rl}, \quad \pi(h_{\varepsilon_i}) = e_{ii} - e_{i'i'}.$$

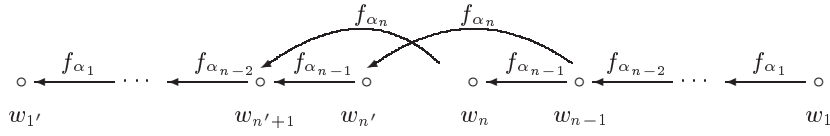The action of the Chevalley generators can be conveniently visualized by the diagrams

$$\mathfrak{g} = \mathfrak{so}(2n+1)$$
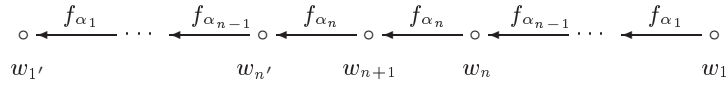
$$\mathfrak{g} = \mathfrak{sp}(2n)$$

$$\mathfrak{g} = \mathfrak{so}(2n)$$

Reverting the arrows one gets the diagrams for positive Chevalley generators of $\mathfrak{g}$.

We introduce a partial ordering on the integer interval I by setting $i \preccurlyeq j$ if and only if there is a (monic) Chevalley monomial $\psi \in U_q(\mathfrak{g}_-)$ such that w_j is equal to ψw_i up to an invertible scalar multiplier, $w_j = \psi w_i$. This monomial, if exists, represents a path from w_i to w_j in the representation diagram, which becomes the Hasse diagram of the poset. Such ψ is unique, which is obvious for the series B and C and still true for D. Indeed, two different paths from w_{n-1} to w_{n+2} yield the products $f_{\alpha_n} f_{\alpha_{n-1}}$ and $f_{\alpha_{n-1}} f_{\alpha_n}$, which are the same due to Serre relations. We denote this monomial by ψ_{ji}. The relation $\prec$ is consistent with the natural ordering on $\mathbb{Z}$, and coincides with it unless $\mathfrak{g} = \mathfrak{so}(2n)$. In the latter case n and n' are incomparable.

In what follows, we also use the monomials ψ^{ij} obtained from ψ_{ji} by reverting the order of factors. It is clear that $\psi^{ij} = \psi^{im}\psi^{mj}$ for any m such that $i \preccurlyeq m \preccurlyeq j$.

Definition 2.1. *We call ψ^{ij} the principal monomial of the pair $i \preccurlyeq j$.*

Their significance will be clear later in the section devoted to the standard filtration of tensor product modules.

Remark that all Chevalley monomials of weight $\varepsilon_j - \varepsilon_i$ are obtained from ψ^{ij} by permutation of factors.

We will also need another partial ordering on I that is relative to $\mathfrak{k}$: write $i \lessdot j$ if w_i and $w_j \in U_q(\mathfrak{k}_-)\mathfrak{k}_- w_i$. Clearly $i \lessdot j$ if and only if $i \prec j$ and w_i, w_j belong to an irreducible $\mathfrak{k}$-submodule in $\mathbb{C}^N$. Let $I_{\mathfrak{k}} \subset I$ be the set of all minimal elements with respect to this ordering and $\bar{I}_{\mathfrak{k}}$ be its complement in I. Elements of $I_{\mathfrak{k}}$ label the highest weight vectors of the irreducible $\mathfrak{k}$-submodules in $\mathbb{C}^N$.

2.1. Reduced Shapovalov inverse. In this section, we recall a construction of Shapovalov inverse reduced to $\text{End}(\mathbb{C}^N) \otimes U_q(\mathfrak{b}_-)$. It is given in [16] for the general linear and orthosymplectic quantum groups (see also [17] for the general case). Note with care that [16,17] deal with a different version of the quantum group. To adapt those results to the current setting, one has to twist the coproduct by $q^{\sum_{i=1}^{n} h_i \otimes h_i}$ and replace q with q^{-1}.

Given $\lambda \in \frac{1}{\hbar}\mathfrak{h}^* \oplus \mathfrak{h}^*$ consider a 1-dimensional $U_q(\mathfrak{b}_\pm)$-module $\mathbb{C}_\lambda$ with the representation defined by the assignment $q^{\pm h_\alpha} \mapsto q^{\pm(\lambda, \alpha)}$, $e_\alpha \mapsto 0$ for $\alpha \in \Pi^+$. Denote by M_λ the Verma module $U_q(\mathfrak{g}) \otimes_{U_q(\mathfrak{b}_+)} \mathbb{C}_\lambda$ with the canonical generator v_λ, [19]. Let M_λ^* denote the opposite Verma module $U_q(\mathfrak{g}) \otimes_{U_q(\mathfrak{b}_-)} \mathbb{C}_{-\lambda}$ of the lowest weight $-\lambda$. There is an invariant pairing $M_\lambda \otimes M_\lambda^* \to \mathbb{C}$, which is equivalent to the contravariant Shapovalov form on M_λ, upon an identification $M_\lambda^* \sim M_\lambda$ through an anti-algebra isomorphism $U_q(\mathfrak{g}_-) \simeq U_q(\mathfrak{g}_+)$, [20]. We also call it Shapovalov form.

Recall that a vector $v \neq 0$ in a $U_q(\mathfrak{g})$-module V is called singular if $e_\alpha v = 0$ for all $\alpha \in \Pi^+$. Singular vectors are defined up to a scalar multiplier. Reduced Shapovalov inverse is a matrix $\hat{F} = \sum_{i=1}^{j} e_{ij} \otimes \hat{f}_{ij} \in \text{End}(\mathbb{C}^N) \otimes \hat{U}_q(\mathfrak{b}_-)$, where the roof means extension over the field of fractions of $U_q(\mathfrak{h})$. This matrix yields a singular vector $\hat{F}(w_j \otimes v_\lambda)$ in $\mathbb{C}^N \otimes M_\lambda$ for all $j \in I$. For generic λ the matrix $\hat{F}$ is a homomorphic image of the Shapovalov inverse lifted to $\hat{U}_q(\mathfrak{g}_+) \otimes \hat{U}_q(\mathfrak{b}_-)$.

