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CONSTRUCTING $SU(2) \times U(1)$ ORBIT SPACE FOR QUTRIT MIXED STATES

ABSTRACT. The orbit space $\mathfrak{P}(\mathbb{R}^8)/G$ of the group

$$G := SU(2) \times U(1) \subset U(3)$$

acting adjointly on the state space $\mathfrak{P}(\mathbb{R}^8)$ of a 3-level quantum system is discussed. The semi-algebraic structure of $\mathfrak{P}(\mathbb{R}^8)/G$ is determined within the Procesi–Schwarz method. Using the integrity basis for the ring of G-invariant polynomials $\mathbb{R}[\mathfrak{P}(\mathbb{R}^8)]^G$, the set of constraints on the Casimir invariants of the group $U(3)$ coming from the positivity requirement for Procesi–Schwarz gradient matrix, $\mathrm{Grad}(z) \geqslant 0$, is analyzed in detail.

§1. INTRODUCTION

Since a very beginning of quantum mechanics, a highly nontrivial interplay between the quantities describing a composite quantum system as a "single whole" and "local characteristics" of its constituents became the subject of intensive studies (holistic v.s. reductionism views). The present note aims to discuss a mathematical aspect of "the whole and the parts" problem in quantum theory considering a model of 3-dimensional quantum system, qutrit. Skipping aside the physical motivation, these mathematical issues can be formulated as follows.

Consider the compact Lie group G acting on a real n-dimensional space V and let $H \subset G$ is its compact subgroup. Assume that the corresponding orbit spaces V/G and V/H admit a realization as semi-algebraic subsets, $Z(V/G)$ and $Z(V/H)$ of $\mathbb{R}^q$ for a certain q. The mathematical version of "the whole and the parts" dilemma can be formulated as the problem of determination of correspondence between sets $Z(V/H)$ and $Z(V/G)$.

In applications to the quantum theory the role of space V plays the space of mixed states of n-dimensional binary quantum system, $\mathfrak{P}(\mathbb{R}^{n^2-1})$.

The groups G and H are associated with the unitary group $U(n)$ and its subgroup, $U(n_1) \times U(n_2) \subset U(n)$, [1] acting in adjoint manner

$$\mathrm{Ad}\,(g)\,\varrho = g \varrho g^{-1} \qquad g \in U(n) \tag{1}$$

on the density matrices $\varrho \in \mathfrak{P}(\mathbb{R}^{n^2-1})$. The action (1) determines the "global orbit space", $\mathfrak{P}(\mathbb{R}^{n^2-1}) \,|U(n)$, and the so-called *entanglement space* $\mathfrak{P}(\mathbb{R}^{n^2-1}) \,|U(n_1) \times U(n_2)$ of a binary $n_1 \times n_2$ system.

The semi-algebraic structure of both orbit spaces admits description in terms of the corresponding ring of G-invariant polynomials, $\mathbb{R}[\mathfrak{P}]^{U(n)}$ and $\mathbb{R}[\mathfrak{P}]^{U(n_1) \times U(n_2)}$. According to the Procesi and Schwarz method [1,2] these semi-algebraic varieties in $\mathbb{R}^q$ are defined by the syzygy ideal for the corresponding integrity basis and the semi-positivity of the so-called gradient matrix, $\mathrm{Grad}(z) \geqslant 0$. As it was discussed recently in [3], the orbit space $\mathfrak{P}(\mathbb{R}^{n^2-1}) \,|U(n)$ representation in terms of the integrity basis for $U(n)$-invariant polynomial ring is completely determined from the physical requirements formulated as the semi-positivity and Hermicity of density matrices. The conditions $\mathrm{Grad}(z) \geqslant 0$ do not bring any new restriction on the elements in the integrity basis for $\mathbb{R}[\mathfrak{P}]^{U(n)}$. In contrast to that case, the algebraic and geometric properties of the entanglement space, are more subtle. It turns that in order to determine the local orbit space $\mathfrak{P}(\mathbb{R}^{n^2-1}) \,|U(n_1) \times U(n_2)$ the additional constraints arising from the semi-positivity of Grad-matrix should be taken into account. Moreover additional inequalities in elements of the integrity basis for $\mathbb{R}[\mathfrak{P}]^{U(n_1) \times U(n_2)}$ provide constraints on the U(n)-invariants. Below, aiming to exemplify this statement the toy model, which mimicry a generic case of a binary composite system will be studied. Namely, we consider the 3-dimensional quantum system, defined by the state space $\mathfrak{P}(\mathbb{R}^8)$, which is a locus in quo of the action of the symmetry group U(3) and its U(2) subgroup $SU(2) \times U(1)$.

