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CALCULATIONS IN EXCEPTIONAL GROUPS, AN UPDATE

Abstract. This paper is a slightly expanded text of our talk at the PCA-2014. There, we announced two recent results, concerning explicit polynomial equations defining exceptional Chevalley groups in microweight or adjoint representations. One of these results is an explicit characteristic-free description of equations on the entries of a matrix from the simply connected Chevalley group $G(\mathrm{E}_7, R)$ in the 56-dimensional representation V. Before, similar description was known for the group $G(\mathrm{E}_6, R)$ in the 27-dimensional representation, whereas for the group of type E_7 it was only known under the simplifying assumption that $2 \in R^*$. In particular, we compute the normalizer of $G(\mathrm{E}_7, R)$ in $\mathrm{GL}(56, R)$ and establish that, as also the normalizer of the elementary subgroup $E(\mathrm{E}_7, R)$, it coincides with the extended Chevalley group $\overline{G}(\mathrm{E}_7, R)$. The construction is based on the works of J.Lurie and the first author on the E_7-invariant quartic forms on V. Another major new result is a complete description of quadratic equations defining the highest weight orbit in the adjoint representations of Chevalley groups of types E_6, E_7 and E_8. Part of these equations not involving zero weights, the so-called square equations (or $\pi/2$-equations) were described by the second author. Recently, the first author succeeded in completing these results, explicitly listing also the equations involving zero weight coordinates linearly (the $2\pi/3$-equations) and quadratically (the π-equations). Also, we briefly discuss recent results by S.Garibaldi and R.M.Guralnick on octic invariants for E_8.

The present paper, which is a direct sequel of [70], is based on our talk at the PCA-2014. In this talk, we reported some recent advances in

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a major project, whose goal is to develop methods of calculations, both manual and computer ones, in exceptional Chevalley groups. In the spirit of this conference, we concentrate on explicit polynomial equations defining exceptinal groups in their minimal representations, minuscule and the adjoint ones. Before passing to our new results, let us briefly outline the state of art in the field.

§1. Introduction

Let Φ be a reduced irreducible root system, since classical groups are well understood, and G_2 is small enough to allow direct matrix calculations, in our project we are mostly interested in the large exceptional cases, where $\Phi = \mathrm{E}_6, \mathrm{E}_7, \mathrm{E}_8$ or F_4. Further, let $G = G(\Phi, R)$ be a Chevalley group of type Φ over a commutative rings R with 1. Usually, we assume G to be simply connected.

Basically, proofs of structure results for G have either ring-theoretic, or representation-theoretic and geometric flavour. They amount either to reduction of the dimension of the ground ring R, or the rank of the root system Φ, or, eventually, to a combination of both. Beneath, both methods make heavy use of [elementary] calculations in the elementary Chevalley group $E(\Phi, R)$ spanned by the elementary unipotents $x_\alpha(\xi)$, where $\alpha \in \Phi$, $\xi \in R$, subject to Steinberg relations.

Localisation methods are well documented in the literature. They were introduced in the study of algebraic groups by Andrei Suslin, who realised that higher analogues of local-global principles of Quillen's type also work at the level of K_1, K_2 and beyond. The first proofs of structure theorems for exceptional groups by Eiichi Abe, Kazuo Suzuki, Giovanni Taddei, Leonid Vaserstein, and others were based on Quillen–Suslin's localisation and patching.

The next major advance was originated by Anthony Bak, who introduced localisation-completion to prove the nilpotency of the [linear] K_1. Over the last 12 years his method was simplified, generalised and expanded in several directions by Roozbeh Hazrat, the second author, and Zuhong Zhang [18, 19], and most notably by Alexei Stepanov [1, 50]. One of the most important aspects was that (unlike [3]) these papers operate at the relative rather than the absolute level from the outset. We reported on recent versions of localisation methods, and some of their applications at the PCA-2010 and PCA-2012, see the conference papers [15] and [16], as also at many further conferences, see in particular [17].

We especially recommend to peruse the recent papers by Alexei Stepanov on his method of universal localisation [46, 47], where he develops a new localisation method that allows to establish results independent on the dimension of the ground ring, in terms of the universal/versal coefficient rings that depend on the roots system alone. All details are contained in his [Russian] Dr. Sci. Thesis (=Habilitation) [48]. In our view, these works inaugurate an entirely new chapter in the development of localisation methods. Another amusing byproduct of these papers, was a further rethinking of some aspects of elementary calculations in [20, 45].

In this talk, we do not discuss localisation methods any further, referring to the papers above.