The entries $\hat{f}_{ij}$ can be expressed through the Chevalley generators as follows. First introduce $f_{ij} \in U_q(\mathfrak{g}_-)$ for all $i < j$, which are closely related to the R-matrix of $U_q(\mathfrak{g})$, [16]. Put $f_{ij} = f_{j'i'} = f_i$ for $i - 1 = j < \frac{N+1}{2}$ and

$$f_{ij} = [f_{j-1}, \ldots [f_{i+1}, f_i]_q \ldots]_q, \quad f_{j'i'} = [\ldots [f_i, f_{i+1}]_q, \ldots f_{j-1}]_q, \quad (2.1)$$

for $i + 1 < j \leqslant \frac{N+1}{2}$ and all $\mathfrak{g}$. Furthermore,

$$f_{nn'} = (q^{-1} - 1)f_n^2, \quad f_{i,n+1} = [f_n, f_{in}]_q, \quad f_{n+1,i'} = [f_{n'i'}, f_n]_q,$$
$$i < n, \quad \mathfrak{g} = \mathfrak{so}(2n+1),$$
$$f_{nn'} = [2]_q f_n, \quad f_{in'} = [f_n, f_{in}]_{q^2}, \quad f_{ni'} = [f_{n'i'}, f_n]_{q^2},$$
$$i < n, \quad \mathfrak{g} = \mathfrak{sp}(2n),$$
$$f_{nn'} = 0, \quad f_{in'} = [f_n, f_{i,n-1}]_q, \quad f_{ni'} = [f_{n'+1,i'}, f_n]_q,$$
$$i < n - 2, \quad \mathfrak{g} = \mathfrak{so}(2n),$$

and finally, for $i, j < n$,

$$f_{ij'} = q^{-\delta_{ij}}[f_{n+1,j'}, f_{i,n+1}]_{q^{\delta_{ij}}}, \ N = 2n + 1$$
$$f_{ij'} = q^{-\delta_{ij}}[f_{nj'}, f_{in}]_{q^{1+\delta_{ij}}}, \ N = 2n.$$

There exists an analog of Poincare–Birghoff–Witt (PBW) basis in $U_q(\mathfrak{g}_-)$ generated by certain elements labeled by R^+, which can be presented as deformed commutators of the Chevalley generators, [18]. The presence of PBW bases allows to identify $U_q(\mathfrak{g}_-)$ with $U(\mathfrak{g}_-)$ as vector spaces (and $U_q(\mathfrak{h})$-modules). This identification makes $U_q(\mathfrak{g}_-)$ a deformation of $U(\mathfrak{g}_-)$. It follows that f_{ij} are deformations of root vectors from $\mathfrak{g}_-$.

Put $\rho_i = (\rho, \varepsilon_i)$ for $i \in I$ and introduce $\eta_{ij} = h_i - h_j + \rho_i - \rho_j - \frac{||\varepsilon_i - \varepsilon_j||^2}{2} \in \mathfrak{h} + \mathbb{C}$, $A_i^j = -\frac{q - q^{-1}}{q^{2\eta_{ij}} - 1}$, for all $i, j \in I$ such that $i \prec j$. We call a sequence $\vec{m} = (m_1, \ldots, m_k)$ a route from m_1 to m_k if $m_1 \prec \ldots \prec m_k$. To every route $\vec{m}$ we assign the products

$$f_{\vec{m}} = f_{m_1, m_2} \ldots f_{m_{k-1}, m_k}, \quad A_{\vec{m}}^j = A_{m_1}^j \ldots A_{m_k}^j,$$

where $m_k \prec j$. Given another route, $\vec{l} = (l_1, \ldots, l_s)$ with $\vec{m} \prec \vec{l}$ meaning $m_k \prec l_1$, there is a route $(\vec{m}, \vec{l}) = (m_1, \ldots, m_k, l_1, \ldots, l_s)$. Define $\tilde{\rho}_i = \rho_i + \frac{||\varepsilon_i||^2}{2}$ for all $i \in I$. Then $\hat{f}_{ij} = 0$ if $i > j$, $\hat{f}_{ii} = 1$ and $\hat{f}_{ij} = \sum_{i \preccurlyeq \vec{m} \prec j} f_{\vec{m},j} A_{\vec{m}}^j q^{\eta_{ij} - \bar{\rho}_i + \bar{\rho}_j}$ for $i < j$, where the summation is done

over all routes $(\vec{m}, j)$ from i to j. Note that the factor $q^{\eta_{ij} - \bar{\rho}_i + \bar{\rho}_j}$ comes from a different version of the quantum group adopted in [6, 17].

Lemma 2.2. *Suppose that* $\alpha \in \Pi_{\mathfrak{k}}^+ \subset R_{\mathfrak{g}}^+$ *and* $(i, j) \in P(\alpha)$. *For all* $\lambda \in \mathfrak{C}_{\mathfrak{k}, reg}^*$, *the specialization* $\hat{f}_{ij}[\eta_{ij}]_q$ *at weight* λ *is a deformation of a classical root vector,* $-f_\alpha \in \mathfrak{g}_-$.

Proof. Present λ as $\alpha = \frac{1}{\hbar}\lambda^0 + \lambda^1 \in \mathfrak{C}_{\mathfrak{k}, reg}^*$, $\lambda^i \in \mathfrak{h}^*$. Observe that a) $e^{2\lambda_i^0} = e^{2\lambda_j^0}$ for all $\alpha = \varepsilon_i - \varepsilon_j \in \Pi_{\mathfrak{k}}^+$ once $\lambda^0 \in \mathfrak{c}_{\mathfrak{k}}^*$ and b) there is no k such that $i \prec k \prec j$ and $e^{2\lambda_i^0} = e^{2\lambda_k^0} = e^{2\lambda_j^0}$ if $\lambda^0 \in \mathfrak{c}_{\mathfrak{k}, reg}^*$. Furthermore, write $\hat{f}_{ij}[\eta_{ij}]_q = -f_{ij} - \sum\limits_{i \prec \vec{m} \prec j} f_{i, \vec{m}, j} A_{\vec{m}, j}^j q^{\bar{\rho}_j - \bar{\rho}_i}$, where the sum is taken over non-empty routes $\vec{m}$. For all k subject to $i \prec k \prec j$, the denominator in $A_k^j|_\lambda = -\frac{q - q^{-1}}{q^{2\eta_{kj}|_\lambda} - 1}$ tends to $e^{2\lambda_k^0 - 2\lambda_j^0} - 1 \neq 0$ as $q \to 1$. Therefore, the sum vanishes modulo $\hbar$, and f_{ij} tends to a classical root vector. $\square$