§2. Qutrit

• **The qutrit state parametrization** • Consider a quantum 3-level system, named the qutrit. Its state, the semi-positive Hermitian, of trace

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[1] The subgroup H is determined by a fixed decomposition of system onto the n_1- and n_2- dimensional subsystems, such that $n = n_1 \times n_2$.

one, matrix ϱ can be parameterized as follows:

$$\varrho = \frac{1}{3}\left(\mathbb{I}_3 + \sqrt{3}\sum_{a=1}^{8}\xi_a\lambda_a\right).\tag{2}$$

Here the real parameters $\{\xi_a\}_{a=1,\ldots,8}$ are components of the 8-dimensional Bloch vector $\boldsymbol{\xi}$ and $\{\lambda_a\}_{a=1,\ldots,8}$ are the Gell-Mann matrices generating the Hermitian basis of the Lie algebra $\mathfrak{su}(3)$:

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad \lambda_8 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

The product of pairs of Gell-Man matrices involves two basic sets of $\mathfrak{su}(3)$ algebra constants:

$$\lambda_a\lambda_b = \frac{2}{3}\delta_{ab} + (d_{abc} + if_{abc})\lambda_c,\tag{3}$$

where d_{abc} and f_{abc} denote components of the completely symmetric and skew-symmetric symbols defined via the anti-commutators $\{,\}$ and commutators $[,]$ of the Gell-Mann matrices:

$$d_{abc} = \tfrac{1}{4}\text{Tr}(\{\lambda_a,\lambda_b\}\lambda_c), \quad f_{abc} = \tfrac{1}{4}\text{Tr}([\lambda_a,\lambda_b]\lambda_c).$$

The matrix ϱ from (2) represents a physical state of qutrit iff the Bloch vector $\boldsymbol{\xi}$ is subject to the following polynomial constraints: [2]

$$\xi_a\xi_a \leqslant 1,\tag{4}$$

$$0 \leqslant \xi_a\xi_a - \frac{2}{\sqrt{3}}d_{abc}\xi_a\xi_b\xi_c \leqslant \frac{1}{3}.\tag{5}$$

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[2]The inequalities (4) and (5) reflect the semi-positivity of qutrit's density matrices, $\varrho \geqslant 0$.

● **The unitary symmetry of qutrit** ● As it was mentioned above the unitary group U(3) acts on $\mathfrak{P}(\mathbb{R}^8)$ in adjoint manner. The Bloch vector $\boldsymbol{\xi}$ transforms under Ad-action as 8-dimensional vector

$$\xi'_a = O_{ab}\xi_b, \qquad O \in SO(8),$$

with the special 8-parametric subgroup of SO(8). [3]

● **The "local symmetry"** SU(2) × U(1) ● Consider the U(2) subgroup of U(3) identified (up to conjugation) by the conventional embedding:

$$U(2) = \left\{ g(u) = \left(\begin{array}{c|c} u & \\ \hline & (\det u)^{-1} \end{array} \right) \mid u \in U(2) \right\} \subset SU(3). \quad (6)$$

According to the embedding (6) and to the Gell-Mann basis choice, the U(2) subgroup is generated by $\lambda_1, \lambda_2, \lambda_3$ (generators of SU(2) subgroup) and λ_8 (generator of U(1) subgroup). An element of U(2) subgroup can be written as

$$g = \exp(i\lambda_1\alpha)\exp(i\lambda_2\beta)\exp(i\lambda_3\gamma)\exp(i\theta\lambda_8), \qquad (7)$$

where the Euler angles α, β, γ parametrize the SU(2) group and angle θ corresponds to the U(1) subgroup phase, $\det u = \exp(i\frac{2}{\sqrt{3}}\theta)$.

§3. Sketch of the Procesi-Schwarz method

The Classical theory of Invariants represents the cornerstone in description of orbit spaces. Based on this theory (see .e.g. [6]) the basic ingredients of the description can be formulated as follows.

Consider the compact Lie group G acting linearly on the real d-dimensional vector space V. Let $\mathbb{R}[V]^{\mathrm{G}}$ is the corresponding ring of the G-invariant polynomials on V. Assume $\mathcal{P} = (p_1, p_2, \ldots, p_q)$ is a set of homogeneous polynomials that form the integrity basis,

$$\mathbb{R}[x_1, x_2, \ldots, x_d]^{\mathrm{G}} = \mathbb{R}[p_1, p_2, \ldots, p_q].$$

Elements of the integrity basis define the polynomial mapping:

$$p : V \to \mathbb{R}^q; \qquad (x_1, x_2, \ldots, x_d) \to (p_1, p_2, \ldots, p_q). \qquad (8)$$

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[3]More details on the algebraic and geometric structures of the SU(3) group can be found in the classical paper [8].

Since p is constant on the orbits of G, it induces a homeomorphism of the orbit space V/G and the image X of p-mapping; $V/G \simeq X$ [7]. In order to describe X in terms of $\mathcal{P}$ uniquely, it is necessary to take into account the *syzygy ideal* :

$$I_{\mathcal{P}} = \{h \in \mathbb{R}[y_1, y_2, \ldots, y_q] : h(p_1, p_2, \ldots, p_q) = 0, \text{ in } \mathbb{R}[V]\}.$$

Let $Z \subseteq \mathbb{R}^q$ denote the locus of common zeros of all elements of $I_{\mathcal{P}}$. Then Z is algebraic subset of $\mathbb{R}^q$ such that $X \subseteq Z$. Denoting by $\mathbb{R}[Z]$ the restriction of $\mathbb{R}[y_1, y_2, \ldots, y_q]$ to Z one can easily verify that $\mathbb{R}[Z]$ is isomorphic to the quotient $\mathbb{R}[y_1, y_2, \ldots, y_q]/I_{\mathcal{P}}$ and thus $\mathbb{R}[Z] \simeq \mathbb{R}[V]^G$. Therefore the subset Z essentially is determined by $\mathbb{R}[V]^G$, but to describe X the further steps are required. According to [1, 2] the necessary information on X is encoded in the structure of $q \times q$ matrix with elements given by the inner products of gradients, $\mathrm{grad}(p_i)$:

$$||\mathrm{Grad}||_{ij} = (\mathrm{grad}\,(p_i), \mathrm{grad}\,(p_j)). \tag{9}$$

Thus, summarizing all above observations, the orbit space can be identified with the semi-algebraic variety, defined as points, satisfying two conditions:

    a) $z \in Z$, where Z is the surface defined by the syzygy ideal for the integrity basis in $\mathbb{R}[V]^G$;

    b) $\mathrm{Grad}(z) \geqslant 0$.

