Записки научных
семинаров ПОМИ
Том 430, 2014 г.

N. Gordeev, U. Rehmann

INTERSECTION AND INCIDENCE DISTANCES BETWEEN PARABOLIC SUBGROUPS OF A REDUCTIVE GROUP

Abstract. Let Γ be a reductive algebraic group and let $P, Q \subset \Gamma$ be a pair of parabolic subgroups. We consider here some properties of intersection and incident distances

$$\mathrm{d_{in}}(P,Q) = \max\{\dim P, \dim Q\} - \dim(P\cap Q),$$
$$\mathrm{d_{inc}}(P,Q) = \min\{\dim P, \dim Q\} - \dim(P\cap Q)$$

(if P, Q are Borel subgroups, both numbers coincide with the Tits distance $\mathrm{dist}(P,Q)$ in the building $\Delta(\Gamma)$ of all parabolic subgroups of Γ). In particular, if $\Gamma = \mathrm{GL}(V)$ and $P = P_v, Q = P_u$ are stabilizers in $\mathrm{GL}(V)$ of linear subspaces $v, u \subset V$ we obtain the formula

$$\mathrm{d_{in}}(P,Q) = -d^2 + a_1 d + a_2$$

where $d = \mathrm{d_{in}}(v,u) = \max\{\dim v, \dim u\} - \dim(v\cap u)$ is the intersection distance between the subspaces v, u, and where a_1, a_2 are integers expressed in terms of $\dim V, \dim v, \dim u$.

Introduction

Let $\mathfrak{X}$ be an algebraic variety and let $X, Y \subset \mathfrak{X}$ be subvarieties (all varieties are always supposed to be irreducible). For X, Y let us define the *intersection distance* between X, Y by the formula

$$\mathrm{d_{in}}(X,Y) = \max\{\dim X, \dim Y\} - \dim X\cap Y.$$

It is easy to see the properties of "distance" for $\mathrm{d_{in}}(\ ,\ )$:

1) $\mathrm{d_{in}}(X,Y) \geqslant 0$ and $\mathrm{d_{in}}(X,Y) = 0 \Leftrightarrow X = Y$.
2) $\mathrm{d_{in}}(X,Y) = \mathrm{d_{in}}(Y,X)$.
3) If $X\cap Y\cap Z \neq \emptyset$ and $\dim X\cap Z \geqslant \dim X\cap Y + \dim Y\cap Z - \dim Y$ then $\mathrm{d_{in}}(X,Z) \leqslant \mathrm{d_{in}}(X,Y) + \mathrm{d_{in}}(Y,Z)$.

(Indeed,

$$\max\{\dim X, \dim Y\} + \max\{\dim Y, \dim Z\} \geqslant \dim Y + \max\{\dim X, \dim Z\}.$$

Key words and phrases: parabolic subgroups, Tits distance, Schubert cells.

Hence

$$\max\{\dim X, \dim Z\} - \dim X \cap Z \leqslant \big(\max\{\dim X, \dim Y\} - \dim X \cap Y\big) + \big(\max\{\dim Y, \dim Z\} - \dim(Y \cap Z)\big).$$

4) $\mathrm{d_{in}}(g(X), g(Y)) = \mathrm{d_{in}}(X, Y)$ for any automorphism $g \in \mathrm{Aut}(\mathfrak{X})$.

5) If $\dim X \leqslant \dim Y$ and $X \cap Y \neq \emptyset$ then

$$\dim Y - \dim X \leqslant \mathrm{d_{in}}(X, Y) \leqslant \min\{\dim Y, \dim \mathfrak{X} - \dim X\}.$$

Remark. If $\mathfrak{X}$ is a connected algebraic group and X, Y, Z are closed connected subgroups, then condition of 3) always holds (in particular, if $\mathfrak{X}$ is a linear space and X, Y, Z are linear subspaces).

We may also define the *incidence distance*

$$\mathrm{d_{inc}}(X, Y) = \min\{\dim X, \dim Y\} - \dim X \cap Y = \mathrm{d_{in}}(X, Y) - |\dim X - \dim Y|.$$

For the incidence distance we obviously have properties 2) and 4). Also, instead of 1), 3), 5) we have

1°) $\mathrm{d_{inc}}(X, Y) \geqslant 0$ and $\mathrm{d_{inc}}(X, Y) = 0 \Leftrightarrow X \subseteq Y$ or $X \supseteq Y$.

