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FREE BOUNDARY PROBLEM OF MAGNETOHYDRODYNAMICS

Abstract. We consider a free boundary problem governing the motion of a finite isolated mass of a viscous incompressible electrically conducting fluid in vacuum. Media is moving under the action of magnetic field and volume forces. We prove solvability of this free boundary problem in an infinite time interval under the additional smallness assumptions imposed on initial data and the external forces.

Dedicated to the 80-th jubilee of V. A. Solonnikov

§1. Statement of the problem

Problems of magnetohydrodynamics in fixed simply connected domains has been studied by O. A. Ladyzhenskaya and V. A. Solonnilov in the classical papers [1, 2]. In the last five years, V. A. Solonnikov and his coauthors investigated various free boundary problems of magnetohydrodynamics [3–7]. In particular, unique solvability in an infinite time interval of a free boundary problem governing the motion of a finite isolated mass of electrically conducting capillary liquid in vacuum is proved in [7], provided that initial data are sufficiently small. The present paper can be regarded as a continuation of [7] and extends the result of this paper to the case of the nonhomogeneous equation.

We consider the motion of a finite isolated mass of a viscous incompressible liquid, which possesses electrical conductivity and capillary properties. It is assumed that the liquid is contained in a bounded variable domain Ω_{1t} which boundary consists of two disjoint components: the free boundary Γ_t and the fixed surface Σ that is also a boundary of a fixed domain D. The domain $\bar{D} \cup \Omega_{1t}$ is surrounded by a bounded vacuum region Ω_{2t} with the

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exterior boundary S. It is assumed that the given surfaces Γ_0, S, and Σ are homeomorphic to a sphere, $\Gamma_0 \cap S = \varnothing$, and $\Gamma_0 \cap \Sigma = \varnothing$.

Let $\boldsymbol{f}$ be the force acting. As the region occupied by the fluid is unknown, we assume that this force is defined in the wider domain $\Omega_{10} \cup \Gamma_0 \cup \Omega_{20}$. The problem consists of finding the variable domains Ω_{it}, $(i = 1, 2)$ together with the velocity vector field $\boldsymbol{v}(x,t)$, the pressure $p(x,t)$, $x \in \Omega_{1t}$, and the magnetic field $\boldsymbol{H}(x,t)$, $x \in \Omega_{1t} \cup \Omega_{2t}$. Equations in Ω_{1t} have the form

$$\begin{gathered}\boldsymbol{v}_t + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v} - \nabla\cdot T(\boldsymbol{v},p) - \nabla\cdot T_M(\boldsymbol{H}) = \boldsymbol{f}, \quad \nabla\cdot\boldsymbol{v}(x,t) = 0, \\ \mu_1\boldsymbol{H}_t + \alpha^{-1}\operatorname{rot}\operatorname{rot}\boldsymbol{H} - \mu_1\operatorname{rot}(\boldsymbol{v}\times\boldsymbol{H}) = 0, \quad \nabla\cdot\boldsymbol{H}(x,t) = 0,\end{gathered} \tag{1.1}$$

where ν is the kinematic viscosity, α-conductivity, μ_1-magnetic permeability in Ω_{1t}. $T(\boldsymbol{v},p) = -pI + \nu S(\boldsymbol{v})$ is the viscous stress tensor, $S(\boldsymbol{v}) = \nabla\boldsymbol{v} + (\nabla\boldsymbol{v})^T = \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)_{i,j=1,2,3}$ is the doubled tensor of small strain, $T_M(\boldsymbol{H}) = \mu(\boldsymbol{H}\otimes\boldsymbol{H} - \frac{1}{2}I|\boldsymbol{H}|^2)$ is the magnetic stress tensor.

Magnetic field in the vacuum region Ω_{2t} satisfies the equations

$$\operatorname{rot}\boldsymbol{H} = 0, \qquad \nabla\cdot\boldsymbol{H}(x,t) = 0. \tag{1.2}$$

Equations (1.1), (1.2) are supplied with the following boundary conditions. On the free boundary we set

$$\begin{gathered}\big(T(\boldsymbol{v},p) + [T_M(\boldsymbol{H})]\big)\boldsymbol{n} = \sigma\boldsymbol{n}\mathcal{H}, \\ \boldsymbol{V}_n = \boldsymbol{v}\cdot\boldsymbol{n}, \quad [\mu\boldsymbol{H}\cdot\boldsymbol{n}] = 0, \quad [\boldsymbol{H}_\tau] = 0, \quad x \in \Gamma_t, \quad t > 0.\end{gathered} \tag{1.3}$$

On fixed boundaries we set

$$\begin{gathered}\boldsymbol{H}(x,t)\cdot\boldsymbol{n}(x) = 0, \quad x \in S, \quad t > 0, \\ \boldsymbol{H}(x,t)\cdot\boldsymbol{n}(x) = 0, \quad \operatorname{rot}_\tau\boldsymbol{H} = 0, \quad x \in \Sigma, \quad t > 0, \\ \boldsymbol{v}(x,t) = 0, \quad x \in \Sigma, \quad t > 0.\end{gathered} \tag{1.4}$$

Finally, we append the initial conditions

$$\boldsymbol{v}(x,0) = \boldsymbol{v}_0(x), \quad x \in \Omega_{10}, \qquad \boldsymbol{H}(x,0) = \boldsymbol{H}_0(x), \quad x \in \Omega_{10} \cup \Omega_{20}. \tag{1.5}$$

Here μ_2 is the magnetic permeability in Ω_{2t}, σ is the coefficient of the surface tension, $\mathcal{H}$ is the doubled mean curvature of Γ_t, $\boldsymbol{V}_n$ is the velocity of evolution of the surface Γ_t in the direction of the exterior normal $\boldsymbol{n}$ to Γ_t, $[u] = u^{(1)} - u^{(2)}$ - jump of $u(x)$ on Γ_t, $u^{(i)} = u|_{x\in\overline{\Omega}_{it}}$. We assume that $\nu, \alpha, \sigma, \mu_1, \mu_2$ are positive constants, the density of the fluid is equal to 1.

Local in time solvability of the problem similar to (1.1)–(1.5) was proved in [3]. In [7] the solvability of (1.1)–(1.5) was proved for $\boldsymbol{f} \equiv 0$ in an infinite time interval (there were used additional assumptions that the initial position of the free boundary is close to a sphere and initial data are sufficiently small). Here we extend this result to the case of nonhomogeneous equation $(1.1)_1$ under smallness assumptions on $\boldsymbol{f}$.

