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N. A. Vavilov

DECOMPOSITION OF UNIPOTENTS FOR E_6 AND E_7: 25 YEARS AFTER

Abstract. In this paper I sketch two new variations of the method of decomposition of unipotents in the microweight representations (E_6, ϖ_1) and (E_7, ϖ_7). To put them in context, I first very briefly recall the two previous stages of the method, an A_5-proof for E_6 and an A_7-proof for E_7, first developed some 25 years ago by Alexei Stepanov, Eugene Plotkin and myself (a definitive exposition was given in my paper "A thirdlook at weight diagrams"), and an A_2-proof for E_6 and E_7 developed by Mikhail Gavrilovich and myself in early 2000. The first new twist outlined in this paper is an observation that the A_2-proof actually effectuates reduction to small parabolics, of corank 3 in E_6 and of corank 5 in E_7. This allows to revamp proofs and sharpen existing bounds in many applications. The second new variation is a D_5-proof for E_6, based on stabilisation of columns with one zero. [I devised also a similar D_6-proof for E_7, based on stabilisation of columns with two adjacent zeroes, but it is too abstruse to be included in a casual exposition.] Also, I list several further variations. Actual detailed calculations will appear in my paper "A closer look at weight diagrams of types (E_6, ϖ_1) and (E_7, ϖ_7)".

In this paper we describe several new variations on decomposition of unipotents in microweight representations of Chevalley groups of types E_6 and E_7. This paper is based on my talks at

- Ischia Group Theory 2014 (Napoli, April 2014),
- Tsukuba Workshop on Infinite-Dimensional Lie Theory and Related Topics (Tsukuba, October 2014).

First, I give a very brief account of the existing versions of decomposition of unipotents for these cases, and then outline two new versions of the method, furnishing accurate statements and constructions, but skipping some of the more cumbersome and unwieldy details of calculations. I make

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no attempt to give a broader historical account of the method itself or its applications. The classical cases in vector/polyvector representations are relatively easy and well understood [17, 41]. On the other hand, other large exceptional groups, of types F_4 and E_8, do not have microweight representations, and demand an entirely different level of technical strain. Thus, here I limit myself exclusively to the microweight representations (E_6, ϖ_1) and (E_7, ϖ_7).

§1. The first 12 years

Let Φ be a reduced irreducible root system of rank $l = \mathrm{rk}(\Phi)$ and R be a commutative ring. Further, let $G(\Phi, R)$ be the simply connected Chevalley group of type Φ over R. We fix a split maximal torus $T(\Phi, R)$ in $G(\Phi, R)$ and parametrisations of the root subgroups X_α, $\alpha \in \Phi$, elementary with respect to this torus. In other words, for each root $\alpha \in \Phi$ we fix an isomorphism

$$x_\alpha : \mathbb{G}_{\mathrm{a}} \longrightarrow X_\alpha, \quad \xi \mapsto x_\alpha(\xi).$$

The elements $x_\alpha(\xi)$, $\alpha \in \Phi$, $\xi \in R$, are called *elementary root unipotents* (or, sometimes, elementary generators). The subgroup

$$E(\Phi, R) = \big\langle x_\alpha(\xi) \mid \alpha \in \Phi,\ \xi \in \Phi \big\rangle \leqslant G(\Phi, R)$$

generated by all root unipotents elementary w.r.t. T is called the *elementary Chevalley group* of type Φ over R.

In general, $E(\Phi, R)$ is a proper subgroup of $G(\Phi, R)$, their difference being measured by the value of K_1-functor $K_1(\Phi, R)$. One of the pivotal results of the whole structure theory of Chevalley groups is the celebrated Suslin–Kopeiko–Taddei normality theorem, asserting that for groups of rank $\geqslant 2$ the elementary subgroup $E(\Phi, R)$ is normal in $G(\Phi, R)$, for all commutative rings R.

In effect, this theorem asserts that for any root $\alpha \in \Phi$, any ring element $\xi \in R$, and any element $g \in G(\Phi, R)$ of the Chevalley group the corresponding *root unipotent* $gx_\alpha(\xi)g^{-1}$ belongs to the elementary group $E(\Phi, R)$. In other words, $gx_\alpha(\xi)g^{-1}$ decomposes as a product of elementary root unipotents.

For classical types, the first proofs, due to Suslin and Kopeiko [19, 20, 5], were based on decomposition of the matrix $gx_\alpha(\xi)g^{-1}$ itself. In fact, they rather decomposed not just root unipotents, but broader classes of *root type unipotents* (roughly, matrices from the Zariski closure of the set of root unipotents, maybe, subject to some additional unimodularity conditions).

For exceptional groups, the first general proof, due to Taddei, was based on localisation [21]. See also [3] by Hazrat and the author for an easier localisation proof of a more general result. A new generation of localisation proofs, in a sense the most general ones, were developed by Stepanov, see [14, 16].

Today, 30++ years after, there are many different approaches to the proof of this theorem, see the overview in [23, 1, 17, 4]. One such very powerful method is decomposition of unipotents, initially proposed in 1987 in the Ph. D. Thesis of Alexei Stepanov [13]. It was immediate to generalise it to other classical groups in vector representations, and to $\mathrm{GL}(n, R)$ in polyvector representations, and such generalisations were already contained in [13, 22]. See [17] for a systematic exposition, and also [41], § 1, for a slightly more general view of the polyvector case.

Essentially, in the simplest form decomposition of unipotents gives finite polynomial expressions of the conjugates

$$gx_\alpha(\xi)g^{-1}, \qquad \alpha \in \Phi,\ \xi \in R,\ g \in G(\Phi, R),$$

as products of factors sitting in proper parabolic subgroups, and, in the final count, as products of elementary generators.

The following result was not stated in this form before [2], but actually it is an immediate corollary of the polynomial expression of arbitrary root unipotents in terms of elementary root unipotents, first enunciated in [44]. The first proof of that decomposition for types E_6 and E_7 obtained in 1989 by Eugene Plotkin and the author [42] relied on extensive computer verifications. That proof was outlined in [23], without explicit verification that the occuring signs coincide. In 1997 the author succeeded in checking this fact by hand. The first complete proof is published in [24] and it is anything but immediate.