Decomposition of unipotents and its offsprings. Another major bunch of methods to calculate in Chevalley groups is based on geometric realisations of these groups, in particular in their minimal representations. Over rings, this approach was pioneered in the groundbreaking works by Hideya Matsumoto and Michael Stein [31, 44]. Soon thereafter, Alexei Stepanov, the second author and Eugene Plotkin [49, 53, 54, 73] developed a working approach towards the proof of the main structure theorems for Chevalley groups over rings, and many further related problems (see the description of the intended scope of the whole project in the introduction to [72]).

However, for exceptional groups, our initial approach requested some knowledge of at least part of the equations defining the groups in their minimal representations. At rock bottom, an explicit knowledge of the system of quadrics [24], defining the highest weight orbit in these representations (see [54]).

Later, elaborating this approach, the second author, Mikhail Gavrilovich and Sergei Nikolenko [62, 63, 71] introduced a new group-theoretic twist to this method, which allowed to obtain much more straightforward proofs, that invoked only the presence of very small classical embeddings, such as $A_2 \subseteq E_6, E_7$ (rather than huge classical embeddings $A_5 \subseteq E_6$, $A_7 \subseteq E_7$ or $D_8 \subseteq E_8$ used before that). Also, these new generation proofs, never invoked any equations on the entries of matrices other than [part of the] linear equations defining the corresponding Lie algebra.

However, later again, the second author noticed that using the equations, one can develop much more powerful versions of decomposition of unipotents, that allow to get to a small rank parabolic by forming very few commutators. This would allow to improve the known bounds in many existing applications, as also afford various new applications. One such new method is the A_3-proof [55, 60], some further variations were hinted in his joint paper with Victoria Kazakevich [64].

What is in this paper? This required another look at the equations defining exceptional groups, and stimulated us to return to the project described in [70]. Here, we outline some recent results, mostly due to the first author, who succeeded in completing previous results of the second author to get final characteristic free answers. Namely, we announce the following new results.

• Complete description of the polynomial equations defining $G(E_7, R)$ as a subgroup of $GL(56, R)$, in the microweight representation, over an *arbitrary* commutative ring R,

• Explicit description of the quadratic equations defining the highest weight orbits of $G(E_6, R)$, $G(E_7, R)$ and $G(E_8, R)$ in the adjoint representation, again over an arbitrary commutative ring.

Also, we describe another momentous recent achievement, due to Skip Garibaldi and Robert Guralnick [13].

• Boosting the Cederwall—Palmkvist [8] construction of octic real polynomials in 248 variables, whose automorphism groups are split or compact real forms of E_8, Garibaldi and Guralnick succeeded in proving that over an arbitrary field K the Chevalley group $G(E_8, K)$ admits a similar characterisation as a subgroup of $GL(248, R)$. However, as all such constructions, this construction encounters serious troubles in characteristics 2 and 3, and it would be quite an exercise to convert it into an explicit characteristic free description of equations defining $G(E_8, R)$, over an arbitrary commutative ring R.

We make no attempt whatsoever to describe applications of these results, on which we are currently working, or their significance in the development of new variants of decomposition of unipotents. We refer the reader to our forthcoming papers: to [29], for a more general view of these equations, involving the geometry of homogeneous spaces and Gröbner bases, to [68] for an instance of application of our results on adjoint representations, and to [61], for cute new versions of the decomposition of unipotents.

Basic notation. Since this paper is a direct companion of [70], we do not reproduce the general setting, or the notation thereof, simply referring to [31, 36, 44, 53, 54, 56, 72] for the definitions and further background references. Instead, we introduce an absolute minimum of notation.

Below Φ denotes an arbitrary simply laced reduced irreducible root system. Actually, for the most part we are interested in the cases E_6, E_7 and E_8 and later on in various sections Φ will be E_7 or E_8. Further, let Π be a fundamental root system in Φ. Let e_α, $\alpha \in \Phi$, be a positive Chevalley system in the corresponding complex simple Lie algebra L. Recall that in that case $[e_\alpha, e_\beta] = N_{\alpha,\beta} e_{\alpha+\beta}$, where the structure constants $N_{\alpha,\beta}$ are equal to $0, \pm 1$, with further requirement that certain constants, where β are fundamental, are actually equal to $+1$. As usual, for two roots $\alpha, \beta \in \Phi$ we denote by (α, β) their inner product, and by $\langle \alpha, \beta \rangle = 2(\alpha, \beta)/(\beta, \beta)$ the corresponding Cartan number.

