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ON BIRATIONAL DARBOUX COORDINATES ON COADJOINT ORBITS OF CLASSICAL COMPLEX LIE GROUPS

Abstract. Any coadjoint orbit of the general linear group can be canonically parameterized using an iteration method, where at each step we turn from the matrix of a transformation A to the matrix of the transformation that is the *projection* of A parallel to an eigenspace of this transformation to a coordinate subspace.

We present a modification of the method applicable to the groups SO(N,$\mathbb{C}$) and Sp(N,$\mathbb{C}$). One step of the iteration consists of two actions, namely, the projection parallel to a subspace of an eigenspace and the simultaneous restriction to a subspace containing a co-eigenspace.

The iteration gives a set of couples of functions p_k, q_k on the orbit such that the symplectic form of the orbit is equal to $\sum_k dp_k \wedge dq_k$. No restrictions on the Jordan form of the matrices forming the orbit are imposed.

A coordinate set of functions is selected in the important case of the absence of nontrivial Jordan blocks corresponding to the zero eigenvalue, which is the case $\dim\ker A = \dim\ker A^2$. This case contains the case of general position, the general diagonalizable case, and many others.

§1. Introduction. Notations. Coorninates on orbits of groups of A-series

I remind the method of the canonical parametrization of (co)adjoint orbit of the general linear group in this section. The method was introduced in [2–4] and suggested by I. M. Gelfand and M. I. Najmark [1]. The method is extended on the matrix groups preserving a bilinear quadratic form in the present paper. The possibility of the extension is based on [1] too, where the triangular decompositions of $SO(N)$ and $Sp(N)$ were applied to the representation theory. A distinguishing feature of the method is

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its insensitivity to the Jordan form of matrices generating the orbit, it is applicable to arbitrary complex orbits.

It must be noted that the assumption $A \in \mathrm{so}(N)$ or $A \in \mathrm{sp}(N)$ puts some restrictions on the Jordan form of A. For example, the rank of any skewsymmetric matrix is even, consequently none of these Lie algebras contains matrices similar to, say, $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

We assume that the Jordan forms of the matrices are compatible with the restrictions prescribed by the Lie algebra under the consideration. It is given that the matrix belongs to the corresponding Lie algebra.

Another difficulty is more serious. In some cases the constructed functions are not independent, consequently they do not form a coordinate set of functions. We need to separate out the independent functions and combine them into the canonical set. This difficulty can happen only in the case if the zero root-space has non-trivial Jordan blocks.

The consideration of the general case lies outside the present work. The classification and birational canonical parametrization of the nilpotent orbits is a subject of a specific paper. We will not construct the coordinates on such orbits here.

The last thing we note is that *the presented formulae are valid in the case of zero eigenvalue yet.* They *give* the birational canonical parametrization of the orbits with the complicated zero root space, *but some orbits will not be parameterised.* These “missed orbits” are some algebraically closed subspaces of the already parameterized orbits. A canonical parametrization of the such subspaces is a subject of a theory of the Hamiltonian systems with constraints.

There are no principal difficulties to calculate the Jordan form of a parameterized orbit, it can be done in the same way as for $\mathrm{gl}(N)$ case. Namely, it is sufficient to determine the maximal ranks of all powers of the constructed matrix $\mathcal{P}_{\mathrm{fin}}$ over all values of the parameters in the explicit formula. It is not difficult because $\mathcal{P}_{\mathrm{fin}}$ is triangular. *The normal Jordan form of the matrix from the orbit coincides with the form of such a J in the normal Jordan form that has the ranks of all powers the same as the maximal ranks of the powers of $\mathcal{P}_{\mathrm{fin}}$.* We will not concentrate on it either.

Before proceeding to the subject it is necessary to introduce basic concepts and notations.

Let V be N-dimensional complex linear space. General linear group $\mathrm{GL}(N)$ acts on V by changes of bases: $(\mathbf{e}) \to (\mathbf{e}F)$, where $(\mathbf{e}) = (e_1, e_2, \ldots, e_N)$, $(\mathbf{e}F) = ((\mathbf{e}F)_1, (\mathbf{e}F)_2, \ldots, (\mathbf{e}F)_N)$:

$$e_i, (\mathbf{e}F)_i \in V, (\mathbf{e}F)_k = \sum_i e_i F_{ik}, F_{ij} \in \mathbb{C}, F \in \mathrm{GL}(N, \mathbb{C}).$$

The algebra $\mathrm{gl}(N, \mathbb{C})$ is the space of all matrices. Non-degenerate pairing $A, B \to \operatorname{tr} AB$ identifies the algebra with its dual, so we do not distinguish the algebra $\mathrm{gl}(N)$ and its dual $\mathrm{gl}^*(N)$, adjoint and coadjoint action of the group. The coadjoint orbits of Lie groups are the classical subject of huge amount of investigations for more than hundred years, see [5–10]. The manifold of all matrices similar to the given one is isomorphic to the coadjoint orbit, they are canonically isomorphic symplectic spaces.

Let us denote the (co)adjoint orbit of some element $J \in \mathrm{gl}(N)$ by $\mathcal{O}(J) := \cup_{F \in \mathrm{GL}(N)} F^{-1} J F$. A symplectic form on the orbit is denoted by $\omega_J : \mathrm{T}\mathcal{O} \times \mathrm{T}\mathcal{O} \to \mathbb{C}$. To introduce the method of the parametrization of the orbit presented in [3, 4] we need more notations.

Notation 1. *Let V be represented as a direct sum of two nonzero subspaces: $V = L \oplus M$. A projection of V along L to M is denoted by*

$$\Pi_M^{\|L} \in \mathrm{Hom}(V, M).$$

Notation 2. *Let V be represented as a direct sum of two nonzero subspaces in two special ways: $L_1 \oplus M = L_2 \oplus M = V$. A linear transformation of V that moves points of L_1 to the points of L_2 parallel to M and leave points of M unchanged is denoted by*

$$\Pi_M^{\|L_1} + \Pi_{L_2}^{\|M} = (\Pi_M^{\|L_2} + \Pi_{L_1}^{\|M})^{-1} \in \mathrm{End}\, V.$$

We denote finite ordered sets of vectors by boldface letters. For example basis $(\mathbf{e})$ can be split on two parts $\mathbf{a}$ and $\mathbf{b}$:

$$(\mathbf{e}) = (e_1, e_2, \ldots, e_N) = (\mathbf{a}, \mathbf{b});\ \mathbf{a} = e_1, e_2, \ldots, e_n;\ \mathbf{b} = e_{n+1}, e_{n+2}, \ldots, e_N.$$

Linear envelopes of the sets of vectors we denote by $\mathscr{L}(\ldots)$, for example $\mathscr{L}(\mathbf{a})$ is the corresponding coordinate subspace.

If the first set $\mathbf{a}$ of the vectors of the basis belongs to L_1 and the other part $\mathbf{b}$ of the basis belongs to M, the transformations $(\Pi_M^{\|L_1} + \Pi_{L_2}^{\|M})^{\pm 1}$

have block-triangle matrices:

$$(\mathbf{a},\mathbf{b})(\Pi_{M}^{\parallel L_1}+\Pi_{L_2}^{\parallel M})^{\pm 1}=(\mathbf{a},\mathbf{b})\begin{pmatrix} \mathrm{I} & 0 \\ \mp Q & \mathrm{I} \end{pmatrix}.$$

Matrix elements of Q form affine coordinates on the algebraically open subset of the Grassmanian of n-dimensional subspaces of V.

Let λ' be an eigenvalue of matrices from the orbit (i.e. an eigenvalue of J). Consider $\ker(A-\lambda'\mathrm{I})\cap\mathrm{im}(A-\lambda'\mathrm{I})^{m-1}:=L$ for any m such that $L\neq 0$. Space L is well defined, by A and $m\in\mathbb{N}$, it is a non-zero subspace of V. We denote a dimension of L by $n=n(\lambda',m)\in\mathbb{N}$, it is the number of Jordan blocks, corresponding to λ', of sizes $m\times m$ and larger. A simple eigenvalue is corresponded to m=1 by this definition.