§4. CONSTRUCTING THE G-INVARIANT POLYNOMIALS

Let $\mathrm{GL}(n, \mathbb{C})$ be the general linear group of degree n over the field $\mathbb{C}$. Assume that $\mathrm{GL}(n, \mathbb{C})$, operates with the polynomials $p(x_1, x_2, \ldots, x_n) \in \mathbb{C}[x_1, x_2, \ldots, x_n]$ as follows:

$$(gp)(x_1, x_2, \ldots, x_n) := p(x_1', x_2', \ldots, x_n'), \quad g \in \mathrm{GL}(n, \mathbb{C}), \tag{10}$$

where

$$x_i' = g_{ij}^{-1} x_j. \tag{11}$$

The polynomials $p(x_1, x_2, \ldots, x_n)$ are called G-invariant if they represent the fixed points of the transformation (10):

$$(gp)(x_1, x_2, \ldots, x_n) := p(x_1, x_2, \ldots, x_n). \tag{12}$$

Here we are concerned with the polynomials in n^2 complex entries of the density matrices $p(\varrho) = p(\varrho_{11}, \varrho_{12}, \ldots, \varrho_{nn})$. To reduce the adjoint action (1) to a linear transformation of the type (11) one can identify the Hermitian density matrix ϱ with the complex vector V of length n^2 and consider

the linear representation of the subgroup $L \subset GL(n, \mathbb{C})$ defined via tensor product of unitary matrix with its complex conjugated one

$$L := \mathrm{U(n)} \otimes \overline{\mathrm{U(n)}}. \tag{13}$$

The invariant polynomials (12) form an algebra over the $\mathbb{C}$, and any such invariant can be expressed as a polynomial of the so-called fundamental invariants, the homogeneous polynomials of fixed degrees. Since the homogeneous invariants of a fixed degree form a vector space, it is sufficient to find a maximal, linearly independent set of homogeneous invariants, i.e., a basis for that vector space. The dimension of this vector space can be extracted from the power series (Poincare series [4]) expansion of the Molien function [5]. In fact, given a compact Lie group G and its representation π, the Molien function can be directly defined by the power series (cf. [5])

$$M_\pi(\mathbb{C}[V]^{\mathrm{G}}, q) = \sum_{k=0}^{\infty} c_k(\pi) q^k. \tag{14}$$

Here $c_k(\pi)$ is the number of linearly independent G-invariant polynomials of degree k on V.

4.1. The Molien function. The Molien function (14) associated to the representation $\pi(g)$ of a compact Lie group G on V admits integral representation [5, 6] (Molien's formula):

$$M_\pi(\mathbb{C}[V]^{\mathrm{G}}, q) = \int\limits_{G} \frac{d\mu(g)}{\det(\mathbb{I} - q\pi(g))}, \quad |q| < 1, \tag{15}$$

where $d\mu(g)$ is the Haar measure for Lie group G. According to the Weyl's Integration Formula [6], an integral over a compact Lie group G can be decomposed into a double integral over a maximal torus T and over the quotient of the group by this torus G/T. If the integrand is a function invariant under conjugation in the group, then the latter integral is "q-independent" and the total integral reduces to an integral over the maximal torus with coordinates x and the additional Weyl factor $A(x)$:

$$M_\pi(\mathbb{C}[V]^{\mathrm{G}}, q) = \int\limits_{T} \frac{d\mu[x] \, A(x)}{\det(\mathbb{I} - q\pi(x))}. \tag{16}$$

The resulting integral is transformed into a complex path integral that can be evaluated by residue theorem.

In what follows we present the Molien functions for the U(3) and its U(2) subgroup acting linearly (13) on complex 9-dimensional space.

•**The Molien function for U(3)** • For the group U(3) the Weyl factor $A(x)$ is squire of Vandermonde determinant calculated for torus coordinates divided by the order of the corresponding Weyl group:

$$A_{\mathrm{SU(3)}}(x_1, x_2, x_3) = \frac{1}{3!} \prod_{i<j}^{3} (x_i - x_j)\overline{(x_i - x_j)},$$

and the Molien function is given by

$$M_{\mathrm{U(3)}}^{(d=9)}(q) = \frac{1}{(1-q)(1-q^2)(1-q^3)} \tag{17}$$

• **The Molien function for SU(2)×U(1)** • For this case $\pi \otimes \bar{\pi}$ representation for maximal torus reads

$$\begin{aligned}
\pi \otimes \bar{\pi} &= (x, x^{-1}, y) \otimes (x^{-1}, x, y^{-1}) \\
&= (1, x^2, xy^{-1}, x^{-2}, 1, x^{-1}y^{-1}, yx^{-1}, xy, 1),
\end{aligned}$$

