Записки научных
семинаров ПОМИ
Том 432, 2015 г.

A. Ya. Kazakov, S. Yu. Slavyanov

REPRESENTATIONS AND USE OF SYMBOLIC COMPUTATIONS IN THE THEORY OF HEUN EQUATIONS

Abstract. A first-order 2×2 system equivalent to the Heun equation is obtained. A deformed Heun equation in symmetric form is presented. Series solutions of this equation are presented. A four-parameter subfamily of deformed confluent Heun equation whose solutions have integral representations is found.

As a preamble to this publication one of the authors (S.Yu.S) wants to say a couple of words about his contacts with Prof. A. M. Vershik who recently had celebrated his 80-years jubilee. We met each other in the city Syktyvkar where a new university was organized. We both gave there lectures for 4th year students. Our host was Ya. Eliashberg a very nice and talented young mathematician. We got separate rooms in a flat belonging to university. Living in the same flat allowed us to talk in the evenings. These talks in which we found common friends were very informative to me. In many cases we shared common ideas. Anatolii Moiseevich invited once Revolt Pimenov who had been ousted from Leningrad being a dissident. I remember our stay in Syktyvkar with great warmth and I am proud that A. M. Vershik is addressing me after "Seryozha."

Heun class equations as equations with four fuchsian singularities are a key instrument in many problems of mathematical and theoretical physics [1]. On the other hand a fuchsian 2×2 system also with four fuchsian singularities are widely used in the theory of Painleve equation [2, 3]. Is there a direct relation between these two objects? Unfortunately there is a negative answer to this question. It is needed in general case to introduce an intermediate equation called in our previous papers as Heun1 equation [4]. Heun1 equation is associated with Heun equation and contains one additional Fuchsian singularity with particular properties. Therefore

Key words and phrases: Heun equation, deformed Heun equation, confluent Heun equation, apparent singularity, integral representations.

it is needed to study the scheme: Heun equation → Heun1 equation → Fuchsian system. The corresponding relations: 1) are not unique, 2) give raise to additional symmetries between equations. We need to regret that the corresponding calculations are very complicated and various CAS are needed to perform them. We used Maple for this purpose.

in this publication we present three topics which seems to be eclectic but they shaw different aspects of the theory of Heun equation.

1. Linear 2×2 system related to Heun equation.
2. A new representation of Heun1 equation and series for its solution.
3. Four-parametric subfamily of confluent Heun equation with added apparent singularity is constructed whose solutions have integral representations.

§1. Heun equation and corresponding 2×2 system

Here we find a particular 2×2 system which is generic for Heun equation. For this purpose we use "polynomial rotation" with matrix R which adds one needed parameter. Assume that we take matrix R as

$$R = \begin{pmatrix} z(z-1) & \rho z \\ 0 & (z-t) \end{pmatrix}. \tag{1}$$

Then the inverse matrix to R is

$$R^{-1} = \sigma^{-1} \begin{pmatrix} z-t & 0 \\ -\rho z & z(z-1) \end{pmatrix}. \tag{2}$$

Let vector $\vec{w}$ be the solution of the 2×2 system

$$\vec{w}' = \sigma^{-1} R^{-1} S \vec{w} = T\vec{w}, \tag{3}$$

where $\sigma(z) = \prod_{j=1}^{3}(z - z_j)$ and

$$S = \begin{pmatrix} \alpha z + e_1 & e_2 \\ e_3 z & \beta \end{pmatrix}. \tag{4}$$

Values $z_1 = 0$, $z_2 = 1$, $z_3 = t$ are locations of finite singularities of the system under consideration. Parameters α, β, e_1, e_2, e_3 obey a particular relation (Fuchs relation) which will be written below.

As a result matrix T takes the form

$$T = \sigma^{-1} \begin{pmatrix} (z-t)(\alpha z + e_1) & e_2(z-t) \\ -z(\rho\kappa z + \rho e_1 + e_3(z-1)) & z(\beta(z-1) - \rho e_2) \end{pmatrix}. \tag{5}$$

Solving system for w_1 we obtain the second order equation

$$w''(z) - (\mathrm{tr}T + T'_{12}T_{12}^{-1})w'(z) + (\det T - T'_{12}T_{12}^{-1}T_{11} - T'_{11})w(z) = 0. \quad (6)$$