Define elements $\check{f}_{ij} = \hat{f}_{ij} \prod\limits_{i \preceq k \prec j} [\eta_{kj}]_q \in U_q(\mathfrak{b}_-)$ for all $i \prec j$. They satisfy the identity

$$e_\alpha \check{f}_{ij} = - \sum_{(l,r) \in P(\alpha)} \delta_{l,i} q^{-(\alpha, \varepsilon_l)} \check{f}_{r,j}[\eta_{ij}]_q \quad \mathrm{mod}\ U_q(\mathfrak{g})\mathfrak{g}_-, \quad \forall \alpha \in \Pi^+, \ (2.2)$$

Fix $(i, j) \in P(\alpha)$ for $\alpha \in R^+$ and suppose that $\lambda = \frac{1}{\hbar}\lambda^0 + \lambda^1$ with $\lambda^i \in \mathfrak{h}^*$ satisfies the condition $[\eta_{ij}|_\lambda]_q = 0 = [\eta_{j'i'}|_\lambda]_q$. Then there is a singular vector $v_{\lambda - \alpha}$ of weight $\lambda - \alpha$ in the Verma module M_λ. One can take $v_{\lambda - \alpha} = \check{f}_{ij} v_\lambda$ provided it is not zero, since $e_\alpha \check{f}_{ij} v_\lambda = 0$ for all $\alpha \in \Pi^+$ by (2.2). If $\check{f}_{ij} v_\lambda = 0$ at some λ, one still can obtain $v_{\lambda - \alpha}$ from $\check{f}_{ij} v_\lambda$ (which is polynomial in $e^{\pm 2(\lambda^0, \alpha)}$, $\alpha \in \Pi^+$, for fixed λ^1 and q) via renormalization, since singular vectors are defined up to a scalar multiplier. In particular, if $\alpha \in \mathfrak{k}$ for some generalized Levi subalgebra $\mathfrak{k}$ and $\lambda \in \mathfrak{C}_{\mathfrak{k}, reg}^*$, then $v_{\lambda - \alpha} \simeq f_\alpha v_\lambda \mod \hbar$, by Lemma 2.2. Note that $\check{f}_{ij} v_\lambda \simeq \check{f}_{j'i'} v_\lambda$ if $i \neq j'$, as follows from the theory of Mickelsson algebras for quantum groups, [21].

§3. STANDARD FILTRATION ON $\mathbb{C}^N \otimes M_\lambda$

In what follows, we work out a tool for our analysis of $\mathbb{C}^N \otimes M_\lambda$, where M_λ is a generalized parabolic Verma module of weight λ. In this section, we do it for the ordinary Verma module $M_\lambda = U_q(\mathfrak{g}) \otimes_{U_q(\mathfrak{b}_+)} \mathbb{C}_\lambda$ with $\lambda \in \frac{1}{\hbar}\mathfrak{h}^* \oplus \mathfrak{h}^*$. An essential part of our technique is a diagram language,

whose elements already appeared in [2, 4] and which is given a systematic treatment here. The case of $\mathfrak{gl}(N)$ was already studied in [6], so we do it for orthogonal and symplectic $\mathfrak{g}$. We consider the standard filtration $V_\bullet = (V_i)_{i=1}^N$, $\{0\} = V_0 \subset V_1 \subset \ldots \subset V_N = \mathbb{C}^N \otimes M_\lambda$, where V_i is generated by $\{e_j \otimes v_\lambda\}$, $j \leqslant i$. Its graded module $\mathrm{gr}V_\bullet$ is a direct sum of V_j/V_{j-1}, which are isomorphic to the Verma modules $M_{\lambda+\varepsilon_j}$ (the proof of [22], Lemma 5, readily adapts to quantum groups).

Given $\beta \in \mathbb{Z}_+\Pi^+$ we define $\Psi_\beta \subset U_q(\mathfrak{g}_-)$ to be the subset of Chevalley monomials of weight β. We assume that a pair $(i, j) \in P(\beta)$ is chosen for this section. Having fixed an order of elementary factors in ψ, we regard it a as path from v_λ to ψv_λ. We associate with ψv_λ a graph H_ψ with nodes $\{v^k\} \in M_\lambda$, $v^j = v_\lambda$, $v^i = \psi v_\lambda$, and arrows being negative Chevalley generators acting on M_λ. For $\psi = \psi^{ij}$, this path is unique in almost all cases (except for type D, where we eliminate the ambiguity by fixing the order as $f_{\alpha_{n-1}}f_{\alpha_n}$). For principal ψ, we are concerned not just with the terminating node ψv_λ, but also in all intermediate nodes. On the contrary, for non-principal ψ, only ψv_λ is important for us, while the specific path is immaterial.

We say that f_α has length 2 if $\alpha = \alpha_n$ and $\mathfrak{g} = \mathfrak{so}(2n)$. All other generators are assigned with length 1. If all factors in ψ have length 1, we write $\psi = \phi_i \ldots \phi_{j-1}$ with $\phi_k \in \{f_\alpha\}_{\alpha\in\Pi^+}$, and we set $v^k = \phi_k v^{k+1}$. Then the diagram H_ψ is set to be

Now suppose that ψ has (exactly one) factor of length 2. Write $\psi = \phi_i \ldots \phi_k \phi_{k+2} \ldots \phi_{j-1}$, where $\phi_k = f_{\alpha_n}$ (there are $j - i - 1$ factors). Then the graph H_ψ is

Here we distinguish two cases. If $\psi = \psi^{ij}$, then $\phi_{k-1} = f_{\alpha_{n-1}}$, and the dashed arrow $f_{\alpha_{n-1}}$ is included in H_ψ. The node v^{k+1} is set to $f_{\alpha_{n-1}}v^{k+2}$. For non-principal ψ, the node v^{k+1} is arbitrary (immaterial) and there is no arrow from v^{k+2} to v^{k+1}.