Let basis $(\mathbf{e})$ of V be split on two parts $(\mathbf{a},\mathbf{b})$. Subset $\mathbf{a}$ consists of $n=n(\lambda',m)$ vectors and subset $\mathbf{b}$ consists of $N-n$ vectors. Consider a new basis $(\mathbf{a}_L,\mathbf{b}):=(\mathbf{a},\mathbf{b})(\Pi_{\mathscr{L}(\mathbf{b})}^{\parallel\mathscr{L}(\mathbf{a})}+\Pi_{L}^{\parallel\mathscr{L}(\mathbf{b})})$. The coordinate subspace of the new basis is a well defined subspace L of the eigenspace of A, consequently in the basis $(\mathbf{a}_L,\mathbf{b})$ transformation $A\in\mathrm{gl}(N)$ has a block-triangular form, and original matrix A in initial basis $(\mathbf{a},\mathbf{b})$, has the following form:

$$A=\begin{pmatrix} \mathrm{I} & 0 \\ Q & \mathrm{I} \end{pmatrix}\begin{pmatrix} \lambda'\mathrm{I} & P \\ 0 & \widetilde{A} \end{pmatrix}\begin{pmatrix} \mathrm{I} & 0 \\ Q & \mathrm{I} \end{pmatrix}^{-1}.$$

This representation takes place if A belongs to some algebraically open subset of the orbit. Functions $P:\mathcal{O}(J)\to\mathbb{C}^{n\times(N-n)}$, $Q:\mathcal{O}(J)\to\mathbb{C}^{(N-n)\times n}$, $\widetilde{A}:\mathcal{O}(J)\to\mathbb{C}^{(N-n)\times(N-n)}$ have been constructed. The following standpoint was presented in [4].

The orbit $\mathcal{O}(J)$ is foliated over Grassmanians. The class of birational trivializations of the foliations was presented. The base that is the Grassmatian was covered by a standard affine maps isomorphic to a linear space, say $H_G=H_G(\lambda',m)$, $\dim H_G=n>0$. A fiber of the foliation has a natural structure of the direct product $H_G^*\times\mathcal{O}(\widetilde{J})$, where $\mathcal{O}(\widetilde{J})$ is an orbit of the smaller dimension $\widetilde{J}=\widetilde{J}(J,\lambda',m)$ and H_G^* is a linear space dual to H_G. The direct product $H_G\times H_G^*$ is a linear space with a natural symplectic structure, consequently we get the covering of the orbit $\mathcal{O}(J)$ by domains. Each domain is a symplectic space $\mathcal{O}(\widetilde{J})\times(H_G\times H_G^*)$ and the symplectic structures of the initial orbit and the covering are naturally coordinated.

In other words, *the linear symplectic space*[1] *has been split off the initial orbit over the algebraically open sets.*

So the map $A \to \widetilde{A}$ is a projection of the open subset of $\mathcal{O}(J)$ onto an orbit $\mathcal{O}(\widetilde{J})$, where a Jordan form of $\widetilde{J}$ is defined by J, λ' and m: the Jordan blocks of J with sizes m and larger become one unit smaller, the other Jordan blocks J and $\widetilde{J}$ coincide. The main results of [3,4] are:

- The map $\mathcal{O}(J) \to \mathcal{O}(\widetilde{J})$, defined over the algebraically open subset of the orbit is *a projection of the trivial symplectic fibration.*
  Its fibre is the linear symplectic space, we denote it by $(\mathcal{E}, \omega_{\mathcal{E}})$.
- The couples P_{ij}, Q_{ji} of the symmetric matrix elements of P, Q are *birational Darboux coordinates on the fibre* $\mathcal{E}$.
- The symplectic form ω_J on the orbit is a sum of the forms on the base $\mathcal{O}(\widetilde{J})$ and the fiber $\mathcal{E}$:

$$\omega_J = \omega_{\widetilde{J}} + \omega_{\mathcal{E}} = \omega_{\widetilde{J}} + \operatorname{tr} dP \wedge dQ.$$

We skip the notions of the pull-backs of projections for short. The process can be iterated to exhaust the dimensions of the target orbits. It gives the Darboux coordinates on the initial orbit $\mathcal{O}(J)$. If we start with the orbit of matrix J with the zero trace we get the parametrization of the orbit of $\mathrm{SL}(N)$ embedded into $\mathrm{sl}(N)$, the orbits of the groups of A-series.

Let us consider a dual pattern that gives us the representation

$$A = \begin{pmatrix} \mathrm{I} & 0 \\ Q^\star & \mathrm{I} \end{pmatrix} \begin{pmatrix} \widetilde{A}^\star & P^\star \\ 0 & \lambda'\mathrm{I} \end{pmatrix} \begin{pmatrix} \mathrm{I} & 0 \\ Q^\star & \mathrm{I} \end{pmatrix}^{-1}.$$

The geometrical interpretation of this formula is following. We split the initial basis $(\mathbf{e}) = (\mathbf{a}^\star, \mathbf{b}^\star)$ in such a way that the number of vectors in $\mathbf{a}^\star$ is a dimension of $L^\star := \mathrm{im}(A - \lambda'\mathrm{I}) + \ker(A - \lambda'\mathrm{I})^{m-1}$. It is evident, that $\dim L^\star + \dim L = N$. A new basis $(\mathbf{a}^\star_{L^\star}, \mathbf{b}^\star)$ in which A becomes a block-triangular matrix $\begin{pmatrix} \widetilde{A}^\star & P^\star \\ 0 & \lambda'\mathrm{I} \end{pmatrix}$ is

$$(\mathbf{a}^\star_{L^\star}, \mathbf{b}^\star) := (\mathbf{a}^\star, \mathbf{b}^\star)(\Pi^{\|\mathscr{L}(\mathbf{a}^\star)}_{\mathscr{L}(\mathbf{b}^\star)} + \Pi^{\|\mathscr{L}(\mathbf{b}^\star)}_{L^\star}).$$

We choose the part $\mathbf{a}^\star_{L^\star}$ of the basis in such a way that its linear envelope $\mathscr{L}(\mathbf{a}^\star_{L^\star})$ contains an image of $A - \lambda'\mathrm{I}$, consequently a matrix of $A - \lambda'\mathrm{I}$ has a lower block zero in this basis. The construction $\mathrm{im}(A - \lambda'\mathrm{I}) + \ker(A - \lambda'\mathrm{I})^{m-1}$ provides the fixed Jordan normal form of $\widetilde{A}^\star$, the same as $\widetilde{A}$.

[1]Hereinafter we denote space $H_G \times H^*_G$ by $\mathcal{E}$.

The construction gives a rational map *from the orbit* $\mathcal{O}$ to $(\mathbb{C}\times\mathbb{C})^{\Sigma_{\mathcal{O}}}$, where $\Sigma_{\mathcal{O}}$ is a total number of all pairs P, Q. To prove that it is birational isomorphism it is sufficient to prove that all functions P, Q are independent.

The Jordan form of A is defined by the numbers $\operatorname{rank}(A-\lambda\mathrm{I})^m$. If these ranks have the same values for all[2] P, Q, the statement will be proved.

The following fact is fundamental for the construction: *the normal Jordan form of the upper-triangular matrix with the scalar diagonal blocks is defined by these blocks only* on the algebraically-open subset of the space of free matrix elements.

It is not difficult to see that our scheme gives the bijection between all normal Jordan forms and the sets of the diagonal blocks. To introduce the isomorphism explicitly let us enumerate the diagonal blocks of the upper-triangular matrix according with their sizes n_k:

$$\lambda'\mathrm{I}_{n_1}, \lambda'\mathrm{I}_{n_2}, \lambda'\mathrm{I}_{n_3}, \ldots, \lambda'\mathrm{I}_{n_{\max-1}}, \lambda'\mathrm{I}_{n_{\max}}, \quad n_k \geqslant n_{k+1},$$

where $\det(A-\lambda'\mathrm{I}) = 0, n_k = n_k(\lambda')$. The minimal value of the diagonal blocks corresponding to λ' is $n_{\max}\times n_{\max}$, we put $n_{\max+1} \stackrel{\text{def}}{=} 0$. The total number of the diagonal blocks corresponding to λ' is denoted by $\max = \max(\lambda')$, I_n is the unit matrix $n\times n$.

The number of the Jordan blocks $m\times m$ corresponding to $\lambda=\lambda'$ is equal to $n_m - n_{m+1}$.

We finish A_n-series now, and turn to the subject of the paper that is B_n, C_n, and D_n-series.

