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ON THE GEOMETRIC PROBABILITY OF ENTANGLED MIXED STATES

Abstract. The state space of a composite quantum system, the set of density matrices $\mathfrak{P}_+$, is decomposable into the space of separable states $\mathfrak{S}_+$ and its complement, the space of entangled states. An explicit construction of such a decomposition constitutes the so-called separability problem. If the space $\mathfrak{P}_+$ is endowed with a certain Riemannian metric, then the separability problem admits a measurement-theoretical formulation. In particular, one can define the "geometric probability of separability" as the relative volume of the space of separable states $\mathfrak{S}_+$ with respect to the volume of all states. In the present note, based on the Peres–Horodecki positive partial transposition criterion, the measurement theoretical aspects of the separability problem are discussed for bipartite systems composed either of two qubits or of qubit-qutrit pairs. The necessary and sufficient conditions for the 2-qubit state separability are formulated in terms of local SU(2)⊗SU(2) invariant polynomials, the determinant of the correlation matrix, and the determinant of the Schlienz–Mahler matrix. Using the projective method of generation of random density matrices distributed according to the Hilbert–Schmidt or Bures measure, the separability (including the absolute separability) probabilities of 2-qubit and qubit-qutrit pairs have been calculated.

§1. Introduction

The word "*entanglement*", the "*verschränkung*", in the original Austrian phrasing, was introduced in a glossary of quantum mechanics by Ervin Schrödinger at the Thirties of last century. The name owes its appearance to a strange type of correlations in composite systems predicted by newly created quantum theory [1]. The existence of "entangled" states in quantum theory seemed very problematic and mysterious since its inception, but at present it is experimentally verified and, moreover, practically used in a variety of quantum engineering applications. Undoubtedly, nowadays the entanglement found its own place among the fundamental notions

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of quantum physics and gains the popularity similar words "energy" and "force" had in XIX-th century.

Being highly counter intuitive and strange occurrence, the entanglement has a transparent mathematical formulation. Mathematics certainly dispels the aura of mystery, reducing the understanding of correlations between parts of composed system to the analysis of a set correctly stated algebraic problems. One of the primary importance, the so-called "separability problem" is formulated as follows. Consider a system composed from two d_A and d_B – dimensional subsystems with the Hilbert spaces $\mathcal{H}^{d_A}$ and $\mathcal{H}^{d_B}$ respectively. According to the axioms of quantum mechanics any state of the composed system is given by the density matrix $\varrho \in \mathfrak{P}_+$, that acts on the Hilbert space of the tensor product form:

$$\mathcal{H}^{d_A\, d_B} = \mathcal{H}^{d_A} \otimes \mathcal{H}^{d_B}.$$

For a given $\mathcal{H}^{d_A} \otimes \mathcal{H}^{d_B}$ factorization an element $\varrho_{\mathrm{sep}} \in \mathfrak{P}_+$ belongs to the subset of *separable* states $\varrho_{\mathrm{sep}} \in \mathfrak{S}_+$ if and only if ϱ_{sep} admits the convex decomposition of r tensor product states with some probability distribution ω_k [2]:

$$\varrho_{\mathrm{sep}} = \sum_{k=1}^{r} \omega_k\, \varrho_k^A \otimes \varrho_k^B\,. \tag{1}$$

The operators ϱ_k^A and ϱ_k^B in (1) denote the density operators of subsystems A and B respectively. The states complementary to the separable ones are named the *entangled*.[1]

The definition (1) is an implicit and therefore the question of whether a given state is separable or entangled is worthy of further attention. Even from the first glance becomes clear that the "separability" question is highly intricate. Moreover, as it was shown by Gurvits (cf. [4, 5]) even for a bipartite system the separability problem is categorized computationally as `NP`-hard.

[1]Note that the representation (1) is not unique and even knowing that state is separable to find its decomposition is not an easy task. Furthermore, speaking about the separability, one has always have in mind that a fixed factorization $\mathcal{H}^{d_A} \otimes \mathcal{H}^{d_B}$ has been picked out. Via the global unitary transformation U acting on the total space, one can switch to another factorization, $U\left(\mathcal{H}^{d_A} \otimes \mathcal{H}^{d_B}\right) U^+$. As result, a former separable state can appear as entangled one and vice versa (cf. discussion in [3]).

Complexity of the problem brings into play alternative approaches. Particularly, considering the state space of quantum mechanical system as object with measure (cf. [6, 7]), the "separability problem" can be reshaped into the probability issue [8, 9].

Below adopting the above approach we consider in details a bipartite systems consisted from 2 and 3-level subsystems. Equipping the state space with a certain measure the relative volume of entangled states with respect to the all possible states will be computed

$$\mathcal{P}_{\mathrm{E}} = \frac{\mathrm{Vol(Space\ of\ entangled\ states)}}{\mathrm{Vol(Space\ of\ all\ states)}}\,. \tag{2}$$

This number defines the geometric probability of entanglement, which can be treated as a certain measure for "capacity of quantumness" of the system.

The article is organized as follows. In sections 2 and 3 the basic elements from mathematical description of finite-dimensional quantum systems are given. Latter, using this background, the notion of the separability probability of states is introduced. Based on the Peres–Horodecki positive partial transposition criterion, the necessary and sufficient conditions for 2-qubit state separability are formulated in terms of the local $\mathrm{SU}(2)\otimes\mathrm{SU}(2)$ scalars, determinants of correlation matrix and Schlienz–Mahler matrix [10]. In the section 3, adopting the projective method of generation of random density matrix the probability aspects of the separability characteristics of 2-qubit and qubit-qutrit pairs are studied. The later include determination of the separability and absolute separability probability, as well as the numerical evaluation of distributions of separable matrices with respect to the determinants of the correlation and the Schlienz-Mahler matrices.

§2. Settings

Below the relevant definitions and notions, including the basic algebraic and geometric characteristics of a composite quantum systems are given in a from suitable for the introduction of the probability of quantum states. Note that only a finite dimensional quantum systems are considered.

2.1. State space. At the beginning of the "Golden Era" of quantum mechanics John von Neuman and Lev Landau, became aware of limitations for applicability of the Schródinger's Ψ-function, introduce the notion of a *"mixed quantum state"* [11, 12]. The mixed state is characterized by the self-adjoint, positive semi definite *"density operator"* acting on the Hilbert space of quantum system. For the non-relativistic n-dimensional system the Hilbert space $\mathcal{H}$ is $\mathbb{C}^n$ and the density operator can be identified with $n \times n$ Hermitian, unit trace, positive semi-definite matrix ϱ. This matrix, termed the *density matrix*, completely specifies the state of n-level quantum system. All possible density matrices form the set $\mathfrak{P}_+$ – the *state space* of n-dimensional quantum system.