3°) If X, Y, Z satisfy the condition in 3) and if $\dim X \geqslant \dim Y \geqslant \dim Z$ then

$$\mathrm{d_{inc}}(X, Z) \leqslant \mathrm{d_{inc}}(X, Y) + \mathrm{d_{inc}}(Y, Z).$$

5°) If $\dim X \leqslant \dim Y$ and $X \cap Y \neq \emptyset$ then

$$0 \leqslant \mathrm{d_{inc}}(X, Y) \leqslant \min\{\dim X, \dim \mathfrak{X} - \dim Y\}.$$

Example 0.1. Let V be a linear space over a field K and let $v, u \subset V$ be linear subspaces where $\dim v = k \leqslant \dim u \leqslant l$. Put $r_0 = r_0(k, l) = \min\{l, n - k\}$, $s_0 = s_0(k, l) = \min\{k, n - l\}$. Then

$$l - k \leqslant \mathrm{d_{in}}(v, u) = l - \dim v \cap u \leqslant r_0, \quad 0 \leqslant \mathrm{d_{inc}}(v, u) = k - \dim v \cap u \leqslant s_0$$

and

$$\mathrm{d_{in}}(v, u) = r_0 \Leftrightarrow \mathrm{d_{inc}}(v, u) = s_0 \Leftrightarrow \begin{cases} v \cap u = \{0\} \\ \quad\text{or} \\ v + u = V. \end{cases}$$

Thus, there are exactly $s_0 + 1$ possible values of $\mathrm{d_{in}}(v, u)$ (respectively, of $\mathrm{d_{inc}}(v, u)$) and r_0 (respectively, s_0) is the maximal value of intersection distance $\mathrm{d_{in}}(v, u)$ (respectively, incidence distance $\mathrm{d_{inc}}(v, u)$) which is reached if and only if the planes v, u are in general position.

Now let $P_v, P_u \leqslant \Gamma = \mathrm{GL}(V)$ be the stabilizers of the planes v, u. Then there are exactly $s_0 + 1$ double cosets $P_u \gamma_i P_v$ and the numeration of representatives γ_i such that

$$\mathrm{d_{in}}(g(v), u) = l - k + i \Leftrightarrow \mathrm{d_{inc}}(g(v), u) = i \Leftrightarrow g \in P_u \gamma_i P_v$$

(see [6], Proposition 1.1).

This example shows that distances between linear subspaces parametrize double cosets of their stabilizers which are maximal parabolic subgroups of GL. In this paper we write out the quadratic polynomial which expresses $\mathrm{d_{in}}(P_v, P_u)$ through the $\mathrm{d_{in}}(v, u)$ (Theorem 4.1). Also, we consider some properties of distances between parabolic subgroups of a reductive algebraic group Γ. In particular, we calculate $\mathrm{d_{in}}(P, Q), \mathrm{d_{inc}}(P, Q)$ for a pair of parabolic subgroups $P, Q \leqslant \Gamma$ using the lengths of some special elements of the Weyl group of Γ which are associated with the pair of parabolic subgroups (Proposition 1.1). Using the formulas for distances we calculate the differences $\mathrm{d_{in}}(P, Q) - \mathrm{dist}(P, Q), \mathrm{d_{inc}}(P, Q) - \mathrm{dist}(P, Q)$ where $\mathrm{dist}(P, Q)$ is the Tits distance ([7], p.4) in the building consisting of all parabolic subgroups (Corollary 1.2). We also consider the interpretation of distances between parabolic subgroups as dimensions of corresponding Schubert cells (Proposition 2.1) and consider the distances between parabolic subgroups in general position (Proposition 3.1).

Notation and terminology. By an algebraic variety we mean a quasi-projective variety (irreducible) over a field K; $\dim \varnothing := -\infty$.

For a subset $X \subset \mathfrak{X}$ of an algebraic variety $\mathfrak{X}$ by $\overline{X}$ we denote the closure of X in $\mathfrak{X}$ with respect to Zariski topology.