Let us split initial basis $(\mathbf{e})$ on three parts $(\mathbf{a},\mathbf{c},\mathbf{b})$ and make two steps of the process, one after another. The first one with the kernel of $A-\lambda'\mathrm{I}$ and the second one with the image of $\widetilde{A}+\lambda'\mathrm{I}$, constructed using, generally speaking, different eigenvalues λ' and $-\lambda'$, that results:

$$A = \begin{pmatrix} \mathrm{I} & 0 & 0 \\ q_1^{\mathrm{gl}} & \mathrm{I} & 0 \\ q_2^{\mathrm{gl}} & 0 & \mathrm{I} \end{pmatrix} \begin{pmatrix} \lambda'\mathrm{I} & p_1^{\mathrm{gl}} & p_2^{\mathrm{gl}} \\ 0 & \widetilde{A}_{I,I} & \widetilde{A}_{I,II} \\ 0 & \widetilde{A}_{II,I} & \widetilde{A}_{II,II} \end{pmatrix} \begin{pmatrix} \mathrm{I} & 0 & 0 \\ q_1^{\mathrm{gl}} & \mathrm{I} & 0 \\ q_2^{\mathrm{gl}} & 0 & \mathrm{I} \end{pmatrix}^{-1},$$

$$\begin{pmatrix} \widetilde{A}_{I,I} & \widetilde{A}_{I,II} \\ \widetilde{A}_{II,I} & \widetilde{A}_{II,II} \end{pmatrix} := \begin{pmatrix} \mathrm{I} & 0 \\ q_3^{\mathrm{gl}} & \mathrm{I} \end{pmatrix} \begin{pmatrix} \widetilde{\widetilde{A}} & p_3^{\mathrm{gl}} \\ 0 & -\lambda'\mathrm{I} \end{pmatrix} \begin{pmatrix} \mathrm{I} & 0 \\ q_3^{\mathrm{gl}} & \mathrm{I} \end{pmatrix}^{-1},$$

[2]From the algebraically open set.

consequently

$$\omega = \omega_{\stackrel{\approx}{J}} + \operatorname{tr} dp_1^{\mathrm{gl}} \wedge dq_1^{\mathrm{gl}} + \operatorname{tr} dp_2^{\mathrm{gl}} \wedge dq_2^{\mathrm{gl}} + \operatorname{tr} dp_3^{\mathrm{gl}} \wedge dq_3^{\mathrm{gl}}.$$

A fixed matrix similar to $\stackrel{\approx}{A}$ is denoted by $\stackrel{\approx}{J}$.

Taking into account symmetries makes it possible to convert the sum of the 2-forms into the canonical expression. A calculation of the ranks $(A - \lambda' \mathrm{I})^m$ shows that the constructed functions are independent at least if $\ker A = \ker A^2$. We return to the main exposition now.

§2. Linear algebra of complex spaces with scalar product. Isotropic subspaces

The groups of the series B, C and D preserve a non-degenerate bilinear form, a scalar product $\langle \dots, \dots \rangle$. A product is symmetric $\langle \xi, \eta \rangle = \langle \eta, \xi \rangle$ for B, D, and antisymmetric $\langle \xi, \eta \rangle = -\langle \eta, \xi \rangle$ for C.

We consider all cases simultaneously, in this sense we often use the word *orthogonal* in the broad sense of the word, for symplectic scalar product too.

Consider one of the groups from the list and denote it by $\mathfrak{G}$, the corresponding algebra we denote by $\mathfrak{g}$. Let an index n runs symmetrically $n = \pm[N/2], \pm([N/2] - 1), \dots, \pm 1$ and takes the value $n = 0$ for odd N.

Definition 1. *A basis is called* ***standard*** *if its Gram matrix*[3] *g is*

$$\begin{pmatrix} 0 & 0 & \tau \\ 0 & 1 & 0 \\ \tau & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \tau \\ -\tau & 0 \end{pmatrix}, \ \text{or} \ \begin{pmatrix} 0 & \tau \\ \tau & 0 \end{pmatrix},$$

where τ is a square anti-diagonal matrix, consisting of units. It is the matrix of an inversion:

$$\tau = \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 1 & 0 \\ \vdots & \vdots & \iddots & \vdots & \vdots \\ 0 & 1 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \end{pmatrix}.$$

[3] A Gram matrix of the set of vectors $f_1, f_2, \dots$ is a matrix of their pairwise products $g_{ij} := \langle f_i, f_j \rangle$.

For all series B, C, and D, it holds $g^T = g^{-1}$. For the orthogonal groups $g^2 = \mathrm{I}$, for the symplectic groups $g^2 = -\mathrm{I}$.

We treat g, τ as symbols of variable sizes, like unit matrix I. It means that *square matrices* g, τ, I *have the sizes that are necessary for the present situation, for the current formula* in the text.

We consider linear algebra over $\mathbb{C}$, there are non-zero isotropic vectors: $\langle \xi, \xi \rangle = 0$ does *not* imply that $\xi = 0$.

Definition 2. *The space L is called* ***isotropic****, if it consists of isotropic vectors: $\xi \in L \Rightarrow \langle \xi, \xi \rangle = 0$.*

If a standard basis is given, an example of isotropic space is a coordinate subspace enveloping several coordinate vectors with the indices of the same sign.

An *orthogonal complement* $L^\perp$ to the space L is called a set of all vectors orthogonal to all vectors of L:

$$\eta \in L^\perp \Leftrightarrow \langle \eta, \xi \rangle = 0 \ \forall \xi \in L.$$

An orthogonal complement is a subspace. For the non-zero isotropic L, $L \subset L^\perp \neq V$, consequently $L + L^\perp = L^\perp \neq V$. Nevertheless

- $(L^\perp)^\perp = L$,
- $\dim L + \dim L^\perp = \dim V$.

It is evident that a dimension of isotropic subspace is not greater than a half of the dimension of the space N. Consequently two isotropic subspaces in a general position do not intersect.

Proposition 1. *A set of the pairs (E, G) of isotropic spaces of the same dimension $n = \dim E = \dim G$ is an algebraic manifold. It has an algebraically-open subset such that*

$$E^\perp \oplus G = G^\perp \oplus E = V.$$

Let $E^\perp \oplus G = V$. Let us denote $W = E^\perp \cap G^\perp$. It is evident that $\dim W = \dim V - 2n$, consequently

$$V = E \oplus W \oplus G.$$

Proposition 2. *A contraction of the scalar product $\langle \ldots, \ldots \rangle$ on W from V is non-degenerated and has the same type as V has.*

Proof. Let $\xi_0 : \forall \eta \in W \langle \xi_0, \eta \rangle = 0$, then $\xi_0 \in W^\perp$. By the definition $W := E^\perp \cap G^\perp$, consequently $\xi_0 \in E^\perp, \xi_0 \in G^\perp$, and $\xi_0 \in (E + W + G)^\perp = V^\perp = 0$. □

Let us fix the splitting $V = E \oplus W \oplus G$, and let L be an isotropic space of the same dimension as E is. Let these spaces be from such algebraically open sets that $L \oplus W \oplus G = L^{\perp} \oplus G = V$. We define some special orthogonal linear transformation of V, that transforms E to L now[4]. This transformation is a couple of consequent projections: we project E to L along $W \oplus G$ first, and project $L \oplus W$ along G to $L^{\perp}$ after that. Subspace G remaines unchanged.

Let us denote by $\mathcal{Q} \in \text{End } V$ a transformation

$$\mathcal{Q} := (\Pi_{W\oplus G}^{\| E} + \Pi_{L}^{\| W\oplus G}) \circ (\Pi_{G}^{\| L\oplus W} + \Pi_{L^{\perp}}^{\| G}).$$

The first transformation (from the left) is identical on $W \oplus G$, it moves E to L parallel to $W \oplus G$. The second transformation is identical on G, it moves $L \oplus W$ to $L^{\perp}$ parallel to G. We note that $L \subset L^{\perp}$, consequently the second transformationv is identicall on L too: $(\Pi_{G}^{\| L\oplus W} + \Pi_{L^{\perp}}^{\| G})\Big|_{L} = id_L \in \text{End } L$.

Theorem 1. *Transformation* $\mathcal{Q} := (\Pi_{W\oplus G}^{\| E} + \Pi_{L}^{\| W\oplus G}) \circ (\Pi_{G}^{\| L\oplus W} + \Pi_{L^{\perp}}^{\| G})$ *preserves the scalar product, it is orthogonal and unimodular:* $\mathcal{Q} \in \mathfrak{G}$.