Usually, we consider Chevalley group $G(\Phi, R)$ in some rational representation V. Mostly, V will be one of the fundamental representations $V = V(\varpi_i)$, where ϖ_i denotes the i-th fundamental weight. We denote by Λ the set of weights of V *with multiplicities* and choose an admissible base of V. Recall that an admissible base consists of weight vectors, and thus can be indexed by the weights $\lambda \in \Lambda$, in the sequel we denote it by e^λ, $\lambda \in \Lambda$. Now, any vector $v \in V$ admits a linear expansion $v = \sum e^\lambda v_\lambda$, where the sum is taken over $\lambda \in \Lambda$. This means that a vector $v \in V$ can be interpreted as a coordinate column $v = (v_\lambda)$, where $\lambda \in \Lambda$.

§2. Equations defining E_7 inside $\mathrm{Sp}(56, R)$: the state of art

Back in 2009 among other things we reported on the (then recent) paper [65], where we listed explicit polynomial equations defining the simply-connected Chevalley group of type E_6 and its normaliser in the 27-dimensional representation. In that paper we described the normaliser of $G_{\mathrm{sc}}(\mathrm{E}_6, R)$ in $\mathrm{GL}(27, R)$, as the largest subgroup consisting of matrices whose first columns are subject to 27 quadratic equations in 27 variables (see also § 7 of [70], especially Theorem 10).

Also, there we presaged similar results for the simply connected Chevalley group of type E_7 and its normaliser in the 56-dimensional representation. Under the simplifying assumption that $2 \in R^*$, this was indeed very similar in spirit to the case of E_6, and relatively straightforward. However, it has taken us much more time and effort to cope with the many additional hurdles occurring in the case, where 2 in not invertible. Actually, the key new idea comes from the works of Jacob Lurie [26] and the first author [27].

Recall that in [65] we worked from the classical description of the Chevalley group E_6 acting on the 27-dimensional module $V = V(\varpi_1)$, in terms of a cubic form on V (see [54, 65, 69] for many more details and references).

There exists a similar, but more complicated description of the simply connected group of type E_7 acting on the 56-dimensional module $V = V(\varpi_7)$. In this case, to determine the group one needs two invariants, a quadratic one, and a quartic one. First of all, in this case the module V is self-dual and supports a unimodular symplectic form h. Further, there exists a four-linear form $f : V \times V \times V \times V \longrightarrow R$ such that G can be identified with the full isometry group of the pair h, f, or, in other words, with the group consisting of all $g \in \mathrm{GL}(V)$ such that $h(gu, gv) = h(u, v)$ and $f(gu, gv, gx, gy) = f(u, v, x, y)$ for all $u, v, x, y \in V$. The similarities of this pair of forms constitute the extended Chevalley group $G = G(E_7, R)$.

The construction of h is immediate. It is much trickier to construct an invariant of degree four. Classically, one constructs not the four-linear form f itself, but rather the corresponding quartic. That the group G of type E_7 preserves a form of degree 4 in 56 variables, was first noted by Éli Cartan, at least in characteristic 0. His explicit construction does not seem to work, but most probably this was simply a misprint. The form itself was also known to Leonard Dickson back in 1901, in the context of the 28 bitangents, and thus of the Weyl group $W(E_7)$.

An extremely elegant elementary construction of such a quartic invariant over a field K of characteristic distinct from 2 was proposed by Hans Freudenthal in 1952. Namely, he identifies V with the space $A(8, K)^2$ of pairs of anti-symmetric 8×8 matrices, and considers the following symplectic inner product, and the following quartic form:

$$h((a_1, b_1), (a_2, b_2)) = \frac{1}{2}(\mathrm{tr}(a_1 b_2^t) - \mathrm{tr}(a_2 b_1^t)),$$

$$Q((a, b)) = \mathrm{pf}(a) + \mathrm{pf}(b) - \frac{1}{4}\,\mathrm{tr}((ab)^2) + \frac{1}{16}\,\mathrm{tr}(ab)^2.$$

Now, for all fields K of characteristics distinct from 2, one can indeed identify the isometry group of this pair with the simply connected Chevalley group G of type E_7 over K, there are two remarkable proofs of this fact, by Michael Aschbacher and Bruce Cooperstein [2, 9]. The construction of this form in the above papers are slightly different. In fact, in [2] this form is constructed in terms of A_6, the essence of this construction is expressed by the equality $56 = 7+21+21+7$). As opposed to that, the construction in [9] is much closer in spirit to the original Freudenthal construction, and is phrased in terms of A_7, where $56 = 28 + 28$. The isometry group of the form Q is generated by G and a diagonal element of order 2, see [9].