2.1.1. *State space as a semi-algebraic variety.* The space of Hermitian matrices is topologically isomorphic to $\mathbb{R}^{n^2}$. Due to the positive semi-definiteness any density matrix ϱ represents a point of semi-algebraic variety, $\mathfrak{P}_+(\mathbb{R}^{n^2-1})$ of affine subspace, defined by a unit trace equation $\mathrm{Tr}\varrho = 1$. Nevertheless of a long story of studies of finite dimensional systems it is very little known about $\mathfrak{P}_+(\mathbb{R}^{n^2-1})$ for arbitrary n. It turns out that even for small n the structure of $\mathfrak{P}_+(\mathbb{R}^{n^2-1})$ is quite cumbersome.[2]

•**Density matrices and universal enveloping algebra** $\mathfrak{U}(\mathfrak{su}(\mathrm{n}))$• The state space has useful algebraic description in terms of the universal enveloping algebra $\mathfrak{U}(\mathfrak{su}(\mathrm{n}))$ of the Lie algebra $\mathfrak{su}(\mathrm{n})$. Let $e_1, e_2, \dots, e_{n^2-1}$ form the basis for $\mathfrak{su}(\mathrm{n})$

$$\mathfrak{su}(\mathrm{n}) = \sum_{i=1}^{\mathrm{n}^2-1} \xi_i \, e_i . \tag{3}$$

Consider elements from $\mathfrak{U}(\mathfrak{su}(\mathrm{n}))$ of the following form:

$$\varrho = \frac{1}{n}\left(\mathbb{I}_{n\times n} + \imath\sqrt{\frac{n(n-1)}{2}}\sum_{i=1}^{\mathrm{n}^2-1} \xi_i \, e_i\right), \tag{4}$$

with a real (n^2-1)-dimensional vector $\boldsymbol{\xi} = (\xi_1, \xi_2, \dots, \xi_{n^2-1})$. As it was mentioned above, expression (4) represents an element of the space of states $\mathfrak{P}_+$ if the vector $\boldsymbol{\xi}$ is subject to a finite set of polynomial inequalities:

$$f_\alpha(\boldsymbol{\xi}) \geqslant 0 . \tag{5}$$

Moreover, it turns that the semi-algebraic set described by (5) admits representation with the polynomial functions f_α that are invariant under

[2]The neighbourhood of a generic point of $\mathfrak{P}_+(\mathbb{R}^{n^2-1})$ is locally isometric to $\left(SU(n)/U(1)^{n-1}\right) \times D^{n-1}$, where the component D^{n-1} is $(n-1)$-dimensional disc (cf. [13]).

the adjoint action of the unitary group SU(n) on $\mathfrak{P}_+(\mathbb{R}^{n^2-1})$. More precisely, consider SU(n)-invariant polynomial ring $\mathbb{R}[\mathfrak{P}_+]^{\mathrm{SU(n)}}$ and a set of homogeneous polynomials $\mathcal{P} = (t_1, t_2, \dots, t_n)$, that form its integrity basis

$$\mathbb{R}[\xi_1, \xi_2, \dots, \xi_{n^2-1}]^{\mathrm{SU(n)}} = \mathbb{R}[t_1, t_2, \dots, t_n]. \tag{6}$$

Then the space of states $\mathfrak{P}_+(\mathbb{R}^{n^2-1})$ for arbitrary n is semi-algebraic subset given by inequalities of the following type

$$p_i(t_1, t_2, \dots, t_n) \geqslant 0\,, \quad i = 1, 2, \dots, s \tag{7}$$

where $p_i \in \mathbb{R}[\mathfrak{P}_+]^{\mathrm{SU(n)}}$. Below, analysing requirements of the Hermicity and semi-positivity for density matrices, the explicit form of inequalities (7) will be given. With this aim a brief digression, devoted to the construction of the integrity basis $\mathcal{P} = (t_1, t_2, \dots, t_n)$ from elements of the universal algebra center $\mathcal{Z}(\mathfrak{su}(\mathrm{n}))$ is in order.

DIGRESSION-1

• SU(n)-invariance • Construction of the adjoint SU(n)-invariants from the elements of $\mathcal{Z}(\mathfrak{su}(\mathrm{n}))$ is well known procedure. Referring to the literature on this subject (see e.g., [14]) we briefly state the results and discuss constraints on these invariants due to the Hermicity and positive semi-definiteness of density matrices. We are looking for polynomials in $\xi_1, \xi_2, \dots, \xi_{n^2-1}$ variables

$$\phi(\boldsymbol{\xi}) = \sum c_{i_1\cdots i_r}\, \xi_{i_1}\xi_{i_2} \dots \xi_{i_r}\,, \tag{8}$$

which are invariant under the adjoint action

$$\phi(\boldsymbol{\xi}) = \phi((\mathbf{Ad}\boldsymbol{g})^{\boldsymbol{T}}\boldsymbol{\xi})\,, \tag{9}$$

where $(\mathrm{Ad}g)^T$ is the transpose matrix of adjoint operator calculated in the basis $e_{i_1}, e_{i_2}, \dots, e_{n^2-1}$:

$$g\, e_i\, g^{-1} = (\mathrm{Ad}g\,)_{ij} e_j\,, \qquad g \in \mathrm{SU(n)}\,. \tag{10}$$

These polynomials are in one to one correspondence with the elements of center $\mathcal{Z}(\mathfrak{su}(\mathrm{n}))$

$$\mathfrak{C}_r = \sum \frac{1}{r!} c_{i_1\cdots i_r} \sum_{\sigma\in S_r} e_{i_{\sigma(1)}} e_{i_{\sigma(2)}} \,\dots\, e_{i_{\sigma(r)}}\,, \tag{11}$$

where S_r is the group of permutations of $1, 2, \dots r$.

Furthermore, the $n-1$ independent Casimir operators $\mathfrak{C}_r$ in (11) serve as a resource for the integrity basis of the polynomial ring $\mathbb{R}[\mathfrak{P}_+]^{\mathrm{SU(n)}}$.

The scalars appeared from above isomorphism are commonly referred as Casimir invariants. The first Casimir invariants up to six order in $\boldsymbol{\xi}$ are given:

$$\mathfrak{C}_2 = (n-1)\,\boldsymbol{\xi}\cdot\boldsymbol{\xi} \tag{12}$$

$$\mathfrak{C}_3 = (n-1)\,(\boldsymbol{\xi}\vee\boldsymbol{\xi}\,)\cdot\boldsymbol{\xi} \tag{13}$$

$$\mathfrak{C}_4 = (n-1)\,(\boldsymbol{\xi}\vee\boldsymbol{\xi}\,)\cdot(\boldsymbol{\xi}\vee\boldsymbol{\xi}\,) \tag{14}$$

$$\mathfrak{C}_5 = (n-1)\,\Big((\boldsymbol{\xi}\vee\boldsymbol{\xi}\,)\vee(\boldsymbol{\xi}\vee\boldsymbol{\xi}\,)\Big)\cdot\boldsymbol{\xi} \tag{15}$$

$$\mathfrak{C}_6 = (n-1)\,(\boldsymbol{\xi}\vee\boldsymbol{\xi}\vee\boldsymbol{\xi}\,)^2 \tag{16}$$

$$\cdots\cdots,$$

where

$$(\boldsymbol{U}\vee\boldsymbol{V}\,)_a := \kappa\, d_{abc}U_aV_b\,,$$

d_{abc} are symmetric structure constants for $\mathfrak{su}(\mathfrak{n})$ and $\kappa = \sqrt{n(n-1)/2}$ is normalization constant. Another, an equivalent set of invariants, useful from a computational point of view, is given by the so-called trace invariants, power series in eigenvalues, $\{\lambda\} = \lambda_1, \lambda_2, \dots, \lambda_n$. of the density matrix

$$t_k := \operatorname{tr}(\varrho^k) = \lambda_1^k + \lambda_2^k + \cdots + \lambda_n^k\,, \qquad k = 1, 2, \dots, n. \tag{17}$$

Below we formulate requirements of Hermicity and semi-positivity of density matrix directly in terms of (17).