Proof. Let us introduce the following notation. Let $\mathbf{a}, \mathbf{c}, \mathbf{b}$ be sets of vectors. Gram matrix of the set $(\mathbf{a}, \mathbf{c}, \mathbf{b})$ we denote as a block-matrix:

$$\begin{pmatrix} \langle \mathbf{a}, \mathbf{a} \rangle & \langle \mathbf{a}, \mathbf{c} \rangle & \langle \mathbf{a}, \mathbf{b} \rangle \\ \langle \mathbf{c}, \mathbf{a} \rangle & \langle \mathbf{c}, \mathbf{c} \rangle & \langle \mathbf{c}, \mathbf{b} \rangle \\ \langle \mathbf{b}, \mathbf{a} \rangle & \langle \mathbf{b}, \mathbf{c} \rangle & \langle \mathbf{b}, \mathbf{b} \rangle \end{pmatrix},$$

where, for example, $\langle \mathbf{a}, \mathbf{c} \rangle$ is a matrix of the pairwise products of the vectors from the sets $\mathbf{a}$ and $\mathbf{c}$: the matrix element $(\langle \mathbf{a}, \mathbf{c} \rangle)_{ij}$ is $\langle a_i, c_j \rangle$.

Consider transformation $\mathcal{Q}$ of the standard basis $(\mathbf{a}, \mathbf{c}, \mathbf{b}) \to (\mathbf{a}, \mathbf{c}, \mathbf{b})\mathcal{Q} =: (\mathbf{a}_L, \mathbf{c}_L, \mathbf{b})$. The corresponding matrix is block-triangular, we denote its blocks by $q, q_{\Box}, \widetilde{q}$:

$$(\mathbf{a}, \mathbf{c}, \mathbf{b}) \begin{pmatrix} \mathrm{I} & 0 & 0 \\ q & \mathrm{I} & 0 \\ q_{\Box} & \widetilde{q} & \mathrm{I} \end{pmatrix} = (\mathbf{a} + \mathbf{c}q + \mathbf{b}q_{\Box}, \mathbf{c} + \mathbf{b}\widetilde{q}, \mathbf{b}) = (\mathbf{a}_L, \mathbf{c}_L, \mathbf{b}).$$

Our aim is to prove that the Gram-matrix does not change. First of all we note that L is isotropic and $\mathbf{a}_L$ belongs to L, consequently $\langle \mathbf{a}_L, \mathbf{a}_L \rangle = 0$.

The second set of the basic vectors $\mathbf{c}_L = \mathbf{c} + \mathbf{b}\widetilde{q}$ compleat $\mathbf{a}_L$ to a basis of $L^{\perp}$, consequently $\mathbf{c}_L \subset L^{\perp}$ and $\langle \mathbf{a}_L, \mathbf{c}_L \rangle = 0$.

[4]It follows from the orthogonality that it transforms $E^{\perp}$ to $L^{\perp}$ too.

Finally the products of vectors from the set $\mathbf{a}+\mathbf{c}q+\mathbf{b}q_{\square}$ on the vectors $\mathbf{b}$ are the same as for the sets $\mathbf{a}$ and $\mathbf{b}$, because $\mathscr{L}(\mathbf{c},\mathbf{b})=\mathscr{L}(\mathbf{b})^{\perp}$. So we check the first line of the Gram-matrix.

Consider the second line. The Gram-matrix is symmetrical or anti-symmetrical, consequently we do not have to check several entries, say $\langle\mathbf{c},\mathbf{a}\rangle,\langle\mathbf{b},\mathbf{a}\rangle$ and $\langle\mathbf{b},\mathbf{c}\rangle$. Consider $\langle\mathbf{c}+\mathbf{b}\widetilde{q},\mathbf{c}+\mathbf{b}\widetilde{q}\rangle$, it keeps its initial value $g=\langle\mathbf{c},\mathbf{c}\rangle$ because only the vectors from the set $\mathbf{b}$ are added to the vectors from the set $\mathbf{c}$, but the added vectors are orthogonal both to the vectors from $\mathbf{b}$, and $\mathbf{c}$: $\mathscr{L}(\mathbf{b})^{\perp}=\mathscr{L}(\mathbf{b},\mathbf{c})$, consequently $\langle\mathbf{c},\mathbf{b}\rangle=0$ and $\langle\mathbf{b},\mathbf{b}\rangle=0$. By the same reason $\langle\mathbf{c}_L,\mathbf{b}\rangle=\langle\mathbf{c}+\mathbf{b}\widetilde{q},\mathbf{b}\rangle=0$.

Formally, one block $\langle\mathbf{b},\mathbf{b}\rangle$ in the third line must be checked, but the vectors from $\mathbf{b}$ have not been changed at all. □

§3. Symmetries of matrices from group $\mathfrak{G}$ and its algebra $\mathfrak{g}$

Group $\mathfrak{G}\ni F$ changes the basis $(\mathbf{a},\mathbf{c},\mathbf{b})\xrightarrow{F}(\mathbf{a},\mathbf{c},\mathbf{b})F,\ F\in\mathfrak{G}$. It keeps the value of the scalar product $\langle\xi,\eta\rangle=\xi^Tg\eta=(F\xi)^TgF\eta$ if and only if

$$F^TgF=g.$$

The differentiation of $F^TgF=g$ gives the condition for $A=\dot{F}F^{-1}\in\mathbb{C}^{N\times N}$ to belong to the algebra $\mathfrak{g}$:

$$A\in\mathfrak{g}\ \Leftrightarrow A^Tg+gA=0.$$

The most important statements have been formulated as three theorems.

Theorem 2. *For any vectors* $\xi,\eta:\ \langle A\xi,\eta\rangle=-\langle\xi,A\eta\rangle$.

Proof.

$$\langle A\xi,\eta\rangle=(A\xi)^Tg\eta=\xi^TA^Tg\eta=-\xi^TgA\eta=-\langle\xi,A\eta\rangle \qquad \square$$

Theorem 3.

$$A^Tg+gA^T=0\Rightarrow\dim\ker(A-\lambda\mathrm{I})^k=\dim\ker(A+\lambda\mathrm{I})^k,\ \forall k,\lambda.$$

Proof.

$$\begin{aligned}\dim\ker(A-\lambda\mathrm{I})^k&=\dim\ker(A^T-\lambda\mathrm{I})^k=\dim\ker(g^{-1}A^Tg-\lambda\mathrm{I})^k\\&=\dim\ker-(A+\lambda\mathrm{I})^k=\dim\ker(A+\lambda\mathrm{I})^k\end{aligned} \qquad \square$$

We see that the eigenvalues of $A \in \mathfrak{g}$ form pairs $\pm\lambda'$, and the structures of their root-spaces coincide. The third important fact is that the orthogonal complement to the eigenspace corresponding to λ' is the co-eigenspace, corresponding to $-\lambda'$:

Theorem 4.

$$\ker^{\perp}(A - \lambda' \mathrm{I}) = \mathrm{im}(A + \lambda' \mathrm{I}).$$

Proof. Let $\xi = (A + \lambda'\mathrm{I})\xi'$, and $A\sigma = \lambda'\sigma$, then $\langle (A + \lambda'\mathrm{I})\xi', \sigma\rangle = \lambda'\langle \xi', \sigma\rangle + \langle A\xi', \sigma\rangle = \lambda'\langle \xi', \sigma\rangle - \langle \xi', A\sigma\rangle = \lambda'\langle \xi', \sigma\rangle - \lambda'\langle \xi', \sigma\rangle = 0$, consequently

$$\mathrm{im}(A + \lambda'\mathrm{I}) \subset \ker(A - \lambda'\mathrm{I})^{\perp}.$$

The structures of the root-spaces of λ' and $-\lambda'$ coincide, consequently the dimensions $\mathrm{im}(A + \lambda'\mathrm{I})$ and $\ker(A - \lambda'\mathrm{I})^{\perp}$ coincide too, that implies

$$\mathrm{im}(A + \lambda'\mathrm{I}) = \ker(A - \lambda'\mathrm{I})^{\perp}, \quad \ker(A - \lambda'\mathrm{I}) = \mathrm{im}(A + \lambda'\mathrm{I})^{\perp} \qquad \square$$

Corollary 1.

$$(\ker(A - \lambda'\mathrm{I}) \cap \mathrm{im}(A - \lambda'\mathrm{I})^{k})^{\perp} = \mathrm{im}(A + \lambda'\mathrm{I}) + \ker(A + \lambda'\mathrm{I})^{k}.$$

Proof. It is evident that $(L \cap M)^{\perp} \supset L^{\perp} + M^{\perp}$, and the dimensions coincide again. $\square$

Proposition 3. *The eigenspaces* $\ker(A - \lambda'\mathrm{I})$ *and* $\ker(A - \lambda''\mathrm{I})$ *are orthogonal if* $\lambda' + \lambda'' \neq 0$.