Theorem 1. *Let R be a commutative ring and $\Phi = \mathrm{E}_6, \mathrm{E}_7$. Then any root element of the form $gx_\alpha(\xi)g^{-1}$, $\alpha \in \Phi$, $\xi \in R$, $g \in G(\Phi, R)$ is a product of at most L elementary root unipotents, where*

- $L = 4 \cdot 16 \cdot 27 = 1728$ *for* $\Phi = \mathrm{E}_6$*,*
- $L = 4 \cdot 27 \cdot 56 = 6048$ *for* $\Phi = \mathrm{E}_7$*.*

Here, 27 and 56 are dimensions of microweight representations of the simply connected Chevalley groups of types E_6 and E_7, respectively. Further, 16 and 27 are dimensions of the [abelian] unipotent radicals U_P of the corresponding maximal parabolic subgroups P, of type P_1 in E_6, and of type P_7 in E_7. Finally, 4 is the inexorable factor occuring as one expresses a root type element from the Levi factors of types $D_5 \leqslant E_6$ or $\mathrm{E}_6 \leqslant \mathrm{E}_7$ as commutators of unipotents from U_P and the opposite unipotent radical U_P^-, i.e. elements of $U_P U_P^- U_P U_P^-$.

Technically, the main step in these proofs – the so called "main lemma" – can be stated as follows. Given a matrix $g \in G$, retrieve enough small unipotents which stabilise columns of g, to ensure they span the whole elementary group $E(\Phi, R)$. To be actually expressed as short products of elementaries, these unipotents have to come from proper subsystem subgroups. That the proofs in [23] and [24] would work at all, seemed to be a miracle. As a matter of fact, in these proofs the unipotents stabilising columns of g were taken from the *largest* possible classical subgroups, of types $\mathrm{A}_5 \leqslant \mathrm{E}_6$ and $\mathrm{A}_7 \leqslant \mathrm{E}_7$. Another rather burdensome aspect of these proofs was the necessity to meticulously control signs of both the action constants and equations defining highest weight orbits in these representations.

Observe, that such sharp polynomial bounds could be very useful in real life applications. Compare, for instance, the polynomial bounds for the width of commutators in elementary generators, obtained by Alexander Sivatsky and Alexei Stepanov [11], where they could rely on decomposition of unipotents in the above strong form, with the hyperexponential bounds in our paper with Alexei Stepanov [18], where we had to restraint ourselves to localisation methods instead. See [2] for a thorough discussion.

§2. Weyl modules and weight diagrams

Actually, these proofs used various tools related to root systems, Weyl groups, Lie algebras, representation theory, geometry of minimal modules, weight diagrams, detailed control of structure constants and equations, etc., which we cannot recall here, in any reasonable way. Instead, we refer to the classical papers by Hideya Matsumoto [7] and Michael Stein [12], where Chevalley groups over rings were first treated with similar techniques, and to our previous papers [23, 10, 43], for background information and many related references. In fact, the present work is a direct sequel of [24, 30, 31, 26, 27, 25, 40, 28], and we assume that the reader has seen at least some of these papers.

However, we have to introduce at least some absolute minimum of notation indispensable for the rest of the paper. Usually, Chevalley groups occur as linear groups, in certain representations. Let $V = V(\varpi)$ be the

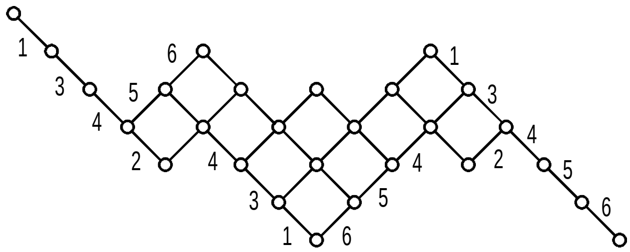

Fig. 1. (E_6, ϖ_1).

Weyl module of the Chevalley group $G(\Phi, R)$ with highest weight ϖ, and let $\pi : G(\Phi, R) \longrightarrow \mathrm{GL}(V)$ be the corresponding rational representation. Sometimes, the image of this representation is denoted by $G_{\varpi}(\Phi, R)$.

Typically, in many common applications $\varpi = \varpi_i$, $i = 1, \dots, l$, is a fundamental weight. In the present paper we are only interested in the *microweight* cases (E_6, ϖ_1) and (E_7, ϖ_7).

Further, let $\Lambda = \Lambda(\varpi)$ be the set of weights of the representation π, with multiplicities, and let v^{λ}, $\lambda \in \Lambda$, be an admissible base of V. All vectors are expressed as coordinate *columns* with respect to the base v^{λ}, $\lambda \in \Lambda$. Now, we can represent a vector $u = (u_{\lambda}) \in V$ by a marked graph as follows. Put the coordinate u_{λ} in the node of the weight diagram, corresponding to the weight λ.

Now, we can in the usual way represent an element $g \in G(\Phi, R)$ by the matrix $(g_{\mu\nu})$, $\mu, \nu \in \Lambda$, whose entry $g_{\mu\nu}$ in the position (μ, ν) equals the coefficient with which v^{μ} occurs in the linear expansion of $\pi(g)v^{\nu}$, with respect to the base v^{λ}, $\lambda \in \Lambda$. Below, we usually identify g with this matrix, and write simply $g = (g_{\mu\nu})$.

The columns of these matrices can be conceived as elements of V, in that case their rows should be interpreted as elements of the dual module V^*. It is very important that the columns and rows of these matrices are not linearly ordered, but partially ordered, in accordance with the weight diagram of π or its dual, respectively.

For a microweight representation $V = V(\varpi)$ one has $\Lambda = W(\Phi)\varpi$. In other words, all weights are extremal and, thus, of multiplicity 1, so that Λ is indeed the set of weights of V, in the usual sense. One can normalise an admissible base v^{λ}, $\lambda \in \Lambda$, in such a way that for any $\alpha \in \Phi$ and any

$\xi \in R$ one has

$$x_\alpha(\xi)v^\lambda = v^\lambda + c_{\lambda\alpha}\xi v^{\lambda+\alpha},$$

where all *action structure constants* $c_{\lambda\alpha}$ are equal to ± 1, see [7]. Usually, one choses the crystal base, with the following positivity property: all structure constants $c_{\lambda\alpha}$ are equal to $+1$ for the fundamental and the negative fundamental roots, i. e. $c_{\lambda\alpha} = +1$, whenever $\alpha \in \pm\Pi$. Existence of such a base is classically known, in [24] and [26] one can find two elementary proofs. Actually, all structure constants in crystal bases for (E_6, ϖ_1) and (E_7, ϖ_7) are tabulated in [38] and [35], respectively.

The most important technical tool in our calculations are weight diagrams. Let $\pi : G(\Phi, R) \longrightarrow \mathrm{GL}(V(\varpi))$ be a representation of a Chevalley group on a Weyl module. For a microweight representation π its *weight diagram*, which in this case coincides with the *crystal graph*, is a marked graph constructed as follows.

• Its nodes correspond to the weights $\lambda \in \Lambda$ of π.

• Two nodes λ and μ are joined by a bond marked i if their difference $\lambda - \mu = \alpha_i$ is the i-th fundamental root.

Not to overcharge our diagrams with arrows, we draw them in such a way that a larger weight always stands to the left of and/or higher than a smaller one, landscape orientation being primary. Moreover, usually we omit at least one of the two equal labels at the opposite sides of a parallelogramm.