Characteristics $p \geqslant 5$ can be treated uniformly, and do not cause serious trouble, whereas characteristic 3 requires some special attention. However, this approach breaks down completely in characteristic 2. Not only the construction itself does not work as stated, it seems that in characteristic 2 the module V does not support any non-trivial symmetric G-invariant four-linear form at all, see [2]. This is related to the fact that in characteristic 2 the squaring of the symplectic form

$$f(u, v, x, y) = h(u, v)h(x, y) + h(u, x)h(v, y) + h(u, y)h(v, x),$$

becomes symmetric, which is not the case for characteristics $\geqslant 3$. As a matter of fact, in [2] Aschbacher constructs another G-invariant four-linear form, which is only symmetric with respect to the *even* permutations.

There are different constructions of the form Q, in particular the celebrated construction by Robert Brown [6] in terms of ternary algebras, that works in characteristics $\neq 2, 3$. Let V be a space with a nondegenerate inner product. Then to define a trilinear form on V is the same as to define thereon an algebra structure. Similarly, four-linear forms on V correspond to ternary algebra structures. There exists a remarkable ternary algebra of dimension 56, which can be constructed in terms of the exceptional Jordan algebra $\mathbb{J}$, see [6, 11] and references there. This algebra consists of 2×2 matrices $\mathbb{J}$ with scalar diagonal entries, $56 = 1 + 27 + 27 + 1$.

These constructions were widely used in the study of the groups by Tonny Springer, Skip Garibaldi, and many others, but as we indicated above,

§3. Equations defining E_7 inside $\mathrm{Sp}(56, R)$: a system of quadrics, and the extended group

First, we explicitly construct an ideal I in the polynomial ring $\mathbb{Z}[x_1, \dots, x_{56}]$, generated by 133 quadratic forms $f_1, \dots, f_{133}$, such that the scheme-theoretic normaliser of the ideal I coinsides with the extended Chevalley group of type E_7.

This construction works as follows. The highest Weyl orbit of equations defining the highest weight orbit consists of *square equations*. For minuscule and adjoint representations, their constructions and numerology were studied by the second author in [54,58,59]. Let us recall the basic construction of [58] in the context of the 56-dimensional representation (E_7, ϖ_7).

Let $\Lambda = W(\mathrm{E}_7)\varpi_7$ be the set of weights of this representation. Usually, we interpret these weights as roots of E_8 such that the last fundamental root α_8 occurs in their expansion with coefficient 1. A subset $\Omega \subseteq \Lambda$ is called a square if $|\Omega| \geqslant 4$ and for any $\lambda \in \Omega$ the difference $\lambda - \mu$ with all other weights $\mu \in \Omega$, except exactly one of them, denoted by λ^*, is a root, whereas the difference $\lambda - \lambda^*$ is not a root (in which case λ^* is necessarily orthogonal to λ). In [58] it is proven that any maximal square Ω consists of exactly 12 roots, and the sum $\lambda + \lambda^*$ does not depend on the choice of $\lambda \in \Omega$. Moreover, for (E_7, ϖ_7) a maximal square Ω is uniquely determined by this sum. In other words, maximal squares are in bijective correspondence with the 126 roots of E_7. More precisely, for a root $\alpha \in \mathrm{E}_7$ we set

$$\Omega(\alpha) = \big\{\lambda \in \Lambda \mid \lambda - \alpha \in \Lambda\big\}.$$

Let Ω be a maximal square. Chose orthogonal roots $\rho, \rho^* \in \Omega$ and define a polynomial $f_{\rho,\rho^*} \in \mathbb{Z}[x_\lambda]_{\lambda\in\Lambda}$ by the equality

$$f_{\rho,\rho^*} = x_\rho x_{\rho^*} - \sum N_{\rho,-\lambda} N_{\rho^*,-\lambda^*} x_\lambda x_{\lambda^*},$$

where the sum is taken over all pairs of orthogonal weights $\{\lambda, \lambda^*\}$ from Ω, except the initial pair $\{\rho, \rho^*\}$. In [58] the equation of the form $f_{\rho,\rho^*}(v) = 0$ imposed on vectors $v = (v_\lambda)_{\lambda\in\Lambda} \in V$ was called a square equation corresponding to a maximal square Ω. Indeed, this equation only depends on the square Ω itself, and not on a arbitrary choice of a pair $\{\rho, \rho^*\}$. For any other pair of orthogonal weights from the same maximal square Ω the left hand-side of this equation remains the same, up to sign. Thus, choosing one pair of orthogonal weights from any maximal square Ω, we get 126 quadratic equations, corresponding to the maximal squares in Λ, or, what is the same, to the 126 roots of E_7. Without confusion, we can denote these polynomials by f_α, $\alpha \in \Phi$.