where x is coordinate on SU(2) group torus and y is coordinate on $U(1)$. The Weyl factor for SU(2) group

$$A_{\mathrm{SU(2)}}(x) := 1 - \frac{x^2 - x^{-2}}{2}$$

implies reduction of the integral in (16) to the double path integral

$$\begin{aligned}
M_{\mathrm{SU(2)}\times\mathrm{U(1)}}^{(d=9)}(q) &= \int \frac{d\,\mu_{\mathrm{SU(2)}} d\,\mu_{\mathrm{U(1)}}}{\det|1 - q\,\pi \otimes \bar{\pi}|} \\
&= \frac{1}{8\pi^2} \frac{1}{(1-q)^3} \oint_{|x|=1} \oint_{|y|=1} \frac{(1-x^2)^2\,xdx\,ydy}{(1-qx^2)(1-qxy)(y-qx)(x-qy)(xy-q)(x^2-q)}.
\end{aligned}$$

Subsequent calculation of the residues of the integrand first with respect to y at poles $P_y = \{qx,\ q/x\}$ and then with respect to x variable at poles $P_x = \{\pm\sqrt{q},\ \pm q\}$ gives finally the rational expression for the Molien function:

$$M_{\mathrm{SU(2)}\times\mathrm{U(1)}}^{(d=9)}(q) = \frac{1}{(1-q)^2(1-q^2)^2(1-q^3)}. \tag{18}$$

4.2. U(3) and SU(2)×U(1)-invariant polynomials. Expressions (17) and (18) for Molien functions indicate that the set fundamental homogeneous polynomials for rings $\mathbb{C}[x]^{\mathrm{SU}(3)}$ consists of three polynomials of degree 1, 2 and 3, while there are five $\mathrm{SU}(2) \times \mathrm{U}(1)$-invariant homogeneous polynomials forming the integrity basis for he ring $\mathbb{C}[x]^{\mathrm{SU}(2)\times\mathrm{U}(1)}$. The latter basis includes one polynomial of degree 1, two polynomials of degree 2 and one polynomial of degree 3.

As the integrity basis for the ring $\mathbb{C}[x]^{\mathrm{SU}(3)}$ can be composed either of the trace invariants $t_k = \mathrm{tr}\left(\varrho^k\right)$, $k = 1, 2, 3$ or of the SU(3) Casimir invariants constructed via correspondence with the elements of the center universal enveloping algebra $\mathfrak{U}(\mathfrak{su}(3))$.

• **Casimir invariants** • Using the Bloch parametrization for the density matrix (2) of qutrit, the quadratic and qubic Casimir invariants are the following polynomials

$$\mathfrak{C}_2 = \xi_i \xi_i \,, \tag{19}$$

$$\mathfrak{C}_3 = \sqrt{3}\, d_{ijk}\, \xi_i \xi_j \xi_k \,, \tag{20}$$

• **SU(2)×U(1)-invariants** • Due to the graded structure of the ring of invariants their constructions reduces to the constructions of the basic homogeneous invariant polynomials. These homogeneous G -invariant polynomials of a given degree are subject to the system of linear homogeneous equations (12). Actually those equations are reduced to the infinitesimal version of the following form [4]

$$e_i f = 0 \,, \quad i = 1, \ldots, m \,,$$
$$g_j f = f \,, \quad i = 1, \ldots, s \,,$$

where $e_1, \ldots, e_m$ form the basis of Borel subgroup $B \subset G$ and $g_1, \ldots, g_s$ is a system of representatives of conjugated classes for the group G with respect to its connected subgroup G^0. Applying this generic scheme one can derive the following set of $\mathrm{SU}(2) \times \mathrm{U}(1)$-invariants:

$$f_1 = \xi_8 \,, \tag{21}$$

$$f_2 = \xi_1^2 + \xi_2^2 + \xi_3^2 \,, \tag{22}$$

$$f_3 = \xi_4^2 + \xi_5^2 + \xi_6^2 + \xi_7^2 \,, \tag{23}$$

$$f_4 = 2\left(-\xi_1\left(\xi_4\xi_6 + \xi_5\xi_7\right) + \xi_2\left(\xi_4\xi_7 - \xi_5\xi_6\right)\right) + \xi_3\left(-\xi_4^2 - \xi_5^2 + \xi_6^2 + \xi_7^2\right). \tag{24}$$

§5. Orbit spaces of qutrit

Before applying the above mentioned method by Processi and Schwarz [1, 2] to the orbit space construction let us reformulate the semi-algebraic description of the qutrit state space $\mathfrak{P}(\mathbb{R}^8)$ in terms of the SU(3) Casimir invariants. In doing so, we mainly follow the ideology presented in [9].

5.1. The global orbit space $\mathfrak{P}(\mathbb{R}^8)/\mathrm{SU}(3)$. Let us start with the semi-algebraic structure of state space $\mathfrak{P}(\mathbb{R}^8)/\mathrm{SU}(3)$.