Proof. Let ξ' and ξ'' be the eigenvectors corresponding to λ' and λ''.

$$\langle A\xi', \xi''\rangle = \lambda'\langle \xi', \xi''\rangle = -\lambda''\langle \xi', \xi''\rangle \Rightarrow (\lambda' + \lambda'')\langle \xi', \xi''\rangle = 0. \qquad \square$$

Corollary 2. *The eigenspaces corresponding to the nonzero eigenvalues are isotropic.*

$\square$

Let us consider zero eigenvalue. From $\ker^{\perp}(A - \lambda'\mathrm{I}) = \mathrm{im}(A + \lambda'\mathrm{I})$, it follows that $\ker^{\perp} A = \mathrm{im}\, A$. We proved the following

Proposition 4. *The kernel of* A *contains the isotropic subspace* $\ker A \cap \mathrm{im}\, A$.

Note that it is zero subspace iff there are no nontrivial Jordan blocks corresponding to $\lambda = 0$.

Let us consider any eigenvalue λ'. All spaces $\ker(A-\lambda'\mathrm{I})\cap\mathrm{im}(A-\lambda'\mathrm{I})^m$ are isotropic for nonzero λ', and if $\lambda'=0$ the spaces are isotropic for $m>1$. If $\lambda'=0$, we take $m>1$ from now.

Let us consider the transformation $\mathcal{Q}$ from Theorem 1. Let the splitting $(\mathbf{a},\mathbf{c},\mathbf{b})$ correspond to the dimension of the space $\ker(A-\lambda'\mathrm{I})\cap\mathrm{im}(A-\lambda'\mathrm{I})^m$, that is the isotropic subspace $L:=\ker(A-\lambda'\mathrm{I})\cap\mathrm{im}(A-\lambda'\mathrm{I})^m$ defining transformation $\mathcal{Q}$. It means that $\dim\mathscr{L}(\mathbf{a})=\dim\mathscr{L}(\mathbf{b})=\dim\ker(A-\lambda'\mathrm{I})\cap\mathrm{im}(A-\lambda'\mathrm{I})^m$. We consider matrix $\mathcal{Q}^{-1}A\mathcal{Q}$ of the transformation A in the basis $(\mathbf{a}_L,\mathbf{c}_L,\mathbf{b})$, that is the result of the action of $\mathcal{Q}=\mathcal{Q}(\lambda',m)$ on the initial basis $(\mathbf{a},\mathbf{c},\mathbf{b})$.

Theorem 5. *Matrix $\mathcal{Q}^{-1}A\mathcal{Q}\in\mathfrak{g}$ is block-triangular:*

$$\mathcal{Q}^{-1}A\mathcal{Q}=\begin{pmatrix}\lambda'\mathrm{I} & \rho & \rho_\square\\ 0 & A_w & \widetilde{\rho}\\ 0 & 0 & -\lambda'\mathrm{I}\end{pmatrix},$$

where $\rho,\rho_\square,\widetilde{\rho}$ are some matrices. Matrix A_w belongs to the algebra[5] *of the same series: ${A_w}^Tg+gA_w=0$, but the size of matrices is smaller.*

Proof. The first set $\mathbf{a}_L$ of the basic vectors belongs to the eigenspace corresponding to λ', consequently the first column is $(\lambda'\mathrm{I},0,0)^T$.

By the definition of $\mathcal{Q}$, the sets $\mathbf{a}_L,\mathbf{c}_L$ of the basic vectors form the basis of $(\ker(A-\lambda'\mathrm{I})\cap\mathrm{im}(A-\lambda'\mathrm{I})^m)^\perp$. By the Corollary 1

$$(\ker(A-\lambda'\mathrm{I})\cap\mathrm{im}(A-\lambda'\mathrm{I})^m)^\perp=\mathrm{im}(A+\lambda'\mathrm{I})+\ker(A+\lambda'\mathrm{I})^m,$$

consequently the envelope of $\mathbf{a}_L,\mathbf{c}_L$ contains the image of $A+\lambda'\mathrm{I}$, so the matrix of the transformation $A+\lambda'\mathrm{I}$ in the basis $\mathbf{a}_L,\mathbf{c}_L,\mathbf{b}$, that is $\mathcal{Q}^{-1}A\mathcal{Q}+\lambda'\mathrm{I}$, has lower $\dim\ker(A-\lambda'\mathrm{I})\cap\mathrm{im}(A-\lambda'\mathrm{I})^m$ lines zero.

The last we have to prove is $A_w^Tg+gA_w=0$. It is a consequence of $\mathcal{Q}\in\mathfrak{G}$ that $\mathcal{Q}A\mathcal{Q}^{-1}\in\mathfrak{g}\Leftrightarrow(\mathcal{Q}A\mathcal{Q}^{-1})^Tg+g\mathcal{Q}A\mathcal{Q}^{-1}=0$.

The situation is similar to the following. The symmetry of any matrix $B:B^T=B$ implies the symmetry of any its square sub-block the position of which is symmetrical with respect to the diagonal. Now we turn back to the algebra $\mathfrak{g}$. It is a consequence of the anti-diagonality of g that from $B^Tg+gB=0$ it follows $B_w^Tg+gB_w=0$ for any square sub-block B_w the position of which is symmetrical with respect to both the diagonal and the anti-diagonal. $\square$

[5]Matrix (symbol) g has "variable size", it is $2\dim\ker(A-\lambda'\mathrm{I})\cap\mathrm{im}(A-\lambda'\mathrm{I})^m>0$ units smaller than g in the previous formulae.

We refolmulate the result taking into account the changing of the Jordan form of any transformation after its restriction on the subspace that contain the co-eigenspace and after the factorization with respect to the subspace of the eigenspace (see [4]).

Let us consider a linear space

$$W := (\mathrm{im}(A+\lambda'\mathrm{I}) + \ker(A+\lambda'\mathrm{I})^m)/(\ker(A-\lambda'\mathrm{I}) \cap \mathrm{im}(A-\lambda'\mathrm{I})^m).$$

To get W *we factorize the space consisting of vectors orthogonal to isotropic L with respect to this L*. The result of such special quotient inherits the scalar product from V. It is not difficult to see that the Gram matrix g of the standard basis of W has the same type as for V but the smaller size. Transformation $A \in \mathfrak{g} \subset \mathrm{End}\, V$ acts naturally on W, we denote this action by A_w.

Theorem 6. *Transformation A_w belongs to algebra $\mathfrak{g}$ of matrices of size $\dim W \times \dim W$. The Jordan normal form of A_w differs from the Jordan form of A by the number of blocks corresponding to $\pm\lambda'$ only. If $\lambda' \neq 0$, the sizes of blocks of the sizes $m \times m$ and larger become one unit smaller. If $\lambda' = 0$, the sizes of the blocks corresponding to the zero eigenvalue of the sizes $m \times m$, $m > 1$ and larger become two units smaller.*

The transformation from the basis $(\mathbf{a}, \mathbf{c}, \mathbf{b})$ to the basis $(\mathbf{a}_L, \mathbf{c}_L, \mathbf{b})$ is performed by the block-triangular matrix $\mathcal{Q}$:

$$(\mathbf{a}, \mathbf{c}, \mathbf{b}) \begin{pmatrix} \mathrm{I} & 0 & 0 \\ q & \mathrm{I} & 0 \\ q_\square & \widetilde{q} & \mathrm{I} \end{pmatrix} = (\mathbf{a}_L, \mathbf{c}_L, \mathbf{b}).$$

It follows from Theorem 1 that the basis $(\mathbf{a}_L, \mathbf{c}_L, \mathbf{b})$ is standard too. It implies that matrix $\mathcal{Q}$ belongs to group $\mathfrak{G}$. Let us determine what it means for the blocks $q, \widetilde{q}, q_\square$, and for the blocks of $\mathcal{Q}^{-1}A\mathcal{Q}$.

§4. Symmetries of block-matrices

Let us introduce an operation of *the conjugation of matrices with respect to the antidiagonal*: $A \to A^{\vdash}$. This conjugation transforms the rows to the columns and conversely too, but the elements preserving their places belong not to the diagonal but to the antidiagonal now[6]:

$$A^{\vdash}_{-i,-j} = A_{-j,-i} \Leftrightarrow A^{\vdash}_{i,j} = A_{-j,-i}.$$

[6]If indices run from 1 to N, then $A^{\vdash}_{N-i+1,N-j+1} = A_{N-j+1,N-i+1} \Leftrightarrow A^{\vdash}_{i,j} = A_{N-j+1,N-i+1}$

This antidiagonal conjugation can be expressed using an inversion and the usual conjugation $A \to A^T$:

$$A^\vdash = (A\tau)^T\tau = \tau A^T\tau,$$

where τ is the matrix of the inversion that is antidiagonal matrix consisting of units.