As in [7], we assume that the initial position of the free boundary Γ_0 can be regarded as a normal perturbation of the sphere S_{R_0}, where R_0 is defined by the relation

$$\frac{4}{3}\pi R_0^3 = |D| + |\Omega_{10}|.$$

More precisely, we assume that

$$\Gamma_0 = \{x = y + \boldsymbol{N}(y)\rho_0(y), \quad y \in S_{R_0}\},$$

where $\boldsymbol{N}(y) = \frac{y}{|y|}$ is the exterior normal to S_{R_0} and ρ_0 is a given small function. The function

$$\boldsymbol{\xi}(t) = \frac{1}{|\Omega_0|}\int\limits_{\Omega_t} x dx = \frac{1}{|\Omega_0|}\int\limits_0^t \left(\int\limits_{\Omega_{1\tau}} \boldsymbol{v}(x,\tau) dx \right) d\tau$$

is the barycenter point of the domain $\Omega_t = \bar{D} \cup \Omega_{1t}$ filled with the liquid of the density 1 (formally, the domain D can be also considered as filled with a liquid). Besides, we assume that $\boldsymbol{\xi}(0) = 0$. We intend to prove that Γ_t is close to a sphere with the center at the point $\boldsymbol{\xi}(t)$. Due to the conservation of the volume, this sphere has the same radius R_0. We are looking for Γ_t in the form

$$\Gamma_t = \{x = y + \boldsymbol{N}(y)\rho(y,t) + \boldsymbol{\xi}(t), \quad y \in S_{R_0}\},$$

where the functions $\rho(y,t)$, $\boldsymbol{\xi}(t)$ are unknown. Assumptions $|\Omega_0| = \frac{4}{3}\pi R_0^3$ and $\int_{\Omega_0} x_i dx = 0$ can be written as the following conditions for ρ_0:

$$\int_{S_1} ((R_0+\rho_0)^3 - R_0^3) dS = 0, \quad \int_{S_1} y_i((R_0+\rho_0)^4 - R_0^4) dS = 0, \quad i = 1,2,3, \tag{1.6}$$

where S_1 denotes the unit sphere.

Henceforth, we use the following notation. By the Sobolev space $W_2^s(\Omega)$, $\Omega \subset \mathbb{R}^n$ with non-integer $s > 0$ we mean the space of functions $u(x)$, $x \in \Omega$

with the finite norm

$$\|u\|^2_{W_2^s(\Omega)} = \|u\|^2_{W_2^{[s]}(\Omega)} + \sum_{|\alpha|=[s]} \int_\Omega \int_\Omega \big|D^\alpha u(x) - D^\alpha u(y)\big|^2 \frac{dxdy}{|x-y|^{n+2(s-[s])}},$$

where $[s]$ denotes the integer part of s and

$$\|u\|^2_{W_2^{[s]}(\Omega)} = \sum_{0\leqslant|\alpha|\leqslant[s]} \int_\Omega \big|D^\alpha u(x)\big|^2 dx$$

is the standard norm in the space $W_2^{[s]}(\Omega)$. The anisotropic Sobolev-Slobodetskii space $W_2^{s,s/2}(Q_T)$ in the cylindrical domain $Q_T = \Omega\times(0,T)$ can be defined as $W_2^{s,0}(Q_T)\cap W_2^{0,s/2}(Q_T)$, where

$$W_2^{s,0}(Q_T) = L_2((0,T), W_2^s(\Omega)),$$
$$W_2^{0,s/2}(Q_T) = L_2(\Omega, W_2^{s/2}(0,T))$$

with the respective norm

$$\|u\|^2_{W_2^{s,s/2}(Q_T)} := \int_0^T \|u(\cdot,t)\|^2_{W_2^s(\Omega)} dt + \int_\Omega \|u(x,\cdot)\|^2_{W_2^{s/2}(0,T)} dx. \qquad (1.7)$$

Sobolev spaces of functions defined on smooth surfaces are introduced in a standard way, with the help of local maps and partition of unity.

Now we formulate the main result.

Theorem 1. *Let* $\boldsymbol{v}_0 \in W_2^{1+l}(\Omega_{10})$*,* $\rho_0 \in W_2^{2+l}(S_{R_0})$*,* $\boldsymbol{H}_0 \in W_2^{1+l}(\Omega_{io})$*,* $i=1,2$*,* $l\in(1/2,1)$*, satisfy compatibility conditions and conditions* (1.6)*. Let* $\boldsymbol{f}\in W_2^{l,l/2}(\Omega\times(0,+\infty))$*,* $\nabla\boldsymbol{f}\in W_2^{l,l/2}(\Omega\times(0,+\infty))$*,* $D^2\boldsymbol{f}\in L_2(\Omega\times(0,+\infty))$*,* $\Omega=\Omega_{10}\cup\Gamma_0\cup\Omega_{20}$*. Also, we assume that the following smallness conditions*

$$\|\boldsymbol{v}_0\|_{W_2^{1+l}(\Omega_{10})} + \|\rho_0\|_{W_2^{2+l}(S_{R_0})} + \sum_{i=1,2} \|\boldsymbol{H}_0\|_{W_2^{1+l}(\Omega_{i0})} \leqslant \epsilon \ll 1, \quad (1.8)$$

$$\|D^2\boldsymbol{f}\|_{L_2(\Omega\times(0,+\infty))} \leqslant \varepsilon,$$
$$\|e^{bt}\nabla\boldsymbol{f}\|_{W_2^{l,l/2}(\Omega\times(0,+\infty))} \leqslant \varepsilon,$$
$$\|e^{bt}\boldsymbol{f}\|_{W_2^{l,l/2}(\Omega\times(0,+\infty))} \leqslant \varepsilon, \quad b>0 \qquad (1.9)$$

hold and at the initial moment of time

$$\mathrm{dist}\{\Gamma_0,\Sigma\} > 3d_0, \quad \mathrm{dist}\{\Gamma_0, S\} > 3d_0, \quad d_0 > (C^*+1)\varepsilon,$$

where C^* *is defined in* (4.41).

