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ON PINSKER FACTORS FOR ROKHLIN ENTROPY

Abstract. In this paper we prove that any dynamical system has a unique maximal factor of zero Rokhlin entropy, the so-called Pinsker factor. It is also proven that if the system is ergodic and this factor has no atoms, then the system is a relatively weakly mixing extension of its Pinsker factor.

§1. Introduction

A few years ago Lewis Bowen [1] introduced a new invariant for actions of sofic groups – *sofic entropy*, and this led to great progress in the problem of classification of Bernoulli shifts up to measure conjugacy and even (for some class of groups) up to orbit equivalence. Sofic groups form a very large class of countable groups, but it is still an open question whether all the countable groups are sofic or not.

So, sofic entropy, potentially, does not work for some groups. There is, however, a very natural alternative: the minimal entropy of a generating partition, the so-called *Rokhlin entropy*. The name comes from the result of Rokhlin stating that the Kolmogorov entropy for aperiodic transformations equals the infimum of the entropies of generating partitions [7]. The fact that this is an invariant is obvious from the definition. It follows from [1] that the Rokhlin entropy equals the entropy of the base space for Bernoulli shifts over sofic groups. It is well known that the sofic entropy is bounded from above by the Rokhlin entropy (see [1, 5]). Seward [8] proved a generalization for Rokhlin entropy of Krieger's and Denker's theorems.

In the realm of Kolmogorov entropy, it is well known that every system has a unique maximal factor of zero entropy, the so-called *Pinsker factor*, and it is known that the system itself is a relatively weakly mixing extension of its Pinsker factor (see [4]). In this paper, we will transfer these

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classical results into the new setting of Rokhlin entropy. Namely, we will prove the following theorem.

Theorem 1. *Every dynamical system contains a Pinsker subalgebra with respect to Rokhlin entropy, that is, a unique maximal subalgebra among subalgebras with zero Rokhlin entropy.*

After that we will use the Furstenberg–Zimmer structure theory to obtain the following result which resembles a classical theorem on Kolmogorov entropy ([4]).

Theorem 2. *Assume that the Pinsker factor of an ergodic dynamical system is nonatomic. Then the initial system is a realively weakly mixing extension of its Pinsker factor.*

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§2. Facts from ergodic theory and descriptive set theory

Our intention here is to remind some facts and notions from ergodic theory we will use in proofs.

A *standard probability space*, or a *Lebesgue space*, is a measurable space arising from any Borel probability measure on a standard Borel space. Such spaces have a lot of convenient properties (see [6, 4]). Consider a standard probability space $\mathbf{X} = (X, \mathscr{X}, \mu)$ (where X is a set, $\mathscr{X}$ is a σ-algebra, and μ is a measure). A σ-subalgebra $\mathscr{A} \subset \mathscr{X}$ is said to be complete if for any $A \in \mathscr{A}$ and $B \in \mathscr{X}$ such that $\mu(A \Delta B) = 0$ (here Δ denotes the symmetric difference operation) we have $B \in \mathscr{A}$. Obviously, any σ-subalgebra has the *completion*. Let $\pi : \mathbf{X} \to \mathbf{Y}$ be a measure-preserving map between standard probability spaces $\mathbf{X} = (X, \mathscr{X}, \mu)$ and $\mathbf{Y} = (Y, \mathscr{Y}, \nu)$. In this situation, the pair $(\mathbf{Y}, \pi)$ is called a *factor*, π is called *a factor map*, and $\mathbf{Y}$ is called a factor space. For any factor there is a correspondent σ-subalgebra: consider the completion $\mathscr{Y}'$ of the σ-subalgebra $\{\pi^{-1}(A) | A \in \mathscr{Y}\}$. Also, for every complete σ-subalgebra in $\mathscr{X}$ there is a factor such that the correspondent σ-subalgebra is exactly the given one. Thus there is a natural correspondence between σ-subalgebras and factors. For more details, see [4].