In Figures 1 and 2 we reproduce the weight diagrams for the two Weyl modules considered in the present paper, the module $V(\varpi_1)$ for $G(\mathrm{E}_6, R)$ and the module $V(\varpi_7)$ for $G(\mathrm{E}_7, R)$.

The most important assignment of weight diagrams is to serve as croquis drawings of *weight graphs*. Recall that weight graphs are defined similarly to weight diagrams, but display edges corresponding to *all* positive roots, rather than just those corresponding to the fundamental ones. Any attempt to draw a weight graph with a few dozen vertices leads to a complete mess. Luckily, this is utterly redundant. Read in conjunction with root tables weight diagrams allow to easily recover all information encoded in weight graphs. Namely, to restitute edges corresponding to any root $\alpha = m_1\alpha_1 + \ldots + m_l\alpha_l$, it suffices to find in the corresponding weight diagram all paths that comprise m_1 edges marked 1, m_2 edges marked 2, etc., in any order.

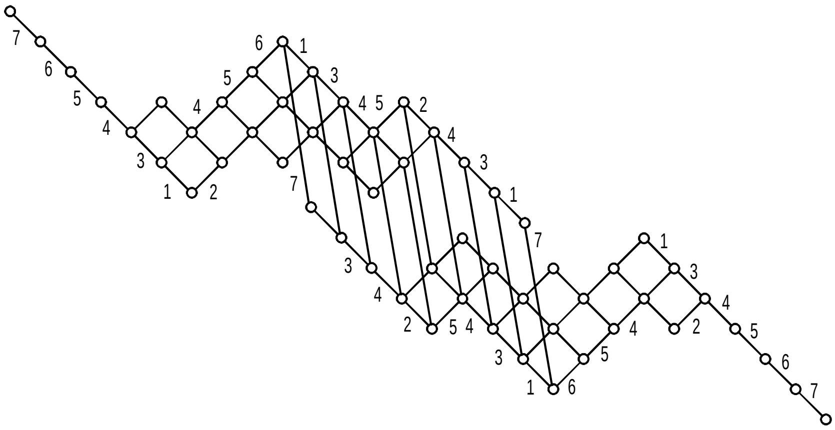

Fig. 2. (E_7, ϖ_7).

For the representations (E_6, ϖ_1) and (E_7, ϖ_7) Weyl orbits on pairs of weights (λ, μ), $\lambda, \mu \in \Lambda$, are distinguished by a single invariant, namely by the distance $d(\lambda, \mu)$ between λ and μ in the weight graph. Weights $\lambda, \mu \in \Lambda$ such that $d(\lambda, \mu) = 1$ are called *adjacent*. This means that their difference $\lambda - \mu \in \Phi$ is a root. Weights $\lambda, \mu \in \Lambda$ such that $d(\lambda, \mu) = 2$ will be called *distant*. This means that their difference $\lambda - \mu \in \Phi$ is not a root, but can be expressed as the sum of two roots. In the case (E_6, ϖ_1) distance $d(\lambda, \mu)$ between weights only takes values 0,1 and 2. Thus, any two roots are either equal, or adjacent, or distant. In the case (E_7, ϖ_7) there is the fourth possible value, $d(\lambda, \mu) = 3$. More precisely, for each weight λ there exists a *unique* weight μ at distance 3 from λ, namely $\mu = -\lambda$. This weight is called *opposite* to λ. In the sequel, when weight of $\Lambda(\varpi_7)$ is interpreted as a certain set of roots in E_8, we usually denote the weight opposite to λ by λ^*, in this case, $\lambda^* = \rho - \lambda$, where ρ is the maximal root of E_8.

§3. The next 12 years

Returning to the setting of § 1, suppose we already know that the elementary subgroup $E(\Phi, R)$ is normal in $G(\Phi, R)$ and are interested in further applications, we do not have to hunt for small unipotents stabilising columns of *arbitrary* matrices $g \in G$. In [17] Alexei Stepanov and I noticed that decomposition of unipotents immediately implies also the standard description of normal (or, in fact, $E(\Phi, R)$-normalised) subgroups of $G(\Phi, R)$.

For let H be a non-central subgroup normalised by $E(\Phi, R)$. Let $g \in H$ be a non-central element. Then g does not commute with some elementary root unipotent $x_\alpha(1)$, $\alpha \in \Phi$. Then by decomposition of unipotents it does not commute with some unipotent $x \in E(\Phi, R)$ such that multiplication by x does not change some column of g^{-1}. Then gxg^{-1} falls into a proper parabolic subgroup. By looking a bit more carefully we can ensure that already $[x^{-1}, g] = x^{-1}gxg^{-1}$ does not fall into that parabolic subgroups, which allows us to recourse to parabolic reduction and conclude that H contains a non-trivial elementary root unipotent. [Recently, Stepanov observed that things are even easier than that, in the above situation $[x^{-1}, g]$ *always* sits in a product of two non-opposite parabolic subgroups, which already suffices to conclude that H contains a non-trivial elementary root unipotent.] At this point, standard description immediately follows by level reduction.

The following twist was proposed in [30], were Mikhail Gavrilovich and I noticed that in microweight representations there is no need whatsoever to worry about being able to stabilise an arbitrary column of a generic element $g \in G(\Phi, R)$. Let, in the above setting, g be any non-central element of H, and let $x_\alpha(1)$ be an elementary root unipotent not commuting with g. Replacing g by $[g, x_\alpha(1)] \in H$, which is itself non-central, we can from the very start assume that g is a root unipotent. [Well, technically the product of a root unipotent $z = gx_\alpha(1)g^{-1}$ by an elementary root unipotent $x_\alpha(-1)$, but that does not change most of the columns of z.]

Now, recollect the definition of a microweigh representation, which essentially amounts to saying that exponents are very short,

$$gx_\alpha(1)g^{-1} = g(e + e_\alpha)g^{-1} = e + ge_\alpha g^{-1},$$

where e_α is the root element of the corresponding Lie algebra. Thus, outside of the principal diagonal z satisfies the linear equations defining the Lie algebra of $G(\Phi, R)$. In particular, all columns of z abound with zeroes – at least 10 of them in each column in the case of (E_6, ϖ_1) and at least 28 of them in each column in the case of (E_7, ϖ_7).

For instance, if we consider the first columns of these matrices, corresponding to the highest weight $\varpi = \varpi_1$ or $\varpi = \varpi_7$, then they have zeroes in the 10 positions occuring to the right of the 5 parallel bonds labelled 1 in Figure 1, and in the 28 positions occuring below the 10 parallel bonds labelled 7 in Figure 2.