For the presentation of the symplectic case we need one more concept. It is the operation that reverses the sign of symbols that have the indices of one sign and preserves the objects with the indices of the opposite sign. The corresponding matrix is a diagonal matrix, one half of which is matrix unit I and the other half is minus unit $-$I. In the cases of the orthogonal groups we do not need such an operation and in the uniform presentation we can set this matrix just unit matrix.

To avoid such an extra-notation we note that matrix $g\tau$ has all the necessary properties. *It is the unit matrix for the cases with the symmetrical scalar product, and for the symplectic cases it is a diagonal matrix that has a left-upper half collected from units and a right-lower half collected from minus-units.* We remind that

Note 1. *The matrices g, $g\tau$, g^2 are "adjustable", like τ or like the unit matrix. Their structures are given, but the sizes depend on the context.*

Now we turn to an important condition $A^Tg + gA = 0$, it is equivalent to $A \in \mathfrak{g}$. It can be written as $A^\vdash = -\tau g A \tau g$, where the involution $A \to \tau g A \tau g$ is identical for the orthogonal cases.

Consider the symplectic cases. Let us represent matrix as four blocks of the half dimension. The involution preserves the diagonal blocks and changes signs of the antidiagonal blocks:

$$\tau g \begin{pmatrix} B & C \\ E & D \end{pmatrix} \tau g = \begin{pmatrix} B & -C \\ -E & D \end{pmatrix}.$$

Let us split matrix A on $9 = 3 \times 3$ blocks in such a way that *the pattern is symmetric with respect to the both diagonals.* For the orthogonal groups $A^Tg + gA = 0$ is equivalent to

$$A = \begin{pmatrix} B & \rho & \rho_\Box \\ E & F & -\rho^\vdash \\ H & -E^\vdash & -B^\vdash \end{pmatrix}, \text{ where } \rho_\Box^\vdash = -\rho_\Box, F^\vdash = -F, H^\vdash = -H.$$

For the symplectic cases $\tau g \neq \mathrm{I}$, and the condition $A^T g + gA = 0$ is equivalent to

$$A = \begin{pmatrix} B & \rho & \rho_{\square} \\ E & F & -\tau g \rho^{\vdash} \\ H & -E^{\vdash}\tau g & -B^{\vdash} \end{pmatrix}, \quad \rho_{\square}^{\vdash} = \rho_{\square}, F^{\vdash} = -\tau g F \tau g, H^{\vdash} = H.$$

We can see that the middle block in the last column is defined by ρ and we will write $-\tau g \rho^{\vdash}$ instead of $\widetilde{\rho}$.

Let us consider lower unitriangular matrix $\mathcal{Q}$. It follows from the symmetry $\mathcal{Q}^T g \mathcal{Q} = g$, that we can change the block $\widetilde{q}$ for $-q^{\vdash}\tau g$ too:

$$\begin{pmatrix} \mathrm{I} & 0 & 0 \\ q & \mathrm{I} & 0 \\ q_{\square} & -q^{\vdash}\tau g & \mathrm{I} \end{pmatrix}.$$

An angle unit $q_{\square}$ does not have a simple symmetry, it is neither symmetrical nor antisymmetrical with respect to the antidiagonal conjugation here. Nevertheless, matrix $q_{\square}$ has a following property:

$$q_{\square} = q_{\boxtimes} - \frac{1}{2} q^{\vdash}\tau g q,$$

where $q_{\boxtimes} = \mp q_{\boxtimes}^{\vdash}$ has the same symmetry as $\rho_{\square}$ has, and the summand with the opposite symmetry in comparison with $\rho_{\square}$ and $q_{\boxtimes}$, symmetry is defined by q, it is $\frac{1}{2} q^{\vdash}\tau g q$.

Summand $-\frac{1}{2} q^{\vdash}\tau g q$ is *the square matrix with the opposite symmetry with respect to the antidiagonal conjugation in comparison with* $\rho_{\square}$ *and* $q_{\boxtimes}$. Matrix $q^{\vdash}\tau g q$ is symmetrical for the orthogonal groups and antisimmetrical for the symplectic groups because $(\tau g)^{\vdash} = \pm \tau g$.

- For the orthogonal groups

$$q_{\boxtimes} := \frac{1}{2}(q_{\square} - q_{\square}^{\vdash}), \quad \frac{1}{2} q^{\vdash}\tau g q = \frac{1}{2} q^{\vdash} q = \frac{1}{2}(q^{\vdash} q)^{\vdash}.$$

- For the symplectic groups

$$q_{\boxtimes} := \frac{1}{2}(q_{\square} + q_{\square}^{\vdash}), \quad \frac{1}{2} q^{\vdash}\tau g q = -\frac{1}{2}(q^{\vdash}\tau g q)^{\vdash}.$$

We can write it uniformly:

$$q_{\boxtimes} = \left(q_{\square} - (g^2) q_{\square}^{\vdash}\right)/2, \quad \left(q^{\vdash}\tau g q\right)^{\vdash} = (g^2)\left(q^{\vdash}\tau g q\right),$$

because $g^2 = \pm \mathrm{I}$. We put g^2 into the brackets to emphasise that τg and (g^2) have the different sizes in the formula, it is just the sign.

An arbitrary[7] element from the orbit $\mathcal{O}^{\mathfrak{g}}$ is represented as:

$$A = \begin{pmatrix} \mathrm{I} & 0 & 0 \\ q & \mathrm{I} & 0 \\ q_\square & -q^\vdash \tau g & \mathrm{I} \end{pmatrix} \begin{pmatrix} \lambda' \mathrm{I} & \rho & \rho_\square \\ 0 & A_w & -\tau g \rho^\vdash \\ 0 & 0 & -\lambda' \mathrm{I} \end{pmatrix} \begin{pmatrix} \mathrm{I} & 0 & 0 \\ * & \mathrm{I} & 0 \\ * & * & \mathrm{I} \end{pmatrix}^{-1}.$$

§5. Canonical coordinates on orbits of series B, C and D

Let us consider the element $A \in \mathfrak{g}$ as an element of the orbit of $\mathrm{GL}(N)$ in $\mathrm{gl}(N)$, where we know the Darboux coordinates (see [3,4]). We represent the single transformation $\mathcal{Q} := (\Pi_G^{\| L\oplus W} + \Pi_{L^\perp}^{\| G}) \circ (\Pi_{W\oplus G}^{\| E} + \Pi_L^{\| W\oplus G})$ from $\mathfrak{G}$ as a couple of the sequential transformations $\Pi_{W\oplus G}^{\| E} + \Pi_L^{\| W\oplus G}$ and $\Pi_G^{\| L\oplus W} + \Pi_{L^\perp}^{\| G}$ each of them from $\mathrm{SL}(N)$.

Let us denote the canonical coordinates corresponding to the first step by $p_1^{\mathrm{gl}}, q_1^{\mathrm{gl}}, p_2^{\mathrm{gl}}, q_2^{\mathrm{gl}}$, and the coordinates corresponding to the second step by $p_3^{\mathrm{gl}}, q_3^{\mathrm{gl}}$:

$$\begin{pmatrix} \mathrm{I} & 0 & 0 \\ q_1^{\mathrm{gl}} & \mathrm{I} & 0 \\ q_2^{\mathrm{gl}} & 0 & \mathrm{I} \end{pmatrix} \begin{pmatrix} \lambda' \mathrm{I} & p_1^{\mathrm{gl}} & p_2^{\mathrm{gl}} \\ 0 & \widetilde{A}_{I,I} & \widetilde{A}_{I,II} \\ 0 & \widetilde{A}_{II,I} & \widetilde{A}_{II,II} \end{pmatrix} \begin{pmatrix} \mathrm{I} & 0 & 0 \\ q_1^{\mathrm{gl}} & \mathrm{I} & 0 \\ q_2^{\mathrm{gl}} & 0 & \mathrm{I} \end{pmatrix}^{-1},$$

$$\begin{pmatrix} \widetilde{A}_{I,I} & \widetilde{A}_{I,II} \\ \widetilde{A}_{II,I} & \widetilde{A}_{II,II} \end{pmatrix} := \begin{pmatrix} \mathrm{I} & 0 \\ q_3^{\mathrm{gl}} & \mathrm{I} \end{pmatrix} \begin{pmatrix} A_w & p_3^{\mathrm{gl}} \\ 0 & -\lambda' \mathrm{I} \end{pmatrix} \begin{pmatrix} \mathrm{I} & 0 \\ q_3^{\mathrm{gl}} & \mathrm{I} \end{pmatrix}^{-1},$$