However, there is also another Weyl orbit of equations. Namely, for a root $\alpha \in \mathrm{E}_7$ we introduce the polynomial $g_\alpha \in \mathbb{Z}[x_\lambda]_{\lambda\in\Lambda}$ defined as

$$g_\alpha = \sum N_{\lambda,\lambda^*} x_\lambda x_{\lambda^*},$$

where the sum is taken over pairs $\{\lambda,\lambda^*\}\in\Omega(\alpha)$. Again, there are 126 such polynomials, but it is easy to verify that the ideal in $\mathbb{Z}[x_\lambda]_{\lambda\in\Lambda}$, generated by all polynomials g_α, $\alpha\in\Phi$, coincides with the ideal generated by the ideal generated by g_α, $\alpha\in\Pi$. Denote $g_i=g_{\alpha_i}$, $i=1,\dots,7$.

Thus, finally we get an ideal I generated by the above quadratic polynomials f_α, $\alpha\in\mathrm{E}_7$ and g_i, $i=1,\dots,7$. Altogether this gives us $126+7=133$ quadratic polynomials, which already is a good omen.

Теорема 1. *The simply connected Chevalley group $G(\mathrm{E}_7,R)$ preserves I.*

This results can be also stated as follows. Denote by $\mathrm{Fix}_R(I)$ the set of R-linear transformations preserving the ideal I,

$$\mathrm{Fix}_R(I)=\{g\in\mathrm{GL}(56,R)\mid f(gx)\in I \text{ for all } f\in I\}.$$

Then $G_{\mathrm{sc}}(\mathrm{E}_7,R)\leqslant\mathrm{Fix}_R(I)$. In fact, quite a bit more can be said in this case.

Теорема 2. *The scheme theoretic stabiliser of I coincides with the extended simply connected Chevalley group $\overline{G}_{\mathrm{sc}}(\mathrm{E}_7,-)$.*

This result can be interpreted as an explicit description of the extended simply connected Chevalley–Demazure group scheme $\overline{G}_{\mathrm{sc}}(\mathrm{E}_7,-)$ of type $\Phi=\mathrm{E}_7$, constructed in [4], see also [57] for other constructions and many further references. For $\Phi=\mathrm{E}_7$, the scheme $\overline{G}_{\mathrm{sc}}(\mathrm{E}_7,-)$ can be interpreted as the Levi factor of the parabolic subscheme of type P_8 in the usual Chevalley—Demazure scheme $G(\mathrm{E}_8,-)$ of type E_8. Moreover, the Levi factor $\overline{G}_{\mathrm{sc}}(\mathrm{E}_7,-)$ acts on the unipotent radical U_8 of P_8. This action gives us an irreducible 56-dimensional representation of $G(\mathrm{E}_7,-)$ with the highest weight ϖ_7.

Our results are also intimately related with the description of $G_{\mathrm{sc}}(\mathrm{E}_7,R)$ as the stabiliser of a system of four-linear forms on $V=V(\varpi_7)$. Namely, in [27] the first author constructed a four-linear form $f:V\times V\times V\times V\longrightarrow R$ such that

$$G_{\mathrm{sc}}(\mathrm{E}_7,R)=\big\{g\in\mathrm{GL}(56,R)\mid f(gu,gv,gw,gz)=f(u,v,w,z),\\ h(gu,gv)=h(u,v) \text{ for all } u,v,w,z\in V\big\}.$$

Here, h is the obvious symplectic inner product on V. Let us briefly describe the form f. If we identify V with U_8 as above, we get a canonical base $(v^\lambda)_{\lambda\in\Lambda}$ of V, where $\Lambda\subseteq\mathrm{E}_8$ is the corresponding unipotent set of roots. In order to define $f(u,v,w,z)$ for all $u,v,w,z\in V$ it suffices to define $f(v^\lambda,v^\mu,v^\nu,v^\rho)$ for all $\lambda,\mu,\nu,\rho\in\Lambda$. Let $\delta\in\mathrm{E}_8$ be the maximal root. Consider the semisimple complex Lie algebra $\mathfrak{e}_8$ of type E_8 with a Chevalley base $(e^\lambda)_{\lambda\in\mathrm{E}_8}$. Note that for $\lambda,\mu,\nu,\rho\in\Lambda$ the element $[[[[e^{-\delta},e^\lambda],e^\mu],e^\nu],e^\rho]$ lies in $\mathbb{Z}e^\delta$. Suppose that

$$[[[[e^{-\delta},e^\lambda],e^\mu],e^\nu],e^\rho]=c(\lambda,\mu,\nu,\rho)e^\delta.$$

We put $f(v^\lambda,v^\mu,v^\nu,v^\rho) = c(\lambda,\mu,\nu,\rho)$. It is easy to check that $f(v^\lambda,v^\mu,v^\nu,v^\rho)=0$ unless $\lambda+\mu+\nu+\rho=2\delta$.