Then, there exists a small ε, such that the problem (1.1)–(1.5) *has a unique solution defined for any $t > 0$, which has the following properties: the free boundary Γ_t is located in the layer $0 < R_0 - d_0 \leqslant |y| \leqslant R_0 + d_0$ and do not intersect the fixed parts of the boundary,*

$$\rho(\cdot,t) \in W_2^{2+l}(S_{R_0}), \quad \rho_t(\cdot,t) \in W_2^{1+l}(S_{R_0}), \quad \boldsymbol{v}(\cdot,t) \in W_2^{1+l}(\Omega_{1t}),$$
$$\boldsymbol{H}^{(i)}(\cdot,t) \in W_2^{1+l}(\Omega_{it}).$$

The solution is decaying exponentially as t tends to $+\infty$.

The plan of the present paper is as follows. In section 2, we use Hanzawa coordinate transform and pass from the free boundary problem to a problem in the domain with a fixed boundary. In section 3, we prove exponential decay for solutions of linear problems. In section 4, we prove the main result for the nonlinear problem.

§2. Reduction to a problem in a fixed domain

In order to reduce the problem (1.1)–(1.5) to a problem set in a fixed domain, we construct the mapping which transforms $\Omega = \Omega_{1t} \cup \Gamma_t \cup \Omega_{2t}$ to $\Omega = \mathcal{F}_1 \cup S_{R_0} \cup \mathcal{F}_2$, where $\mathcal{F}_1$ is the domain bounded by Σ and S_{R_0}, $\mathcal{F}_2 := \Omega/\overline{\mathcal{F}}_1$, $\partial\mathcal{F}_2 = S \cup S_{R_0}$. We introduce this mapping by the relation

$$x = y + \boldsymbol{N}^*(y)\rho^*(y,t) + \chi(y)\boldsymbol{\xi}(t) \equiv e_{\rho,\boldsymbol{\xi}}(y), \quad y \in \Omega, \qquad (2.1)$$

where $\chi(y)$ is a smooth non-negative cut-off function, which is equal to 1 if y belongs to the layer $R_0 - d_0 \leqslant |y| \leqslant R_0 + d_0$ and vanishes outside the layer $R_0 - 2d_0 \leqslant |y| \leqslant R_0 + 2d_0$. $\boldsymbol{N}^*(y)$ and $\rho^*(y,t)$ are sufficiently regular extensions of $\boldsymbol{N}$ and ρ from S_{R_0} into Ω such that $\rho^*(y,t) = 0$ near S and Σ and C^1-norm of ρ^* is small. We denote by $\mathcal{L}(y,\rho^*,\boldsymbol{\xi})$ the Jacobi matrix of the transform (2.1), $L := \det \mathcal{L}$, and $\widehat{\mathcal{L}} := L\mathcal{L}^{-1}$.

With the help of the transformation (2.1), we pass from the free boundary problem (1.1)–(1.5) to a nonlinear problem in the fixed domain $\Omega = \mathcal{F}_1 \cup S_{R_0} \cup \mathcal{F}_2$, for the unknown functions $\boldsymbol{u}(y,t) = \boldsymbol{v} \circ e_{\rho,\boldsymbol{\xi}}$, $q(y,t) = p \circ e_{\rho,\boldsymbol{\xi}} - \frac{2\sigma}{R_0}$, $\boldsymbol{h}(y,t) = \widehat{\mathcal{L}}(y,\rho^*,\boldsymbol{\xi})(\boldsymbol{H} \circ e_{\rho,\boldsymbol{\xi}})$. The given function $\boldsymbol{f}$ is transformed to

$$\boldsymbol{f}\,(e_{\rho,\boldsymbol{\xi}},t) = \boldsymbol{f}(y) + \int\limits_0^1 \nabla \boldsymbol{f}(y + s(\boldsymbol{N}^*\rho^* + \chi\boldsymbol{\xi}),t)ds\,(\boldsymbol{N}^*(y)\rho^*(y,t) + \chi(y)\boldsymbol{\xi}(t))\,.$$

We separate linear and nonlinear parts in the same way as it was done in [3,7] and arrive at the problem, which can be decomposed in two parts.

The first part with linear terms depending on u, q, and ρ is as follows:

$$\begin{aligned}
&\boldsymbol{u}_t - \nu\nabla^2\boldsymbol{u} + \nabla q = \boldsymbol{f}(y) + (\boldsymbol{f}(e_{\rho,\xi}, t) - \boldsymbol{f}(y)) + \boldsymbol{l}_1(\boldsymbol{u}, q, \boldsymbol{h}, \rho), y \in \mathcal{F}_1,\\
&\nabla\cdot\boldsymbol{u} = l_2(\boldsymbol{u}, \rho), \quad y \in \mathcal{F}_1,\\
&\boldsymbol{u}(y,t)\Big|_{y\in\Sigma} = 0, \quad \nu\Pi_0 S(\boldsymbol{u})\boldsymbol{N} = \boldsymbol{l}_3(\boldsymbol{u}, \rho), y \in S_{R_0},\\
&-q + \nu\boldsymbol{N}\cdot S(\boldsymbol{u})\boldsymbol{N}(y) + \sigma B_0\rho = l_4(\boldsymbol{u}, \boldsymbol{h}, \rho) + l_5(\rho), y \in S_{R_0}\\
&\rho_t - \boldsymbol{u}\cdot\boldsymbol{N}(y) + \frac{1}{|\Omega_0|}\int_{\mathcal{F}_1}\boldsymbol{u}dy\cdot\boldsymbol{N}(y) = l_6(\boldsymbol{u}, \rho), \quad y \in S_{R_0},\\
&\boldsymbol{u}(y,0) = \boldsymbol{u}_0(y), \quad y \in \mathcal{F}_1, \quad \rho(y,0) = \rho_0(y), \quad y \in S_{R_0},
\end{aligned} \tag{2.2}$$