• **Hermicity of ϱ in terms of the SU(n)-invariants** • Since ϱ is a Hermitian matrix all solutions (eigenvalues $\{\lambda\}$) of the characteristic equation

$$\det\|\lambda - \varrho\| = \lambda^n - S_1\lambda^{n-1} + S_2\lambda^{n-2} - \cdots + (-1)^n S_n = 0\,, \tag{18}$$

are real numbers. In accordance with the classical result a certain information on the properties of the roots can be extracted from the so-called *Bézoutian*, the matrix $\mathrm{B} = \Delta^T\Delta$, constructed from the Vandermonde matrix

$$\Delta = \begin{pmatrix} 1 & \lambda_1 & \lambda_1^2 & \dots & \lambda_1^{n-1} \\ 1 & \lambda_2 & \lambda_2^2 & \dots & \lambda_2^{n-1} \\ 1 & \lambda_3 & \lambda_3^2 & \dots & \lambda_3^{n-1} \\ \vdots & \vdots & \vdots & \vdots\vdots\vdots & \\ 1 & \lambda_n & \lambda_n^2 & \dots & \lambda_n^{n-1} \end{pmatrix}. \tag{19}$$

The entries of the Bézoutian are simply the trace invariants:

$$\mathrm{B}_{ij} = t_{i+j-2}\,. \tag{20}$$

The Bézoutian accommodate information on number of distinct roots (via its rank), numbers of real roots (via its signature), as well as the Hermicity condition. A real characteristic polynomial has all its roots *real and distinct* if and only if the Bézoutian is positive definite. Here we are interesting only in situation of a generic density matrices (the space of degenerate matrices with coinciding roots are measure zero sets). For this case the positivity of Bézoutian reduces to the requirement,

$$\det \|\mathrm{B}\| > 0\,. \tag{21}$$

Since $\det \|\mathrm{B}\| = (\det \|\Delta\|)^2$, the determinant of the Bézoutian is nothing else as the *discriminant* of the characteristic equation (18)

$$\mathrm{Disc} = \prod_{i>j} (\lambda_i - \lambda_j)^2\,, \tag{22}$$

rewritten in terms of the trace polynomials

$$\mathrm{Disc}(t_1, t_2, \dots, t_n) := \det \|\mathrm{B}\|. \tag{23}$$

Dependence of discriminant on trace invariants only up to order n pointed in left side of (23) assumes that all higher trace invariants t_k with $k > n$ in (20) are expressed via polynomials in $t_1, t_2, \dots, t_n$, (the Cayley–Hamilton Theorem).

•Semi-positivity of ϱ in terms of the SU(n)-invariants• Positive semi-definiteness implies the non-negativity of roots of (18):

$$\lambda_k \geqslant 0\,, \qquad k = 1, 2, \dots, n. \tag{24}$$

Inequalities (24) are not computationally efficient, the eigenvalues $\{\lambda\}$ are non-polynomial SU(n)-invariants. Fortunately, it it is known (see e.g. [15, 16] and references therein), that instead of (24) the equivalent set of inequalities can be formulated in terms the first n-symmetric polynomials in eigenvalues of ϱ:

$$S_k \geqslant 0\,, \qquad k = 1, 2, \dots, n. \tag{25}$$

Opposite to eigenvalues, the coefficients S_k are SU(n)-invariant polynomial functions of density matrix elements and thus are expressible in terms of the trace invariants. An elegant expression for S_k is given by the following determinant

$$S_k = \frac{1}{k!} \det \begin{pmatrix} t_1 & 1 & 0 & \cdots & 0 \\ t_2 & t_1 & 2 & \cdots & 1 \\ t_3 & t_2 & t_1 & \cdots & \\ \vdots & \vdots & \vdots & \vdots\vdots\vdots & k-1 \\ t_k & t_{k-1} & t_{k-2} & \cdots & t_1 \end{pmatrix} . \tag{26}$$

Summarizing, the algebraic set of inequalities in SU(n)-invariants describing the state space $\mathfrak{P}_+(\mathbb{R}^{n^2-1})$, as the semi-algebraic variety of the affine subspace $\mathrm{Tr}\varrho = 1$, read:

$$\mathrm{Disc} \geqslant 0 \,, \qquad \text{Hermicity} \,, \tag{27}$$

$$S_k \geqslant 0 \,, \qquad \text{Semi-positivity} \,. \tag{28}$$

Now we are in position to pose the following question: *Is the space of separable states $\mathfrak{S}_+$ the semi-algebraic set as well?* Nevertheless, of many efforts performed during last decades, a complete answer for a generic case is unknown yet. But, for a simplest bipartite system $2 \otimes 2$, composed from pair of 2-dimensional subsystems, qubits, the space of separable states $\mathfrak{S}^{2\otimes 2}$ admits nice description as a basic semi-algebraic variety. Next paragraph is devoted to the detailed demonstration of this particular result.

2.1.2. *Decomposing state space: separable vs. entangled.* As it was mentioned in the Introduction, due to the quantum superposition principle, an arbitrary state of a composite system is described by the element of the tensor products of density operators of its subsystems. For a given factorization of system into the parts, the state space $\mathfrak{P}_+(\mathbb{R}^{n^2-1})$ decomposes into the separable $\mathfrak{S}_+$ and entangled components. Further more, since the property of separability is independent of the choice of basis in each subsystem, it was conjectured that $\mathfrak{S}_+$ (see discussion in Chen and Dokovic [17]) represents the so-called basic closed semi-algebraic set, which is defined by polynomial inequalities in variables, that are invariant under independent action of the unitary transformations of each subsystems. Below, starting with the necessary definitions, the description of $\mathfrak{S}_+$ for a pair of qubits will be given.

A generic 15-parameter density matrix for composite $2 \otimes 2$ system consistent from 2-qubits reads

$$\varrho = \frac{1}{4} \left[\mathbb{I}_4 + \boldsymbol{a} \cdot \boldsymbol{\sigma} \otimes \mathbb{I}_2 + \mathbb{I}_2 \otimes \boldsymbol{\sigma} \cdot \boldsymbol{b} + c_{ij}\, \boldsymbol{\sigma}_i \otimes \boldsymbol{\sigma}_j \right] . \tag{29}$$

Representation (29) is often named as Fano [18] decomposition of 2-qubits state with parameters $\boldsymbol{a}$ and $\boldsymbol{b}$ assigned to the Bloch vectors of the reduced density matrices ϱ_A and ϱ_B extracted from ϱ by taking the partial traces over second and first qubit respectively:

$$\varrho_A = \mathrm{Tr}_B\, \varrho, \qquad \varrho_B = \mathrm{Tr}_A\, \varrho. \tag{30}$$

Nine real coefficients c_{ij} are usually collected in the *"correlation matrix"*, $\|\mathrm{C}\|_{ij} = c_{ij}$. As follows from its name, the C-matrix contains information on interactions between parts of the composed system.

• The separability criterion • Perhaps the most useful tool for qualifying separability is the famous Peres-Horodecki criterion [19–21], which is based on the idea of the partial transposition. The partial transpose ϱ^{T_B} of 2-qubits density matrix is defined as

$$\varrho^{T_B} = I \otimes T\varrho, \tag{31}$$

where T is the standard transposition operation. Under the transposition the Pauli matrices change as $T(\sigma_1, \sigma_2, \sigma_3) \to (\sigma_1, -\sigma_2, \sigma_3)$.

The states whose partial transposition preserves its positivity are termed as Positive Partial Transpose (PPT)-states. It is easy to verify that any separable state is PPT. The opposite is not true, even for low dimensional bipartite systems. The counterexamples for 3×3 shows that there are entangled states with a positive partial transpose. However, for composite binary systems of type 2×2 and 2×3 the Peres-Horodecki criterion asserts that the state ϱ is separable if and only if its partial transposition ϱ^{T_B} is positive as well.[3]

Intuitively it is clear that entanglement in composite systems is function of the "relative orientation" of its subsystems only, any "local characteristics" of subsystems are unessential for the separability property. To give a rigorous sense to this view the second digression on the so-called local invariance possessing by composite systems is in order.