A *dynamical system*, or a G-*space*, is a pair $(\mathbf{X}, T)$ consisting of a standard probability space $\mathbf{X} = (X, \mathscr{X}, \mu)$ and an action of a countable group G on $\mathbf{X}$ by measure-preserving transformations. Two dynamical systems $(\mathbf{X}, T)$ and $(\mathbf{Y}, S)$ with the same acting group are said to be *isomorphic*, or *conjugate*, if there is a measure-preserving isomorphism $\varphi : \mathbf{X} \to \mathbf{Y}$ that is equivariant with respect to the actions: $\varphi(T^g(x)) = S^g(\varphi(x))$ for every g and for almost every x. A *factor* of a dynamical system $(\mathbf{X}, T)$ is a pair consisting of a system $(\mathbf{Y}, S)$ and an equivariant measure-preserving map $\pi : \mathbf{X} \to \mathbf{Y}$. An *extension* of a dynamical system $(\mathbf{Y}, S)$ is a pair consisting of a system $(\mathbf{X}, T)$ and an equivariant measure-preserving map $\pi : \mathbf{X} \to \mathbf{Y}$. Two extensioms $((\mathbf{X}_1, T_1), \pi_1)$ and $((\mathbf{X}_2, T_2), \pi_2)$ are said to be isomorphic if there is an equivariant measure-preserving a.e. bijection $\psi : \mathbf{X}_1 \to \mathbf{X}_2$ such that $\pi_1 = \pi_2 \circ \psi$.

A σ-subalgebra $\mathscr{A}$ is called an invariant subalgebra if for every set $A \in \mathscr{A}$ we have $T^g(A) \in \mathscr{A}$. It is not hard to see that on the correspondent factor space a factor action S of the group G can be defined. Thus there is a natural correspondence between factors and invariant σ-subalgebras (see [4]).

Let $\mathbf{X} = (X, \mathscr{X}, \mu)$ be a standard probability space. A *partition* α is a countable or finite collection of disjoint measurable subsets of $\mathbf{X}$ covering the whole space. A partition α is said to be measurable with respect to a σ-subalgebra $\mathscr{A}$ if all its elements belong to $\mathscr{A}$. Consider a dynamical system $(\mathbf{X}, T)$. We will say that a σ-subagebra is generated by a partition if it is the smallest complete invariant σ-subalgebra with respect to which the partition is measurable. A partition is said to be *generating* if it generates the σ-algebra $\mathscr{X}$.

Let $(\mathbf{X}, T)$, $(\mathbf{Y}, S)$ be two dynamical systems, and let $\pi : \mathbf{X} \to \mathbf{Y}$ be a factor map. For any partition α of $\mathbf{X}$ measurable with respect to the σ-subalgebra correspondent to the factor $(\mathbf{Y}, \pi)$, there is a unique, up to a set of zero measure, partition β of $\mathbf{Y}$ such that $\alpha = \pi^{-1}(\alpha)$.

Lemma 1. *Let $(\mathbf{X}, T)$ (with $\mathbf{X} = (X, \mathscr{X}, \mu)$) be a dynamical system, let $\mathscr{A}$ be an invariant σ-subalgebra, and let α be an $\mathscr{A}$-measurable partition of $\mathbf{X}$. Let β be a partition of $\mathbf{Y}$ such that $\alpha = \pi^{-1}(\beta)$. Let $(\mathbf{Y}, S)$ be the correspondent factor action, and let $\pi : \mathbf{X} \to \mathbf{Y}$ be the factor map. The following assertions are equivalent:*

1. *The partition α generates the σ-subalgebra $\mathscr{A}$.*

2. *The partition* β *generates the* σ*-algebra* $\mathscr{Y}$.
3. *The partition* β *separates points of* $\mathbf{Y}$*, that is, there exists a conull set* $Y' \subset Y$ *such that for any* $y_1, y_2 \in Y'$, $y_1 \neq y_2$*, there is an element* $g \in G$ *such that* $S^g(y_1)$ *and* $S^g(y_2)$ *lie in different parts of* β.