The part with linear terms depending on h is as follows:

$$\begin{aligned}
&\mu_1\boldsymbol{h}_t + \alpha^{-1}\operatorname{rot}\operatorname{rot}\boldsymbol{h} = \boldsymbol{l}_7(\boldsymbol{h}, \boldsymbol{u}, \rho), \quad \nabla\cdot\boldsymbol{h} = 0, \quad y \in \mathcal{F}_1,\\
&\operatorname{rot}\boldsymbol{h} = \operatorname{rot}\boldsymbol{l}_8(\boldsymbol{h}, \rho), \quad \nabla\cdot\boldsymbol{h} = 0, \quad y \in \mathcal{F}_2,\\
&[\mu\boldsymbol{h}\cdot\boldsymbol{N}] = 0, \quad [\boldsymbol{h}_\tau] = \boldsymbol{l}_9(\boldsymbol{h}, \rho) = [\boldsymbol{A}(\boldsymbol{h}, \rho)], \quad y \in S_{R_0},\\
&\boldsymbol{h}(y,t)\cdot\boldsymbol{n}(y) = 0, \quad y \in S\cup\Sigma, \quad \operatorname{rot}_\tau\boldsymbol{h} = 0, \quad y \in \Sigma,\\
&\boldsymbol{h}(y,0) = \boldsymbol{h}_0(y), \quad y \in \mathcal{F}_1\cup\mathcal{F}_2.
\end{aligned} \tag{2.3}$$

In the above relations, $\Pi_0\boldsymbol{w} = \boldsymbol{w} - \boldsymbol{N}(\boldsymbol{w}\cdot\boldsymbol{N})$, the expression $\sigma B_0\rho$ is the first variation of $\sigma(\mathcal{H} + \frac{2}{R_0})$ with respect to ρ and has the form

$$B_0\rho = -\frac{1}{R_0^2}(\Delta_{S_1}\rho + 2\rho),$$

Δ_{S_1} is the Laplacean defined on the unit sphere S_1. By $l_1, l_2, \dots, l_9$ we denote nonlinear terms, which are the same as in [7], where one can find their expressions.

Henceforth, we use the following notation

$$\begin{aligned}
X_{(t_1,t_2)}(\boldsymbol{u}, q, \rho, \boldsymbol{h}) = {}& \|\boldsymbol{u}\|_{W_2^{2+l,1+l/2}(\mathcal{F}_1\times(t_1,t_2))} + \|\nabla q\|_{W_2^{l,l/2}(\mathcal{F}_1\times(t_1,t_2))}\\
&+ \|\rho\|_{W_2^{l/2}(t_1,t_2;W_2^{5/2}(S_{R_0}))} + \|\rho_t\|_{W_2^{l+3/2,l/2+3/4}(S_{R_0}\times(t_1,t_2))}\\
&+ \sum_{i=1}^{2}\|\boldsymbol{h}^{(i)}\|_{W_2^{2+l,1+l/2}(\mathcal{F}_i\times(t_1,t_2))},
\end{aligned}$$

where $t_1 \geqslant 0$ and $t_2 > t_1$ may be finite or infinite.

To prove Theorem 1, we need to establish the existence result for the problem (2.2), (2.3) in an infinite time interval and exponential decay of the solution in corresponding Sobolev norms.

Theorem 2. *Let* $\boldsymbol{u}_0 \in W_2^{1+l}(\mathcal{F}_1)$, $\rho_0 \in W_2^{2+l}(S_{R_0})$, $\boldsymbol{h}_0^{(i)} \in W_2^{1+l}(\mathcal{F}_i)$, $i = 1,2$, $\boldsymbol{f} \in W_2^{l,l/2}(\Omega \times (0,+\infty))$, $\nabla \boldsymbol{f} \in W_2^{l,l/2}(\Omega \times (0,+\infty))$, $D^2\boldsymbol{f} \in L_2(\Omega \times (0,+\infty))$ *with a certain* $l \in (1/2,1)$. *Let the compatibility conditions, conditions* (1.6), *and the following smallness conditions*

$$\|\boldsymbol{u}_0\|_{W_2^{1+l}(\mathcal{F}_1)} + \|\rho_0\|_{W_2^{2+l}(S_{R_0})} + \sum_{i=1,2} \|\boldsymbol{h}_0^{(i)}\|_{W_2^{1+l}(\mathcal{F}_i)} \leqslant \epsilon \ll 1, \tag{2.4}$$

$$\|D^2\boldsymbol{f}\|_{L_2(\Omega\times(0,+\infty))} \leqslant \varepsilon, \quad \|e^{bt}\nabla\boldsymbol{f}\|_{W_2^{l,l/2}(\Omega\times(0,+\infty))} \leqslant \varepsilon,$$
$$\|e^{bt}\boldsymbol{f}\|_{W_2^{l,l/2}(\Omega\times(0,+\infty))} \leqslant \varepsilon, \quad b>0 \tag{2.5}$$

be satisfied. Then problem (2.2), (2.3) *has a unique solution with the following regularity properties:*

$$\boldsymbol{u} \in W_2^{2+l,1+l/2}(Q^1_\infty), \quad \nabla q \in W_2^{l,l/2}(Q^1_\infty),$$
$$\rho \in W_2^{l/2}(0,+\infty; W_2^{5/2}(S_{R_0})), \quad \rho_t \in W_2^{l+3/2,l/2+3/4}(G_\infty),$$
$$\boldsymbol{h}^{(i)} \in W_2^{2+l,1+l/2}(Q^i_\infty),$$

where $Q^i_\infty = \mathcal{F}_i \times (0,+\infty)$, $G_\infty = S_{R_0} \times (0,+\infty)$, $\boldsymbol{h}^{(i)} = \boldsymbol{h}|_{x\in\mathcal{F}_i}$, $i=1,2$. *The solution satisfies the inequality*

$$\begin{aligned} X_{(0,+\infty)}&\left(e^{at}\boldsymbol{u}, e^{at}q, e^{at}\rho, e^{at}\boldsymbol{h}\right) \\ &\leqslant c\Big(\|\boldsymbol{u}_0\|_{W_2^{1+l}(\mathcal{F}_1)} + \|\rho_0\|_{W_2^{2+l}(S_{R_0})} + \sum_{i=1}^{2} \|\boldsymbol{h}_0^{(i)}\|_{W_2^{1+l}(\mathcal{F}_i)} \\ &\qquad + \|e^{at}\boldsymbol{f}\|_{W^{l,l/2}(\Omega\times(0,+\infty))} + \|e^{at}\nabla\boldsymbol{f}\|_{W_2^{l,l/2}(\Omega\times(0,+\infty))}\Big), \end{aligned} \tag{2.6}$$

with a certain small $0 < a < b$.