Let G be a group and $H \leqslant G$ be a subgroup. We denote by G/H the set of left cosets $\{g_\alpha H\}$ and by $G \backslash H$ the set of right cosets $\{H g_\beta\}$.

§1. Distances between parabolic subgroups of a reductive algebraic group

Now let Γ be a reductive algebraic group which is defined and split over a field K. Let Π be a fixed simple root system corresponding to Γ and let $R = \langle \Pi \rangle$ be the corresponding root system, let R^+, R^- be the sets of positive and negative roots and let $W = W(R)$ be the Weyl group. If $w \in W$ by $\dot{w}$ we denote any preimage of w in Γ (cf. [4], 2.5) and by $l(w)$

we denote the length of w with respect to Π. Further, let B be the Borel subgroup that corresponds to Π and let $P \supset B$ be a parabolic subgroup containing B. Then for $w \in W$ the group $\dot{w}P\dot{w}^{-1}$ does not depend on the choice of the preimage $\dot{w}$ of w and therefore we may and we will denote the group $\dot{w}P\dot{w}^{-1}$ by the symbol $w(P)$.

First of all, we consider the connection of the intersection distance $\mathrm{d_{in}}(P,Q)$ between parabolic subgroups $P, Q \subset \Gamma$ and the Tits distance $\mathrm{dist}(P,Q)$. Namely, let $\Delta(\Gamma)$ be the set of all parabolic subgroups of Γ. Then $\Delta(\Gamma)$ is a building ([7], 5.2) where between any two elements $P, Q \in \Delta(\Gamma)$ the distance $d = \mathrm{dist}\, PQ$ is defined as follows: there exists a *stretched gallery of Borel subgroups of length d from P to Q* (cf. [7], p. 4), i.e., a sequence of Borel groups $B_i, i = 0, \dots, d$, such that $B_0 \subset P, B_d \subset Q$, and $\langle B_{i-1}, B_i\rangle$ is, for each $i = 1, \dots, d$, a minimal (non Borel) parabolic subgroup of Γ and such that there is no gallery of strictly smaller length with this property. Consider the distance $\mathrm{dist}\, B'B''$ for a pair of Borel subgroups B', B'' of Γ. We may assume $B' = B$ and $B'' = gBg^{-1}$ for some $g \in \Gamma$. If $g \in B\dot{w}B$ where $w \in W$ then $B'' = b\dot{w}B\dot{w}^{-1}b^{-1}$ for some $b \in B$. Since the distance is invariant with respect to the action of Γ on $\Delta(\Gamma)$ by conjugation we get $\mathrm{dist}\, BB'' = \mathrm{dist}\, B\ w(B)$. On the other hand, $\mathrm{dist}\, B\ w(B) = l(w)$ where $l(w)$ is the length of the element w ([3], 15.3). The length $l(w)$, in turn, is equal to the set of positive roots corresponding to B which become negative after the action of w ([4]), that is,

$$\begin{aligned}\mathrm{dist}\, B\ w(B) &= l(w) = \dim B - \dim(B \cap w(B))\\ &= \mathrm{d_{in}}(B, w(B)) = \mathrm{d_{in}}(B, B') = \mathrm{d_{inc}}(B, B').\end{aligned}$$

Hence the intersection distance between Borel subgroups coincides with the building distance between these subgroups.

Now let us calculate $\mathrm{d_{in}}(P,Q)$ and $\mathrm{d_{inc}}(P,Q)$. We may assume $P = w(P_X)$ where P_X is a standard parabolic subgroup corresponding to a set $X \subset \Pi$ and $Q = P_Y$ for some standard parabolic subgroup P_Y where $Y \subset \Pi$ and $w \in D_{Y,X}$ where $D_{Y,X} \subset W$ is a set of double coset representatives of Q, P(see, [4], 2.7, 2.8). Below we take

$$D_{Y,X} = \{w \in W \mid w^{-1}(Y) \subset R^+ \text{ and } w(X) \subset R^+\}$$

the set of distinguished double coset representatives, that is, representatives of the smallest length (see, [4] 2.7.3 and 2.8.1 iii)). Let $Z = w(X) \cap Y$, and denote by w_X, w_Y, w_Z the elements of *maximal length* in the Weyl group W_X, W_Y, W_Z corresponding to the sets $X, Y, Z \subset \Pi$.