We also consider a graph V_{ij}, which is a part of the natural representation diagram of $U_q(\mathfrak{g}_-)$ that includes all paths from w_i to w_j. We transpose it to make a vertical graph oriented from top w_i to bottom w_j.

We denote by $\mathrm{Arr}(v^k)$ the set of arrows originated at v^k and similarly $\mathrm{Arr}(w_m)$ the set of arrows from w_k. By construction, an arrow from node m to node k has length $k - m$.

Finally, we define tensor product $D_\psi = H_\psi \otimes V_{ij}$ as a graph on a two-dimensional lattice whose nodes are $w_k^m = w_k \otimes v^m \in \mathbb{C}^N \otimes M_\lambda$ and arrows are $\mathrm{Arr}(w_k^m) = \mathrm{Arr}(w_k) \otimes \mathrm{id} \bigcup \mathrm{id} \otimes \mathrm{Arr}(v^m)$, The diagram is oriented so that H_ψ-arrows and V_{ij}-arrows are directed, respectively, leftward and downward; the origin w_i^j is in the right upper corner. We need only the triangular part of the diagram including the nodes v_m^k with $k + m \geqslant i + j$. The set $\{w_k^k\}_{k=i}^j$ is called principal diagonal. With $\psi = \psi^{ij}$, the node w_k^k on the principal diagonal is $w_k \otimes \psi^{kj} v_\lambda$, $k = i, \ldots, j$. Here is an example of diagram D_ψ with all arrows of length 1:
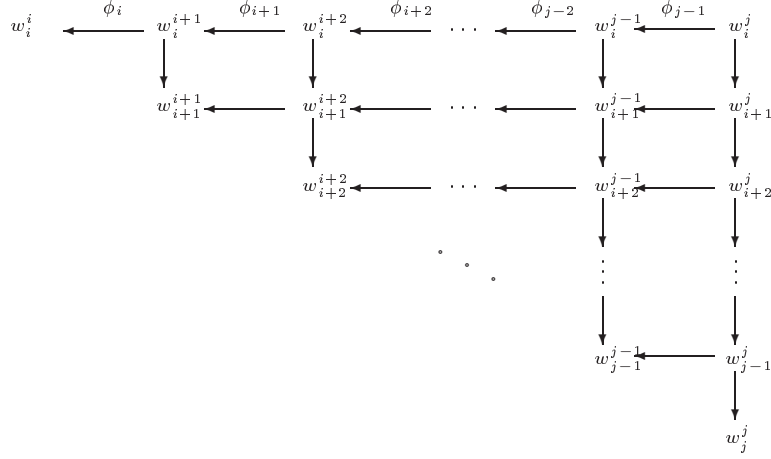

Fig. 1

The arrows represent the action of the Chevalley generators on the tensor factors $\mathbb{C}^N$ (vertical) and M_λ (horizontal). The following property of this action readily follows from the coproduct of the Chevalley generators: suppose that $\phi \in \mathrm{Arr}(v^m)$ and $\phi \notin \mathrm{Arr}(w_k)$. If $v^r = \phi v^m$, then $\phi(w_k^m) = v_k^r$,

i.e., the horizontal arrow yields the action of ϕ on the entire tensor product. In general, $\phi(w_k^m) = v_k^r \mod \mathbb{C}v_s^m$, where $w_s = \phi w_k$.

Suppose that nodes of a column segment BC (with C the bottom node) belong to a $U_q(\mathfrak{g})$-submodule $M \subset \mathbb{C}^N \otimes M_\lambda$. Let ϕ be a Chevalley generator assigned to a horizontal arrow with the origin at this column. Consider the following situations:

(1) The length of ϕ is 1.
    (a) There is no vertical ϕ-arrow with the origin at C.
    (b) There is a vertical ϕ-arrow with the origin at C.
(2) The length of ϕ is 2, and the size of BC is 2 or greater. Let C' and C'' be the nodes 1 and 2 steps up, respectively.
    (a) There is no vertical ϕ-arrow with the origin at C and at C'.
    (b) There is a vertical ϕ-arrow with the origin either at C or at C'.

1.a)   1.b)   2.a)   2.b)
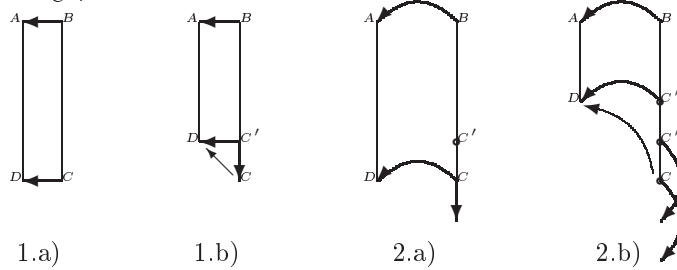
Definition 3.1. *We call the transition from column BC to column AD an elementary move or simply move of the length equal to the length of ϕ-arrow. The elementary moves 1.a) and 2.a) are called left. The elementary moves 1.b) and 2.b) are called diagonal.*

Lemma 3.2 (Elementary moves)**.** *Under the conditions above, the column segment AD lies in M.*

Proof. Clear. $\square$

We will use elementary moves to reach a node or collection of nodes in the diagram starting from the rightmost column, which is assumed to be in a submodule M. That way we prove that the target nodes are in M.

Let $D'_\psi \subset D_\psi$ denote the subgraph whose nodes form the triangle lying above the principal diagonal, i.e. $\{w_k^m\}_{k+m>i+j}$.