§3. Exponential decay for solutions to linear problems

In this section, we consider a linear problem, which arises if all nonlinear terms in (2.2), (2.3) are omitted. This problem consists of two parts: hydrodynamical and magnetic. First, we consider the hydrodynamical problem

$$\begin{aligned}
&\boldsymbol{v}_t - \nu\nabla^2\boldsymbol{v} + \nabla p = \boldsymbol{f}(y,t), \quad \nabla\cdot\boldsymbol{v} = 0, \quad y \in \mathcal{F}_1,\\
&\Pi_0 S(\boldsymbol{v})\boldsymbol{N} = 0,\\
&-p + \nu\boldsymbol{N}\cdot S(\boldsymbol{v})\boldsymbol{N} + \sigma B_0\rho = 0, \quad y \in S_{R_0},\\
&\rho_t - \big(\boldsymbol{v} - |\Omega_0|^{-1}\int_{\mathcal{F}_1}\boldsymbol{v}(y,t)dy\big)\cdot\boldsymbol{N} = 0, \quad y \in S_{R_0},\\
&\boldsymbol{v}(y,t) = 0, \quad y \in \Sigma,\\
&\boldsymbol{v}(y,0) = \boldsymbol{v}_0(y), \quad y \in \mathcal{F}_1, \quad \rho(y,0) = \rho_0(y), \quad y \in S_{R_0}.
\end{aligned} \tag{3.1}$$

Let the initial data in (3.1) satisfy the natural compatibility conditions

$$\nabla\cdot\boldsymbol{v}_0(y) = 0, \quad y\in\mathcal{F}_1, \quad \Pi_0 S(\boldsymbol{v}_0)\boldsymbol{N}(y)\big|_{S_{R_0}} = 0, \quad \boldsymbol{v}_0\Big|_{\Sigma} = 0, \tag{3.2}$$

and the orthogonality conditions

$$\int_{S_{R_0}}\rho_0(y)dS = 0, \quad \int_{S_{R_0}} y_i\rho_0(y)dS = 0, \quad i = 1,2,3, \tag{3.3}$$

obtained by linearization of (1.6).

Theorem 3. *Let $\boldsymbol{v}_0 \in W_2^{1+l}(\mathcal{F}_1)$, $\rho_0 \in W_2^{2+l}(S_{R_0})$, $\boldsymbol{f} \in W_2^{l,l/2}(\mathcal{F}_1 \times (0,T))$, $T \in (0,+\infty]$ conditions (3.2) and (3.3) be satisfied. Also, we assume that the given function $\boldsymbol{f}$ is decaying exponentially as $t \to +\infty$ and*

$$\|e^{a_1 t}\boldsymbol{f}\|_{W_2^{l,l/2}(\mathcal{F}_1\times(0,T))} < +\infty, \qquad a_1 > 0. \tag{3.4}$$

Then, the problem (3.1) has a unique solution: $\boldsymbol{v} \in W_2^{2+l,1+l/2}(Q_T^1)$, $\nabla p \in W_2^{l,l/2}(Q_T^1)$, $\rho \in W_2^{l/2}(0,T;W_2^{5/2}(S_{R_0}))$, $\rho_t \in W_2^{l+3/2,l/2+3/4}(G_T)$, $Q_T^1 = \mathcal{F}_1 \times (0,T)$, $G_T = S_{R_0} \times (0,T)$, which is subject to the estimate

$$\begin{aligned}
&\|e^{at}\boldsymbol{v}\|_{W_2^{2+l,1+l/2}(Q_T^1)} + \|e^{at}\nabla p\|_{W_2^{l,l/2}(Q_T^1)} + \|e^{at}\rho\|_{W_2^{l/2}(0,T;W_2^{5/2}(S_{R_0}))}\\
&\quad + \|e^{at}\rho_t\|_{W_2^{l+3/2,l/2+3/4}(G_T)} + \sup_{t<T}\|e^{at}\boldsymbol{v}(\cdot,t)\|_{W_2^{1+l}(\mathcal{F}_1)}\\
&\quad + \sup_{t<T}\|e^{at}\rho(\cdot,t)\|_{W_2^{2+l}(S_{R_0})}\\
&\leqslant c\big(\|\boldsymbol{v}_0\|_{W_2^{1+l}(\mathcal{F}_1)} + \|\rho_0\|_{W_2^{2+l}(S_{R_0})} + \|e^{at}\boldsymbol{f}\|_{W_2^{l,l/2}(Q_T^1)}\big)
\end{aligned} \tag{3.5}$$

with a certain constant $0 < a < a_1$.

Proof. Existence of a solution to the hydrodynamical linear problem with the above stated regularity properties is known (see [3, 7]). To prove estimate (3.5), we first deduce the energy estimate. For this purpose, we multiply the first equation in (3.1) by $\boldsymbol{v}$, integrate over $\mathcal{F}_1$, and integrate by parts. We arrive at the relation

$$\frac{1}{2}\frac{d}{dt}\|\boldsymbol{v}(\cdot,t)\|^2_{L_2(\mathcal{F}_1)} + \frac{\nu}{2}\|S(\boldsymbol{v})\|^2_{L_2(\mathcal{F}_1)} + \int\limits_{\partial\mathcal{F}_1}\big(-\nu S(\boldsymbol{v})\boldsymbol{N}\cdot\boldsymbol{v} + p\boldsymbol{v}\cdot\boldsymbol{N}\big)ds = \int\limits_{\mathcal{F}_1}\boldsymbol{f}\cdot\boldsymbol{v}dy. \tag{3.6}$$

Due to the boundary conditions in (3.1), the surface integral in (3.6) is equal to

$$\int\limits_{S_{R_0}}\sigma B_0\rho\Big(\rho_t + \frac{1}{|\Omega_0|}\int\limits_{\mathcal{F}_1}\boldsymbol{v}(y,t)dy\cdot\boldsymbol{N}\Big)ds = \int\limits_{S_{R_0}}\sigma\rho_t B_0\rho ds + \int\limits_{S_{R_0}}\sigma B_0\rho\boldsymbol{\xi}^{'}(t)\cdot\boldsymbol{N}ds. \tag{3.7}$$

The first term at the right-hand side of (3.7) can be rewritten in the form

$$-\frac{\sigma}{R_0^2}\int_{S_1}\big(\Delta_{S_1}\rho + 2\rho\big)\rho_t ds = \frac{\sigma}{2R_0^2}\frac{d}{dt}\int_{S_1}\big(|\nabla_\omega\rho|^2 - 2\rho^2\big)ds = \frac{1}{2}\frac{d}{dt}M(t),$$

where

$$M(t) = \frac{\sigma}{R_0^2}\int_{S_1}\big(|\nabla_\omega\rho|^2 - 2\rho^2\big)ds.$$