Proposition 1.1. *Let* $P = w(P_X), Q = P_Y, w \in D_{Y,X}$ *where* $D_{Y,X}$ *is the set of distinguished double coset representatives. Then*

$$\begin{aligned} \mathrm{d_{in}}(P,Q) &= \max\{\dim\, P, \dim Q\} - \dim B - l(w_Z) + l(w) \\ &= \max\{l(w_X), l(w_Y)\} - l(w_Z) + l(w), \\ \mathrm{d_{inc}}(P,Q) &= \min\{\dim\, P, \dim Q\} - \dim B - l(w_Z) + l(w) \\ &= \min\{l(w_X), l(w_Y)\} - l(w_Z) + l(w). \end{aligned}$$

Proof. According to the theorem of Howlett ([4], Proposition 2.8.11)

$$w(P_X) \cap P_Y = \bigcup_{w' \in W_Z} (B \cap w(B))\dot{w}'(B \cap w(B)).$$

The cell $(B \cap w(B))\dot{w}_Z(B \cap w(B))$ is of the maximal dimension among the above double cosets, hence

$$\dim w(P_X) \cap P_Y = \dim(B \cap w(B))\dot{w}_Z(B \cap w(B)). \tag{1.1}$$

Further, let U_Z be the unipotent radical of the Borel subgroup corresponding to the root system generated by Z. Then $U_Z \subset B \cap w(B)$ and only the root subgroups of U_Z cannot be moved from the left side of $(B \cap w(B))\dot{w}_Z(B \cap w(B))$ to the right side. Thus,

$$\dim(B \cap w(B))\dot{w}_Z(B \cap w(B)) = \dim U_Z + \dim(B \cap w(B)). \tag{1.2}$$

Further,

$$\dim U_Z = l(w_Z); \quad \dim B \cap w(B) = \dim B - l(w). \tag{1.3}$$

The assertion follows now from (1.1)–(1.3), the definitions of $\mathrm{d_{in}}(\,,\,), \mathrm{d_{inc}}(\,,\,)$ and the equalities $\dim P - \dim B = l(w_X),\ \dim Q - \dim B = l(w_Y)$. □

Now we denote by P_Z (resp. Q_Z) the parabolic subgroups corresponding to the set $Z \subset w(\Pi) \cap \Pi$ with respect to the root system generated by $w(\Pi)$ (resp. Π). Hence $B \subset Q_Z,\ w(B) \subset P_Z$. Obviously,

$$\dim P_Z = \dim Q_Z = \dim B + l(w_Z). \tag{1.4}$$

By [4], 2.8.4 we obtain

$$(P \cap Q)R_u(P) = P_Z,\ (P \cap Q)R_u(Q) = Q_Z. \tag{1.5}$$

Corollary 1.2.

$$\begin{aligned} \mathrm{d_{in}}(P,Q) &= \mathrm{dist}PQ + (\max(\dim\, P, \dim Q) - \dim(P \cap Q)R_u(P)), \\ \mathrm{d_{inc}}(P,Q) &= \mathrm{dist}PQ + (\min(\dim\, P, \dim Q) - \dim(P \cap Q)R_u(P)). \end{aligned}$$

Proof. Since $l(w) = \text{dist}\, PQ$, the statement of the Corollary follows from Proposition 1.1, and from (1.4), (1.5). □

§2. The dimension of Schubert cells

Here we preserve the assumptions for $\Gamma, X, Y \subset \Pi, , P_X, P_Y, D_{Y,X}$. Consider the maps

$$\varphi_{/X} : \Gamma \to \Gamma/P_X, \quad \varphi_{\backslash Y} : \Gamma \to P_Y\backslash\Gamma.$$

The homogeneous spaces $\Gamma/P_X, P_Y\backslash\Gamma$ are smooth projective varieties.