consequently

$$\rho = p_1^{\mathrm{gl}} + p_2^{\mathrm{gl}} q_3^{\mathrm{gl}}, \rho_\square = p_2^{\mathrm{gl}}, q = q_1^{\mathrm{gl}}, q_\square = q_2^{\mathrm{gl}}, -\tau g \rho^\vdash = p_3^{\mathrm{gl}}, -q^\vdash \tau g = q_3^{\mathrm{gl}},$$

and $p_1^{\mathrm{gl}} = \rho + \rho_\square q^\vdash \tau g$. We can calculate the increment of the symplectic form in going from space V to space W:

$$\begin{aligned} &\mathrm{tr}\, dp_1^{\mathrm{gl}} \wedge dq_1^{\mathrm{gl}} + \mathrm{tr}\, dp_2^{\mathrm{gl}} \wedge dq_2^{\mathrm{gl}} + \mathrm{tr}\, dp_3^{\mathrm{gl}} \wedge dq_3^{\mathrm{gl}} \\ &\qquad = 2\, \mathrm{tr}\, d\rho \wedge dq + \mathrm{tr}\, d\rho_\square \wedge dq_\square + \mathrm{tr}\, d\rho_\square q^\vdash \tau g \wedge dq. \end{aligned}$$

We use the equality

$$\mathrm{tr}\, d\rho_\square \wedge (q_\square + (g^2) q_\square^\vdash)/2 = \mathrm{tr}\, d\rho_\square \wedge \frac{1}{2} q^\vdash \tau g q = 0,$$

[7]The element from the algebraically open set of matrices having Gaussian expansion on the product of the upper- and lower- triangular factors.

to replace $q_\square$ with its (anti)symmetrical part $q_\boxtimes := (q_\square - (g^2)q_\square^\vdash)/2$. It is correct because the trace of the product of symmetrical and antisymmetrical matrices is equal to zero.

Matrix $q_\boxtimes = (q_\square - (g^2)q_\square^\vdash)/2$ is (anti)symmetrical, and we can calculate the trace $\operatorname{tr} d\rho_\square \wedge dq_\square$ using only one half of the pairs of matrix elements $\rho_\square$ and $q_\square - (g^2)q_\square^\vdash = 2q_\boxtimes$, it is just the values that we need along with q and ρ, for the renewal (reconstruction) of the initial matrix. They form the coordinate set of functions.

To formulate the main result of the present article we introduce a couple of rectangular matrices P and Q now.

Matrix P consists of two blocks, one is rectangular and the other block is square. The rectangular block $2\rho + \rho_\square q^\vdash \tau g$ is $n \times (N - 2n)$. The square block adjoined to the right side of the rectangular one is $\rho_\square$, its size is $n \times n$.

Matrix Q consists of the blocks symmetrical to the blocks of P. The upper block is q, its size is $(N - 2n) \times n$. The square block adjoint to the lower side of the rectangular one is $q_\square - (g^2)q_\square^\vdash - \eth$, where $\eth = \operatorname{adiag} q_\square$ is the antidiagonal matrix[8]:

$$P = \begin{pmatrix} 2\rho + \rho_\square q^\vdash \tau g & , \rho_\square \end{pmatrix}, Q = \begin{pmatrix} q \\ q_\square - (g^2)q_\square^\vdash - \eth \end{pmatrix}.$$

The version of the present formulae for P and Q for the orthogonal groups[9]:

$$P = \begin{pmatrix} 2\rho + \rho_\square q^\vdash & , \rho_\square \end{pmatrix}, Q = \begin{pmatrix} q \\ q_\square - q_\square^\vdash \end{pmatrix}.$$

The version of the formulae for P and Q for the symplectic groups:

$$P = \begin{pmatrix} 2\rho + \rho_\square q^\vdash \begin{pmatrix} \mathrm{I} & 0 \\ 0 & -\mathrm{I} \end{pmatrix} & , \rho_\square \end{pmatrix}, Q = \begin{pmatrix} q \\ q_\square + q_\square^\vdash - \eth \end{pmatrix}.$$

Note 2. *About the antidiagonal term* $\eth$.

Let us illustrate the subtraction of the antidiagonal $\eth$. We consider 2×2 case, the square blocks only, just for the illustration:

$$\rho_\square = \begin{pmatrix} a & b \\ c & a \end{pmatrix}, q_\square = \begin{pmatrix} d & e \\ g & f \end{pmatrix}, q_\boxtimes = \frac{1}{2}(q_\square + q_\square^\vdash) = \begin{pmatrix} (h+f)/2 & e \\ g & (h+f)/2 \end{pmatrix}.$$

[8]The subtraction of $\eth$ divides the antidiagonal of the lower square block of Q by two, in the symplectic case.

[9]We do not subtract $\eth$ because in the orthogonal cases we do not use the antidiagonal elements.

The goal is to introduce variables $p_1, q_1, p_2, q_2, p_3, q_3$ in such a way that the value tr $d\rho_\square \wedge dq_\boxtimes$ takes the canonical form $dp_1 \wedge dq_1 + dp_2 \wedge dq_2 + dp_3 \wedge dq_3$. The direct calculation gives:

$$d\rho_\square \wedge dq_\boxtimes = da \wedge d(h+f) + db \wedge dg + dc \wedge de.$$

We have introduced

$$P = \rho_\square,\ \eth = \text{adiag}\ q_\square = \begin{pmatrix} 0 & e \\ g & 0 \end{pmatrix},\ Q = q_\square + q_\square^\vdash - \eth = \begin{pmatrix} h+f & e \\ g & h+f \end{pmatrix},$$

and have claimed that the canonical coordinates are the pairs $(P_{-1,-1}, Q_{-1,-1}) = (a, h+f)$, $(P_{-1,1}, Q_{1,-1}) = (b, g)$, $(P_{1,-1}, Q_{-1,1}) = (c, e)$.

We have proved that any matrix A from the algebraically open subset of the of the orbit $\mathcal{O}^{\mathfrak{g}}(J)$ can be written as $\mathcal{Q}_{fin}\mathcal{P}_{fin}\mathcal{Q}_{fin}^{-1}$, where[10] $\mathcal{P}_{fin} = \mathcal{P}_{fin}(P_{ij}^k, Q_{ij}^k)$ and $\mathcal{Q}_{fin} = \mathcal{Q}_{fin}(Q_{ij}^k)$ are matrices constructed during the iteration process. We enumerate the steps of the process by $k:\ k = 1, 2, \ldots$

We have constructed the map $\mathcal{O}^{\mathfrak{g}} \to (\mathbb{C}\times\mathbb{C})^{\Sigma_\mathcal{O}} \ni \cup_{k,i,j}(P_{ij}^k, Q_{ij}^k)$, where $\Sigma_\mathcal{O}$ is a number of all pairs of the constructed functions.

Note 3. *Matrices from the algebraically-closed sets may be out of the orbit. As an example we put $P_{ij}^k = 0\ \forall i, j, k$. We get diagonalizable matrices, they can not be from a non-diagonalizable orbit.*

Nevertheless, the Jordan form is constant on the algebraically open subset of $(\mathbb{C}\times\mathbb{C})^{\Sigma_\mathcal{O}}$. The image of the inverse map belongs to some orbit. Unfortunately, *in some special cases it is not the orbit from which we started.* Sometimes the functions constructed by the orbit are not independent. Additional symmetries on P_{ij}^k, Q_{ij}^k must be put on the construction of the map *the orbit.* The loss of simplicity happens, if the root-space of the zero eigenvalue has a complicated Jordan structure.

Theorem 7 (Concluding theorem)**.** *Canonical coordinates on the algebraically-open domain of the orbit $\mathcal{O}^{\mathfrak{g}}(J)$, where* $\dim\ker J = \dim\ker J^2$*, are the symmetrical pairs of elements of matrices P and Q, namely P_{ij} and Q_{ji}, where*

- *the indices i, j of the coordinate pairs* ***for the orthogonal groups*** *satisfy an inequality $i + j < 0$,*

[10]Matrix $\mathcal{Q}_{fin}$ is collected just from Q_{ji}^k. Any matrix element of $\mathcal{P}_{fin}$ is linear with respect to P_{ij}^k's and the coefficients of these linear functions are the products of several matrix elements Q_{ji}^k, linear with respect to each Q_{ji}^k.