The bulk of the above system of quadratic forms consists of the second order partial derivatives of the [regular part of] the form f. In other words, $G_{\mathrm{sc}}(\mathrm{E}_7,R)$ is the group of linear transformations preserving both f and h, whereas $\overline{G}_{\mathrm{sc}}(\mathrm{E}_7,R)$ is the group of similarities of that pair of forms.

In the following theorem, all normalisers and transporters are taken inside $\mathrm{GL}(56,R)$.

Теорема 3. *Let R be any commutative ring, then*

$$N(E(\mathrm{E}_7,R))=N(G(\mathrm{E}_7,R))=\mathrm{Tran}(E(\mathrm{E}_7,R),G(\mathrm{E}_7,R))=G(R).$$

One of the possible applications of these results, we have in mind, would be another approach to the proof of the standard description of automorphisms of Chevalley groups, without assumption that $2\in R^*$. For classical groups such a description was well known for some decades, but for exceptional groups the situation looked very different, since all published proofs either imposed irrelevant extra conditions, or contained serious gaps. Recently, this problem was almost completely settled by Elena Bunina, see [7] and references therein. She works in the adjoint representation and one of the most problematic parts is to verify that a conjugation in the corresponding general linear group $\mathrm{GL}(n,R)$ is indeed a conjugation inside $G_{\mathrm{ad}}(\Phi,R)$ itself. But if the abstract and the algebraic normalisers coincide, as they do by Theorem 3, then all such conjugations normalising $G_{\mathrm{ad}}(\Phi,R)$ are indeed honest inner automorphisms. Of course, with this end one has either to rewrite all proofs of [7] and previous works for microweight representations, or to prove an analogue of our Theorem 3 for adjoint representations. In the next section we report recent progress regarding the second of these tasks.

§4. Equations defining the highest weight orbit in the adjoint representation

In this section we describe another important recent advance. Namely, just a few months before the Conference, in [28] the first author finally completed an explicit description of the equations on the highest weight orbit in the adjoint representation of Chevalley groups of types E_6, E_7 and E_8, started by the second author in [54, 58]. Namely, he introduced two further types of equations, on top of the *square equations* considered in [58, 59].

This is extremely important, since the equations listed in [54, 58] in themselves do not suffice to construct decomposition of unipotents in the adjoint representations of Chevalley groups of types E_l, $l = 6, 7, 8$. Now, we are finally in a position to complete the project started in [54, 58]

To state these equations, we need to recall some further bits of notation (see [54, 56, 58] for many more details). Let, as before Φ be a reduced irreducible simply laced root system. Actually, we are most interested in the cases $\Phi = E_6, E_7, E_8$, but it all works verbatim also for the classical cases. Recall that for all cases, apart from $\Phi = A_l$, the adjoint representation $V = V(\omega)$ is fundamental, to wit, $(\Phi, \omega) = (E_6, \varpi_2)$, (E_7, ϖ_1) or (E_8, ϖ_8). The set of weights Λ of this representation consists of roots and zero weight, of multiplicity l. Denote the zero weights by $0_1, \dots, 0_l$. The corresponding base vectors will be denoted simply $e^1, \dots, e^l$. Thus, a vector $v \in V$ can be uniquely expressed in the form

$$v = \sum_{\lambda \in \Lambda} e^{\lambda} v_{\lambda} \sum_{\alpha \in \Phi} e^{\lambda} v_{\lambda} + \sum_{i=1}^{l} e^{i} \hat{v}_i.$$

There are three Weyl orbits of quadratic equations defining the highest weight orbit in V, the ones not involving zero-weight coordinates, the ones that contain linear terms in these coordinates, and the ones that contain quadratic terms.

• **$\pi/2$-equations.** First, let $\alpha, \beta \in \Phi$ be two orthogonal roots. As in § 3 let us consider all other unordered pairs of roots with the same sum:

$$S_{\pi/2}(\alpha, \beta) = \big\{\{\gamma, \delta\} \mid \gamma + \delta = \alpha + \beta,\ \{\gamma, \delta\} \neq \{\alpha, \beta\}\big\}.$$

Consider the following equation on a vector $v = (v_\lambda) \in V$, where $\lambda \in \Lambda$:

$$v_\alpha v_\beta = \sum_{\{\gamma,\delta\} \in S_{\pi/2}(\alpha,\beta)} N_{\alpha,-\gamma} N_{\beta,-\delta} v_\gamma v_\delta. \tag{1}$$

Actually, these equations are renamed $\pi/2$-equations in [28].