Calculation of included in (6) coefficients gives

$$T'_{12}T_{12}^{-1} = \frac{1}{z} + \frac{1}{z-1}$$

$$\mathrm{tr}T = -\frac{e_1}{z} + \left(\alpha + e_1 + \frac{\rho e_2}{t-1}\right)\frac{1}{z-1} + \left(\beta - \frac{\rho e_2}{t-1}\right)\frac{1}{z-t}$$

$$\det T = \sigma^{-1}(\alpha\beta z + e_1\beta + e_2 e_3)$$

$$T'_{12}T_{12}^{-1}T_{11} - T'_{12} = -\alpha(z-t).$$

Then second order equation (6) takes the form

$$\begin{gathered} w''(z) + \left(\frac{1-e_1}{z} + \left(\frac{1+e_1+\alpha-\widetilde{\rho}e_2}{z-1}\right) + \frac{\beta+\widetilde{\rho}e_2}{z-t}\right)w'(z) \\ +\sigma^{-1}(\alpha\beta z - \alpha(z-t) + e_1\beta + e_2 e_3)w(z) = 0, \end{gathered} \quad (7)$$

where for simplicity we denoted

$$\rho = \widetilde{\rho}(1-t).$$

One can see that equation (7) coincides with standard Heun equation up to redefining the parameters

$$y''(z) + \sum_{j=1}^{3}\frac{1-\theta_j}{z-z_j}y'(z) + \left(\frac{\alpha(\beta-1)}{z(z-1)} + \sigma^{-1}H\right)y_z = 0, \quad (8)$$

where parameters α, β, e_1, e_2, $\widetilde{\rho}$ are related to characteristic exponents at fuchsian singularities and parameter e_3 is related to accessory parameter H.

The Fuchs relation for characteristic exponents in (7) reads

$$\alpha - 1 + \beta + e_1 - e_1 - \alpha + \widetilde{\rho}e_2 + 1 - \beta - \widetilde{\rho}e_2 = 1$$

is satisfied automatically.

Hence, we found fuchsian system 2×2 equivalent to Heun equation.

§2. Solutions of the deformed Heun equation

Here we focus on equations which are denoted as deformed Heun equations and on its solutions.

Beyond routine computation the symmetrized presentation of a deformed Heun equation is given (to our knowledge a new one).

If an apparent singularity at the point $z_0 = q$ is added to the four regular singularities $z_1 = 0$, $z_2 = 1$, $z_3 = t$, $z_4 = \infty$ in Heun equation, we obtain a *deformed Heun equation* [4,5] and will use the notation *Heun1* for such an equation. We assume that all finite singularities are real and that $0 < q < 1/2$, $t > 1$. With a proper normalization of parameters *Heun1* equation reads

$$\begin{aligned}\sigma(z)w'' + \left\{\sum_{j=1}^{3}\theta_j\sigma_j(z) - \sigma(z)\frac{1}{z-q}\right\}w' \\ + \left\{\alpha\beta(z-t) - \sigma_3(t)H + \frac{\mu\sigma_3(q)(z-t)}{z-q}\right\}w = 0,\end{aligned} \tag{9}$$

where $\sigma(z) = z(z-1)(z-t)$, $\sigma_j(z) = \sigma(z)/(z-z_j)$. Whenever a Heun equation depends on 6 parameters, a *Heun1* equation (15) depends on 8 parameters with parameters q, μ added to the list. Parameters α, β, θ_j must satisfy the Fuchs condition

$$\sum_{j=1}^{3}\theta_j - \alpha - \beta = 2. \tag{10}$$

Further on we assume that

$$\theta_1 > 1. \tag{11}$$

In addition the following necessary condition (absence of logarithmic terms) holds

$$\sigma_3(t)H = \sigma(q)\mu^2 + (\sigma_3(q) + \tau(q))\mu + \alpha\beta(q-t), \tag{12}$$

where $\tau(q) = \sum_{j=1}^{3}\theta_j\sigma_j(q)$. Hence, the actual number of parameters is diminished by one while accessory parameter H can be excluded resulting in

$$\begin{aligned}\sigma(z)\left(w'' - \frac{w'}{z-q}\right) + \left(\sum_{j=1}^{3}\theta_j\sigma_j(z)\right)w' + \alpha\beta z w \\ - \left\{\sigma(q)\left(\mu^2 - \frac{\mu}{z-q}\right) + \left(\sum_{j=1}^{3}\theta_j\sigma_j(q)\right)\mu + \alpha\beta q\right\}w = 0.\end{aligned} \tag{13}$$

If we introduce the operator

$$D = \frac{d}{dz}$$

equation (13) can be presented in a symmetrical way

$$P(D, z) - P(\mu, q) = 0. \tag{14}$$

Our purpose would be to study solutions of equation (13) as series in the vicinity of apparent singularity $z = q$.