But now something extraordinary happens. Since all these components are equal to zero, any positive root whose linear expansion contains α_1 or α_7, respectively, – in other words, any root from the unipotent radical of P_1 in E_6 or P_7 in E_7 – performs exactly one nontrivial addition. In fact, only the highest weight coordinate v_ϖ is affected, since all other coordinates v_λ which could be added somewhere by $x_\alpha(\xi)$ – or, what is the same, coordinates corresponding to the weights $\lambda \in \Lambda$ such that $\lambda + \alpha \in \Lambda$, – are themselves equal to zero.

This means, there are plenty of ways to stabilise such a column with small unipotents, starting with something as teeny as products of *two* elementary root unipotents in A_2 or even $2\,\mathrm{A}_1$. This makes that part of the proof as short and easy, as the proofs for classical cases, expounded in [17]. In fact, there is no more need to control signs of the structure constants, or to invoke any equations, other then the linear equations defining the Lie algebra of G. Of course, now it is quite a bit trickier to prove the main lemma asserting that we have enough such unipotents, to still eventually get a nontrivial elementary root unipotent inside H.

Actually, later I tried to elaborate that proof, to achieve simultaneous stabilisation of several columns, to achieve reduction to smaller rank parabolics. Such a reduction is necessary in many further applications, including possible applications at the level of K_2. One of the tricks I proposed, was the so called A_3-proof, see [25, 28], which allowed to simultaneously stabilise *two* columns of matrices from E_6 and E_7, some of whose entries vanish.

There were also other similar attemps, in particular, those consisting in varying not only the classical subsystem but also the parabolic subgroup therein, outlined in our papers with Victoria Kazakevich [32, 33]. In some cases they allowed to simultaneously stabilise a column and a row of matrices from E_6 and E_7 in the corresponding representations.

However, as we see in the next sections, at that time we missed something very essential, that the A_2-proof itself supplies reduction not to maximal parabolic, but to some rather deep ones.

§4. A_2-proof for E_6 reappraised: paghi uno, prendi tre

In [30] we constructed root elements x of type A_2 that stabilise *one* column of a root type unipotent g in groups of types E_6 and E_7. Here, we show that in fact such an x *automatically* stabilises *three* adjacent columns of g for the case of E_6 and *six* such adjacent columns[1] for the case of E_7.

Thus, instead of reduction to maximal parabolics of types P_1 or P_7 this proof gives at no extra cost reduction to much smaller parabolics. With this observation, some proofs in [30] could be made even shorter, then they actually are. More importantly, greater ease to extract unipotents from small rank parabolics provides entirely new promise of applications. I advertise two of such forthcoming applications in the last section.

The following result asserts that the same small unipotent x coming from A_2 that was used in [30] to stabilise one column of a root unipotent g, in fact stabilises three columns. To reconcile notation with the subsequent argument, we state it in the form asserting that x stabilises the first three columns of g. Since the Weyl group $W(\mathrm{E}_6)$ is transitive on triples of pairwise adjacent weights, this does not incur any loss of generality.

Theorem 2. *Let $g \in G(\mathrm{E}_6, R)$ be a root type unipotent. Then there exists a non-trivial root type unipotent $x = x_\alpha(\xi)x_\beta(\zeta)$ of type A_2 such that*

$$(xg)_{*\varpi_1} = g_{*\varpi_1}, \quad (xg)_{*,\varpi_1-\alpha_1} = g_{*,\varpi_1-\alpha_1}, \quad (xg)_{*,\varpi_1-\alpha_1-\alpha_3} = g_{*,\varpi_1-\alpha_1-\alpha_3}.$$

Proof. In the following calculations we realise the 27-dimensional module $V(\varpi_1)$ as an internal Chevalley module in the standard parabolic subgroup P_7 of the simply connected Chevalley group $G(\mathrm{E}_7, R)$. In other words, we identify $V(\varpi_1)$ with the unipotent radical $V = U_7$, equipped with the conjugation action of the [algebraic] commutator subgroup of the Levi factor L_7.

Thus, the roots of E_6 are depicted by their Dynkin form in E_6, whereas the weights of the 27-dimensional module V are depicted by their Dynkin form in E_7. Under the above identification the weights of V are precisely the roots of E_7, such that α_7 occurs in their expansions with the coefficient 1. As usual, we denote the set of all such roots by Λ. It is easy to check that in this realisation the vectors $v^\alpha = x_\alpha(1)$, $\alpha \in \Lambda$, constitute a crystal base of V (see [24, 26] for proofs and further details).

Recall that the maximal number of roots of E_6, forming mutual angles $\pi/3$, equals 5. Let us fix such a set, maximal with respect to the chosen order on E_6:

$$\beta_1 = \begin{smallmatrix}12321\\2\end{smallmatrix}, \quad \beta_2 = \begin{smallmatrix}12321\\1\end{smallmatrix}, \quad \beta_3 = \begin{smallmatrix}12221\\1\end{smallmatrix}, \quad \beta_4 = \begin{smallmatrix}12211\\1\end{smallmatrix}, \quad \beta_5 = \begin{smallmatrix}12210\\1\end{smallmatrix}.$$

[1] There are two Weyl orbits on sextuples of adjacent columns in (E_7, ϖ_7), the one in question are sextuples in P_3-position, those that cannot be completed to a heptuple. The other orbit consists of sextuples in $P_1 \cap P_2$-position.

Further, consider the following three series of weights:

$$\gamma_1 = \begin{smallmatrix}001111\\1\end{smallmatrix}, \quad \gamma_2 = \begin{smallmatrix}001111\\0\end{smallmatrix}, \quad \gamma_3 = \begin{smallmatrix}000111\\0\end{smallmatrix}, \quad \gamma_4 = \begin{smallmatrix}000011\\0\end{smallmatrix}, \quad \gamma_5 = \begin{smallmatrix}000001\\0\end{smallmatrix},$$

$$\delta_1 = \begin{smallmatrix}011111\\0\end{smallmatrix}, \quad \delta_2 = \begin{smallmatrix}011111\\1\end{smallmatrix}, \quad \delta_3 = \begin{smallmatrix}012111\\1\end{smallmatrix}, \quad \delta_4 = \begin{smallmatrix}012211\\1\end{smallmatrix}, \quad \delta_5 = \begin{smallmatrix}012221\\1\end{smallmatrix},$$

$$\varepsilon_1 = \begin{smallmatrix}111111\\0\end{smallmatrix}, \quad \varepsilon_2 = \begin{smallmatrix}111111\\1\end{smallmatrix}, \quad \varepsilon_3 = \begin{smallmatrix}112111\\1\end{smallmatrix}, \quad \varepsilon_4 = \begin{smallmatrix}112211\\1\end{smallmatrix}, \quad \varepsilon_5 = \begin{smallmatrix}112221\\1\end{smallmatrix}.$$