• **$2\pi/3$-equations.** As above, let $\alpha, \beta \in \Phi$ be two orthogonal roots. Consider all pairs of roots $\{\gamma, \delta\}$ such that $\gamma + \delta = \alpha$ and γ, δ are not orthogonal to β. Note that if γ is orthogonal to β and $\gamma + \delta = \alpha$, then

$(\delta,\beta) = (\alpha-\gamma,\beta) = 0$, so δ is also orthogonal to β. Also, $0 = (\alpha,\beta) = (\gamma+\delta,\beta) = (\gamma,\beta)+(\delta,\beta)$. It follows, that for such a pair $\{\gamma,\delta\}$ one of the roots γ or δ forms angle $2\pi/3$ with β, while the other forms angle $\pi/3$. Put

$$S_{2\pi/3}(\alpha,\beta) = \big\{\{\gamma,\delta\} \mid \gamma+\delta=\alpha,\ (\gamma,\beta)\neq 0\big\}.$$

Consider the following equation on a vector $v = (v_\lambda)_{\lambda\in\Lambda} \in V$:

$$v_\alpha \cdot \sum_{s=1}^{l} \langle\beta,\alpha s\rangle \hat{v}_s = -\sum N_{\gamma,\delta} v_\gamma v_\delta, \tag{2}$$

where the sum on the right hand side is taken over all pairs $\{\gamma,\delta\} \in S_{2\pi/3}(\alpha,\beta)$, such that γ forms angle $\pi/3$ with β. We call this equation the $2\pi/3$-equation corresponding to the pair (α,β).

• **π-equations.** As always, let $\alpha,\beta\in\Phi$ be two orthogonal roots. Consider all pairs of roots $\{\gamma,\delta\}$ such that $\gamma=-\delta$ and γ,δ are not orthogonal to α and β. Two possibilities can possibly occur.

∘ Either $(\gamma,\alpha)=(\gamma,\beta)$, and then automatically $(\delta,\alpha)=(\delta,\beta)$. Switching γ and δ if necessary, we may without loss of generality assume that the angles γ forms with α and β are both equal to $2\pi/3$. In this case we set

$$S_\pi^-(\alpha,\beta) = \big\{(\gamma,\delta) \mid \gamma+\delta=0,\ \gamma+\alpha,\gamma+\beta\in\Phi\big\}.$$

∘ Or exactly one of the angles between γ or δ and α equals $2\pi/3$. Again, switching γ and δ we may assume that γ forms angle $2\pi/3$ with α, and then automatically the angles between γ and β and between δ and α will be equal to $\pi/3$, whereas the angle between δ and β is equal to $2\pi/3$. In this case we set

$$S_\pi^+(\alpha,\beta) = \big\{(\gamma,\delta) \mid \gamma+\delta=0,\ \gamma+\alpha,\delta+\beta\in\Phi\big\}.$$

Define the sign $\varepsilon(\gamma,\delta)$ of a pair (γ,δ) to be equal to $+1$ if $(\gamma,\delta)\in S_\pi^+(\alpha,\beta)$ and to -1 if $(\gamma,\delta)\in S_\pi^-(\alpha,\beta)$

Consider the following equation on a vector $v = (v_\lambda)_{\lambda\in\Lambda} \in V$:

$$\sum_{s=1}^{l}\langle\alpha,\alpha_s\rangle v_s \cdot \sum_{s=1}^{l}\langle\beta,\alpha_s\rangle v_s = \sum \varepsilon(\gamma,\delta) v_\gamma v_\delta, \tag{3}$$

where the sum on the right hand side is taken over all $(\gamma,\delta)\in S_\pi^+(\alpha,\beta)\cup S_\pi^-(\alpha,\beta)$.

Now we are all set to state the main result of [28].

Теорема 4. *The set of vectors $v \in V$ satisfying the equations* (1), (2), (3) *for all $\alpha, \beta \in \Phi$, $\alpha \perp \beta$, is invariant under the action of the group $G_{\mathrm{ad}}(\Phi, R)$.*

The proof of this result in [28] consists in a plethora of delicate computations with roots and structure constants, similar in spirit to those in [54, 58], but much more meticulous. In particular, it follows that the first column of any matrix $g \in E(\Phi, R)$ satisfies the above equations.

§5. The octic E_8 invariant

Let us mention another amazing recent development. Namely, in [8] Martin Cederwall and Jakob Palmkvist constructed an *octic* polynomial in 248 variables, invariant under the action of E_8 in the adjoint representation. We are embarrassed to concede that we were not aware of that paper (published in a physics journal) when compiling the bibliography on explicit realisations of exceptional groups in [66] and only learned of its existence from the 1st version of the remarkable preprint by Skip Garibaldi and Robert Guralnick [12].