Lemma 3.3. *Suppose that $\psi = \psi^{ij}$ is a principal monomial. Then the linear span of D'_ψ lies in V_{j-1}.*

Proof. Suppose that all horizontal arrows in D'_ψ have length 1, as e.g. for $\mathfrak{g} = \mathfrak{so}(2n+1)$, and $\mathfrak{g} = \mathfrak{sp}(2n)$. Consider the diagram D_ψ on Fig.3.a, where D'_ψ is the triangle ABC. The column BC belongs to V_{j-1} by construction. All arrows have length 1. Applying elementary diagonal moves we prove that ABC is in V_{j-1}.

Now suppose there is a horizontal arrows of length 2. Assuming $i \leqslant n - 1$, $n' + 1 \leqslant j$, consider the diagram D_ψ where the triangle D'_ψ is denoted by ABC (cf. Fig.3.b). The rightmost column BC belongs to V_{j-1} by construction. For each node in the trapezoid $JBCL$ there is a horizontal arrow of length 1. Those arrows are distinct from vertical arrows for all nodes in the line $L'M \subset JBCL$. Apply the corresponding left moves to the columns rested on $L'M$. This operation proves that trapezoid $HBCI$ is in V_{j-1}. Then apply the diagonal move of length 2 to the column JL and get $FG \subset V_{j-1}$. All arrows in the triangle ADE have length 1, therefore $ADE \subset V_{j-1}$, via diagonal moves.

The case $i = n$, $n' + 1 \leqslant j$ dysplayed on Fig.3.a is similar to already considered: all horizontal arrows within D'_ψ are of length 1. The case $i \leqslant n - 1$, $n' = j$ is displayed on Fig.3.b: Apply the diagonal move of length 1 to the column BC' and get $DE \subset V_{j-1}$. Then apply the diagonal move of length 2 to BC' and get $EG \subset V_{j-1}$. Thence the entire triangle AFG is in M.   □

Proposition 3.4. *Suppose* $\psi \in \Psi_\beta$, $(i,j) \in P(\beta)$, *and* $\psi \neq \psi^{ij}$. *Then* $w_i \otimes \psi v_\lambda \in V_{j-1}$.

Proof. Consider a factorization $\psi = \psi'\psi^{mj}$, where m is some integer satisfying $i \prec m \preceq j$ and $\psi' \in \Psi_{\varepsilon_i - \varepsilon_m}$. Choose m to be the smallest possible. In the factorization $\psi^{ij} = \psi^{im}\psi^{mj}$ let ϕ be the rightmost Chevalley factor in ψ^{im}, while ϕ' the rightmost factor in ψ'. Due to the choice of m, $\phi \neq \phi'$. Further we consider algebras of types B, C separately from D.

In diagrams of types B and C, all arrows have length 1, Fig.3.a. All nodes in the north-east rectangle $CDIH$ are the same as in $D_{\psi^{ij}}$. Therefore $CDGF$ is in V_{j-1}, by Lemma 3.3. Since $\phi' \neq \phi$, the left move via ϕ' maps CF onto BE, modulo $CF \subset V_{j-1}$, proving $BE \subset V_{j-1}$. Applying diagonal moves to BE we get the triangle $ADE \subset V_{j-1}$ including the node A, which is $w_i^i = w_i \otimes \psi v_\lambda$. Now we look at the type D. We can assume that $i \leqslant n - 1, n' + 1 \leqslant j$, since otherwise this case reduces to already considered. If the length of ϕ' is 1, the reasoning is the same as above. The only difference is that one may have to use a diagonal move of length
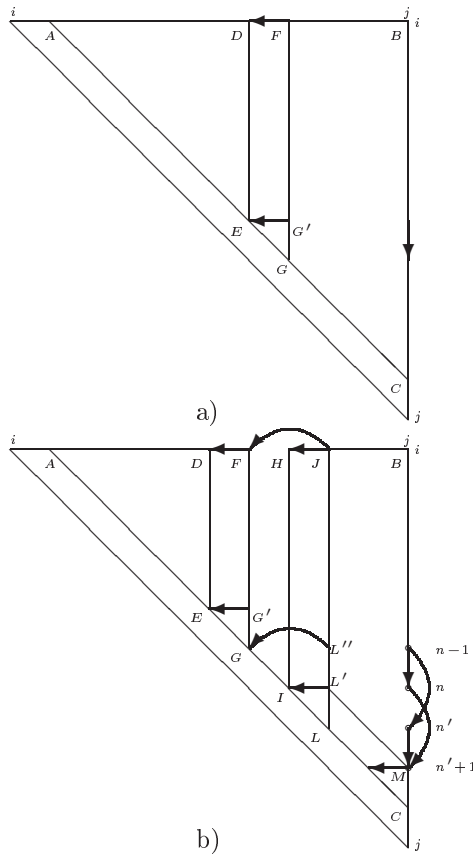
Fig. 2

2 in transition from BE to A, see Fig.3.b. If the length of ϕ' is 2, then the transition to BE is performed via ϕ' applied to $CF' \subset V_{j-1}$, as shown on Fig.3.c. This proves that $BE \subset V_{j-1}$. Further, all horizontal arrows in the triangle ABE are of length 1 (the factor f_{α_n} enters ψ only once). This situation is similar to the types B and C considered earlier. Thus, the node $A = w_i^i = w_i \otimes \psi v_\lambda$, belongs to V_{j-1}. $\qquad\square$
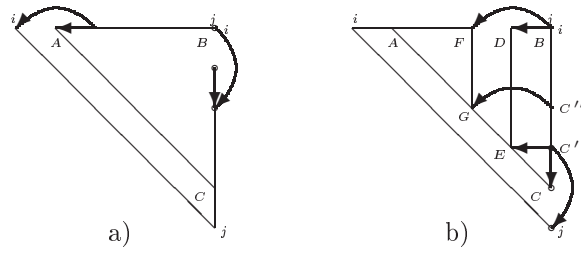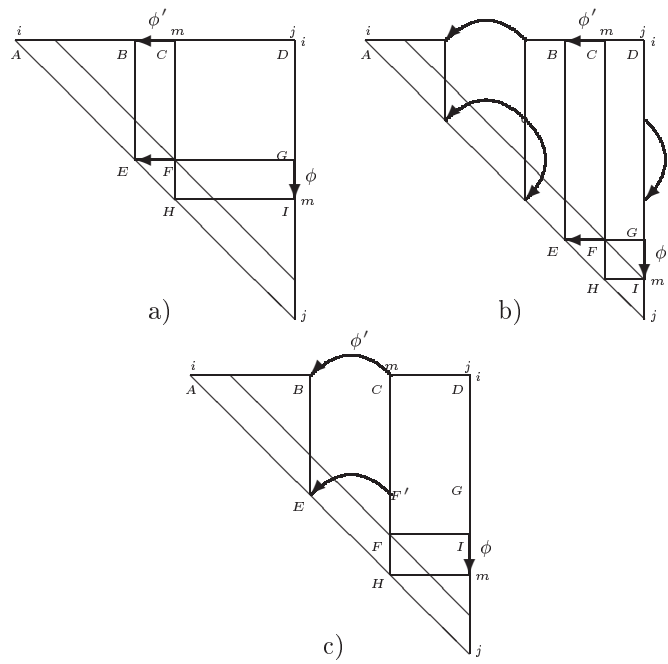
Fig. 3