§3. Basic facts on Rokhlin entropy

In what follows, all the σ-subalgebras are complete.

Let $\mathbf{X} = (X, \mathscr{X}, \mu)$ be a standard probability space, and let $(\mathbf{X}, T)$ be a dynamical system. The Shannon entropy of a parition $\alpha = \{A_1, A_2, \ldots\}$ is given by the formula $H(\alpha) = -\sum_{i=0} \mu(A_i) \log(\mu(A_i))$ with the usual convention $0 \log 0 = 0$. The Rokhlin entropy of an invariant σ-subalgebra $\mathscr{A}$ is the infimum of the Shannon entropies of generating partitions measurable with respect to this subalgebra. We will denote it by $h(\mathscr{A})$. The Rokhlin entropy $h(\mathbf{X}, T)$ of a dynamical system is defined as the Rokhlin entropy of the σ-algebra $\mathscr{X}$. We will say that a σ-algebra $\mathscr{A}$ is generated by a set $\{\mathscr{A}_i\}$ of σ-algebras, and denote it by $\mathscr{A} = \bigvee \mathscr{A}_i$, if $\mathscr{A}$ is the minimal σ-algebra containing these subalgebras. In this situation we will also say that $\mathscr{A}$ is the join of the σ-algebras $\{\mathscr{A}_i\}$.

Lemma 2. *Let* $\{\mathscr{A}_1, \mathscr{A}_2, \ldots\}$ *be a countable or finite set of invariant* σ*-subalgebras and* $\mathscr{A}$ *be their join. Then* $h(\mathscr{A}) \leqslant \sum h(\mathscr{A}_i)$.

Proof. The claim is obviously true if the sum of the entropies is infinite. Otherwise take any $\varepsilon > 0$. By definition, for any i there is a generating partition α_i for $\mathscr{A}_i$ with Shannon entropy smaller than $h(\mathscr{A}_i) + \varepsilon/2^i$. By the completeness of the space of partitions with respect to the Rokhlin metric (see [7]), the join $\bigvee_i \alpha_i$ exists and $H(\bigvee_i \alpha_i) \leqslant \sum_i H(\alpha_i) < \sum_i h(\mathscr{A}_i) + \varepsilon$. This is, obviously, a generating partition for $\mathscr{A}$. □

Lemma 3. *Let* I *be a linearly ordered set and* $\{\mathscr{A}_i\}_{i \in I}$ *be a monotone sequence of* σ*-subalgebras, that is,* $\mathscr{A}_i \subseteq \mathscr{A}_j$ *for* $i \leqslant j$*. Then there is a countable subset* J *in* I *such that* $\bigvee_{i \in I} \mathscr{A}_i = \bigvee_{j \in J} \mathscr{A}_i$.

Proof. The set $\mathscr{X}$ endowed with the metric $d(A, B) = \mu(A \Delta B)$ is a complete separable semimetric space. It is not hard to see that the join of a monotone sequence of σ-subalgebras corresponds to the closure of the union of the correspondent subsets in the semimetric space $(\mathscr{X}, d)$. Now we can refine a countable subsequence with the same closure of the union, since our space is a separable semimetric space. □

Proof of Theorem 1. Consider the set of all invariant σ-subalgebras of zero Rokhlin entropy ordered by inclusion. It follows from the two previous lemmas and Zorn's lemma that there is at least one maximal zero-entropy algebra. It is easy to see that this algebra is unique, since two maximal subalgebras can be joined, and their join will differ from both of them and have zero entropy, leading to a contradiction. $\square$

§4. The Pinsker factor and relative weak mixing

Let $(\mathbf{X}, T)$ (with $\mathbf{X} = (X, \mathscr{X}, \mu)$) be a G-space and $(\mathbf{Y}, R)$ (with $\mathbf{Y} = (Y, \mathscr{Y}, \nu)$) be a factor of $(\mathbf{X}, T)$. The system $(\mathbf{X}, T)$ is called relatively weakly mixing over $(\mathbf{Y}, S)$ if its relatively independent joining with any ergodic system over the common factor $(\mathbf{Y}, S)$ is ergodic.