In order to study solutions at the point $z = q$ it is convenient to substitute $z = x + q$ with following mapping of singularities $x_1 = -q$, $x_0 = 0$, $x_2 = 1 - q$, $x_3 = t - q$ and obtain

$$\begin{aligned}(x^3 + r(q)x^2 + s(q)x + \sigma(q))\left\{w'' - \frac{w'}{x}\right\} + \left\{u(q)x^2 + v(q)x + \tau(q)\right\}w' \\ + \alpha\beta(x+q) - (\sigma(q)(\mu^2 - \frac{\mu}{x}) + \mu\tau(q) + \alpha\beta q)w = 0,\end{aligned} \tag{15}$$

where

$$s(q) = \sum_{j=1}^{3}\sigma_j(q), \quad r(q) = \sum_{j=1}^{3}q_j, \quad q_1 = 0,\ q_2 = 1,\ q_3 = t;$$

$$u(q) = \sum_{j=1}^{3}\theta_j, \quad \tau(q) = \sum_{j=1}^{3}\theta_j\sigma_j(q), \quad v(q) = \sum_{j=1}^{3}\theta_j q_j.$$

The local solution at apparent singularity $x = 0$ is sought in the form

$$w(x) = \sum_{k=0}^{\infty} a_k x^k. \tag{16}$$

The first equation arising at equating the terms after substituting series (16) into equation (15) would be

$$a_1 = \mu a_0.$$

Two linearly independent solutions w_1 and w_2 can be introduced.

Either A:

$$a_0 = 0, \quad a_1 = 0, \quad a_2 = 1 \quad \to w_1(x)$$

or B:

$$a_0 = 1, \quad a_1 = \mu, \quad a_2 = 0, \quad \to w_2(x).$$

In case A the system for coefficients a_k reads

$$\begin{aligned}\sigma(q)(k+1)(k+3)a_{k+3} + ((k+2)(ks(q)+\tau(q)) + \sigma(q)\mu)a_{k+2}\\ +((k+1)((k-1)r(q)+v(q)) - \sigma(q)\mu^2 - \tau(q)\mu)a_{k+1}\\ +(k(k-2+u(q)) + \alpha\beta)a_k = 0.\end{aligned} \tag{17}$$

In particular

$$3\sigma(q)a_3 + (2\tau(q) + \sigma(q)\mu)a_2 = 0$$

$$8\sigma(q)a_4 + (3(s(q)+\tau(q)) + \sigma(q)\mu)a_3 + (2v(q) - \tau(q)\mu - \sigma(q)\mu^2)a_2 = 0.$$

Fourth-term recursive relation (17) may be considered as a Pouincare-Perron type difference equation. Its characteristic equation reads

$$\sigma(q)\lambda^3 + s\lambda^2 + r\lambda + 1 = 0. \tag{18}$$

Characteristic exponents λ_j are

$$\lambda_1 = \frac{-1}{q}, \quad \lambda_2 = \frac{-1}{q-1}, \quad \lambda_3 = \frac{-1}{q-t}. \tag{19}$$

Hence, the radius of convergence R of series (16) would be

$$R = \min\{q, 1, t\} = q.$$

In the same way other solutions in vicinities of singularities is constructed. The corresponding four-term recurrent systems for coefficients is found similar to that for a_k.

§3. Integral representations for solutions of the confluent Heun class equation

This section is devoted to the confluent Heun equation with single added apparent singularity. This equation can be written as

$$\begin{aligned}w''(z) + \left[a + \frac{\beta}{z} + \frac{\gamma}{z-1} - \frac{1}{z-\lambda}\right] w'(z)\\ + \frac{1}{z(z-1)}\left[a\sigma z + L + \frac{\chi}{z-\lambda}\right] w(z) = 0.\end{aligned} \tag{20}$$

We shall call it cHeun1 in what follows. Here parameter L takes a special value,

$$L = -\frac{(a(\sigma-\chi)(\lambda^3-\lambda^2-\lambda)+\beta\,\chi(\lambda-1)+\chi\,\gamma\,\lambda+\chi^2-2\,\chi\,\lambda+\chi}{\lambda\,(\lambda-1)}, \qquad (21)$$

which guarantees that point $z=\lambda$ is an apparent singular point of the equation. The residue of the coefficient at $w'(z)$ at the point $z=\lambda$ equals to the negative integer (here it is -1), this value defines the order of the apparent singular point (here it equals to 1). Note, that apparent singularity is determined by the following condition: equation has singularity at this point, but all its solutions are holomorphic near this point. Comparing equation (20) and confluent Heun equation

$$w''(z)+\left[a+\frac{\beta}{z}+\frac{\gamma}{z-1}\right]w'(z)+\frac{\delta z+\chi}{z(z-1)}w(z)=0, \qquad (22)$$

one can conclude, that if $z=\lambda$ coincides with regular singularity of the equation (namely, $z=0$ or $z=1$), equation (20) can be reduced to the equation (22) by substitution $w(z)=(z-\lambda)^{\zeta}\cdot u(z)$ at suitable value ζ. So, studying solutions and monodromy of equation (20) we get simultaneously information about corresponding objects related to the equation (22).