• **The semi-positivity of density matrix**• The equations (4) and (5) defining the semi-positivity of the qutrit density matrix in terms of the Bloch vector $\boldsymbol{\xi}$ can be rewritten via two SU(3) Casimir invariants $\mathfrak{C}_2$ and $\mathfrak{C}_3$ as follows

$$0 \leqslant \mathfrak{C}_2 \leqslant 1, \tag{25}$$

$$0 \leqslant 3\mathfrak{C}_2 - 2\mathfrak{C}_3 \leqslant 1. \tag{26}$$

• **The Hermicity of density matrix**• The inequalities (25) and (26) should be completed by the reality condition of eigenvalues of the qutrit density matrix. This condition can be also expressed as polynomial inequality in two Casimirs. This inequality is the non-negativity requirement for the discriminant of the characteristic equation $\det(\lambda - \varrho) = 0$ for the qutrit density matrix ϱ:

$$\mathrm{Disc} := \mathfrak{C}_2^3 - \mathfrak{C}_3^2 \geqslant 0. \tag{27}$$

Thus the intersection of the strip determined by the linear inequalities (25) and (26) with the domain (27) defines the image of qutrit state space under the polynomial mapping. This intersection represents the curvilinear triangle ABC on the $(\mathfrak{C}_2, \mathfrak{C}_3)$-plane depicted on the Figure 1.

Now we show the triangle ABC represents the coset space $\mathfrak{P}(\mathbb{R}^8)/\mathrm{SU}(3)$ for the qutrit state space. Indeed, since the determinant of the Procesi–Schwarz $\mathrm{Grad}_{\mathrm{SU}(3)}$-matrix

$$\mathrm{Grad}_{\mathrm{SU}(3)} = \begin{pmatrix} 4\mathfrak{C}_2 & 6\mathfrak{C}_3 \\ 6\mathfrak{C}_3 & 9\mathfrak{C}_2^2 \end{pmatrix} \tag{28}$$

is proportional to the discriminant (27)

$$\det \|\mathrm{Grad}_{\mathrm{SU}(3)}\| = 36(\mathfrak{C}_2^3 - \mathfrak{C}_3^2),$$

the semi-positivity of Grad-matrix, that determines the orbit space $\mathfrak{P}(\mathbb{R}^8)/\mathrm{SU}(3)$ coincides with the Hermicity requirement of the qutrit density matrix.
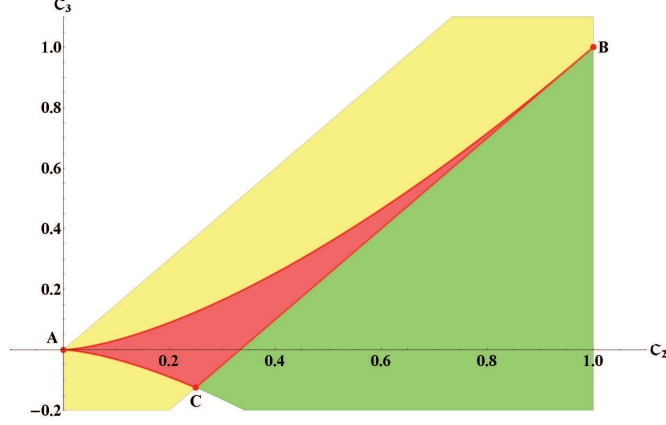
Fig. 1. Triangle ABC as qutrit's global orbit space on the
Casimir's $(\mathfrak{C}_2, \mathfrak{C}_3)$-plane.

5.2. The orbit space $\mathfrak{P}/\mathrm{SU}(2) \times \mathrm{U}(1)$. Let us start with the observation
that the $\mathrm{SU}(3)$ Casimir invariants can be expressed in terms of the four
$\mathrm{SU}(2) \times \mathrm{U}(1)$-invariants (21)–(24) as

$$\mathfrak{C}_2 = f_1^2 + f_2 + f_3 \,, \quad \mathfrak{C}_3 = f_1 \left(f_2 - \frac{1}{2} f_3 \right) - \frac{3\sqrt{3}}{4} f_4 - f_1^3 . \qquad (29)$$

Because we are interested in the projection of orbit space $\mathfrak{P}/\mathrm{SU}(2) \times \mathrm{U}(1)$
to the space $\mathfrak{P}(\mathbb{R}^8)/\mathrm{SU}(3)$, it is constructive to use relations (29) and to
build the integrity basis that contains $\mathfrak{C}_2$ and $\mathfrak{C}_3$ as its elements of the
second and third degree,

$$\mathcal{P}^{\mathrm{SU}(2) \times \mathrm{U}(1)} := \{ f_1, \, f_2, \, \mathfrak{C}_2, \, \mathfrak{C}_3 \} \,.$$

Let us present the 4×4 Grad-matrix for the integrity basis $\{ f_1, f_2, \mathfrak{C}_2, \mathfrak{C}_3 \}$,
in the the block form