- *the indices i, j of the coordinate pairs* ***for the symplectic groups*** *satisfy an inequality $i + j \leqslant 0$.*

Proof. We have proved that the symplectic form has a canonical form in the constructed functions P_{ij}^k, Q_{ji}^k. What remains is to prove that for the algebraically-open subset of the space of the parameters P_{ij}^k, Q_{ji}^k, formula $A = \mathcal{Q}_{fin}\mathcal{P}_{fin}\mathcal{Q}_{fin}^{-1}$ gives a matrix with the preassigned orbit.

It is not difficult to see that the Jordan form of any upper-triangular matrix with fixed scalar diagonal blocks is constant on the algebraically-open subset of the space of free matrix elements. So, we have to prove the independence of the functions P_{ij}^k, Q_{ji}^k only.

Let us make the iteration process *using pairs of non-zero eigenvalues only.* At the end we get a zero-dimensional orbit corresponding to the zero eigenvalue if it exists, it does not produce the functions P_{ij}, Q_{ji}.

The independence of the constructed functions follows from the variation of the Jordan form of the upper-triangular matrix $\mathcal{P}_{fin}$ during the iteration. To control the Jordan form it is useful to take into account:

- If matrices from the algebra $\mathfrak{g}$ are similar to each other (i.e. they have the same Jordan structure), they belong to the same orbit $\mathcal{O}^{\mathfrak{g}}$. It means that the conjugating matrix can be taken from the corresponding group $\mathfrak{G}$.
- The matrices have the same Jordan form iff their functions $\varkappa$ of two variables λ and m coincide: $\varkappa(\lambda, m|A) = \operatorname{rank}(A - \lambda \mathrm{I})^m$.
- The variation of the Jordan form during the iteration for the eigenvalues $\pm\lambda' \neq 0$ is the same as considered in [4]. □

5.1. Examples. In the first example we demonstrate that the matrix elements of P, Q can be dependent in the nilpotent case.

Example 1. *The set of functions P_{ij}, Q_{ji} constructed on the orbit $\mathcal{O}(J)$, where*

$$J = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \in \mathrm{so}(5)$$

is not independent. From the identity $J^2=0$ *follows that* $\rho=(\rho_{-2,0},\rho_{-1,0})=0$*, consequently*

$$P=\begin{pmatrix} (\rho_\Box)_{-2,1}q_{0,-1} & (\rho_\Box)_{-2,1} & * \\ -(\rho_\Box)_{-2,1}q_{0,-2} & * & * \end{pmatrix},$$

$$Q=\begin{pmatrix} q_{0,-2} & q_{0,-1} \\ (q_\Box)_{1,-2}-(q_\Box)_{2,-1} & * \\ * & * \end{pmatrix},$$

so the values P_{ij}, Q_{ji} *are dependent:*

$$P_{-2,0}Q_{0,-2}=-P_{-1,0}Q_{0,-1}=P_{-2,1}Q_{0,-1}Q_{0,-2}.$$

Nevertheless, it is not difficult to construct a canonical coordinate set $(p_1^{so},q_1^{so}),(p_2^{so},q_2^{so})$ using these P_{ij},Q_{ji}:

$$p_1^{so}=P_{-2,0}, q_1^{so}=2Q_{0,-2}; \quad p_2^{so}=P_{-2,1}, q_2^{so}=Q_{-2,1}-Q_{0,-1}Q_{0,-2}.$$

We do not consider such cases in this paper. As a regular example of the theory let us calculate the canonical coordinates and check their canonicity in the simplest case when the term $\rho_\Box q^\vdash \tau g$ does not vanish. It is the case $n=1$, nonzero matrices $\rho_\Box, q_\boxtimes$ have the size 1×1 that implies the symmetric property. Matrix g is antisymmetrical matrix 2×2:

$$g=\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad g\tau=\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We set $A_w=0$ for short, because 2×2 case is not interesting, it coincides with sl$(2,\mathbb{C})$-case, any $B\in$ sl$(2,\mathbb{C})$ satisfies $B^Tg+gB=0$.

We parameterize an orbit $\mathcal{O}^{sp}(J)$ of the symplectic group Sp$(4,\mathbb{C})$, where $J=\mathrm{diag}(\lambda',0,0,-\lambda')\neq 0$:

$$A=\begin{pmatrix} 1 & 0 & 0 & 0 \\ q_1 & 1 & 0 & 0 \\ q_2 & 0 & 1 & 0 \\ q_\Box & q_2 & -q_1 & 1 \end{pmatrix}\begin{pmatrix} \lambda' & \rho_1 & \rho_2 & \rho_\Box \\ 0 & 0 & 0 & \rho_2 \\ 0 & 0 & 0 & -\rho_1 \\ 0 & 0 & 0 & -\lambda' \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ q_1 & 1 & 0 & 0 \\ q_2 & 0 & 1 & 0 \\ q_\Box & q_2 & -q_1 & 1 \end{pmatrix}^{-1}.$$

The offered coordinates on the orbit are

$$(p_1^{sp},p_2^{sp},p_0^{sp})=(2\rho_1-\rho_\Box q_2, 2\rho_2+\rho_\Box q_1,\rho_\Box),$$

and $(q_1^{sp},q_2^{sp},q_0^{sp})=(q_1,q_2,q_\Box)$, we will skip the subindex "*sp*" for the q-coordinates below.

Note that the coordinate with the subindex zero came from "the antidiagonal" of the square 1×1 matrix, there is no sum like $q_\Box+q_\Box^\vdash$ here. They are just matrix elements $(\mathcal{Q}^{-1}A\mathcal{Q})_{-2,2}$ and $(\mathcal{Q})_{2,-2}$.

We immerse an orbit $\mathcal{O}^{\mathrm{sp}}(J) \ni A$ of the symplectic group to the "moore roomy" orbit of the general linear group $\mathcal{O}^{\mathrm{sl}}(J)$, where the canonical parametrization $p_i, q_i,\ i = 0, 1 \dots 4$ is known:

$$A_v = \begin{pmatrix} 1 & 0 & 0 & 0 \\ q_1 & 1 & 0 & 0 \\ q_2 & 0 & 1 & 0 \\ q_0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \lambda' & p_1 & p_2 & p_0 \\ 0 & & & \\ 0 & & \widetilde{A} & \\ 0 & & & \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ q_1 & 1 & 0 & 0 \\ q_2 & 0 & 1 & 0 \\ q_0 & 0 & 0 & 1 \end{pmatrix}^{-1},$$

$$\begin{pmatrix} & & \\ & \widetilde{A} & \\ & & \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ q_3 & q_4 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & p_3 \\ 0 & 0 & p_4 \\ 0 & 0 & -\lambda' \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ q_3 & q_4 & 1 \end{pmatrix}^{-1}.$$

Equating of the matrix elements, particularly $q_0 = q_\Box$, gives

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ q_1 & 1 & 0 & 0 \\ q_2 & 0 & 1 & 0 \\ q_\Box & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \lambda' & (\rho_1, \rho_2, \rho_\Box) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ q_2 & -q_1 & 1 \end{pmatrix}^{-1} \\ 0 & \widetilde{A} \end{pmatrix} \begin{pmatrix} * \end{pmatrix}^{-1},$$

$$\widetilde{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ q_3 & q_4 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & p_3 \\ 0 & 0 & p_4 \\ 0 & 0 & -\lambda' \end{pmatrix} \begin{pmatrix} * \end{pmatrix}^{-1}$$
$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ q_2 & -q_1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & \rho_2 \\ 0 & 0 & -\rho_1 \\ 0 & 0 & -\lambda' \end{pmatrix} \begin{pmatrix} * \end{pmatrix}^{-1},$$

consequently,

$$p_1 = \rho_1 - q_2\rho_\Box = \frac{1}{2}(p_1^{sp} - p_0^{sp} q_2^{sp}), p_2\rho_2 + q_1\rho_\Box = \frac{1}{2}(p_2^{sp} + p_0^{sp} q_1^{sp}),$$

$$p_3 = \widetilde{\rho}_1 = \rho_2 = \frac{1}{2}(p_2^{sp} - p_0^{sp} q_1^{sp}), p_4 = \widetilde{\rho}_2 = -\rho_1 = \frac{1}{2}(p_1^{sp} + p_0^{sp} q_2^{sp}),$$

$$q_3 = q_2 = q_2^{sp}, q_1 = -q_4 = q_1^{sp}, p_0 = \rho_\Box = p_0^{sp}, q_0 = q_\Box = q_0^{sp}.$$

One can see that

$$\sum_{i=0}^{4} dp_i \wedge dq_i = \sum_{j=0}^{2} dp_i^{sp} \wedge dq_i^{sp}.$$

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