It has been shown in [7] that if the orthogonality conditions (3.3) are fulfilled at the initial moment of time, then the same conditions are fulfilled for the solution $\rho(y,t)$ of the problem (3.1) at any time $t > 0$. Due to this fact, ρ is orthogonal to the first and the second eigenfunctions of Laplace-Beltarmi operator Δ_{S_1}. It implies that $M(t)$ is positively defined:

$$M(t) \geqslant C\|\rho(\cdot,t)\|^2_{W_2^1(S_1)}, \tag{3.8}$$

while the second term at the right-hand side of (3.7) is equal to zero. Consequently, (3.6) takes the form

$$\frac{1}{2}\frac{d}{dt}\Big(\|\boldsymbol{v}(\cdot,t)\|^2_{L_2(\mathcal{F}_1)} + M(t)\Big) + \frac{\nu}{2}\|S(\boldsymbol{v})\|^2_{L_2(\mathcal{F}_1)} = \int\limits_{\mathcal{F}_1}\boldsymbol{f}\cdot\boldsymbol{v}dy. \tag{3.9}$$

To add the dissipative term for ρ, we use the so-called "free energy" method, introduced by M. Padula [8,9]. It was proved that if $\rho\in W_2^{1/2,0}(G_T)$ has the time derivative $\rho_t \in L_2(G_T)$ and satisfies the orthogonality condition

$$\int\limits_{S_{R_0}} \rho(y,t)ds = 0,$$

then, there exists a vector field $\boldsymbol{w}(\cdot,t) \in W_2^1(\mathcal{F}_1)$ such that $\boldsymbol{w}_t(\cdot,t) \in L_2(\mathcal{F}_1)$,

$$\nabla\cdot\boldsymbol{w} = 0, \quad y\in\mathcal{F}_1, \quad t>0, \qquad \boldsymbol{w}\big|_{\Sigma} = 0, \qquad \boldsymbol{w}\cdot\boldsymbol{N}\Big|_{S_{R_0}} = \rho,$$

and

$$\begin{aligned} &\|\boldsymbol{w}(\cdot,t)\|_{W_2^1(\mathcal{F}_1)} \leqslant c\|\rho(\cdot,t)\|_{W_2^{1/2}(S_{R_0})}, \\ &\|\boldsymbol{w}(\cdot,t)\|_{L_2(\mathcal{F}_1)} \leqslant c\|\rho(\cdot,t)\|_{L_2(S_{R_0})}, \\ &\|\boldsymbol{w}_t(\cdot,t)\|_{L_2(\mathcal{F}_1)} \leqslant c\big(\|\rho_t(\cdot,t)\|_{L_2(S_{R_0})} + \|\rho(\cdot,t)\|_{W_2^{1/2}(S_{R_0})}\big). \end{aligned} \tag{3.10}$$

We multiply the first equation in (3.1) by $\boldsymbol{w}$, integrate over $\mathcal{F}_1$, and integrate by parts. Taking into account boundary conditions, we obtain

$$\frac{d}{dt}\int\limits_{\mathcal{F}_1}\boldsymbol{v}\cdot\boldsymbol{w}dy+\frac{\nu}{2}\int\limits_{\mathcal{F}_1}S(\boldsymbol{v}):S(\boldsymbol{w})dy-\int\limits_{\mathcal{F}_1}\boldsymbol{v}\cdot\boldsymbol{w}_tdy+M(t)=\int\limits_{\mathcal{F}_1}\boldsymbol{f}\cdot\boldsymbol{w}dy. \tag{3.11}$$

We multiply (3.11) by a small positive number γ and add it to (3.9), it leads to

$$\frac{1}{2}\frac{d}{dt}\Big(E(t)\Big) + D(t) = \int\limits_{\mathcal{F}_1}\boldsymbol{f}\cdot\boldsymbol{v}dy + \gamma\int\limits_{\mathcal{F}_1}\boldsymbol{f}\cdot\boldsymbol{w}dy, \tag{3.12}$$

where

$$E(t) = \|\boldsymbol{v}(\cdot,t)\|^2_{L_2(\mathcal{F}_1)} + 2\gamma\int\limits_{\mathcal{F}_1}\boldsymbol{v}\cdot\boldsymbol{w}dx + M(t),$$

$$D(t)=\tfrac{\nu}{2}\|S(\boldsymbol{v})\|^2_{L_2(\mathcal{F}_1)}+\gamma\tfrac{\nu}{2}\int\limits_{\mathcal{F}_1}S(\boldsymbol{v}):S(\boldsymbol{w})dx-\gamma\int\limits_{\mathcal{F}_1}\boldsymbol{v}\cdot\boldsymbol{w}_tdx+\gamma M(t).$$

For sufficiently small γ, estimates (3.10) and the Korn inequality imply (it can be demonstrated in the same way as in [7])

$$1/2\big(\|\boldsymbol{v}(\cdot,t)\|^2_{L_2(\mathcal{F}_1)} + M(t)\big) \leqslant E(t) \leqslant 3/2\big(\|\boldsymbol{v}(\cdot,t)\|^2_{L_2(\mathcal{F}_1)} + M(t)\big) \tag{3.13}$$

and

$$D(t) \geqslant \alpha\big(\|\boldsymbol{v}(\cdot,t)\|^2_{W_2^1(\mathcal{F}_1)} + M(t)\big), \quad \alpha>0. \tag{3.14}$$

Now, we multiply (3.12) by e^{ct} with a certain $0 < c \leqslant 2a_1$, which gives

$$\frac{d}{dt}\Big(\frac{1}{2}e^{ct}E(t)\Big) - \frac{c}{2}e^{ct}E(t) + e^{ct}D(t) = \int\limits_{\mathcal{F}_1} e^{ct}\boldsymbol{f}\cdot(\boldsymbol{v}+\gamma\boldsymbol{w})dy. \qquad (3.15)$$

At first, we fix γ in such a way that (3.13), (3.14) hold, then we choose so small c that

$$D(t) - \frac{c}{2}E(t) \geqslant \alpha_1\left(\|\boldsymbol{v}(\cdot,t)\|^2_{L_2(\mathcal{F}_1)} + M(t)\right), \qquad \alpha_1 > 0. \qquad (3.16)$$

In what follows, it is convenient to use the following notation

$$e^{ct}E(t) = \mathcal{U}^2(t), \qquad e^{ct}\big(D(t) - \frac{c}{2}E(t)\big) = \mathcal{R}^2(t).$$