Our proof starts with the observation that

$$\varpi_1 - \varepsilon_1 = (\varpi_1 - \alpha_1) - \delta_1 = \gamma_{12} - \gamma_2 = \beta_1,$$

$$\varpi_1 - \varepsilon_2 = (\varpi_1 - \alpha_1) - \delta_2 = \gamma_{12} - \gamma_1 = \beta_2.$$

As we mentioned in the previous section, outside of the principal diagonal the entries of a root element $g \in G(\mathrm{E}_6, R)$ are subject to the linear equations defining the Lie algebra of $G(\mathrm{E}_6, R)$. In particular,

$$g_{\varpi_1,\varepsilon_1} = \pm g_{\varpi_1-\alpha_1,\delta_1} = \pm g_{\gamma_{12},\gamma_2}$$

and similarly

$$g_{\varpi_1,\varepsilon_2} = \pm g_{\varpi_1-\alpha_1,\delta_2} = \pm g_{\gamma_{12},\gamma_1}.$$

However, unlike the original A_2-proof [30], now it is essential that we add 0, rather than *twice* something. Thus, as in the A_5-proof [24] now we have to keep an eye on signs. The relevant entries come from the multiples of the root elements $e_{-\beta_1}$ and $e_{-\beta_2}$. However (see, for instance, [24] or [26]), their signs are the same, as the signs of the corresponding entries of the opposite root elements e_{β_1} and e_{β_2}.

Looking at the last two lines of [38], Table 10 (or, for that matter, Tables 13 or 16 therein), we see that

$$e_{\beta_1} = e_{\varpi_1,\varepsilon_1} + e_{\varpi_1-\alpha_1,\delta_1} + e_{\gamma_{12},\gamma_2} + e_{\gamma_{13},\gamma_3} + e_{\gamma_{14},\gamma_4} + e_{\gamma_{15},\gamma_5},$$

$$e_{\beta_2} = -e_{\varpi_1,\varepsilon_2} - e_{\varpi_1-\alpha_1,\delta_2} - e_{\gamma_{12},\gamma_1} + e_{\gamma_{23},\gamma_3} + e_{\gamma_{24},\gamma_4} + e_{\gamma_{25},\gamma_5}.$$

Actually, the first of these elements is already calculated in [24], Proposition 1. Besides, we could easily calculate the second one *by hand*, with the same recipe.

Combining the two above observations, we see that

$$g_{\varpi_1,\varepsilon_1} = g_{\varpi_1-\alpha_1,\delta_1} = g_{\gamma_{12},\gamma_2} = \xi, \qquad g_{\varpi_1,\varepsilon_2} = g_{\varpi_1-\alpha_1,\delta_2} = g_{\gamma_{12},\gamma_1} = \zeta.$$

Now, it is clear that if $\xi \neq 0$, then $x_{\beta_1}(\zeta)x_{\beta_2}(-\xi)$ is the desired unipotent. On the other hand, if $\xi = 0$, then the first three columns of g are not modified already by the action of $x_{\beta_1}(1)$. □

Inspecting the above proof, we see that the fact that g is a root element is never used as such, we only invoke *some* linear equations on its entries. Combining this with the observation immediately preceding the theorem, we see that we have in fact established the following technical but somewhat more general result, see [29] for a comprehensive exposition.

Theorem 3. *Let $\lambda, \mu, \nu \in \Lambda(\varpi_1)$ be three pair-wise adjacent weights. Assume that the entries of $g \in G_{\varpi_1}(\mathrm{E}_6, R)$ satisfy the following linear equations.*

- *$g_{\rho\sigma} = 0$ for $\sigma = \lambda, \mu, \nu$ and any $\rho \in \Lambda(\varpi_1)$ such that $d(\rho, \sigma) \geqslant 2$.*
- *There exist roots $\alpha, \beta \in \Phi$ such that*

$$g_{\lambda,\lambda-\alpha} = g_{\lambda,\mu-\alpha} = g_{\lambda,\nu-\alpha} = \xi, \qquad g_{\lambda,\lambda-\beta} = g_{\lambda,\mu-\beta} = g_{\lambda,\nu-\beta} = \zeta.$$

Then there exists a non-trivial root unipotent $x = x_{\beta_1}(\zeta)x_{\beta_2}(\xi)$ of type A_2 such that

$$(xg)_{*\lambda} = g_{*\lambda}, \qquad (xg)_{*\mu} = g_{*\mu}, \qquad (xg)_{*\nu} = g_{*\nu}.$$

§5. A_2-proof for E_7 reappraised: paghi uno, prendi sei

For A_7, a similar reassessment of the A_2-proof [30], leads to the following *astounding* result. The bottom line is that forming *one* commutator of a root element g with a small unipotent x of type A_2, whose entries are chosen to stabilise one column, we *automatically* stabilise *six* of them. In other words, implementing reduction to a maximal parabolic P_7 we immediately precipitate to a rank 2 parabolic $P_3 \cap P_4 \cap P_5 \cap P_6 \cap P_7$.

Theorem 4. *Let $g \in G(\mathrm{E}_7, R)$ be a root type unipotent. Then there exists a non-trivial root type unipotent $x = x_\alpha(\xi)x_\beta(\zeta)$ of type A_2 such that*

$$\begin{aligned}
(xg)_{*\varpi_7} &= g_{*\varpi_7},\\
(xg)_{*,\varpi_7-\alpha_7} &= g_{*,\varpi_7-\alpha_7},\\
(xg)_{*,\varpi_7-\alpha_7-\alpha_6} &= g_{*,\varpi_7-\alpha_7-\alpha_6},\\
(xg)_{*,\varpi_7-\alpha_7-\alpha_6-\alpha_5} &= g_{*,\varpi_7-\alpha_7-\alpha_6-\alpha_5},\\
(xg)_{*,\varpi_7-\alpha_7-\alpha_6-\alpha_5-\alpha_4} &= g_{*,\varpi_7-\alpha_7-\alpha_6-\alpha_5-\alpha_4},\\
(xg)_{*,\varpi_7-\alpha_7-\alpha_6-\alpha_5-\alpha_4-\alpha_3} &= g_{*,\varpi_7-\alpha_7-\alpha_6-\alpha_5-\alpha_4-\alpha_3}.
\end{aligned}$$

Proof. Here we describe the 56-dimensional module $V = V(\varpi_7)$ for the simply connected Chevalley group $G(\mathrm{E}_7, R)$ as an internal Chevalley module in the parabolic subgroup P_8 of the Chevalley group $G(\mathrm{E}_8, R)$. More precisely, it is interpreted as the section $V = U_8/[U_8, U_8]$ of the derived series of the unipotent radical U_8, equipped with the conjugation action of the [algebraic] commutator subgroup of the Levi subgroup L_8.