Solutions of the Heun class equations have not integral representations in general situation, this fact greatly complicates their analytic study. Evidently, if coefficients of the equation contain a small or large parameter one can use corresponding asymptotical technique in order to describe monodromy properties of the equation. Moreover, there are some special situations when monodromy of the equation can be described in details. For instance, if one of the regular singularity is the apparent one for the equation (7) or (22), the analytic description of theirs monodromy was obtained in [6,7]. Note, that equation can be identified in these situations as an hypergeometric class equation, in particular there are integral representations of solutions [8,9]. The aim of the present paper is search of situations, when confluent Heun equation with single added apparent singularity has solutions expressed by contour integrals. Equation (20) has 6 free parameters a, β, γ, σ, λ, χ, we shall find 4-parameter situations, when solutions have integral representations. Note, that our results correspond to the arbitrary values of the characteristic parameters a, β, γ and accessory parameter χ, so the "genuine" singularities of the equation will be in general situation.

3.1. Auxiliary system of equations and its integral symmetry. So, we look for equation (20) whose solutions can be expressed by contour integrals. Let consider the next system of equations

$$(zA + B)W'(z) = CW(z), \tag{23}$$

where A, B, C – are constants 3×3 matrices, $W(z) = (w_1(z),\, w_2(z),\, w_3(z))^T$ is 3-vector function. If matrix C has zero column (let it be the third column), the system (23) is rewritten as

$$W'(z) = (zA + B)^{-1} CW(z), \tag{24}$$

and it can be decomposed into a system for functions $w_1(z)$, $w_2(z)$, and separate equation for $w_3(z)$. So, the function $w_3(z)$ can be expressed in terms of $w_1(z)$, $w_2(z)$. The system for functions $w_1(z)$, $w_2(z)$ can be reduced to the scalar differential equation of the second order for function $w_2(z)$, and suitable choice of the matrix coefficients A, B, C generates equation (20). Then one component of the vector-function $W(z)$ will be a solution of the equation (20), and other components will be expressed through it.

The following statement can be proven by integration by parts, (see details in [15]).

Theorem. Let $Y(t)$ be a solution of the system

$$(tA + B)Y'(t) = (C + \mu A)Y(t), \tag{25}$$

branching in a vicinity of the regular singular point, L be a Pochhammer contour, embracing point $t = z$ and this regular singularity, then vector-function

$$W(z) = \int\limits_L (z - t)^{-1-\mu} Y(t)dt, \tag{26}$$

is a solution of the system (23), branching in a vicinity of the same regular singularity on complex plane z.

The matrix $C + \mu A$ has in general situation nonzero third column, therefore the system (25) can not be reduced to the scalar differential equation of the second order. Let U be a constant 3×3 - matrix. Consider gauge transform

$$Y(t) = U^{-1}V(t). \tag{27}$$

Then 3-vector-function $V(t) = (v_1(t), v_2(t), v_3(t))^T$ is a solution of the system

$$V'(t) = U(tA + B)^{-1}(C + \mu A)U^{-1}V(t). \tag{28}$$

Assume that the next relations hold,

$$\left[U(tA+B)^{-1}(C+\mu A)U^{-1}\right]_{13}=\left[U(tA+B)^{-1}(C+\mu A)U^{-1}\right]_{23}=0. \quad (29)$$

Then system (28) can be split into the system for the functions $v_1(t)$, $v_2(t)$ and equation for the function $v_3(t)$. Respectively, system for the functions $v_1(t)$, $v_2(t)$ can be reduced to the scalar differential equation of the second order for the function $v_2(t)$. Then function $v_3(t)$ can be expressed in terms of function $v_2(t)$ by integration. We consider this situation in what follows.

Relations (29) generate a rather complicated system of algebraic equations for the parameters of matrices A, B, C, U and for the parameters of the initial equation (20). This system can be analyzed with help of computer algebraic system like Maple or Mathematica, its particular solution generates the equation (20) which solutions have integral representations.

3.2. Equation which solutions have integral representations. The full description of needed calculations is too cumbersome. They can be restored from the script in Maple, which can be found in Appendix. This script does not contain results of step by step calculations, the results of calculations are given for some strings only, where coefficients of equations are calculated.

In accordance with results of the script, coefficients of initial equation are given by the following relations. Function $w(z) = w_2(z)$ is a solution of the equation

$$w''(z) + M_1(z)w'(z) + N_1(z)w(z) = 0, \quad (30)$$

$$M_1(z) = a - (z-\lambda)^{-1} + \frac{-\theta_1+1}{z} + \frac{-\theta_2+1}{z-1}, \quad (31)$$

$$N_1(z) = \frac{1}{z(z-1)}\left[-a\theta_2 z + \theta_1\theta_2 + \frac{\lambda\theta_2}{z-\lambda}\right]. \quad (32)$$

This equation is a special case of equation (20). It has 4 free parameters a, θ_1, θ_2, λ, which define the behavior of solutions in vicinities of singular points, and accessory parameter equals to $\lambda\theta_2$. Position of the apparent singularity $z = \lambda$ and another parameter of equation can be expressed in terms of these free parameters.