Let $\mathcal{D} = P_Y \dot{w} P_X$ be a double coset and let $\text{Sch}_w(Y/X) = \overline{\varphi_{/X}(\mathcal{D})}$, $\text{Sch}_w(Y \setminus X) = \overline{\varphi_{\backslash Y}(\mathcal{D})}$ denote the corresponding Schubert varieties in Γ/P_X, $P_Y\backslash\Gamma$ respectively.

Proposition 2.1.

$$\dim \text{Sch}_w(Y/X) = \begin{cases} \text{d}_{\text{in}}(P_X, w^{-1}(P_Y)) = \text{d}_{\text{in}}(w(P_X), P_Y) \\ \qquad \textit{if } \dim P_Y \geqslant \dim P_X, \\ \text{d}_{\text{inc}}(P_X, w^{-1}(P_Y)) = \text{d}_{\text{inc}}(w(P_X), P_Y) \\ \qquad \textit{if } \dim P_Y < \dim P_X. \end{cases}$$

$$\dim \text{Sch}_w(Y\backslash X) = \begin{cases} \text{d}_{\text{in}}(w(P_X), P_Y) = \text{d}_{\text{in}}(P_X, w^{-1}(P_Y)) \\ \qquad \textit{if } \dim P_Y \leqslant \dim P_X, \\ \text{d}_{\text{inc}}(w(P_X), P_Y) = \text{d}_{\text{inc}}(P_X, w^{-1}(P_Y)) \\ \qquad \textit{if } \dim P_Y > \dim P_X. \end{cases}$$

Proof. Let $P = P_X$, $Q = w^{-1}(P_Y) = \dot{w}^{-1}P_Y\dot{w}$, $S = P \cap Q$ and let $Q = \cup_\alpha q_\alpha S$ be the decomposition into the union of left cosets. Then fibers of the map

$$(\varphi_{/X})_{|\,\dot{w}Q} : \dot{w}Q = P_Y\dot{w} \to \Gamma/P$$

are equal to sets $\{\dot{w}q_\alpha S\}$ (indeed, for $q_1, q_2 \in Q$ the equality $\varphi_{/X}(\dot{w}q_1) = \varphi_{/X}(\dot{w}q_2)$ is equivalent to $\dot{w}q_1 = \dot{w}q_2p$ for some $p \in P$ and, therefore, $p = q_2^{-1}q_1 \in Q \cap P = S$). Hence

$$\dim \text{Sch}_w(Y/X) = \dim Q - \dim S = \dim w^{-1}(P_Y) - \dim(P_X \cap w^{-1}(P_Y)) \tag{2.1}$$

Now the statement for $\dim \text{Sch}_w(Y/X)$ follows from (2.1) and the definitions of distances. By the same arguments we get the statement for $\dim \text{Sch}_w(Y\backslash X)$. □

Corollary 2.2. *For every $w \in W$ we have*

$$\dim \mathrm{Sch}_w(Y/X) - \dim \mathrm{Sch}_w(Y\backslash X) = \dim P_Y - \dim P_X.$$

Proof. This follows directly from Proposition 2.1 and the definition of $\mathrm{d_{in}}(\ ,\ )$ and $\mathrm{d_{inc}}(\ ,\ )$. □

§3. Parabolic subgroups in general position

We say that two closed subgroups P, Q of an algebraic group Γ are *in general position* if

$$\dim P \cap Q = \dim P + \dim Q - \dim \Gamma > 0 \ \text{ or } \ \dim P \cap Q = 0. \qquad (3.1)$$

The condition of (3.1) is equivalent to

$$\mathrm{d_{in}}(P,Q) = \min\{\max\{\dim \Gamma/P,\ \dim \Gamma/Q\}, \max\{\dim P, \dim Q\}\}. \quad (3.2)$$

Proposition 3.1. *Let $P = gP_Xg^{-1}, Q = P_Y$ where $g \in \Gamma$. The following statements are equivalent:*

i. *the groups P, Q are in general position;*
ii. $\underline{\mathrm{d_{in}}}(P,Q) = \max\{\dim \Gamma/P, \dim \Gamma/Q\}$*;*
iii. $\overline{QP} = \Gamma$*;*
iv. $g \in P_Y\dot{w}P_X$ *where $P_Y\dot{w}P_X$ is a single open double coset of P_X, P_Y.*

Proof. The equivalence i.⇔ ii. follows from (3.2) and the fact that $\dim P, \dim Q > \frac{1}{2}\dim \Gamma$.