Identity (3.15) reads

$$\frac{1}{2}\frac{d}{dt}\big(\mathcal{U}^2(t)\big) + \mathcal{R}^2(t) = \int\limits_{\partial\mathcal{F}_1} e^{ct}\boldsymbol{f}\cdot(\boldsymbol{v}+\gamma\boldsymbol{w})dy. \qquad (3.17)$$

We estimate the right-hand side of (3.17) by the Hölder inequality, making use of (3.10) and (3.8)

$$\int\limits_{\mathcal{F}_1} e^{ct}\big|\boldsymbol{f}\cdot(\boldsymbol{v}+\gamma\boldsymbol{w})\big|dy \leqslant e^{ct}\|\boldsymbol{f}\|_{L_2(\mathcal{F}_1)}\big(\|\boldsymbol{v}\|_{L_2(\mathcal{F}_1)} + \gamma\|\boldsymbol{w}\|_{L_2(\mathcal{F}_1)}\big)$$
$$\leqslant C_1 e^{\frac{c}{2}t}\|\boldsymbol{f}(\cdot,t)\|_{L_2(\mathcal{F}_1)}\mathcal{U}(t).$$

Consequently, (3.17) implies

$$\frac{d}{dt}\big(\mathcal{U}(t)\big) \leqslant C_1 e^{\frac{c}{2}t}\|\boldsymbol{f}(\cdot,t)\|_{L_2(\mathcal{F}_1)}.$$

It follows that

$$\mathcal{U}(t) \leqslant C_1\int\limits_0^t e^{\frac{c}{2}\tau}\|\boldsymbol{f}(\cdot,\tau)\|_{L_2(\mathcal{F}_1)}d\tau + \mathcal{U}(0). \qquad (3.18)$$

By (3.8), (3.18), we obtain the exponential decay for the solution in L_2 norms:

$$\|\boldsymbol{v}(\cdot,t)\|^2_{L_2(\mathcal{F}_1)} + \|\rho(\cdot,t)\|^2_{W^1_2(S_{R_0})}$$
$$\leqslant C_2 e^{-c^*t}\Big(\|\boldsymbol{v}_0\|^2_{L_2(\mathcal{F}_1)} + \|\rho_0\|^2_{W^1_2(S_{R_0})} + \int\limits_0^t \|e^{\frac{c^*\tau}{2}}\boldsymbol{f}(\cdot,\tau)\|^2_{L_2(\mathcal{F}_1)}d\tau\Big), \quad (3.19)$$

where the constant C_2 is independent of t, $c^* < c$. Multiplying (3.18) by $e^{-\frac{1}{2}(c-\beta)t}$, where $c - \beta > 0$, we have

$$\begin{aligned} &\mathcal{U}(t)e^{-\frac{1}{2}(c-\beta)t} \\ &\leqslant C_1 \int_0^t e^{-\frac{1}{2}(c-\beta)(t-\tau)} e^{\frac{\beta}{2}\tau} \|\boldsymbol{f}(\cdot,\tau)\|_{L_2(\mathcal{F}_1)} d\tau + e^{-\frac{1}{2}(c-\beta)t}\mathcal{U}(0). \end{aligned} \tag{3.20}$$

Inequality (3.20) implies the estimate of the integral

$$\int_0^T \Big(e^{-\frac{1}{2}(c-\beta)t}\mathcal{U}(t)\Big)^2 dt = \int_0^T e^{\beta t} E(t) dt$$

by the quantity

$$\int_0^T \|e^{\frac{\beta}{2}t}\boldsymbol{f}(\cdot,t)\|^2_{L_2(\mathcal{F}_1)} dt + \mathcal{U}^2(0).$$

Together with (3.8), it yields

$$\begin{aligned} &\int_0^T e^{\beta t} \Big(\|\boldsymbol{v}(\cdot,t)\|^2_{L_2(\mathcal{F}_1)} + \|\rho(\cdot,t)\|^2_{W_2^1(S_{R_0})}\Big) dt \\ &\leqslant C_3 \left(\|\boldsymbol{v}_0\|^2_{L_2(\mathcal{F}_1)} + \|\rho_0\|^2_{W_2^1(S_{R_0})} + \int_0^T \|e^{\frac{\beta}{2}t}\boldsymbol{f}(\cdot,t)\|^2_{L_2(\mathcal{F}_1)} dt\right) \end{aligned} \tag{3.21}$$

with a certain positive $\beta < c \leqslant 2a_1$.

Now, it is convenient to introduce the functions

$$\tilde{\boldsymbol{v}} = e^{at}\boldsymbol{v}, \qquad \tilde{p} = e^{at}p, \qquad \tilde{\rho} = e^{at}\rho, \qquad \tilde{\boldsymbol{f}} = e^{at}\boldsymbol{f},$$

where $0 < a \leqslant \frac{\beta}{2} < a_1$, As a consequence of (3.1), we obtain

$$\begin{gathered} \tilde{\boldsymbol{v}}_t - \nu\nabla^2\tilde{\boldsymbol{v}} + \nabla\tilde{p} = a\tilde{\boldsymbol{v}} + \tilde{\boldsymbol{f}}, \quad \nabla\cdot\tilde{\boldsymbol{v}} = 0, \quad y \in \mathcal{F}_1, \\ \Pi_0 S(\tilde{\boldsymbol{v}})\boldsymbol{N} = 0, \\ -\tilde{p} + \nu\boldsymbol{N}\cdot S(\tilde{\boldsymbol{v}})\boldsymbol{N} + \sigma B_0\tilde{\rho} = 0, \\ \tilde{\rho}_t = \big(\tilde{\boldsymbol{v}} - |\Omega_0|^{-1}\textstyle\int_{\mathcal{F}_1}\tilde{\boldsymbol{v}}(y,t)dy\big)\cdot\boldsymbol{N} + a\tilde{\rho}, \quad y \in S_{R_0}, \\ \tilde{\boldsymbol{v}}(y,t) = 0, \quad y \in \Sigma, \\ \tilde{\boldsymbol{v}}(y,0) = \boldsymbol{v}_0(y), \quad y \in \mathcal{F}_1, \quad \tilde{\rho}(y,0) = \rho_0(y), \quad y \in S_{R_0}. \end{gathered}$$