Further, script contains the coefficients of equation for the function $v(t) = v_2(t)$,

$$v''(t) + M_2(t)v'(t) + N_2(t)v(t) = 0, \quad (33)$$

$$M_2(t) = a + \frac{1-\theta_1-\mu}{t} - \frac{1}{t-\widetilde{\lambda}}, \quad (34)$$

$$N_2(t) = \frac{1}{t}\left[-a\mu + \frac{\widetilde{\chi}}{t-\widetilde{\lambda}}\right], \tag{35}$$

where

$$\widetilde{\chi} = \frac{\mu\,\sigma\,(a\lambda + \alpha\,\theta_1 - \theta_1)}{a\sigma\,\lambda + \alpha\,\sigma\,\theta_1 - \mu\,\lambda - \sigma\,\theta_1 - \lambda\theta_2}, \tag{36}$$

$$\begin{aligned}\widetilde{\lambda} = & \big[a\lambda\big(\mu\sigma(a\lambda+\alpha\theta_1) - \lambda(\mu+\theta_2)^2 + \sigma\theta_2(a\lambda+\alpha\theta_1) - \sigma\theta_1(\mu+\theta_2)\big)\big]^{-1} \\ & \times\sigma\big(a^2\lambda^2\sigma\theta_1 + 2a\lambda\sigma{\theta_1}^2(\alpha-1) + \sigma{\theta_1}^3(\alpha-1)^2 - a\lambda^2(\mu+\theta_1)(\mu+\theta_2) \\ & -\alpha\,\lambda\,\mu^2\theta_1 - \alpha\,\lambda\,\theta_1(\mu\theta_1 + \mu\theta_2 + \theta_1\theta_2) + \lambda\,\theta_1(\mu+\theta_1)(\mu+\theta_2)\big).\end{aligned} \tag{37}$$

As it follows from these relations, equation (33) is a confluent hypergeometric equation with added apparent singularity of the first order. This equation was discussed recently in [7], where its monodromy was calculated. In particular, it was shown, that solutions of this equation can be expressed by contour integrals.

Equation for the function $v_3(t)$ can be written as

$$t(t-1)\frac{d}{dt}v_3(t) = S_1(t)v_1(t) + S_2(t)v_2(t) + t(\theta_2+\mu)v_3(t),$$

where $S_{1,2}(t)$ are polynomials of the second degree in t. We omit theirs explicit expressions for the brevity. Taking into account relations (26), (27), we conclude, that solutions of the equation (30) have integral representations.

Remark 1. Equation (33) has single regular singularity $t = 0$. But the equation for the function $v_3(t)$ has regular singularity $t = 1$, therefore in accordance with (26), (27) solution $w_2(z)$ has two branching points $z = 0$, $z = 1$.

Remark 2. Parameters σ, μ, included in the relations (36), (37), are not connected with parameters of equation (30). Parameter σ defines matrix elements of matrix U. Parameter μ defines the kernel of integral transform (26) and coefficients of the equation (33).

3.3. Summary and concluding remarks. Existence of ntegral representations of solutions of equation (30) means that its monodromy can be expressed in explicit terms.

Equation (20) has rich set of symmetries, which connect solutions of the equations with different parameters.

1. There are elementary symmetries, which arise at simple transforms of the equation (20). Firstly, they appear at substitution of variables

$z = 1 - s$, which interchanges singular points. Evidently, such substitution changes the coefficients of the equation(20), but its structure stays unchanged. Further, there are s-homotopic substitutions $w(z) = z^{1-\beta}u(z)$, $w(z) = (1-z)^{1-\gamma}v(z)$. These substitutions transform solutions holomorphic at a singular point into solutions branching at this singularity and vice versa. There is symmetry of the equation (20) connected with substitution $w(z) = \exp(-az)v(z)$, which transforms asymptotic behavior of solutions in a vicinity of irregular singularity $z = \infty$.

2. Equation (20) has integral symmetries. Euler integral symmetry was studied in [10]. Another integral symmetry whose kernel is solution of the confluent hypergeometric equation was obtained in [11].

3. There are gauge symmetries of the equation (20), see details in [12].

We emphasize that these symmetries connect solutions of the equations. It means, that these symmetries can be expressed in terms of monodromy of the corresponding equations, see details in [13, 14]. Respectively, monodromy can be explicitly calculated for the equations, which are results of the application the described symmetries to equation (30).

Here is the summary of our results. It is derivation Heun class equation, confluent Heun equation with added apparent singularity, whose solutions can be described by contour integrals. This equation has 4 free parameters, which determine the characteristics of the singular points and accessory parameter. This fact is different from the situations discussed in articles [6–9], where one of singular points was an apparent singularity. This is currently the most general Heun class equation, whose solutions can be expressed in terms of contour integrals. There were some other attempts to describe Heun class equations with solutions expressed by contour integrals. First of all, there are situations when Heun equation can be reduced to the hypergeometric one [16]. This situation generates Heun equations with 2 free parameters. Direct factorization of the Heun equation [17] generates Heun equation with 3 free parameters.