Thus, the roots of E_7 are depicted by their Dynkin forms in E_7, whereas the weights of the module V are depicted by their Dynkin form in E_8. Here, the weights are precisely the roots of E_8, such that α_8 appears in their expansions with coefficient 1. As in the case of E_6, we denote the set of all such weights by Λ. Since $[U_8, U_8] = X_\rho$, where, as above,

$$\rho = \begin{smallmatrix}2465432\\3\end{smallmatrix}$$

is the maximal root of E_8, the vectors $v^\alpha = x_\alpha(1)X_\rho$, $\alpha \in \Lambda$, form a base of V.

Let us describe all weights of $V(\varpi_7)$. Consider the following series of weights

$$\gamma_1 = \begin{smallmatrix}1111111\\0\end{smallmatrix}, \quad \gamma_2 = \begin{smallmatrix}0111111\\0\end{smallmatrix}, \quad \gamma_3 = \begin{smallmatrix}0011111\\0\end{smallmatrix}, \quad \gamma_4 = \begin{smallmatrix}0001111\\0\end{smallmatrix},$$
$$\gamma_5 = \begin{smallmatrix}0000111\\0\end{smallmatrix}, \quad \gamma_6 = \begin{smallmatrix}0000011\\0\end{smallmatrix}, \quad \gamma_7 = \begin{smallmatrix}0000001\\0\end{smallmatrix}.$$

Recall that the maximal number of root of E_7, forming mutual angles $\pi/3$, equals 7. Let us fix such a set, maximal with respect to the chosen order on E_7:

$$\beta_1 = \begin{smallmatrix}234321\\2\end{smallmatrix}, \quad \beta_2 = \begin{smallmatrix}134321\\2\end{smallmatrix}, \quad \beta_3 = \begin{smallmatrix}124321\\2\end{smallmatrix}, \quad \beta_4 = \begin{smallmatrix}123321\\2\end{smallmatrix},$$
$$\beta_5 = \begin{smallmatrix}123221\\2\end{smallmatrix}, \quad \beta_6 = \begin{smallmatrix}123211\\2\end{smallmatrix}, \quad \beta_7 = \begin{smallmatrix}123210\\2\end{smallmatrix}.$$

Then all weights of $V(\varpi_7)$ look as follows:

- 7 weights γ_i,
- 21 weights $\gamma_{ij} = \beta_i + \gamma_j = \beta_j + \gamma_i$, $i \neq j$,
- 21 weights $\gamma^*_{ij} = \rho - \gamma_{ij}$,
- 7 weights $\gamma^*_i - \rho - \gamma_i$.

The proof is similar to the proof of Theorem 2, and starts with the observation that

$$\gamma_7^* - \gamma_{17}^* = \gamma_6^* - \gamma_{16}^* = \gamma_5^* - \gamma_{15}^* = \gamma_4^* - \gamma_{14}^* = \gamma_3^* - \gamma_{13}^* = \gamma_{17}^* - \gamma_2 = \beta_1,$$

$$\gamma_7^* - \gamma_{27}^* = \gamma_6^* - \gamma_{26}^* = \gamma_5^* - \gamma_{25}^* = \gamma_4^* - \gamma_{24}^* = \gamma_3^* - \gamma_{23}^* = \gamma_{12} - \gamma_1 = \beta_2.$$

We skip all actual verifications, that the signs agree, etc. With that end we have to recourse to [35], Table 10 (or, for that matter, Tables 12, 14 or 16). All details are reproduced in my forthcoming paper [29]. □

§6. D_5-proof for E_6

In this section we construct root type unipotents stabilising a column of an element $g \in G_{\varpi_1}(\mathrm{E}_6, R)$, provided this column has [at least] one zero. These unipotents will be constructed in terms of subsets conjugate to the following eight-element subset

$$\beta_1 = \begin{smallmatrix}12321\\2\end{smallmatrix} \qquad \beta_2 = \begin{smallmatrix}12321\\1\end{smallmatrix} \qquad \beta_3 = \begin{smallmatrix}12221\\1\end{smallmatrix} \qquad \beta_4 = \begin{smallmatrix}12211\\1\end{smallmatrix}$$

$$\beta_{-4} = \begin{smallmatrix}11221\\1\end{smallmatrix} \qquad \beta_{-3} = \begin{smallmatrix}11211\\1\end{smallmatrix} \qquad \beta_{-2} = \begin{smallmatrix}11111\\1\end{smallmatrix} \qquad \beta_{-1} = \begin{smallmatrix}11111\\0\end{smallmatrix}$$

In the sequel this set is denoted by $\Omega \subseteq \mathrm{E}_6$. Up to conjugacy by an element of the Weyl group, Ω can be characterised as a maximal subset with the following property, see [27], Corollary 1. A root $\beta_i \in \Omega$ is orthogonal to β_{-i}, and forms the angle $\pi/3$ with all roots β_j, $j \neq \pm i$. In other words, its weight diagram is of type D_4. Obviously, Ω is higher than any of its Weyl conjugates. Thus, it is uniquely characterised as the senior subset of this shape.

Theorem 5. *Let $g \in G(\mathrm{E}_6, R)$. Assume that $g_{\mu\lambda} = 0$ for a pair of distant weights λ, μ. Then there exists a non-trivial root type unipotent*

$$z = x_{\beta_1}(z_1)x_{\beta_2}(z_2)\dots x_{\beta_{-2}}(z_{-2})x_{\beta_{-1}}(z_{-1})$$

of type D_5 such that $(xg)_{*\lambda} = g_{*\lambda}$. The parameters of z may be chosen to be equal to $\pm g_{\nu\lambda}$, where $\nu \in \Omega$.

Proof. For the product

$$z = x_{\beta_1}(z_1)x_{\beta_2}(z_2)\dots x_{\beta_{-2}}(z_{-2})x_{\beta_{-1}}(z_{-1})$$

to be an element of root type, its coefficients $z_1, \dots, z_{-1} \in R$ should lie on a quadric in the eight-dimensional affine space defined by the equation

$$z_1 z_{-1} \pm z_2 z_{-2} \pm z_3 z_{-3} \pm z_4 z_{-4} = 0.$$

The signs can be specified, and this is done in [24, 27]. The toughest part of the proof of this theorem consists exactly in the verification that all signs agree, and this can be done either by explicit computer calculations, or by the methods of [24, 26, 27].