$$\mathrm{Grad}_{\mathrm{SU}(2) \times \mathrm{U}(1)} = \begin{pmatrix} \mathcal{A}, & \mathcal{B} \\ \mathcal{B}^T, & \mathcal{D} \end{pmatrix} . \qquad (30)$$
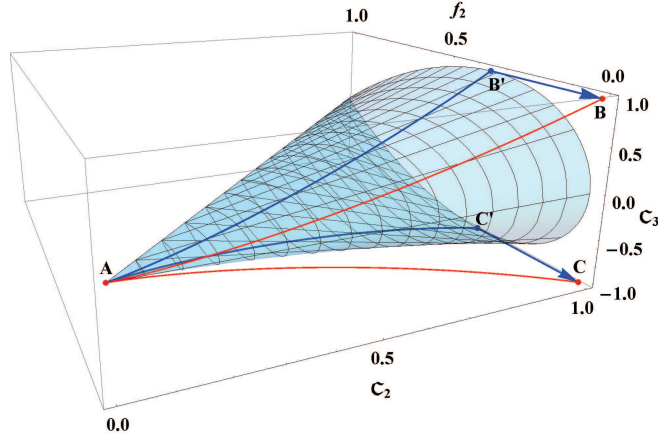
Fig. 2. Domain $\mathrm{Grad}_{\mathrm{SU}(2) \times \mathrm{U}(1)} \geqslant 0$ and its projection to $(\mathfrak{C}_2, \mathfrak{C}_3)$-plane for $f_1 = 0$.

Here $\mathcal{A} := \mathrm{diag}(1, 4f_2)$, is the 2×2 diagonal matrix $\mathcal{D}$ that coincides with the Grad-matrix (28) and

$$\mathcal{B} := \begin{pmatrix} 2f_1, & \frac{3}{2}(3f_2 - f_1^2 - \mathfrak{C}_2) \\ 4f_2, & 3f_1(f_2 + \mathfrak{C}_2) + 2\mathfrak{C}_3 \end{pmatrix}. \tag{31}$$

It is easy to see that the semi-positivity of matrix (30) is reduced to the non-negativity condition of its determinant:

$$\det \|\mathrm{Grad}_{\mathrm{SU}(2) \times \mathrm{U}(1)}\| \geqslant 0 \tag{32}$$

Furthermore, from the expression

$$\begin{aligned}
\det \|\mathrm{Grad}_{\mathrm{SU}(2) \times \mathrm{U}(1)}\| = {}& 4 \left(\mathfrak{C}_2 + 3f_2 - f_1^2 \right) \\
& \times \Big[-9f_1^2 \left(\mathfrak{C}_2^2 + 3f_2^2 \right) - 12\mathfrak{C}_3 f_1 (\mathfrak{C}_2 - 3f_2) + 3f_1^4 (2\mathfrak{C}_2 + 3f_2) \\
& + 27f_2 (\mathfrak{C}_2 - f_2)^2 - 4\mathfrak{C}_3^2 + 4\mathfrak{C}_3 f_1^3 - f_1^6 \Big].
\end{aligned} \tag{33}$$

it follows that domain of the Grad-matrix non-negativity is the 4-dimensional body bounded by two 3-dimensional hypersurfaces that we denote by Σ_+ and Σ_-. The explicit parametrization of $\Sigma_\pm$ can be found by solving

the equation

$$- 9f_1^2 \left(\mathfrak{C}_2^2 + 3f_2^2\right) - 12\mathfrak{C}_3 f_1 \left(\mathfrak{C}_2 - 3f_2 - \frac{1}{3}f_1^2\right)$$
$$+ 3f_1^4(2\mathfrak{C}_2 + 3f_2) + 27f_2(\mathfrak{C}_2 - f_2)^2 - 4\mathfrak{C}_3^2 - f_1^6 = 0 \tag{34}$$

with respect to $\mathfrak{C}_3$. Thereby, the $\Sigma_\pm$ hypersurfaces are given by equations:

$$\mathfrak{C}_3 = \frac{3}{2}\left(f_1(3f_2 - \mathfrak{C}_2) + \frac{f_1^3}{3} \mp \sqrt{3f_2}\left(-\mathfrak{C}_2 + f_2 + f_1^2\right)\right). \tag{35}$$

According to (35), the Σ_+ and Σ_- intersect if

$$\sqrt{3f_2}\left(f_2 + f_1^2 - \mathfrak{C}_2\right) = 0. \tag{36}$$

Thus, $\Sigma_\pm$ hypersurfaces intersect along the following 2-dimensional surfaces Δ_1 and Δ_2:

(1) Δ_1 surface:

$$f_2 = 0, \qquad \mathfrak{C}_3 = \frac{3}{2}f_1\left(\frac{f_1^2}{3} - \mathfrak{C}_2\right), \tag{37}$$

(2) Δ_2 surface:

$$f_2 + f_1^2 - \mathfrak{C}_2 = 0, \qquad \mathfrak{C}_3 = 3f_1\left(\mathfrak{C}_2 - \frac{4}{3}f_1^2\right). \tag{38}$$

To make description of orbit space more transparent, consider its 3-dimensional cross sections for different values of the "local" invariant f_1:

$\bullet$ $\mathfrak{P}/\mathrm{SU}(2) \times \mathrm{U}(1)$ **for** $f_1 = 0$ $\bullet$ The 3-dimensional slice of the "local" orbit space fixed by the local invariant $f_1 = 0$ is drawn on Figure 2. From this picture one can see that the projection of the "cone of semipositivity" of the Grad-matrix to the $(\mathfrak{C}_2, \mathfrak{C}_3)$-plane reproduces exactly the ABC triangle, the orbit space $\mathfrak{P}(\mathbb{R}^8)/\mathrm{SU}(3)$ depicted on Figure 1.