Note, that gauge transforms discussed in [12] acting on solutions of Heun equations with 2 added apparent singularities generate another free parameter. It would be interesting to obtain an analogue of presented here results for the Heun equations with two added apparent singularities.

3.4. Appendix. Here is enclosed a Maple script that implements the calculations for the construction of the equation(30). It contains details of calculation of the coefficients of this equation, which is a special case of the equation (20). Further, this script describes the calculation of the coefficients of the equation (33). In accordance with the script, solutions of the equation (30) can be expressed as contour integrals.

> $with(PolynomialTools);\ with(LinearAlgebra);\ with(DEtools);$

> $c_1 := -a; c_2 := -\dfrac{a\cdot\lambda\cdot(\theta_2+\mu)}{a\cdot\lambda+\alpha\cdot\theta_1-\theta_1}; c_3 := 0;\ c_4 := -\dfrac{a\cdot\lambda+\alpha\cdot\theta_1-\theta_1}{\lambda};$

> $c_5 := -\mu; c_6 := 0\,; c_9 := 0; c_7 := -\dfrac{a\cdot\varrho\cdot b_3+\mu+\theta_1+\theta_2}{b_3};$

> $c_8 := -\dfrac{\lambda\cdot(\theta_2+\mu)\,(a\cdot\varrho\cdot b_3+\mu+\theta_1)}{(a\cdot\lambda+\alpha\cdot\theta_1-\theta_1)\cdot b_3};$ *#fixing the parameters of the matrix C*

> $C1 := \{(1,1)=c[1], (1,2)=c[2], (1,3)=c[3], (2,1)=c[4], (2,2)=c[5],$

> $(2,3)=c[6], (3,1)=c[7], (3,2)=c[8], (3,3)=c[9]\};\ C := Matrix(3, C1);$

> $b_1 := 1; b_2 := 0; b_4 := 1;\ b_5 := -\dfrac{\alpha\mu\lambda+\alpha\lambda\theta_2+a\lambda+\alpha\theta_1-\mu\lambda-\lambda\theta_2-\theta_1}{a\lambda+\alpha\theta_1-\theta_1}; b_6 := \alpha\cdot b_3;$

> $\xi := \dfrac{\lambda(\theta_2+\mu)}{(a\lambda+\alpha\theta_1-\theta_1)\,b_3};$ *#fixing the parameters of the matrix B*

> $B1 := \{(1,1)=b_1, (1,2)=b_2, (1,3)=b_3, (2,1)=b_4, (2,2)=b_5, (2,3)=b_6,$

> $(3,1)=\varrho\cdot b_1+\xi\cdot b_4, (3,2)=\varrho\cdot b_2+\xi\cdot b_5, (3,3)=\varrho\cdot b_3+\xi\cdot b_6\};\ B := Matrix(3, B1);$

> $\#u_1 := 0; u_2 := -\dfrac{\nu\cdot\lambda\cdot(\theta_2+\mu)\,u_4}{(a\cdot\lambda+\alpha\theta_1-\theta_1)\,b_3}; u_3 := \nu\cdot u_4;\ u_5 := \sigma\cdot u_4;$

> $u_6 := -\dfrac{b_3\,(\sigma a\cdot\lambda+\sigma\alpha\theta_1-\sigma\theta_1-\mu\cdot\lambda-\lambda\cdot\theta_2)\,u_4}{\lambda\cdot(\theta_2+\mu)};$ *#fixing the parameters of the matrix U*

> $u_9 :=$

> $-\dfrac{1}{(\theta_2+\mu)\cdot\lambda\cdot\nu\cdot u_4^2}\Big(\Big(a\nu\cdot\lambda\cdot u_4^2u_8+\alpha\nu\theta_1u_4^2u_8-\mu\nu\cdot\lambda\cdot u_4^2u_7-\nu\cdot\lambda\cdot\theta_2u_4^2u_7$

> $-\nu\theta_1u_4^2u_8-a\cdot\lambda-\alpha\theta_1+\theta_1\Big)\cdot b_3;$

> $U1 := \{(1,1)=u_1, (1,2)=u_2, (1,3)=u_3, (2,1)=u_4, (2,2)=u_5, (2,3)=u_6, (3,1)=u_7,$

> $(3,2)=u_8, (3,3)=u_9\};\ U := Matrix(3, U1);$

> $A1 := \{(1,1)=0, (1,2)=0, (1,3)=0, (2,1)=0, (2,2)=1, (2,3)=0,$

> $(3,1)=0, (3,2)=0, (3,3)=1\};\ A := Matrix(3, A1);$ *#fixing of the matrix A*

> $AZ := z\cdot A+B;$ *# calculation the parameters of the matrix z·A +B*

> $D1 := Determinant(AZ);\ AI := MatrixInverse(AZ);$ *#its Determinant and Inverse*