As usual, conjugating by an element of the Weyl group $W(\mathrm{E}_6)$ we can from the very start replace any pair (λ,μ) of distant weights by any other such pair, say by $(\varpi_1,-\varpi_6)$. For the argument below it will be convenient for us to rename weights of $\Lambda(\varpi_1)$. The weights $\varpi_1,\upsilon,-\varpi_6$, where $\upsilon=\substack{012221\\1}$, constitute the highest triad of pair-wise distant weights, see [24, 34, 38]. The remaining 24 weights $\Lambda\setminus\{\varpi_1,\upsilon,-\overline{\omega}\}$ are naturally subdivided into 3 octets[2], each forming the diagram of type D_4:

$$\lambda_1=\substack{000011\\0},\quad \lambda_2=\substack{000111\\0},\quad \lambda_3=\substack{001111\\0},\quad \lambda_4=\substack{011111\\0},$$

$$\lambda_{-4}=\substack{001111\\1},\quad \lambda_{-3}=\substack{011111\\1},\quad \lambda_{-2}=\substack{012111\\1},\quad \lambda_{-1}=\substack{012211\\1},$$

$$\mu_1=\substack{111111\\0},\quad \mu_2=\substack{111111\\1},\quad \mu_3=\substack{112111\\1},\quad \mu_4=\substack{1122111\\1},$$

$$\mu_{-4}=\substack{122111\\1},\quad \mu_{-3}=\substack{122211\\1},\quad \mu_{-2}=\substack{123211\\1},\quad \mu_{-1}=\substack{123211\\2},$$

$$\nu_1=\substack{112221\\1},\quad \nu_2=\substack{122221\\1},\quad \nu_3=\substack{123221\\1},\quad \nu_4=\substack{123221\\2},$$

$$\nu_{-4}=\substack{123321\\2},\quad \nu_{-3}=\substack{123321\\2},\quad \nu_{-2}=\substack{124321\\2},\quad \nu_{-1}=\substack{134321\\2}.$$

We set

$$\begin{aligned} z={}&x_{\beta_1}(g_{\lambda_4,\varpi})x_{\beta_2}(g_{\lambda_3,\varpi})x_{\beta_3}(g_{\lambda_2,\varpi})x_{\beta_4}(g_{\lambda_1,\varpi})\\ &\times x_{\beta_{-4}}(g_{\lambda_{-1},\varpi})x_{\beta_{-3}}(g_{\lambda_{-2},\varpi})x_{\beta_{-2}}(g_{\lambda_{-3},\varpi})x_{\beta_{-1}}(g_{\lambda_{-4},\varpi}) \end{aligned}$$

where, for breavity sake we write simply ϖ instead of ϖ_1. Let us inspect, how left multiplication by z modifies the first column of g. Multiplication by z yields $8\cdot 6=48$ additions within each column.

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[2] Andrei Lavrenov suggested that the most natural way to dub these weights would be to use the three *ættir* of Futhark. Eventually, I am going to follow that suggestion, but for now I keep the interim notation used in the slides of my talks.

• First of all, entries in the positions $(-\varpi_6,\varpi)$, (λ_i,ϖ) and (υ,ϖ) are not affected at all.

• Entries in the positions (μ_i,ϖ) are affected by 1 addition each, since $\mu_i=-\varpi_6+\beta_{-i}$. However, by assumption $g_{-\varpi_6,\varpi}=0$ so that the extra summand is 0. This accounts for 8 additions.

• Entries in the positions (ν_i,ϖ) are very much affected, for each of them *four* extra summands occur, each of the form $\pm g_{\lambda_i,\varpi}g_{\lambda_j,\varpi}$. Let us reproduce the corresponding fragment of the matrix of signs of $V(\varpi_1)$, see [38], Table 7. The entry of this matrix in the position (λ,μ) is the sign, with which e_α adds v^λ to v^μ. Actually, using the algorithm proposed in [24, 26], these signs could have been easily calculated by hand.

	1112221	1122221	1123221	1223221	1123321	1223321	1224321	1234321
0000011	+	+	+	+	0	0	0	0
0000111	−	−	0	0	+	+	0	0
0001111	+	0	−	0	−	0	+	0
0011111	−	0	0	−	0	−	−	0
0101111	0	+	+	0	+	0	0	+
0111111	0	−	0	+	0	+	0	−
0112111	0	0	−	−	0	0	+	+
0112211	0	0	0	0	−	−	−	−

Let us summarise, what it means precisely in terms of additions to the eight entries $g_{\mu_i,\varpi}$:

$$\begin{pmatrix} -g_{\lambda_1,\varpi} & g_{\lambda_2,\varpi} & -g_{\lambda_3,\varpi} & g_{\lambda_4,\varpi} & 0 & 0 & 0 & 0 \\ -g_{\lambda_{-2},\varpi} & g_{\lambda_{-1},\varpi} & 0 & 0 & -g_{\lambda_3,\varpi} & g_{\lambda_4,\varpi} & 0 & 0 \\ -g_{\lambda_{-3},\varpi} & 0 & g_{\lambda_{-1},\varpi} & 0 & -g_{\lambda_2,\varpi} & 0 & g_{\lambda_4,\varpi} & 0 \\ -g_{\lambda_{-4},\varpi} & 0 & 0 & g_{\lambda_{-1},\varpi} & 0 & -g_{\lambda_2,\varpi} & g_{\lambda_3,\varpi} & 0 \\ 0 & -g_{\lambda_{-3},\varpi} & g_{\lambda_{-2},\varpi} & 0 & -g_{\lambda_1,\varpi} & 0 & 0 & g_{\lambda_4,\varpi} \\ 0 & -g_{\lambda_{-4},\varpi} & 0 & g_{\lambda_{-2},\varpi} & 0 & -g_{\lambda_1,\varpi} & 0 & g_{\lambda_3,\varpi} \\ 0 & 0 & -g_{\lambda_{-4},\varpi} & g_{\lambda_{-3},\varpi} & 0 & 0 & -g_{\lambda_1,\varpi} & g_{\lambda_2,\varpi} \\ 0 & 0 & 0 & 0 & -g_{\lambda_{-4},\varpi} & g_{\lambda_{-3},\varpi} & -g_{\lambda_{-2},\varpi} & g_{\lambda_{-1},\varpi} \end{pmatrix} \begin{pmatrix} g_{\lambda_{-1},\varpi} \\ g_{\lambda_{-2},\varpi} \\ g_{\lambda_{-3},\varpi} \\ g_{\lambda_{-4},\varpi} \\ g_{\lambda_4,\varpi} \\ g_{\lambda_3,\varpi} \\ g_{\lambda_2,\varpi} \\ g_{\lambda_1,\varpi} \end{pmatrix}.$$

As we see, quite appropriately, in six of the affected positions non-zero signs alternate and the occuring extra summands cancel pair-wise, which accounds for further $6 \cdot 4 = 24$ additions.