For non-vanishing values of f_1 the attainable area of the Casmir invariants $(\mathfrak{C}_2, \mathfrak{C}_3)$ is shrinking. To illustrate this effect, we give below the corresponding pictures for positive, $f_1 = 2/5$ and negative, $f_1 = -2/5$ values of invariant f_1.

$\bullet$ $\mathfrak{P}/\mathrm{SU}(2) \times \mathrm{U}(1)$ **for** $f_1 = 2/5$ $\bullet$ For this value the "cone of semipositivity" is drawn on the Figure 3. For non-zero values of f_1 the vertex of "cone of semipositivity" intersects the Casmir invariants $(\mathfrak{C}_2, \mathfrak{C}_3)$-plane at the point D that differs from the point A. The line DE is projection of the surface Δ_2 with $f_1 = 2/5$. With growing f_1 the line DE moves towards BC and for $f_1 = 1/2$ it covers the last. To make more vivid illustration
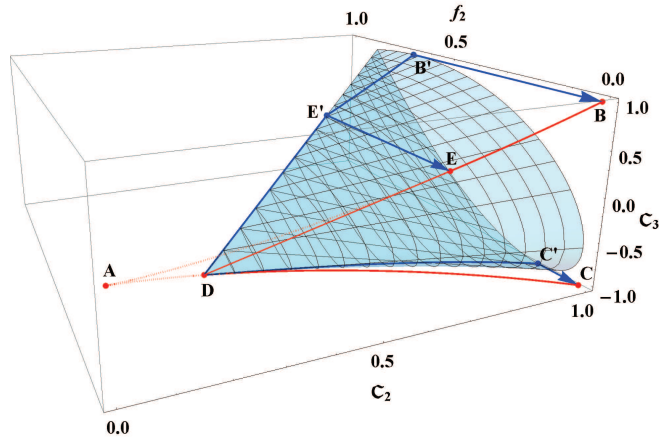
Fig. 3. Domain $\mathrm{Grad}_{\mathrm{SU}(2)\times\mathrm{U}(1)} \geqslant 0$ and its projection to $(\mathfrak{C}_2, \mathfrak{C}_3)$-plane for $f_1 = 2/5$.
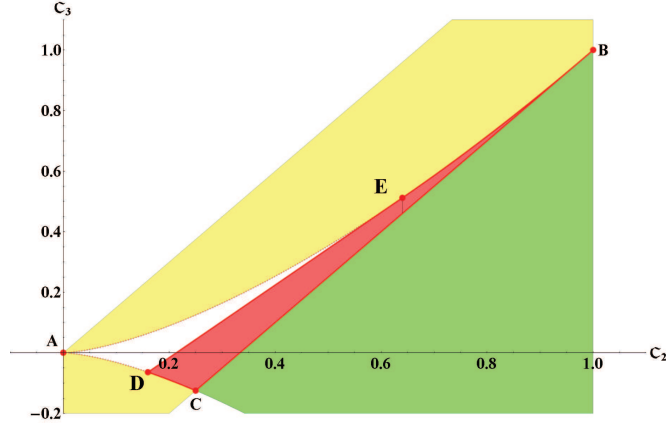

Fig. 4. DCBE is the image of $\mathfrak{P}/\mathrm{SU}(2) \times \mathrm{U}(1)$ on SU(3) orbit space for fixed $f_1 = 2/5$.

the shrinking area of the allowed SU(3) Casimirs invariants is shown on Figure 4.
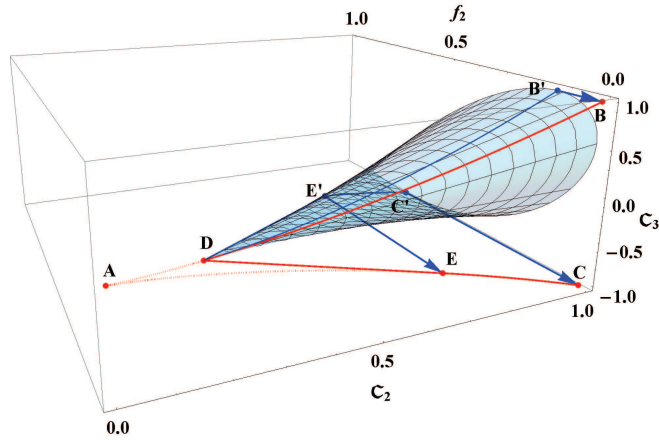
Fig. 5. Domain $\mathrm{Grad}_{\mathrm{SU}(2)\times\mathrm{U}(1)} \geqslant 0$ and its projection to $(\mathfrak{C}_2, \mathfrak{C}_3$-plane for $f_1 = -2/5$.
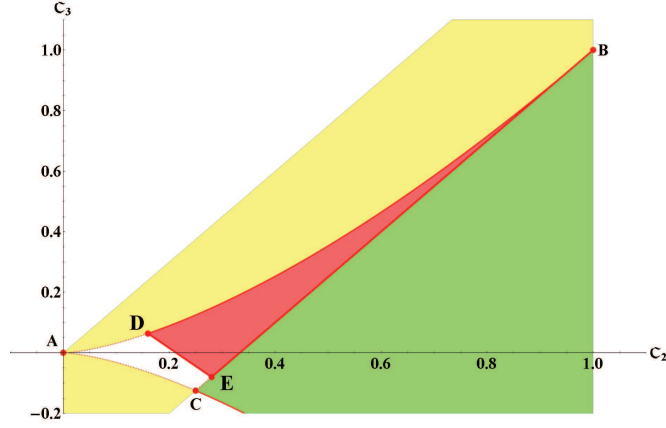

Fig. 6. DCBE is the image of $\mathfrak{P}/\mathrm{SU}(2) \times \mathrm{U}(1)$ on $\mathrm{SU}(3)$ orbit space for $f_1 = -2/5$.