Fig. 4

For $i \preccurlyeq j$ denote by $|i - j|$ the distance (the number of arrows in a path) from i to j on the Hasse diagram of the natural representation of $U_q(\mathfrak{g}_-)$.

Proposition 3.5. *Suppose that $i, j \in I$ are such that $i \prec j$. Then*

$$w_i \otimes \psi^{ij} v_\lambda \quad = \quad (-1)^{|i-j|} q^{-(\lambda, \vartheta_{ij})} w_j \otimes v_\lambda \quad \mod V_{j-1}. \qquad (3.3)$$

Proof. Suppose that $\alpha \in \Pi^+$ and $(i, k) \in P(\alpha)$. By Lemma 3.3, the node $w_i \otimes \psi^{kj} v_\lambda \in D'_\psi$ lies in V_{j-1} Applying $\Delta f_\alpha = f_\alpha \otimes q^{-h_\alpha} + 1 \otimes f_\alpha$ to $w_i \otimes \psi^{kj} v_\lambda$ we get

$$w_i \otimes \psi^{ij} v_\lambda = q^{-(\lambda, \alpha) - (\alpha, \varepsilon_j - \varepsilon_k)} w_k \otimes \psi^{kj} v_\lambda$$

$$= q^{-(\lambda, \vartheta_{ij} - \vartheta_{kj})} w_j \otimes \psi^{kj} v_\lambda \quad \mod V_{j-1}$$

for all $k \preccurlyeq j$. Here we used $f_\alpha w_i = w_k$ and $f_\alpha \psi^{kj} = \psi^{ij}$ for all $k \preccurlyeq j$. Proceeding recursively along the path from i to j with the boundary condition $\vartheta_{jj} = 0$ we complete the proof. $\qquad\square$

3.1. Generalized parabolic Verma modules. Fix a generalized Levi subalgebra $\mathfrak{k} \subset \mathfrak{g}$ and a weight $\lambda \in \mathfrak{C}^*_{\mathfrak{k}, \mathrm{reg}}$. Let $M^\mathfrak{h}_\lambda$ denote the Verma module of highest weight λ. For each $\alpha \in \Pi^+_\mathfrak{k}$, there is a singular vector $v_{\lambda - \alpha} \in M^\mathfrak{h}_\lambda$ generating a submodule $M^\mathfrak{h}_{\lambda - \alpha} \subset M^\mathfrak{h}_\lambda$, cf. Section 2.1. Set $M^\mathfrak{k}_\lambda$ to be the quotient of $M^\mathfrak{h}_\lambda$ by the submodule $\sum_{\alpha \in \Pi^+_\mathfrak{k}} M^\mathfrak{h}_{\lambda - \alpha}$.

We denote by $V^\mathfrak{k}_\bullet = (V^\mathfrak{k}_i)^N_{i=1}$ a filtration of $\mathbb{C}^N \otimes M^\mathfrak{k}_\lambda$ by the modules $V^\mathfrak{k}_i$ generated by $w_k \otimes v_\lambda$, $k = 1, \ldots, i$. For $\mathfrak{k} = \mathfrak{h}$ it is the standard filtration considered in the previous sections. Clearly $V^\mathfrak{k}_\bullet$ is obtained from $V^\mathfrak{h}_\bullet$ through the projection $\mathbb{C}^N \otimes M^\mathfrak{h}_\lambda \to \mathbb{C}^N \otimes M^\mathfrak{k}_\lambda$. Further we show that $V^\mathfrak{k}_j / V^\mathfrak{k}_{j-1}$ vanishes once $j \in \bar{I}_\mathfrak{k}$ and q is close to 1.

Proposition 3.6. *For each $\lambda \in \mathfrak{C}^*_{\mathfrak{k}, \mathrm{reg}}$ there is a neighborhood Ω of 1 in $\mathbb{C}$ such that the submodule $V^\mathfrak{k}_j$ is generated by $w_i \otimes v_\lambda$, $i \leqslant j$, $i \in I_\mathfrak{k}$, for all $q \in \Omega$.*

Proof. For all j denote by $V'_j \subset V^\mathfrak{k}_j$ the submodule generated by all $w_i \otimes v_\lambda$ with $i \leqslant j$ and $i \in I_\mathfrak{k}$. We aim to prove that $V'_j = V^\mathfrak{k}_j$.