So far, we have only used signs of the structure costants, now it comes to equations. As in [24], to conclude the proof we should recall that our column is not an arbitrary element of $V(\varpi)$, but a column of a matrix $g \in G_{\varpi}(\Phi, R)$. As is well known, any such column lies in the highest weight orbit, and, thus, satisfies 27 five-term quadratic equations of the form

$$x_1 x_{-1} \pm x_2 x_{-2} \pm x_3 x_{-3} \pm x_4 x_{-4} \pm x_5 x_{-5} = 0,$$

known as Borel–Freudenthal equations, see [24, 27, 34, 38] for an explicit choice of signs in these equations, which is absolutely vital for what follows. We are interested in those equations that involve $g_{-\varpi_6,\tau}$. Obviously, for the first column of our matrix, where $g_{-\varpi_6,\varpi} = 0$, they would reduce to similar *four-term* equations. Looking at the first and the last rows of the above matrix, we see – *un miracolo*! – that we get exactly one of these equations (actually, the lowest one in terms of the natural order on weights) twice, with correct signs! That accounts for other $2 \cdot 4 = 8$ additions.

• The diagonal entry in the positions (ϖ, ϖ) is the most affected one, since there are 8 occuring extra summands (and that was it, since $8+24+8+8=48$). Similarly, looking at the last column of the sign matrix we see that the signs in the eight relevant positions alternate as follows $+-+-$ and then $-+-+$. In other words, multiplication by z adds to $g_{\varpi,\varpi}$ the following expression

$$\begin{aligned} &g_{\lambda_1,\varpi} g_{\mu_4,\varpi} - g_{\lambda_2,\varpi} g_{\mu_3,\varpi} + g_{\lambda_3,\varpi} g_{\mu_2,\varpi} - g_{\lambda_4,\varpi} g_{\mu_1,\varpi} \\ &\qquad - g_{\lambda_{-1},\varpi} g_{\mu_{-4},\varpi} + g_{\lambda_{-2},\varpi} g_{\mu_{-3},\varpi} - g_{\lambda_{-3},\varpi} g_{\mu_{-2},\varpi} + g_{\lambda_{-4},\varpi} g_{\mu_{-1},\varpi}. \end{aligned}$$

But this expression is clearly 0, since both the first and the second lines are equal to 0 by the same reason as above, they are the chunks of the corresponding Borel–Freudenthal equations obtained by obliterating terms involving $g_{-\varpi_6,\varpi} = 0$. We are done. □

§7. Further variations and final remarks

In fact, the three new versions of decomposition of unipotents I sketched in this talk, are just a prelude and an anticipation of a much broader prospect. At this point I came across a dozen or two of similar variations for E_6 and E_7, and I am absolutely positive that further systematic search should unearth many more such explicit reductions to small rank parabolics, also in other representations, and for other groups. Let me mention some of the simplest ones, which are already there.

• D_6-proof: stabilisation of a column with two adjacent zeros in E_7.

• A_3-proof revisited: stabilisation of a column with 5 zeros in P_5 positions in GL_{27}.

• (A_3, P_2)-proof: simultaneous stabilisation of a column and a row in a root element of E_6.

• A_4-proof: stabilisation of a column with 5 zeros in $P_2 \cap P_6$ positions in GL_{27}.

• A_5-proof revisited: simultaneous stabilisation of two columns in elements of E_6 and GL_{27}.

Finally, let me mention some of the most immediate possible applications of these methods.

• **Description of overgroups of subsystem subgroups.** More precisely, let $\Delta \subseteq \Phi$ be a [sufficiently large] root subsystem. What are the intermediate subgroups $E(\Delta, R) \leqslant H \leqslant G(\Phi, R)$? Before this work, there was not a single instance, where this problem was fully solved in an exceptional group over an arbitrary commutative ring. Consult our paper with Alexander Shchegolev [45] for precise conjectures and references to the known results for classical groups.

One of the main technical steps in the proof would be extraction of unipotents from an element $g \in H$ sitting in a proper parabolic P, with the use of unipotents from $E(\Delta, R)$. Since we do not wish to exclude cases when some irreducible components of Δ have rank 2 [or even 1, for that matter], it would be highly expedient to be able to limit ourselves to the case, where g sits in a *small* parabolic.

Presently, Alexander Shchegolev and I have virtually completed the analysis of intermediate subgroups for the simplest such exceptional embedding $\mathrm{A}_7 \leqslant \mathrm{E}_7$, where the answer is stated in terms of one ideal of R. The proof crucially depends on our Theorem 4.

• **Description of subnormal subgroups**. It is well known that this problem is essentially equivalent to description of subgroups of $G(\Phi, R)$, normalised by the relative elementary subgroup $E(\Phi, R, A)$, for an ideal $A \trianglelefteq R$. The standard answer to this last problem looks as follows. There exists an $m = m(\Phi)$ such that for any subgroup $H \leqslant G(\Phi, R)$ normalised by $E(\Phi, R, A)$, there exists an ideal $I \trianglelefteq R$ such that $E(\Phi, R, A^m I) \leqslant H \leqslant C(\Phi, R, I)$.

The most important technical aspects of this problem is to find the smallest such m. This amounts to forming consecutive commutators with elements of $E(\Phi, R, A)$, to extract root unipotents. Clearly, the possibility of getting into small parabolics by forming a single commutator is instrumental in minimising m.

Again, before this work, there was not a single case, where this problem was fully solved for exceptional groups. Equipped with Theorems 2 and 4, Zuhong Zhang recently succeeded in vanquishing the cases E_6 and E_7, with the bound $m = 7$, very close to the actual bounds for classical groups.

Finally, let me mention the most ambitious possible application.

• **Structure of isotropic reductive groups**. Another extremely important unsolved problem is to obtain the standard description of normal subgroups in twisted forms ${}_{\varphi}G(\Phi, R)$ of Chevalley groups of types E_6 and E_7 over an arbitrary commutative ring R, provided that they contain a split subgroup of type A_2. Let us list the forms in question, see [9].

◦ For E_6, this is the twisted Chevalley group of type ${}^2\mathrm{E}_6$ – the above problem is not solved even for quasi-split forms! – plus two *inner* forms of relative rank 2, with Tits indices $\mathrm{E}_{6,2}^{28}$ and $\mathrm{E}_{6,2}^{16}$, plus two further *outer* forms of relative rank 2, with Tits indices ${}^2\mathrm{E}_{6,2}^{16'}$ and ${}^2\mathrm{E}_{6,2}^{16''}$.

◦ For E_7 these are the forms of relative ranks 2, 3 and 4, with Tits indices $\mathrm{E}_{7,2}^{31}$, $\mathrm{E}_{7,3}^{28}$ and $\mathrm{E}_{7,4}^{9}$, respectively.

I am convinced that the methods discussed here could launch a viable approach towards the solution of that problem, which could then constitute a realistic alternative to or a beneficial reinforcement of the localisation methods developed in this setting by Anastasia Stavrova, Victor Petrov, Alexander Luzgarev [8, 6] and others.

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St. Petersburg State University
E-mail: `nikolai-vavilov@yandex.ru`

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