[3]More generally, consider a family of bipartite so-called $k \times l$-states ϱ, i.e., states whose partial traces are matrices with $\mathrm{rank}\varrho_A = k$ and $\mathrm{rank}\varrho_A = l$ respectively. For such $k \times l$-states it is was proved that ϱ is separable if it is PPT and $(k-1)(l-1) \leqslant 2$ [19,20].

DIGRESSION-2

• **The local unitary invariance** • The characterization of entanglement for 2-qubits, as well as more general multipartite systems, admit formulation in terms of invariants of the so-called local groups [22]. To introduce this notion consider a generic multipartite system composed from r-subsystems each with $d_1, d_2, \dots, d_r$ levels respectively. The special subgroup of the unitary group SU(n) with $n = d_1 \times d_2 \times \cdots \times d_r$:

$$\mathrm{SU}(d_1) \otimes \mathrm{SU}(d_2) \otimes \cdots \otimes \mathrm{SU}(d_r), \tag{32}$$

acting on the state space, is termed as the group of *local unitary transformations* (LUT). This action introduces the equivalence relations on $\mathfrak{P}_+(\mathbb{R}^{n^2-1})$ and defines its orbital decomposition. Two states of composite system connected by the LUT transformations (32) have the same non-local properties. Any characteristics of entanglement is a function of the LUT-invariants. Particularly, the separability criterion in terms of the corresponding polynomial LUT-invariants can be given. Before presenting an algebraic formulation of the separability criterion, we pass to a basic description of LUT-invariants (see e.g. [22–26].

• **SU(2)⊗ SU(2) invariants** • The LUT-invariants of the mixed two-qubit system are those polynomials in the elements of state which are constant under the adjoint SU(2)⊗ SU(2) group action. These invariants and the corresponding ring $\mathcal{R}^{\mathrm{SU(2)\otimes SU(2)}}$ have been subject of intensive studies . In this general setting, $\mathcal{R}^{\mathrm{SU(2)\otimes SU(2)}}$ necessarily has the Cohen-Macaulay property, i.e., there exists a homogeneous system of parameters $K_1, K_2, \dots, K_n$, for some n, such that $\mathcal{R}^{\mathrm{SU(2)\otimes SU(2)}}$ is finitely generated as a free module over $\mathbb{C}[K_1, K_2, \dots, K_n]$. It is known that the polynomial ring of $\mathrm{SU}(2) \otimes \mathrm{SU}(2)$ invariants admits the Hironaka decomposition, namely [24]

$$\mathcal{R}^{\mathrm{SU(2)\otimes SU(2)}} = \bigoplus_{k=0}^{15} J_k\, \mathbb{C}[K_1, K_2, \dots, K_{10}], \tag{33}$$

where ten primary algebraically independent polynomials K_r have degrees $\deg K = (1,2,2,2,3,3,4,4,4,6)$; and fifteen secondary linearly independent invariants $J_k,\ k = 0,1,2,\dots,15$ are polynomials of degrees $\deg J = (4,5,6,6,6,7,7,8,8,9,9,9,10,11,15)$ with $J_0 = 1$.

The integrity basis SU(2)⊗ SU(2) invariants in the enveloping algebra $\mathfrak{U}(\mathfrak{su}(\mathrm{n}))$ is known [24, 27]. Following the Quesne's notations, the invariants (up to fourth order), necessary for our analysis, are listed below in assumption of summation over all repeated indices from one to three:

3 invariants of second degree

$$C^{(200)} = a_i a_i\,, \quad C^{(020)} = b_i b_i\,, \quad C^{(002)} = c_{ij} c_{ij}\,, \tag{34}$$

2 invariants of third degree

$$C^{(003)} = \frac{1}{3!}\epsilon_{ijk}\epsilon_{\alpha\beta\gamma} c_{i\alpha} c_{j\beta} c_{k\gamma}, \quad C^{(111)} = a_i c_{ij} b_j, \tag{35}$$

4 invariants of fourth degree

$$C^{(004)} = c_{i\alpha} c_{i\beta} c_{j\alpha} c_{j\beta}, \tag{36}$$

$$C^{(202)} = a_i a_j c_{i\alpha} c_{j\alpha}\,, \tag{37}$$

$$C^{(022)} = b_\alpha b_\beta c_{i\alpha} c_{i\beta}, \tag{38}$$

$$C^{(112)} = \epsilon_{ijk}\epsilon_{\alpha\beta\gamma} a_i b_\alpha c_{j\beta} c_{k\gamma}, \tag{39}$$

Now we will show that two LUT-invariants, namely, $C^{(003)}$ and $C^{(112)}$ play especial role in an algebraic form of the Peres–Horodecki separability criterion.

• **Separability in terms of local invariants** • As it follows from the Peres–Horodecki, the density matrices ϱ for 2-qubits are separable if the coefficients $S_k^{T_B}$ of characteristic equation for the corresponding partially transposed matrices ϱ^{T_B} are non-negative:

$$S_k^{T_B} \geqslant 0\,, \qquad k = 2, 3, 4. \tag{40}$$

As calculations show the second coefficient of characteristic equation is invariant under the partial transposition (31):

$$S_2^{T_B} = S_2, \tag{41}$$

while higher coefficients change as follows

$$S_3^{T_B} = S_3 + \det \|\mathrm{C}\|, \tag{42}$$

$$S_4^{T_B} = S_4 + \det \|\mathrm{M}\|, \tag{43}$$

where M stands for the Schlienz–Mahler matrix [10]:

$$\mathrm{M}_{ij} := c_{ij} - a_i b_j. \tag{44}$$

Comparing with (35) one can easily verify that both determinants $\det \|\mathrm{C}\|$ and $\det \|\mathrm{M}\|$ are invariant under the local group SU(2) ⊗ SU(2):

$$\det\|\mathrm{C}\| = C^{003}, \quad \det\|\mathrm{M}\| = C^{003} - \frac{1}{2}C^{112}. \tag{45}$$

It is interesting that the equations (42) and (43) allow to formulate sufficient conditions for 2-qubits entanglement.

• **Sufficient conditions for 2-qubits entanglement** • Consider a pair of qubits in a generic mixed state (29). Then from (41)–(43) it follows that: *Any density matrix ρ, obeying the inequalities*

$$\det{}^2\|\mathrm{M}\| > 1, \qquad \det{}^2\|\mathrm{C}\| > 1, \tag{46}$$

with necessity is the entangled matrix. The density matrices from the complementary domain

$$-1 \leqslant \det\|\mathrm{M}\| \leqslant 1, \quad -1 \leqslant \det\|\mathrm{C}\| \leqslant 1, \tag{47}$$

are separable as well as entangled ones.

The above described separability vs. entanglement conditions are invariant under LUT-group action, but can be changed under generic unitary transformations. However, noting the maximally mixed state $\varrho_0 \sim \mathbb{I}_{n\times n}$ remains the separable one under an arbitrary $U(n)$ transformations, one can expect an existence of states in its neighbourhoods that posses the separability properties independent of chosen basis. Below a short review of such states characterization is given.