The statement is trivial for $j = 1$. Suppose it is true for all $i < j$. If $j \in I_\mathfrak{k}$, then $V^\mathfrak{k}_j$ is generated by $w_j \otimes v_\lambda$ and by $V^\mathfrak{k}_{j-1} = V'_{j-1}$, hence the proof. Suppose that $j \in \bar{I}_\mathfrak{k}$. Choose the greatest i such that $i < j$. Then $(i, j) \in P(\alpha)$ for some $\alpha \in \Pi^+_\mathfrak{k}$. By Lemma 2.2 there exists an open set $\Omega \subset \mathbb{C}$ containing 1 such that the principal term in $\check{f}_{ij} v_\lambda \simeq v_{\lambda - \alpha}$ is not zero for all $q \in \Omega$. Then $w_j \otimes v_\lambda \simeq w_i \otimes \psi^{ij} v_\lambda \simeq w_i \otimes \check{f}_{ij} v_\lambda = 0$ modulo $V^\mathfrak{k}_{j-1}$, by

Propositions 3.5 and 3.4. By the induction assumption, we conclude that $w_j \otimes v_\lambda \in V'_{j-1}$ and $V_j^{\mathfrak{k}} = V'_{j-1} = V'_j$. $\hspace{2cm}$ □

Corollary 3.7. *The graded module* $\mathrm{gr}V_\bullet^{\mathfrak{k}}$ *is isomorphic to the direct sum* $\oplus_{j \in I_{\mathfrak{k}}} V_j^{\mathfrak{k}}/V_{j-1}^{\mathfrak{k}}$.

Recall that the tensor $\mathcal{R}_{21}\mathcal{R}$ commutes with $\Delta(x)$ for all $x \in U_q(\mathfrak{g})$, [15].

Proposition 3.8. *The invariant operator* $\mathcal{Q} = (\pi \otimes \mathrm{id})(\mathcal{R}_{21}\mathcal{R})$ *preserves the standard filtration. It is scalar on each graded component* $V_j^{\mathfrak{k}}/V_{j-1}^{\mathfrak{k}}$, $j \in I_{\mathfrak{k}}$, *with the eigenvalue*

$$x_j = q^{2(\lambda+\rho,\varepsilon_j)-2(\rho,\varepsilon_1)+||\varepsilon_j||^2-||\varepsilon_1||^2}, \tag{3.4}$$

unless $V_j^{\mathfrak{k}}/V_{j-1}^{\mathfrak{k}} \neq \{0\}$.

Proof. The operator $\mathcal{Q}$ can be presented as $\Delta(z)(z^{-1} \otimes z^{-1})$, for a certain central element z, [23]. Therefore $\mathcal{Q}$ is a scalar multiple on every submodule and factor module of highest weight of $V_N^{\mathfrak{h}}$. Now we do induction on j. The submodule $V_1^{\mathfrak{h}}$ is of highest weight, thence it is $\mathcal{Q}$-invariant. Suppose that $V_{j-1}^{\mathfrak{h}}$ is $\mathcal{Q}$-invariant for $j > 1$. Since $\mathcal{Q}$ is scalar on $V_j^{\mathfrak{h}}/V_{j-1}^{\mathfrak{h}}$, the submodule $V_j^{\mathfrak{h}}$ is $\mathcal{Q}$-invariant.

The eigenvalue of $\mathcal{Q}$ on $V_j^{\mathfrak{h}}/V_{j-1}^{\mathfrak{h}}$ is determined by its highest weight and equal to (3.4), for all $j \in I$, [2]. So the proposition is proved for $\mathfrak{k} = \mathfrak{h}$. The general case is obtained from this by taking projection to $\mathbb{C}^N \otimes M_\lambda^{\mathfrak{k}}$ and applying Corollary 3.7. $\hspace{2cm}$ □

It follows that $\mathcal{Q}$ satisfies the polynomial equation $\prod_{j \in I_{\mathfrak{k}}}(\mathcal{Q} - x_j) = 0$ on $\mathbb{C}^N \otimes M_\lambda^{\mathfrak{k}}$. We will not address the issue if $V_j^{\mathfrak{k}}/V_{j-1}^{\mathfrak{k}}$ survive for all $j \in I_{\mathfrak{k}}$ as we bypass it in what follows.

§4. Representations of quantum conjugacy classes

In this section we extend the ground field $\mathbb{C}$ to the local ring $\mathbb{C}[[\hbar]]$ of formal power series in $\hbar$. The quantum group $U_\hbar(\mathfrak{g})$ is a completion of the $\mathbb{C}[q, q^{-1}]$-algebra $U_q(\mathfrak{g})$ in the $\hbar$-adic topology via the extension $q = e^\hbar$. Its Cartan subalgebra $U_\hbar(\mathfrak{h})$ can be generated by $h_\alpha \in \mathfrak{h}$ instead of $q^{\pm h_\alpha}$.

Assuming that $\mathfrak{k}$ is fixed, we suppress the corresponding superscripts and write simply $M_\lambda = M_\lambda^{\mathfrak{k}}$ and $V_\bullet = V_\bullet^{\mathfrak{k}}$.

Proposition 4.1. *Suppose that* $\lambda \in \mathfrak{C}_{\mathfrak{k},\mathrm{reg}}^*$. *Then* M_λ *is* $\mathbb{C}[[\hbar]]$*-free.*

Proof. The proof is similar to [3], Proposition 6.2, where it is done for a regular pseudo-parabolic Verma module over $U_q\big(\mathfrak{sp}(n)\big)$. The crucial observation is that for all $\alpha \in \Pi_\mathfrak{k}^+$ and $\lambda \in \mathfrak{C}_{\mathfrak{k},\mathrm{reg}}^*$ the vectors $\hat{f}_{ij}(\lambda)$ with $(i,j) \in P(\alpha)$ can be included in a PBW basis in $U_\hbar(\mathfrak{g}_-)$ if the ring of scalars is $\mathbb{C}[\![\hbar]\!]$. This follows from Lemma 2.2. $\qquad\square$

Proposition 4.1 implies that the algebra $\mathrm{End}(M_\lambda)$ is also $\mathbb{C}[\![\hbar]\!]$-free. We are going to realize a quantized conjugacy class of the point $\lim_{\hbar\to 0} q^{2h_\lambda} \in T$ as a subalgebra in $\mathrm{End}(M_\lambda)$.