> $AB := D1\cdot AI;$

> $T := MatrixMatrixMultiply[Z](AB, C);$ *# calculation of the matrix T=Det (z·A +B) ·(z·A +B)⁻ ·C*

> $eq1 := D1\cdot diff(w_1(z), z) - T[1,1]\cdot w_1(z) - T[1,2]\cdot w_2(z);$

> *# first string for the first system:* $D1\cdot\dfrac{d}{dz}w_1(z) - (T_{11}\cdot w_1(z)+T_{12}\cdot w_2(z))$

> $w_1(z) := \dfrac{(D1\cdot diff(w_2(z), z) - T[2,2]\cdot w_2(z))}{T[2,1]};$

> *#second string for the first system:* $w_1(z) = \dfrac{\left(D1\cdot\dfrac{d}{dz}w_2(z) - T_{22}\cdot w_2(z)\right)}{T_{21}}$

> $eq1 := factor(eq1);$

\> $sp := dcoeffs\left(eq1, w_2(z)\right);$
#calculation of the coefficients in the first string after substitution $w_1(z) = \ldots$

\> $M1 := factor(sp[2]/sp[1]);$ #first coefficient in equation: $\frac{d^2}{dz^2} w_2(z) + M1 \cdot \frac{d}{dz} w_2(z) + N1 \cdot w_2(z) = 0$

\> $N1 := factor(sp[3]/sp[1]);$ # second coefficient in equation: $\frac{d^2}{dz^2} w_2(z) + M1 \cdot \frac{d}{dz} w_2(z) + N1 \cdot w_2(z) = 0$

\> $M1\_exp := convert(M1, parfrac, z);$ #rational expansion of the M1

\> $M1\_exp1 := a - \frac{1}{z-\lambda} + \frac{-\theta_2 + 1}{z-1} + \frac{-\theta_1 + 1}{z};\quad Np := z \cdot (z-1) \cdot N1;$

\> $NP\_exp := convert(Np, parfrac, z);$ # rational expansion of the $z \cdot (z-1) \cdot N1(z)$

\> $NP\_exp := -a\,\theta_2 z + \theta_1 \theta_2 + \frac{\lambda\,\theta_2}{z-\lambda};$

\> $Cv := C + \mu \cdot A;\quad AT := t \cdot A + B;$ #calculation of the matrices $(C+\mu \cdot A)$ and $(t \cdot A+B)$

\> $D2 := Determinant(AT);$ #calculation of the $\det(t \cdot A+B)$

\> $AI := MatrixInverse(AT);\quad AB := D2 \cdot AI;\quad Z := MatrixInverse(U);$
\# calculation of the inverse matrices

\> $Tv := MatrixMatrixMultiply[Z](AB, Cv);$
#calculation of the matrix $Tm = \det(t \cdot A+B) \cdot (t \cdot A+B)^{-1} \cdot \left(C + \mu \cdot A\right)$

\> $Prod1 := MatrixMatrixMultiply[Z](U, Tv);$
#calculation of the matrix $\det(t \cdot A+B) \cdot U \cdot (t \cdot A+B)^{-1} \cdot \left(C + \mu \cdot A\right)$

\> $Tm := MatrixMatrixMultiply[Z](Prod1, Z);$
#calculation of the matrix $\det(t \cdot A+B) \cdot U \cdot (t \cdot A+B)^{-1} \cdot \left(C + \mu \cdot A\right) \cdot U^{-1}$

\> $factor(Tm[1,3]);\ factor(Tm[2,3])$ # checking: $Tm_{13} = 0,\quad Tm_{23} = 0$

\> $eq2 := D2 \cdot \left(diff\left(v_1(t), t\right)\right) - Tm[1,1] \cdot v_1(t) - Tm[1,2] \cdot v_2(t);$

\> #first string for the second system: $D2 \cdot \frac{d}{dt} v_1(t) - \left(Tm_{11} \cdot v_1(t) + Tm_{12} \cdot v_2(t)\right)$

\> $v_1(t) := \frac{\left(D2 \cdot diff\left(v_2(t), t\right) - Tm[2,2] \cdot v_2(t)\right)}{Tm[2,1]};$

\> #second string for the second system: $v_1(t) = \frac{\left(D2 \cdot \frac{d}{dt} v_2(t) - Tm_{22} \cdot v_2(t)\right)}{Tm_{21}}$

\> $eq2 := factor(eq2);$ # second equation after substitution $v_1(t) = \ldots$

\> $sp2 := dcoeffs\left(eq2, v_2(t)\right);$ #coefficients of the second equation