When the "local" invariant f_1 lies in the interval $(0, -1]$, an alternative mechanism of shrinking of the triangle ABC triangle is realized:

$\bullet$ $\mathfrak{P}/\mathrm{SU}(2) \times \mathrm{U}(1)$ **for** $f_1 = -2/5$ $\bullet$ For this case the the "cone of semi-positivity" is depicted on the Figure 5. When f_1 takes negative values the points D and E move toward the point B and all coincide for $f_1 = -1$.

The Figure 6 exemplifies the effect of shrinking of the allowed SU(3) Casimirs invariants domain for negative value $f = -2/5$.

Finally, the 3-dimensional of slices of the orbit space $\mathfrak{P}/\mathrm{SU}(2) \times \mathrm{U}(1)$ for different values of f_1 are presented on the Figure 7.
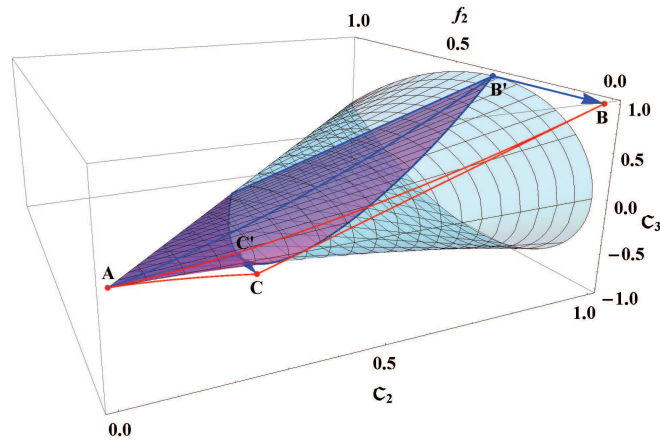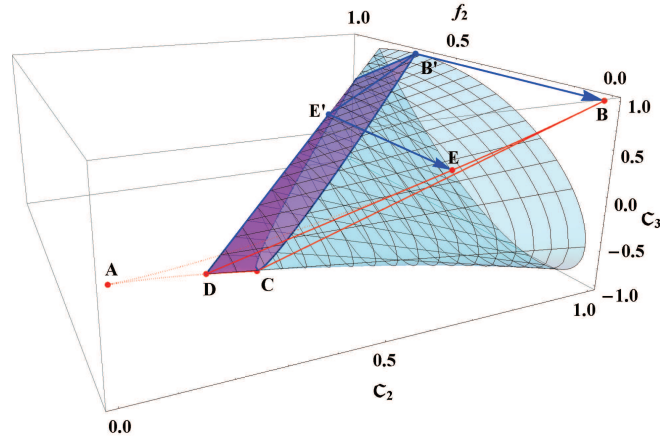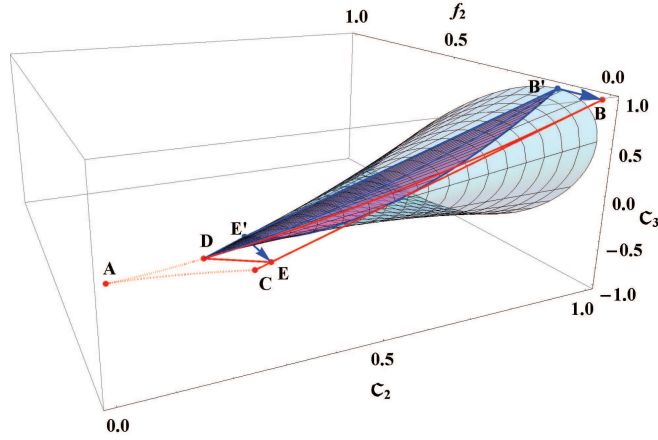
Fig. 7. $\mathfrak{P}/\mathrm{SU}(2) \times \mathrm{U}(1)$ slices for $f_1 = 2/5$ (top), $f_1 = 0$ and $f_1 = -2/5$ (bottom).

§6. Conclusion

In the present note we analyze the $\mathrm{SU}(2){\times}\mathrm{U}(1)$-orbit space of qutrit treating it as simplified analogue of the entanglement space of a composite system. The orbit space is described as a semi-algebraic variety in $\mathbb{R}^4$, defined by a set of polynomial inequalities in $\mathrm{SU}(2){\times}\mathrm{U}(1)$ adjoint invariants. These inequalities follow from the simultaneous semi-positivity of two matrices, the qutrit density matrix and the Procesi–Schwarz Grad-matrix, constructed with the aid of fundamental set of $\mathrm{SU}(2){\times}\,\mathrm{U}(1)$-invariants. It was discussed in details how the semi-positivity of the Grad-matrix for $\mathrm{SU}(2){\times}\,\mathrm{U}(1)$-invariants provides new restrictions on the geometry of orbit space in contrast to the case of the $\mathrm{SU}(3)$ orbit space.

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