Let $(\mathbf{Y}, R)$ be a G-space. Consider a metric compact space (Z, d) with a Borel probability measure η that is invariant under the action of isometries, and let $(g, y) \mapsto S^{g,y}$ (where $y \in \mathbf{Y}$ and $g \in G$) be a measurable family of isometries such that $S^{g,R^h(y)} \circ S^{h,y} = S^{gh,y}$ for any $g, h \in G$ and almost every $y \in Y$, and $S^{e,y} = \mathrm{id}$ for almost every y (where e is the group identity and id is the identity map). Then we can define an action of the group G on the set $Y \times Z$ endowed with the product measure by the formula $Q^g : (y, z) \mapsto (R^g(y), S^{g,y}(x))$. Obviously, this defines a dynamical system which is naturally an extension of $(\mathbf{Y}, R)$. Any extension of $(\mathbf{Y}, R)$ isomorphic to an extension obtained in this way is called an *isometric extension.* The following theorem is the famous Furstenberg–Zimmer dichotomy (see [9, 3, 4]).

Theorem 3 (Furstenberg, Zimmer)**.** *Assume that we have an ergodic extension of an ergodic dynamical system. Then exactly one of the following assertions holds:*

1. *The extension is weakly mixing.*
2. *There is an intermediate isometric extension.*

In order to prove Theorem 2, we will need the following lemma.

Lemma 4. *Let $(\mathbf{X}, T)$ be an ergodic isometric extension of an ergodic system $(\mathbf{Y}, S)$ such that the space $\mathbf{Y}$ has no atoms. Then $h(\mathbf{X}, T) \leqslant h(\mathbf{Y}, S)$.*

Proof. Let $h(\mathbf{Y}) < \infty$, since otherwise the assertion is obvious. Let $\mathbf{Y} = (Y, \mathscr{Y}, \nu)$ and $\mathbf{X} = (X, \mathscr{X}, \mu)$. Without loss of generality we may assume that $X = Y \times Z$ where (Z, d) is a metric compact space. Take $\varepsilon > 0$. Let β be a generating partition for $(\mathbf{Y}, S)$ such that $H(\beta) < \varepsilon/2$. Take β' to be the correspondent partition of $\mathbf{X}$. By Lemma 1, there is a conull set $X' \subset X$ such that for any two points $(y_1, z_1), (y_2, z_2) \in X'$ with $y_1 \neq y_2$ there is an element $g \in G$ such that $T^g(y_1, z_1)$ and $T^g(y_2, z_2)$ lie in different parts of β'. Since $\mathbf{Y}$ has no atoms and is a standard probability space, we have a sequence $\{C_i\}$ of positive-measure subsets in X such that $\nu(C_i) \to 0$. Take any point z_0 from the support of the measure on the set Z. Obviously, any ball $B_r(z_0)$ of positive radius has a positive measure. Let $\{A_i\}$ be the sequence of measurable subsets in X of the form $A_i = C_i \times B_{1/i}(z_0) \subset X \times Z$. Consider the sequence of paritions $\alpha_i = \{A_i, X \setminus A_i\}$. It is easy to see that $H(\alpha_i) \to 0$. Take a subsequence $n_i \to \infty$ such that $\sum_i H(\alpha_{n_i}) < \varepsilon/2$. Then the partition $\beta' \vee \bigvee_i \alpha_{n_i}$ dynamically separates points from a conull subset of $\mathbf{X}$. Thus it is a generating partition, and its entropy is smaller than $h(\mathbf{Y}, S) + \varepsilon$. We are done, since ε can be taken arbitrarily small. □

Proof of Theorem 2. Assume to the contrary that the Pinsker factor has an intermediate isometric extension. By the previous lemma, this extension also has zero entropy, a contradiction. □

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