\> $M2 := factor(sp2[2]/sp2[1]);$ # first coefficient in the equation: $\frac{d^2}{dt^2} v_2(t) + M2 \cdot \frac{d}{dt} v_2(t) + N2 \cdot v_2(t) = 0$

\> $N2 := factor(sp2[3]/sp2[1]);$ # second coefficient in the equation: $\frac{d^2}{dt^2} v_2(t) + M2 \cdot \frac{d}{dt} v_2(t) + N2 \cdot v_2(t) = 0$

> *M2_exp := convert(M2, parfrac, t); # rational expansion of the M2(t)*

>

$$M2\_exp := a - \Big(\big(a\lambda\mu\sigma + a\lambda\sigma\theta_2 + \alpha\mu\sigma\theta_1 + \alpha\sigma\theta_1\theta_2 - \lambda\mu^2 - 2\lambda\mu\theta_2 - \lambda\theta_2^2 - \mu\sigma\theta_1 - \sigma\theta_1\theta_2\big)a\lambda\Big) \Big/ \Big(a^2\lambda^2\mu\sigma t - a^2\lambda^2\sigma^2\theta_1 + a^2\lambda^2\sigma t\theta_2 + a\alpha\lambda\mu\sigma t\theta_1 - 2a\alpha\lambda\sigma^2\theta_1^2 + a\alpha\lambda\sigma t\theta_1\theta_2 - \alpha^2\sigma^2\theta_1^3 + a\lambda^2\mu^2\sigma - a\lambda^2\mu^2 t + a\lambda^2\mu\sigma\theta_1 + a\lambda^2\mu\sigma\theta_2 - 2a\lambda^2\mu t\theta_2 + a\lambda^2\sigma\theta_1\theta_2 - a\lambda^2 t\theta_2^2 - a\lambda\mu\sigma t\theta_1 + 2a\lambda\sigma^2\theta_1^2 - a\lambda\sigma t\theta_1\theta_2 + \alpha\lambda\mu^2\sigma\theta_1 + \alpha\lambda\mu\sigma\theta_1^2 + \alpha\lambda\mu\sigma\theta_1\theta_2 + \alpha\lambda\sigma\theta_1^2\theta_2 + 2\alpha\sigma^2\theta_1^3 - \lambda\mu^2\sigma\theta_1 - \lambda\mu\sigma\theta_1^2 - \lambda\mu\sigma\theta_1\theta_2 - \lambda\sigma\theta_1^2\theta_2 - \sigma^2\theta_1^3\Big) + \frac{-\mu - \theta_1 + 1}{t}$$

> $dent := factor\left(\dfrac{denom(M2)}{t}\right)$; *#calculation of the new apparent singularity*

> *eqr := dent = 0; lambda_new := solve(eqr, t); # new apparent singularity*

>

$$lambda\_new := \Big(\sigma\big(a^2\lambda^2\sigma\theta_1 + 2a\alpha\lambda\sigma\theta_1^2 + \alpha^2\sigma\theta_1^3 - a\lambda^2\mu^2 - a\lambda^2\mu\theta_1 - a\lambda^2\mu\theta_2 - a\lambda^2\theta_1\theta_2 - 2a\lambda\sigma\theta_1^2 - \alpha\lambda\mu^2\theta_1 - \alpha\lambda\mu\theta_1^2 - \alpha\lambda\mu\theta_1\theta_2 - \alpha\lambda\theta_1^2\theta_2 - 2\alpha\sigma\theta_1^3 + \lambda\mu^2\theta_1 + \lambda\mu\theta_1^2 + \lambda\mu\theta_1\theta_2 + \lambda\theta_1^2\theta_2 + \sigma\theta_1^3\big)\Big) \Big/ \Big(a\lambda\big(a\lambda\mu\sigma + a\lambda\sigma\theta_2 + \alpha\mu\sigma\theta_1 + \alpha\sigma\theta_1\theta_2 - \lambda\mu^2 - 2\lambda\mu\theta_2 - \lambda\theta_2^2 - \mu\sigma\theta_1 - \sigma\theta_1\theta_2\big)\Big)$$

> *NmP := factor(t·N2);*

> *convert(NmP, parfrac, t); # rational expansion of the t·N2(t)*

> *chi_new := factor(residue(NmP, t = lambda_new)); # residue of the t·N2(t) at t=lambda_new*

> $chi\_new := \dfrac{\mu\sigma(a\lambda + \alpha\theta_1 - \theta_1)}{a\lambda\sigma + \alpha\sigma\theta_1 - \lambda\mu - \lambda\theta_2 - \sigma\theta_1}$

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St.Petersburg State University
of Technology and Design,
St.Petersburg State University
of AeroSpace Instrumentation,
St.Petersburg, Russia

Поступило 21 октября 2014 г.

St.Petersburg State University,
St.Petersburg, Russia
E-mail: slav@SS2034.spb.edu