• **Absolute separability** • The property of separability vs. entanglement is sensitive to the way of how the system is decomposed into parts. Being generically depended on a fixed factorization, it has exceptions to the rule. M. Kuŝ and K. Zyczkowski in [28] drew attention to the states of n-dimensional quantum system, that are *absolute separable*, i.e., that there is U(n)-invariant subspace $\mathcal{A}\mathfrak{S}_+ \subset \mathfrak{S}_+$

$$\mathcal{A}\mathfrak{S}_+ = \{\varrho \in \mathfrak{S}_+ \mid \mathrm{U}\varrho\,\mathrm{U}^+ \in \mathfrak{S}_+,\ \forall\,\mathrm{U} \in \mathrm{U}(n)\}. \tag{48}$$

What is condition for state to be an absolute separable one ? The answer to this question 2-qubit system was found by Verstraete et al. [29], who showed that a necessary and sufficient condition is given by a quadratic inequality on the eigenvalues of density matrix. Later, for the case of a bipartite system formed from qudits, the similar system of inequalities in the eigenvalues of density matrix has been derived by R.Hildebrand [30]. Particularly, for $2\otimes 2$ and $2\otimes 3$ the inequalities read

$$\lambda_1 - \lambda_3 \quad \leqslant \quad 2\sqrt{\lambda_2\lambda_4}, \tag{49}$$

$$\lambda_1 - \lambda_5 \quad \leqslant \quad 2\sqrt{\lambda_4\lambda_6}. \tag{50}$$

The algebraic description of state space and particularly the separable states presented here is well adapted for an extraction of quantitative characteristics of the entanglement. Now few applications exemplifying this thesis will be given.

§3. Probabilistic view on entanglement

Here probabilistic aspects of the entanglement is discussed within the semi-algebraic description given previous sections. Adopting the probability approach [8, 9, 31–33]. the probabilistic characteristics for 2-qubits and qubit-qutrit system will be presented. Since a standard methods from the theory of probability require existence of measure, below we start with the introduction Riemannian structures on $\mathfrak{P}_+(\mathbb{R}^{n^2-1})$.

3.1. The Riemannian geometry of states. There is no way to single out a unique measure in state space. Various physical and mathematical argumentation have been drawn for introduction of different metrics on $\mathfrak{P}_+(\mathbb{R}^{n^2-1})$. Few popular distances between two density matrices ϱ_1 and ϱ_2, commonly used in the literature, are

- the *trace distance*

$$D_{\mathrm{tr}}(\varrho_1, \varrho_2) = \mathrm{tr}\left(\sqrt{(\varrho_1 - \varrho_2)^2}\right) , \tag{51}$$

- the *Hilbert–Schmidt distance*

$$D_{\mathrm{HS}}(\varrho_1, \varrho_2) = \sqrt{\mathrm{tr}\,[(\varrho_1 - \varrho_2)^2]}\,, \tag{52}$$

- the *Bures distance*

$$D_{\mathrm{B}}(\varrho_1, \varrho_2) = \sqrt{2\left(1 - \mathrm{tr}\left[(\varrho_1^{1/2}\varrho_2\varrho_1^{1/2})^{1/2}\right]\right)}\,, \tag{53}$$

These distances naturally appear in different approaches, e.g., the Bures distance [34] originates from the statistical distance between quantum states [35] and quantum fidelity [36]. Each of them possesses certain advantages as well drawbacks and often the derived results strongly depend on made choice. Below, in order to analyse this type dependence we use the measures corresponding two of them, (52) and (53). The derivation of the corresponding measures on $\mathfrak{P}_+(\mathbb{R}^{n^2-1})$ can be done as follows.

• **The Hilbert–Schmidt measure•** Considering the distance (52) between two infinitesimally close points ϱ and $\varrho + \mathrm{d}\varrho$ we get the flat metric

$$\mathrm{g}_{\mathrm{HS}} = \mathrm{tr}\,(\mathrm{d}\varrho \otimes \mathrm{d}\varrho)\ , \tag{54}$$

which in the Bloch coordinates (4) for system of 2-qubits takes (up to scale factor) the standard Euclidean form in $\mathbb{R}^{15}$:

$$\mathrm{g}_{\mathrm{HS}} = \mathrm{d}\xi_1 \otimes \mathrm{d}\xi_1 + \mathrm{d}\xi_2 \otimes \mathrm{d}\xi_2 + \cdots + \mathrm{d}\xi_{15} \otimes \mathrm{d}\xi_{15} . \tag{55}$$

The measure corresponding to (55)

$$\mathrm{d}\mu_{\mathrm{HS}} := \mathrm{d}\xi_1 \wedge \mathrm{d}\xi_2 \wedge \cdots \wedge \mathrm{d}\xi_{15} \,, \tag{56}$$

admits the following decomposition

$$\mathrm{d}\mu_{\mathrm{HS}} = \mathrm{d}\mu_{{}^\star\Delta_4} \times \mathrm{d}\nu_{U(4)/U(1)^4} \,, \tag{57}$$

where $\mathrm{d}\mu_{{}^\star\Delta_4}$ is the measure on the ordered 3-dimensional simplex[4] in $\mathbb{R}^4$ and $\mathrm{d}\nu_{U(4)/U(1)^4}$ is the measure on the coset $U(4)/U(1)^4$, induced from the conventional Haar measure on the unitary group $U(4)$. Note, that the decomposition (57) follows from the principal axis transformation applied to the density matrices. Since density matrices are Hermitian, for each ϱ there exist a unitary matrix $U \in U(4)$, such that

$$\varrho = U \Lambda U^\dagger . \tag{58}$$

Because the adjoint action on the diagonal matrix Λ has a stability group H_Λ , the matrix U is not unique, U belongs to the coset homeomorphic to $U(4)/H_\Lambda$. To make the representation (58) one to one the diagonal elements of matrix Λ

$$\Lambda = \left(\begin{array}{cccc} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{array} \right) . \tag{59}$$

are restricted to the ordered simplex ${}^\star\Delta_4$ by fixation of the descending order

$$1 \geqslant \lambda_1 \geqslant \lambda_2 \geqslant \lambda_3 \geqslant \lambda_4 \geqslant 0.$$

The stability group H_Λ , depends on the matrix Λ and all possible types of H_Λ are listed in the Table 1.

From the Table 1 on can conclude that the measure is determined from the case with minimal isotropy group, $U(1)^4$. Thus, passing to a new coordinates via transformation (58) the measure gets the form (57)

$$\mathrm{d}\mu_{{}^\star\Delta_4} = \prod_{i>j} (\lambda_i - \lambda_j)^2 \, \mathrm{d}\lambda_1 \wedge \cdots \wedge \mathrm{d}\lambda_4 , \tag{60}$$

[4]The ordered simplex ${}^\star\Delta_4$ is the standard simplex Δ_4 factorized by the action of permutation group S_4.