In [8] an invariant octic polynomial is constructed for the (split and compact) real and complex cases, using the realisation of E_8 in terms of D_8 and its half-spin representation, $248 = 120 + 128$. The construction itself, or the resulting polynomial, are too complicated to be included in the casual exposition (in fact, during the talk itself, we demonstrated the corresponding page of [8]).

The situation was much clarified by Garibaldi and Guralnick in [12] and especially in the 2nd version of that preprint [13], which is about 50% longer, and contains many additional details, and references.

Теорема 5. *Let G be a simple algebraic group of type* E_8 *over a field K and put q for a nonzero G-invariant quadratic form on* $\mathrm{Lie}(G)$. *Then there exists a homogeneous polynomial F of degree* 8 *on* $\mathrm{Lie}(G)$ *that is G-invariant and does not belong to Kq^4. For any each such F,*

- *the [set-theoretic] stabiliser of F in* $\mathrm{GL}(248, K)$ *is generated by G and the eighth roots of unity;*
- *if* $\mathrm{char}(K) \neq 2, 3$, *the scheme-theoretic stabiliser of F in* $\mathrm{GL}(248, K)$ *and the scheme-theoretic stabiliser of KF in* $\mathrm{PGL}(248, K)$ *coincide with [the image of] G.*

It would be a formidable task to generalise this result to arbitrary commutative rings R, in the spirit of [65,67], deriving explicit matrix equations.

As it seems, it may be already quite a challenge to get rid of the assumption $6 \in R^*$. In fact, Garibaldi and Guralnick conjecture that the scheme-theoretic stabiliser of KF in $\mathrm{PGL}(248, K)$ is always smooth, and that the scheme-theoretic stabiliser of F in $\mathrm{GL}(248, K)$ is smooth in characteristic 3. However, it is clearly not smooth in characteristic 2, so that to obtain an explicit characteristic free description of the polynomial equations defining $G(\mathrm{E}_8, R)$ inside $\mathrm{SO}(248, R)$ one will have to find other invariants, inevitably non-symmetric ones.

One extremely important aspect of [13] is that it makes clear that the very special role played by characteristics 2 and 3 is explained not just by the fact that 2 and 3 divide $d!$, where d is the degree of a polynomial invariant, but also by the very exceptional behaviour of Lie algebras in characteristics 2 and 3. From a classical paper by Alexei Rudakov [37] we know that Chevalley algebras in characteristics $p \geqslant 5$ are rigid, which they are not in characteristics 2 and 3.

§6. Final remarks

Due to the time constraints, in this talk we could not touch many further related recent results, and possible generalisations.

In the introduction, we have already mentioned the marvellous recent papers by Alexei Stepanov, especially [46], which clearly indicate an entirely new stage of maturity of the whole theory, and pave way for a whole new range of applications.

Presently, one of the main pursuits of the St. Petersburg school of algebraic groups consists in a systematic endeavour to generalise all existing results from Chevalley groups to arbitrary – sufficiently isotropic! – reductive groups over rings. This enterprise was started by the truly remarkable contribution by Victor Petrov and Anastasia Stavrova [32], who defined the elementary subgroup of such groups, and proved its normality. To do that, they had to develop large fragments of the general structure theory of such groups, thus effectively expanding parts of the classical Borel–Tits structure theory [5] to the Demazure—Grothendieck setting [10], and to develop localisation techniques in the spirit of Quillen—Suslin [51, 52], in this more general context.

Further advances in this direction were mostly due to Stavrova herself, with collaboration of the first author and Ekaterina Kulikova (Sopkina) [22, 30, 41, 43]. Presently, Stavrova, Stepanov and the first author carry through the project whose objective is to describe subgroups of isotropic reductive groups normalised by the elementary subgroup (at lest when $6 \in R^*$), see [42].

Another major advance, was an improvement of factorisations and stability theorems at the level of K_1 and K_2, for exceptional embeddings, obtained by Sergei Sinchuk in his Thesis [39] (see also his paper [40], for an outline of some of these results, without detailed proofs). This was the first considerable improvement of the stability results for exceptional groups obtained by Michael Stein and Eugene Plotkin [33–35, 44] some decades ago.

Finally, we would like to mention the recent amazing paper by Andrei Lavrenov [23], which contains a complete solution of an outstanding open problem, centrality of the symplectic K_2. Before, centrality was only known in the linear case, from the beautiful another presentation by Wilberd van der Kallen [21] (soon thereafter expanded by Marat Tulenbaev from commutative to almost commutative rings). The work by Lavrenov was the first significant progress for decades, and hopefully, combined with the ideas of Petrov and Sinchuk, it will eventually lead to a proof of K_2 centrality also for large exceptional groups.

Certainly, each of these subjects would deserve a separate talk, and a separate survey.

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