Consider the image of the algebra $\mathbb{C}_\hbar[G]$ in $\mathrm{End}(M_\lambda)$ under the composition homomorphism

$$\mathbb{C}_\hbar[G] \to U_q(\mathfrak{g}) \to \mathrm{End}(M_\lambda).$$

Here the algebra $U_q(\mathfrak{g})$ is extended over $\mathbb{C}[\![\hbar]\!]$. This representation induces a character, χ_λ, of the center of $\mathbb{C}_\hbar[G]$. It annihilates the ideal in $\mathbb{C}_\hbar[G]$ generated by the kernel χ_λ and by the entries of the minimal polynomial of $\mathcal{Q}$ as a linear operator on $\mathbb{C}^N \otimes M_\lambda$. The center of $\mathbb{C}_\hbar[G]$ is generated by

$$\begin{aligned}
\tau_k &= \mathrm{Tr}_q(\mathcal{Q}^k) := \mathrm{Tr}\big((\pi(q^{2h_\rho}) \otimes 1)\mathcal{Q}^k\big) \in U_\hbar(\mathfrak{g}), \quad k = 1, 2, \dots, \\
\tau^- &= \mathrm{Tr}_q(\mathcal{Q}_+) - \mathrm{Tr}_q(\mathcal{Q}_-), \quad \text{for} \quad \mathfrak{g} = \mathfrak{so}(2n).
\end{aligned}$$

Here $\mathcal{Q}_\pm$ are the images of $\mathcal{R}_{21}\mathcal{R}$ in $\mathrm{End}(W_\pm) \otimes U_q(\mathfrak{g})$, were $W_\pm \subset \wedge^n(\mathbb{C}^n)$ are finite dimensional irreducible modules of highest weights $\sum\limits_{i=1}^{n-1} \varepsilon_i \pm \varepsilon_n$. In the classical limit, this invariant separates two $SO(2n)$-conjugacy classes whose eigenvalues are all distinct from ± 1. They are flipped by any inversion $x_i \leftrightarrow x_i^{-1}$, $i = 1, \dots, n$, and amount to an $O(2n)$-conjugacy class. If ± 1 is in the spectrum, the $O(2n)$-conjugacy class is also an $SO(2n)$-class. In this case, τ^- is redundant.

Theorem 4.2. *Let $\mathfrak{k} \subset \mathfrak{g}$ be a generalized Levi subalgebra, $\lambda \in \mathfrak{C}_{\mathfrak{k},\mathrm{reg}}^*$, and $M_\lambda = M_\lambda^\mathfrak{k}$ the corresponding generalized parabolic Verma module. Then*

i) *the annihilator of M_λ in $\mathbb{C}_\hbar[G]$ is generated by*

$$\left(\prod_{i\in I_{\mathfrak{k}}}(\mathcal{Q}-x_i)\right)_{ij}, \qquad\qquad i,j=1,\ldots,N,$$

$$\chi_\lambda(\tau_k)-\sum_{i=1}^N x_i^k\prod_{\alpha\in \mathrm{R}_+}\frac{q^{(\lambda+\rho+\varepsilon_i,\alpha)}-q^{-(\lambda+\rho+\varepsilon_i,\alpha)}}{q^{(\lambda+\rho,\alpha)}-q^{-(\lambda+\rho,\alpha)}}, \quad k=1,\ldots,N,$$

$$\chi_\lambda(\tau^-)-\prod_{i=1}^n\big(q^{2(\lambda+\rho,\varepsilon_i)}-q^{-2(\lambda+\rho,\varepsilon_i)}\big), \qquad \mathfrak{g}=\mathfrak{so}(2n),$$

where x_i is given by (3.4),

ii) *the image of $\mathbb{C}_\hbar[G]$ in $\mathrm{End}(M_\lambda)$ is an equivariant quantization of $\mathbb{C}_\hbar[O_x]$, $x=\lim_{\hbar\to0}q^{2h_\lambda}$,*

iii) *this quantization is independent of the choice of initial point and is an exact representation of the unique quantum conjugacy class of x.*

Proof. The statements i) and ii) for all types of classes are proved in [2–5], for certain regular $\mathfrak{k}=\mathfrak{k}_0$. For arbitrary $\mathfrak{k}$ there is an element σ of the Weyl group such that $\mathrm{R}_{\mathfrak{k}}^+=\sigma(\mathrm{R}_{\mathfrak{k}_0}^+)$. The shifted action $\lambda_0\mapsto\sigma(\lambda_0+\rho)-\rho=\lambda$ takes $\mathbb{C}_{\mathfrak{k}_0,\mathrm{reg}}^*$ to $\mathbb{C}_{\mathfrak{k},\mathrm{reg}}^*$. It preserves the central characters and takes the set of eigenvalues of $\mathcal{Q}$ on $\mathbb{C}^N\otimes M_{\lambda_0}^{\mathfrak{h}}$ to eigenvalues on $\mathbb{C}^N\otimes M_\lambda^{\mathfrak{h}}$. Moreover, $\sigma\{x_i\}_{i\in I^{\mathfrak{k}_0}}=\{x_i\}_{i\in I^{\mathfrak{k}}}$ as σ relates the orderings $\prec$ relative to $\mathfrak{k}_0$ and $\mathfrak{k}$. This implies that the annihilator of $M_\lambda^{\mathfrak{k}_0}$ in $\mathbb{C}_\hbar[G]$ vanishes on $M_\lambda^{\mathfrak{k}}$, that is, there is an equivariant homomorphism $\mathbb{C}_\hbar[G/K_0]\to\mathrm{End}(M_\lambda^{\mathfrak{k}})$. In order to complete the proof, we need to show that this homomorphism is an embedding.

Since $\mathbb{C}_\hbar[G/K_0]$ is a direct sum of $\mathbb{C}[[\hbar]]$-finite isotypic $U_\hbar(\mathfrak{g})$-components and $\mathrm{End}(M_\lambda^{\mathfrak{k}})$ is $\mathbb{C}[[\hbar]]$-free, the image of $\mathbb{C}_\hbar[G/K_0]$ is $\mathbb{C}[[\hbar]]$-free. The algebra $\mathbb{C}[G/K_0]$ has no proper invariant ideals, hence the kernel of the map $\mathbb{C}_\hbar[G/K_0]\to\mathrm{End}(M_\lambda^{\mathfrak{k}})$ is zero. This completes the proof. $\qquad\square$

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