Eigenvalues	Stability group H_Λ	$\dim(\frac{U(4)}{H_\Lambda})$	$\dim(\Lambda)$
$\lambda_1 > \lambda_2 > \lambda_3 > \lambda_4 > 0$	$U(1)^4$	12	3
$\lambda_1 = \lambda_2 > \lambda_3 > \lambda_4 > 0$	$U(2) \otimes U(1)^2$	10	2
$\lambda_1 > \lambda_2 = \lambda_3 > \lambda_4 > 0$	$U(1) \otimes U(2) \otimes U(1)$	10	2
$\lambda_1 > \lambda_2 > \lambda_3 = \lambda_4 > 0$	$U(1)^2 \otimes U(2)$	10	2
$\lambda_1 > \lambda_2 = \lambda_3 = \lambda_4 > 0$	$U(1) \otimes U(3)$	6	1
$\lambda_1 = \lambda_2 > \lambda_3 = \lambda_4 > 0$	$U(2) \otimes U(2)$	8	1
$\lambda_1 = \lambda_2 = \lambda_3 > \lambda_4 > 0$	$U(3) \otimes U(1)$	6	1
$\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 \geqslant 0$	$\mathrm{U}(4)$	0	0

Table 1. Stability groups and dimensions of $U(4)/H_\Lambda$ cosets.

with the discriminant of the characteristic equation for ϱ as the Jacobian and the measure on the coset $SU(4)/U(1)^4$, depending on $4^2 - 4$ angles:

$$d\mu_{_{\mathrm{SU(4)/U(1)^4}}} = \omega_1 \wedge \omega_2 \wedge \cdots \wedge \omega_{12}, \tag{61}$$

where $\omega_1, \dots, \omega_{12}$ are the left-invariant 1-forms on U(4) projected to the coset $SU(4)/U(1)^4$. As a result, the Hilbert–Schmidt measure (56) induces the following joint distribution function in the simplex of the density matrix eigenvalues:

$$P_{_{\mathrm{HS}}}(\boldsymbol{\lambda}) = C_n^{\mathrm{HS}}\, \delta\left(1 - \sum_{i=1}^{n} \lambda_i\right) \prod_{i=1}^{n} \Theta(\lambda_i) \prod_{i>j} (\lambda_i - \lambda_j)^2\,, \tag{62}$$

where the normalization constant C_n reads

$$C_n^{\mathrm{HS}} := \frac{\Gamma(n^2)}{\prod\limits_{j=0}^{n-1} \Gamma(n-j)\Gamma(n-j+1)}\,.$$

It is important to note that the distribution (62) may be considered as a special case of the family of measures induced by the partial tracing [31–33]. Below, we will use this observation for the numerical analysis of the geometric probability.

• **The Bures measure•** The infinitesimal form of Bures distance (53) lead to the metric:

$$\mathrm{g}_{_{\mathrm{Bures}}} = \frac{1}{2}\,\mathrm{Tr}\,(G\mathrm{d}\varrho)\,, \tag{63}$$

where G is defined from the equation $\mathrm{d}\varrho = G\varrho + \varrho G$ [37,38].

It is known (see e.g., [36,39]) that the Bures probability distribution in the simplex of eigenvalues reads

$$P_{\text{Bures}}(\boldsymbol{\lambda}) = C_n^{\text{Bures}} \delta\left(1 - \sum_{i=1}^{n} \lambda_i\right) \prod \Theta(\lambda_i) \frac{\mathrm{d}\lambda_i}{\sqrt{\lambda_i}} \prod_{i<j} \frac{(\lambda_i - \lambda_j)^2}{\lambda_i + \lambda_j}, \qquad (64)$$

where

$$C_n^{\text{Bures}} = 2^{n^2-n} \frac{\Gamma(n^2/2)}{\pi^{n/2} \prod_{j=1}^{n} \Gamma(j+1)}\,.$$

is a normalization constant.

3.2. Probability of separability. Now, introducing the measure on the space of states, we are in position to define the probability characteristics of entanglement. The simplest one is the probability of finding the separable states among all possible states, distributed in accordance with the introduced measure on the state space.

• **Geometric probability of separability** • Consider a bipartite system consistent from the pair of qubits or qubit-qutrit. Taking into account the semi-algebraic structure of state space one can define the separability probability as

$$\mathcal{P}_{\text{sep}} = \frac{\int_{\mathfrak{P}_+ \cap\ \widetilde{\mathfrak{P}}_+} \mathrm{d}\mu}{\int_{\mathfrak{P}_+} \mathrm{d}\mu}\,. \qquad (65)$$

The denominator in (65) represents the volume of total state space $\mathfrak{P}_+$, while integral in nominator expresses the volume of separable states over the intersection $\mathfrak{P}_+ \cap\ \widetilde{\mathfrak{P}}_+$ of $\mathfrak{P}_+$ and its image $\widetilde{\mathfrak{P}}_+$ under the partial transposition map. The set $\mathfrak{P}_+ \cap \widetilde{\mathfrak{P}}_+$ represents the subset of $\mathfrak{P}_+$ invariant under the partial transposition map:

$$\mathfrak{P}_+ \cap\ \widetilde{\mathfrak{P}}_+ = \{\rho \in \mathfrak{P}_+ \,|\, \mathrm{I} \otimes \mathrm{T}\rho \in\ \mathfrak{P}_+\}\,,$$

Below in our computations the measure $\mathrm{d}\mu$ in integrals (65) is assumed to either the Hilbert–Schmidt or the Bures form. Since the volume of state space is known for both metrics, the Hilbert–Schmidt [40] and the Bures [36], the problem of determination of separability probability reduces to the evaluation of the integral over the set $\mathfrak{P}_+ \cap\ \widetilde{\mathfrak{P}}_+$.

Postponing for a future studies of generic properties of (65), we will discuss how to evaluate the separability probability for pairs of qubits and qubit-qutrit. The straightforward numerical calculation of the multidimensional integral over the set $\mathfrak{P}_+ \cap \widetilde{\mathfrak{P}}_+$ represents hard computational problem. To avoid very cumbersome computations one can use a reliable remedy, the Monte-Carlo method.

3.3. Generation of density matrices ensembles. The basic ingredient of the Monte-Carlo approach is the generation of a specific random variable. To generate the random density matrices from the Hilbert–Schmidt and and Bures ensemble the ideology of the method of induced measures (cf. [31–33] and [41–43]) can be used. To proceed let us start at first with the generation of the so-called *Ginibre ensemble* [44], i.e., the set of complex matrices whose elements have real and imaginary parts distributed as independent normal random variables.

• The Ginibre ensemble • Let $\mathrm{M}(\mathbb{C}, n)$ is the space of $n \times n$ matrices whose entries are complex numbers. Assume that the elements of $Z \in \mathrm{M}(\mathbb{C}, n)$ are independent identically distributed standard normal complex random variables

$$p(z_{ij}) = \frac{1}{\pi} \exp(-|z_{ij}|), \quad i, j = 1, 2, \ldots, n.$$

The joint probability distribution

$$P(Z) = \prod_{i,j=1}^{n} p(z_{ij}) = \frac{1}{\pi^{n^2}} \exp\left(-\mathrm{Tr}\left(Z^\dagger Z\right)\right) \tag{66}$$

and linear measure on $\mathrm{M}(\mathbb{C}, n)$ determines the Ginibre's measure of probability distribution:

$$\mathrm{d}\mu_{\mathrm{G}}(Z) = P(z)\mathrm{Tr}\left(\mathrm{d}Z^\dagger \mathrm{d}Z\right). \tag{67}$$

Having the random Ginibre matrices one can use a simple prescriptions for generation of elements from both, the Hilbert–Schmidt and the Bures ensembles.

•The Hilbert–Schmidt ensemble • In order to generate the Hilbert–Schmidt states

$$P(\varrho)_{\mathrm{HS}} \approx \Theta(\varrho)\delta(1-\varrho), \tag{68}$$

consider a square $n \times n$ complex random matrix Z from the Ginibre ensemble. Then it is easy to convinced that the matrix

$$\varrho_{\mathrm{HS}} = \frac{Z^\dagger Z}{\mathrm{Tr}\,(Z^\dagger Z)}, \tag{69}$$

is by construction the Hermitian, semi-positive, unit norm matrix and belongs to the Hilbert–Schmidt ensemble (68).