Consider the implication iv. ⇒ ii. We may assume $g = \dot{w}$ for some $w \in W$. Note, if $P_Y\dot{w}P_X$ is an open double coset then

$$\mathrm{Sch}_w(Y/X) = \Gamma/P_X, \quad \mathrm{Sch}_w(Y\backslash X) = P_Y\backslash\Gamma \qquad (3.3)$$

and we have the implication iv. ⇒ ii. from Proposition 2.1.

Now let us prove the implication ii. ⇒ iv. Let $g \in P_Y\dot{w}P_X$ for some $w \in W$ and let $P = gP_Xg^{-1}, Q = P_Y$ be in general position. Then the group $P = \dot{w}P_X\dot{w}^{-1}, Q = P_Y$ are also in general position. Hence we may assume $g = \dot{w}$. We have to check that $P_Y\dot{w}P_X$ is the open double coset of Γ. If $\mathcal{D}_1 = P_Y\dot{w}_1P_X \neq \mathcal{D}_2 = P_Y\dot{w}_2P_X$ then $\varphi_{/X}(\mathcal{D}_1) \cap \varphi_{/X}(\mathcal{D}_2) = \emptyset$ and $\varphi_{\backslash Y}(\mathcal{D}_1) \cap \varphi_{\backslash Y}(\mathcal{D}_2) = \emptyset$. Moreover, there is only one open double coset. Hence the equalities (3.3) hold only for the case when $P_Y\dot{w}P_X$ is this open double coset. Now iv. follows from Proposition 2.1.

Hence we have proved the equivalence ii.⇔ iv.

Again we may assume for $g \in P_Y\dot{w}P_X$ that $g = \dot{w}$.

$$\overline{P_Y\dot{w}P_X\dot{w}^{-1}} = \Gamma \Leftrightarrow P_Y\dot{w}P_X \ \text{ is an open double coset.}$$

Therefore we have iii.⇔ iv. □

§4. Intersection distance between planes and their stabilizers

Let $\Gamma = \mathrm{GL}(V)$, $\dim V = n$ and let $P_v, P_u \leqslant G$ be stabilizers of a k-dimensional linear subspace $v \subset V$ and an l-dimensional linear subspace $u \subset V$. Let $n = \dim V$.

Here we compare the intersection distances $\mathrm{d_{in}}(v,u)$ and $\mathrm{d_{in}}(P_v, P_u)$.

Theorem 4.1. *Let $k \leqslant l$. Then*

$$\mathrm{d_{in}}(P_v, P_u) = \begin{cases} -\mathrm{d_{in}}(v,u)^2 + \mathrm{d_{in}}(v,u)(n+l-k) - (l-k)l \\ \qquad \text{if } k+l \leqslant n, \\ -\mathrm{d_{in}}(v,u)^2 + \mathrm{d_{in}}(v,u)(n+l-k) - (l-k)(n-k) \\ \qquad \text{if } k+l \geqslant n. \end{cases}$$

Proof.

Lemma 4.2.

$$\max\{\dim P_v, \dim P_u\} = \begin{cases} \dim P_v = n^2 - k(n-k) = n^2 + k^2 - kn \\ \qquad \text{if } k+l \leqslant n, \\ \dim P_u = n^2 - l(n-l) = n^2 + l^2 - ln \\ \qquad \text{if } k+l \geqslant n. \end{cases}$$

Proof. The expressions on the right for the dimensions of P_v, P_u are obvious. Further, we have

$$k + l \leqslant n \Leftrightarrow l^2 - k^2 = (l-k)(k+l) \leqslant (l-k)n = ln - kn. \qquad \square$$

Suppose $k + l \leqslant n$.

Let $0 \leqslant V_1 \leqslant V_2 \leqslant \cdots \leqslant V_n$ be a fixed flag and let P_i be the stabilizer of V_i.