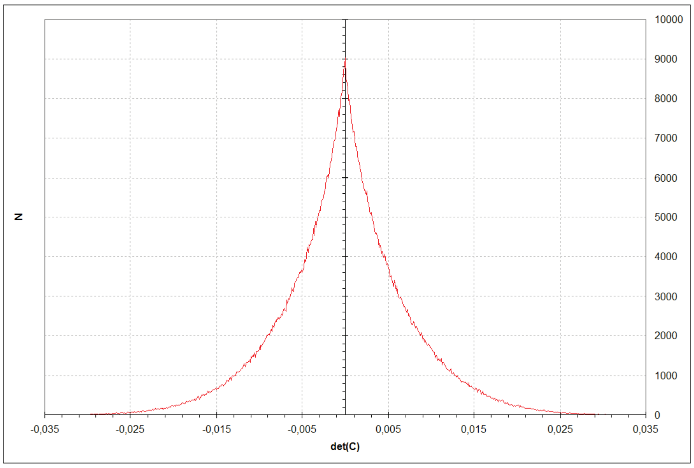

Fig. 1. Distribution of separable states with respect to the correlation measure $\det \|\mathrm{C}\|$ for 10^6 matrices from the Hilbert–Schmidt ensemble.

• **The Bures ensemble** • The density matrix distributed in accordance with the Bure measure can be generated using the Ginibre ensemble as well. Following [42] consider the random matrix

$$\varrho_{\mathrm{B}} = \frac{(\mathbb{I}+U)ZZ^{+}(\mathbb{I}+U^{+})}{\operatorname{Tr}\left[(\mathbb{I}+U)ZZ^{+}(\mathbb{I}+U^{+})\right]}, \tag{70}$$

where the complex matrix Z belongs to the Ginibre ensemble, while U is a unitary matrix distributed according to the Haar measure on unitary group $U(N)$. By straightforward calculation one can verify that matrices ϱ_{B} are distributed in accordance with the Bures measure.

3.4. Numerical results. • **Distribution of separable matrices** • Now having algorithm for generation of the Hilbert–Schmidt and the Bures matrices one can analyse the character of distribution of separable matrices in both ensembles. Considering 2-qubits system, the distributions of separable density matrices with given entanglement characteristics, determinants of the correlation and Schlienz-Mahler matrices, $\det \|\mathrm{C}\|$ and $\det \|\mathrm{M}\|$ have been found. The results of our calculations are given on Figure 1 and Figure 2.

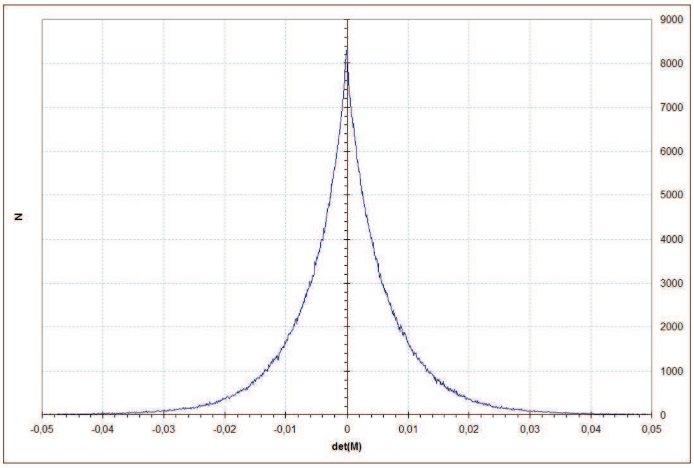

Fig. 2. Distribution of separable states with respect to the Schlienz-Mahler entanglement measure det $\|\mathrm{M}\|$ for 10^6 random Hilbert–Schmidt matrices.

• **Probabilities and conjectures** • Finally we give the values of probabilities for 2-qubits and qubit-qutrit composite systems, whose density matrices are distributed according to the the Hilbert–Schmidt and the Bures measure.

Generating the random density matrices as it was described above and then counting the number of matrices satisfying the PPT conditions:

$$S_k^{T_B} \geqslant 0, \quad k = 1, 2, \ldots, 6.$$

the separability probability for two measures were found. The results are as follows. For the Hilbert–Schmidt measure the separability probabilities are

$$\mathcal{P}_{\mathrm{H-S}}^{2\otimes 2} = 0.2424, \tag{71}$$

$$\mathcal{P}_{\mathrm{H-S}}^{2\otimes 3} = 0.0373, \tag{72}$$

while for the Bures measure computations give

$$\mathcal{P}_{\mathrm{B}}^{2\otimes 2} = 0.073, \tag{73}$$

$$\mathcal{P}_{\mathrm{B}}^{2\otimes 3} = 0.001. \tag{74}$$

Apart from this, the probabilities of absolute separable states for 2-qubits and qubit-qutrit system have been determined. The problem in this case reduces to the calculations of the integrals over the domain of ordered simplex given by the inequalities (49) and (50)

$$\mathcal{P}^{2\otimes 2}_{\text{Measure}} = \int P_{\text{Measure}}(\boldsymbol{\lambda})\Theta(2\sqrt{\lambda_2\lambda_4} - \lambda_1 + \lambda_3), \tag{75}$$

$$\mathcal{P}^{2\otimes 3}_{\text{Measure}} = \int P_{\text{Measure}}(\boldsymbol{\lambda})\Theta(2\sqrt{\lambda_4\lambda_6} - \lambda_1 + \lambda_5). \tag{76}$$

These integrals were evaluated using the `MATHEMATICA` package for the Hilbert–Schmidt (62) and the Bures distributions (64). Summarizing, all results, including the percentage of the absolute separable states, are collected in the Table 2

System	Separable		Abs. Sep
H-S metric			
$2\otimes 2$	24.24 %	23,874174 %	0.365826 %
$2\otimes 3$	3.73 %	2,753321 %	0.976679 %
Bures metric			
$2\otimes 2$	7.3 %	7,2838208 %	0.0161792 %
$2\otimes 3$	0.1 %	0,1 %	-

Table 2. Probabilities for 2-qubits and qubit-qutrit.

§4. Concluding remarks

In the present note the algebraic description of low-dimensional binary composite systems, pairs of qubits and qubit-qutrit has been given in a way well adapted to a computational purposes. Based on this formulations few probabilistic aspects of entanglement have been discussed. Here it is in order a short comment on the results of our numerical experiments with separability probability. Particularly, concerning the separability probability, for the case of Hilbert–Schmidt measure, one can note existence of intriguing simple rational approximations:

$$\mathcal{P}^{2\otimes 2}_{\mathrm{H-S}} = 0.2424 \approx \frac{8}{33} = \frac{2^3}{3*11}\,, \tag{77}$$

$$\mathcal{P}^{2\otimes 3}_{\mathrm{H-S}} = 0.0373 = \frac{16}{429} = \frac{2^4}{3*11*13}\,, \tag{78}$$

in agreement with results conjectured by P. B. Slater few years ago [45,46]. It is curious whether this observation has some deep background or it is an accidental fact of a precise approximation of probabilities by simple rational numbers.

Another interesting unclear feature found is a big value of absolute separability probability for $2\otimes 3$ system with the Hilbert–Schmidt measure, comparing with a 2-qubit system. Finally, it is also to worth to mention a strong dependence of entanglement characteristics on the choice of measure (cf. [47]).

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References

1. E. Schrödinger, *Die gegenwärtige situation in der Quantenmechanik.*— Die Naturwissenschaften **23** (1935), 807–812, 823–828, 844–849.
2. R. F. Werner, *Quantum states with Einstein–Podolsky–Rosen correlations admitting a hidden-variable model.* — Phys. Rev. **A 40** (1989) 4277–4281.
3. W. Thirring, R. A. Bertlmann, P. Köhler, H. Narnhofer, *Entanglement or separability; the choice how to factorize the algebra of a density matrix.* — Eur. Phys. D **64** (2011) 181–196.
4. L. Gurvits, *Classical deterministic complexity of Edmond's problem and quantum entanglement.* in: Proceedings of the Thirty-Fifth Annual ACM Symposium on Theory of Computing, 1019 (electronic), ACM, New York, 2003.
5. L. M. Ioannou, *Computational complexity of the quantum separability problem.* — Quantum Information & Computation **7**, No. 8 (2007) 335–370. arXiv:quant-ph/0603199v7
6. E. A. Morozova, N. N. Chentsov, *Markov invariant geometry on state manifolds.* — Itogi Nauki Tehniki **36** (1990), 69–102.
7. D. Petz, C. Sudar, *Geometries of quantum states.* — J. Math. Phys. **37** (1996) 2662–2673.

8. K. Zyczkowski, P. Horodecki, A. Sanpera, M. Lewenstein, *Volume of the set of separable states.* — Phys. Rev. **A 58** (1998), 883–892.
9. K. Zyczkowski, *Volume of the set of separable states* II. — Phys. Rev. **A 60** (1999) 3496–3507.
10. J. Schlienz, G. Mahler, *Description of entanglement.* — Phys. Rev. **A 52** (1995) 4396–4404.
11. J. von Neumann, Warscheinlichtkeitstheoritischer aufbau der Quantemechanik, Göttingen Nachrichten, 245–272 (1927).
12. L. D. Landau, *Das dämpfungsproblem in der Wellenmechanik.* — Zeitschrift für Physik **45** (1927) 430–441.
13. J. Dittmann, *On the Riemannian metrics on the space of density matrices.* — Rep. Math. Phys. **36** (1995) 309–315.
14. D. P. Zelobenko, *Compact Lie groups and their representations.* — Translations of Mathematical Monographs, Vol. 40, AMS (1978).
15. S. M. Deen, P. K. Kabir, G. Karl, *Positivity constraints on density matrices.* — Phys. Rev. **D4** (1971) 1662–1666.
16. V. Gerdt, A. Khvedelidze, Yu. Palii, *Constraints on* $\mathrm{SU}(2)\otimes\mathrm{SU}(2)$ *invariant polynomials for entangled qubit pairs.* — Yad. Fiz. **74**, No. 6 (2001), 919–955.
17. Lin Chen, Dragomir Z. Dokovic, *Dimensions, lengths and separability in finite-dimensional quantum systems.* — J. Math. Phys. **54** (2013) 022201–13.
18. U. Fano, *Description of states in quantum mechanics by density matrix and operator techniques.* — Rev. Mod. Phys. **29** (1957) 74–93.
19. A. Peres, *Separability criterion for density matrices, Phys.* — Rev. Lett **77** (1996), 1413–1415.
20. M. Horodecki, P. Horodecki, R. Horodecki, *Separability of mixed states: necessary and sufficient conditions.* — Phys. Lett. **A 223** (1996), 1–8.
21. P. Horodecki, *Separability criterion and inseparable mixed states with positive partial transpose.* — Phys. Lett. **232** (1977), 333–339.
22. N. Linden, S. Popescu, *On multi-particle entanglement.* — Fortschr. Phys. **46** (1998), 567–578.
23. M. Grassl, M. Rötteler, T. Beth, *Computing local invariants of qubit systems.* — Phys. Rev. **A 58** (1998), 1833–1859.
24. R. C. King, T. A. Welsh, P. D. Jarvis, *The mixed two-qubit system and the structure of its ring of local invariants.* — J. Phys. A: Math. Theor. **40** (2007) 10083–10108.
25. V. Gerdt, Yu. Palii, A. Khvedelidze, *On the ring of local invariants for a pair of entangled qubits.* — J. Math. Sci. **168** (2010) , 368–378.
26. V. Gerdt, A. Khvedelidze, D. Mladenov, Yu. Palii, $SU(6)$ *Casimir invariants and* $\mathrm{SU}(2)\otimes\mathrm{SU}(3)$ *scalars for a mixed qubit-qutrit states.* — Zap. Nauchn. Semin. POMI **387** (2001), 102–121.
27. C. Quesne, $SU(2)\otimes SU(2)$ *scalars in the enveloping algebra of* $SU(4)$. — J. Math. Phys. **17** (1976), 1452–1467.
28. M. Kus, K. Źyczkowski, *Geometry of entangled states.* — Phys. Rev. **A 63** (2001), 032307–21.
29. F. Verstraete, K. Audenaert, B. D. Moor, *Maximally entangled mixed states of two qubits.* — Phys. Rev. **64** (2001), 012316–6.
30. R. Hildebrand, *Positive partial transpose spectra.* — **A 76** (2007), 052325-5.

31. E. Lubkin, *Entropy of an n-system from its correlation with a k-reservoir.* — J. Math. Phys. **19** (1978), 1028–1031.
32. S. Lloyd, H. Pagels, *Complexity as thermodynamic depth.* — Ann. Phys. NY **188** (1998), 186–213.
33. D. N. Page, *Average entropy of a subsystem.* — Phys. Rev. Lett. **71** (1993), 1291–1294.
34. D. J. C. Bures, *An extension of Kakutani's theorem on infinite product measures to the tensor product of semidefinite ω^*-algebras.* — Trans. Am. Mat. Soc. **135** (1969), 199–212.
35. S. L. Braunstein, C. M. Caves, *Statistical distance and the geometry of quantum states.* — Phys. Rev. Lett. **72** (1994) 3439–3443.
36. H.-J. Sommers, K. Zyczkowski, *Bures volume of the set of mixed quantum states.* — J. Phys. A: Math. Theor. **36** (2003) 10083–10100.
37. A. Uhlmann, *Density operators as an arena for differential geometry.* — Rep. Math. Phys. **33** (1993), 253–263.
38. M. Hübner, *Explicit computation of the Bures distance for density matrices.* — Phys. Lett. **A 163** (1992) 239–242.
39. M. J. Hall, *Random quantum correlations and density operator distributions.* — Phys. Lett. **A 242** (1998), 123–129.
40. K. Zyczkowski, H.-J. Sommers, *Hilbert–Schmidt volume of the set of mixed quantum states.* — J. Phys. A: Math. Theor. **36** (2003), 10115–10130.
41. S. L. Braunstein, *Geometry of quantum inference.* — Phys. Lett. **A 219** (1966) 169–174.
42. K. Zyczkowski, H.-J. Sommers, *Induced measures in the space of mixed states.* — J. Phys. A: Math. Theor. **34** (2001) 7111–7125).
43. V. Al. Osipov, H.-J. Sommeres, K. Zyczkowski, *Random Bures mixed states and the distribution of their purity.* — J. Phys. A: Math. Theor.**43** (2010), 055302–22.
44. J. Ginibre, *Statistical ensembles of complex, quaternion, and real matrices.* — J. Math. Phys. **6** (1965) 440–339.
45. P. B. Slater, *Dyson indices and Hilbert–Schmidt separability functions and probabilities.* — J. Phys. A: Math. Theor. **40** (2007), 14279–14308.
46. P. B. Slater, *Eigenvalues, separability and absolute separability of two-qubit states.* — J. Geom. Phys. **59** (2009), 17–31.
47. J. Batle, M. Casas, A. Plastino, A. R. Plastino, *Metrics, entanglement, and mixedness in the space of two-qubits.* — Phys. Lett. **A 353** (2006), 161–165.

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