For any $i = 0, \ldots, s_0 = \min\{k, n-l\} = k$ (cf. Example 0.1) there is some $\gamma_i \in GL(V)$ such that $\dim(\gamma_i(V_k) \cap V_l) = k - i$ or, equivalently, $d := \mathrm{d_{in}}(\gamma_i(V_k), V_l) = l - k + i$.

Further, let $\mathrm{G}_{k,n}$ be the Grassmann variety of k-planes in V. Then we have the map

$$\varphi_{l/k} : P_l \gamma_i P_k \to \mathrm{G}_{k,n} \approx \Gamma / P_k$$

given by the formula $\varphi_{l/k}(g) = g(V_k)$ for every $g \in P_l\gamma_i P_k$. Hence (cf. Example 0.1)) we have

$$v \in \mathrm{Im}\ \varphi_{l/k} \Leftrightarrow \mathrm{d_{in}}(v, V_l) = d \Leftrightarrow \dim(v \cap V_l) = k - i. \tag{4.1}$$

Now let us define a sequence of non-negative integers

$$a_1 = a_2 = \cdots = a_{k-i} = n - k - d,\ a_{k-i+1} = \cdots = a_k = 0,$$

and let us put

$$\mathrm{Sch}(a_1, \ldots, a_k) = \overline{\{v \in \mathrm{G}_{k,n} \mid \dim(v \cap V_{n-k-a_j+j}) = j\}}. \tag{4.2}$$

Then $\mathrm{Sch}(a_1, \ldots, a_k)$ is a Schubert cell in $\mathrm{G}_{k,n}$ ([5], 1.5) and

$$\begin{aligned} \dim \mathrm{Sch}(a_1, \ldots, a_k) &= k(n-k) - \sum_{j=1}^{k} a_j \\ &= k(n-k) - (k-i)[(n-k) - d] \\ &= k(n-k) - (l-d)[(n-k) - d] \\ &= -d^2 + d(n-k+l) - (l-k)(n-k). \end{aligned} \tag{4.3}$$

The conditions $\dim(v \cap V_{n-k-a_j+j}) = j$ of (4.2) are equivalent to

$$\begin{aligned} \dim(v \cap V_{d+j}) = j \quad \text{for} \quad j = 1, \ldots, k-i, \\ \dim(v \cap V_{n-k+j}) = j \quad \text{for} \quad j = k-i+1, \ldots, k. \end{aligned} \tag{4.4}$$

The conditions of the second equation of (4.4) hold for any generic k-plane of V and all conditions of the first equation of (4.4) follow from the last one : $\dim(v \cap V_l) = k - i$.

Hence (4.1) and (4.2) imply

$$\mathrm{Sch}(a_1, \ldots, a_k) = \overline{\mathrm{Im}\ \varphi_{l/k}}. \tag{4.5}$$

Further, $P_k = P_X, P_l = P_Y$ for some $X, Y \subset \Pi$ (with the notations from § 1) and $\varphi_{/X} = \varphi_{l/k}$. Now Proposition 2.1, Lemma 4.2, assumption $k + l \leqslant n$, and (4.3), (4.5) imply

$$\mathrm{d_{inc}}(P_k, \gamma_i^{-1}(P_l)) = -d^2 + d(n-k+l) - (l-k)(n-k). \tag{4.6}$$

From Lemma 4.3, (4.6) and the definition of the intersection distance we get

$$\begin{aligned} \mathrm{d_{in}}(P_k, \gamma_i^{-1}(P_l)) &= \mathrm{d_{in}}(\gamma_i(P_k), P_l) \\ &= \mathrm{d_{inc}}(P_k, \gamma_i^{-1}(P_l)) + (\dim P_k - \dim P_l) \\ &= -d^2 + d(n-k+l) - (l-k)(n-k) + (l(n-l) - k(n-k)) \\ &= -d^2 + d(n-k+l) - l(l-k). \end{aligned} \tag{4.7}$$

Now we may assume $u = V_l, v = g(V_k), P_u = P_l, P_v = g(P_k) = gP_kg^{-1}$ for some $g \in P_l\gamma_iP_k$. We have

$$\mathrm{d_{in}}(v,u) = \mathrm{d_{in}}(\gamma_i(V_k), V_l) = d; \quad \mathrm{d_{in}}(P_v, P_u) = \mathrm{d_{in}}(\gamma_i(P_k), P_l). \tag{4.8}$$

Our assertion for the case $k + l \leqslant n$ follows from (4.7) and (4.8). The case $k + l > n$ can be done by similar arguments. □

Corollary 4.3. *The parabolic subgroups P_v, P_u are in general position in Γ if and only if the subspaces v, u are in general position in V.*

Proof. The parabolic subgroups P_v, P_u are in general position in G if and only if

$$\mathrm{d_{in}}(P_v, P_u) = \max\{\dim G/P_v, \dim G/P_u\} = \max\{k(n-k), l(n-l)\}$$

(Proposition 3.1, ii.). The planes v, u are in general position in V if and only if

$$\mathrm{d_{in}}(v,u) = \min\{l, n-k\}.$$

Suppose $k + l \leqslant n$. Then $l \leqslant n - k$ and $k(n-k) \leqslant l(n-l)$. Further, by Theorem 4.1,

$$\mathrm{d_{in}}(P_v, P_u) = l(n-l) = -\mathrm{d_{in}}(v,u)^2 + \mathrm{d_{in}}(v,u)(n+l-k) - l(l-k) \Leftrightarrow$$

$$\Leftrightarrow \mathrm{d_{in}}(v,u) = l \text{ or } \mathrm{d_{in}}(v,u) = n-k.$$

Since $l \leqslant n - k$ we have the assertion. The case $k + l \geqslant n$ can be done in a similar way. □

Corollary 4.4. *Let $v', u' \leqslant V$ be planes of the dimension $k \leqslant l$ respectively and let $P_{v'}, P_{u'}$ be their stabilizers. Then*

$$\mathrm{d_{in}}(P_{v'}, P_{u'}) < \mathrm{d_{in}}(P_v, P_u) \Leftrightarrow \mathrm{d_{in}}(v', u') < \mathrm{d_{in}}(v,u).$$

Proof. The function $f(d) = -d^2 + d(n+l-k)$ is strictly increasing on the interval $d \in [l-k, \min\{l, n-k\}]$. □

Remark 4.5. Note that each of the both formulas of the Theorem can be obtained from the other by the substitution $k \to n - l$, $l \to n - k$.

Acknowledgments

Most of this work was done during the visits of N.G. to the Department of Mathematics of the Bielefeld University. The first named author gratefully acknowledges the support by the Department of Mathematics of the Bielefeld University, by the grants RFFI 14-01-00820 and 13-01-91150-ГФЕН$_{-}a$, and by the SPbGU project 6.38.191.2014.

References

1. A. Borel, *Linear Algebraic groups. 2nd enl.ed.* Graduate texts in mathematics **126**. Springer-Verlag, New York Inc.1991.
2. N. Bourbaki, *Éléments de Mathématique. Groupes et Algèbres de Lie.* Chap. IV, V, VI, 2ème édition. Masson, Paris 1981.
3. R. W. Carter, *Simple Groups of Lie Type.* Reprint of the 1972 original. Wiley Classics Library. John Wiley and Sons, New York, 1989.
4. R. W. Carter, *Finite Groups of Lie Type. Conjugacy Classes and Complex Characters.* John Wiley&Sons, Chichester er al., 1985.
5. Ph. Griffiths, J. Harris, *Principles of algebraic geometry.* John Wiley&Sons, New York, 1978
6. N. Gordeev, U. Rehmann, *On linearly Kleiman groups.* — Transf. Groups **18**, No. 3 (2013), 685–709.
7. J.Tits, *Buildings of Spherical Type and and Finite BN-Pairs.* — Lect. Note Maths. **386**, Springer-Verlag (1974).

Department of Mathematics,
Russian State Pedagogical University,
Moijka 48, St.Petersburg 191186, Russia
E-mail: nickgordeev@mail.ru

Поступило 23 сентября 2014 г.

Department of Mathematics,
Bielefeled University,
Universitätsstrasse 25,
D-33615 Bielefeld, Germany
E-mail: rehmann@math.uni-bielefeld.de