By estimate of a solution to the hydrodynamical linear problem [7], we have

$$
\begin{aligned}
&\|\tilde{\boldsymbol{v}}\|_{W_2^{l+2,l/2+1}(Q_T^1)} + \|\nabla\tilde{p}\|_{W_2^{l,l/2}(Q_T^1)} \\
&+ \|\tilde{\rho}\|_{W_2^{l/2}(0,T;W_2^{5/2}(S_{R_0}))} + \|\tilde{\rho}_t\|_{W_2^{l+3/2,l/2+3/4}(G_T)} \\
&\leqslant C_4\Big(\|\tilde{\boldsymbol{v}}\|_{L_2(Q_T^1)} + a\|\tilde{\boldsymbol{v}}\|_{W_2^{l,l/2}(Q_T^1)} + \|\tilde{\boldsymbol{f}}\|_{W_2^{l,l/2}(Q_T^1)} \\
&+ a\|\tilde{\rho}\|_{W_2^{l+3/2,l/2+3/4}(G_T)} + \|\boldsymbol{v}_0\|_{W_2^{l+1}(\mathcal{F}_1)} + \|\rho_0\|_{W_2^{l+2}(S_{R_0})}\Big),
\end{aligned}
\tag{3.22}
$$

with the constant C_4 independent of T. We apply interpolation inequalities for the terms $\|\tilde{\boldsymbol{v}}\|_{W_2^{l,l/2}(Q_T^1)}$ and $\|\tilde{\rho}\|_{W_2^{l+3/2,l/2+3/4}(G_T)}$ at the right-hand side of (3.22) and use (3.21) to estimate $\|\tilde{\boldsymbol{v}}\|_{L_2(Q_T^1)}$, $\|\tilde{\rho}\|^2_{W_2^1(G_T)}$. As a result, we obtain (3.5) with a certain $a < a_1$. □

Now we consider the homogeneous magnetic problem

$$
\begin{aligned}
&\mu_1\boldsymbol{H}_t + \alpha^{-1}\operatorname{rot}\operatorname{rot}\boldsymbol{H} = 0, \quad \nabla\cdot\boldsymbol{H} = 0, \quad y \in \mathcal{F}_1, \\
&\operatorname{rot}\boldsymbol{H} = 0, \quad \nabla\cdot\boldsymbol{H} = 0, \quad y \in \mathcal{F}_2, \\
&[\mu\boldsymbol{H}\cdot\boldsymbol{N}] = 0, \quad [\boldsymbol{H}_\tau] = 0, \quad y \in S_{R_0}, \\
&\boldsymbol{H}\cdot\boldsymbol{n} = 0, \quad y \in S\cup\Sigma, \quad \operatorname{rot}_\tau\boldsymbol{H} = 0, \quad y \in \Sigma, \\
&\boldsymbol{H}(y,0) = \boldsymbol{H}_0(y), \quad y \in \mathcal{F}_1\cup\mathcal{F}_2.
\end{aligned}
\tag{3.23}
$$

Theorem 4. *For arbitrary* $\boldsymbol{H}_0 \in W_2^{1+l}(\mathcal{F}_i)$, $i = 1,2$, *satisfying the compatibility conditions*

$$
\begin{aligned}
\nabla\cdot\boldsymbol{H}_0(x) = 0, \quad y \in \mathcal{F}_1\cup\mathcal{F}_2, \quad \operatorname{rot}\boldsymbol{H}_0(x) = 0, \quad y \in \mathcal{F}_2, \\
[\mu\boldsymbol{H}_0\cdot\boldsymbol{N}] = 0, \quad [\boldsymbol{H}_{0\tau}] = 0, \quad y \in S_{R_0}, \\
\boldsymbol{H}_0\cdot\boldsymbol{n} = 0, \quad \operatorname{rot}_\tau\boldsymbol{H}_0 = 0, \quad y \in \Sigma, \qquad \boldsymbol{H}_0\cdot\boldsymbol{n} = 0, \quad y \in S,
\end{aligned}
\tag{3.24}
$$

the problem (3.23) *has a unique solution, and the inequality*

$$
\sum_{i=1}^{2}\Big(\|e^{at}\boldsymbol{H}^{(i)}\|_{W_2^{2+l,1+l/2}(Q_T^i)} + \sup_{t<T}\|e^{at}\boldsymbol{H}^{(i)}(\cdot,t)\|_{W_2^{1+l}(\mathcal{F}_i)}\Big) \leqslant C_5\sum_{i=1}^{2}\|\boldsymbol{H}_0^{(i)}\|_{W_2^{1+l}(\mathcal{F}_i)}
\tag{3.25}
$$

holds, where a is a certain positive constant and C_5 is a constant independent of T.

Existence has been proved in [3,7]. Below, we sketch the proof of (3.25).

By $\mathcal{H}^m(\Omega)$, $m = 1, 2$ we denote the spaces of vector fields from $W_2^m(\Omega)$, satisfying the conditions (3.24) (without the condition $\operatorname{rot}_\tau \boldsymbol{H} = 0$, $y \in \Sigma$ for $m = 1$).

Problem (3.23) can be written in the form of the Cauchy problem

$$\boldsymbol{H}_t + \mathcal{A}\boldsymbol{H} = \boldsymbol{0}, \quad \boldsymbol{H}|_{t=0} = \boldsymbol{H}_0, \tag{3.26}$$

where the operator $\mathcal{A}$ is defined on the space $\mathcal{H}^2$ as follows:

$$\mathcal{A}\boldsymbol{H} = P_{\mathcal{H}^0}\mu^{-1} \operatorname{rot} \mathcal{E}\alpha^{-1} \operatorname{rot} \boldsymbol{H}.$$

Here $\mathcal{E}$ is an extension operator from $\mathcal{F}_1$ into Ω defined on the space of the divergence free vector fields $\boldsymbol{w}(x)$ with $\boldsymbol{w} \cdot \boldsymbol{N}|_{S_{R_0}} = 0$, $\boldsymbol{w}_\tau|_\Sigma = 0$, and such that

$$(\mathcal{E}\boldsymbol{w})_\tau|_S = 0, \qquad \|\mathcal{E}\boldsymbol{w}\|_{W_2^{1+l}(\Omega)} \leqslant c\|\boldsymbol{w}\|_{W_2^{1+l}(\mathcal{F}_1)}.$$

$P_{\mathcal{H}^0}$ is the orthogonal projection on $\mathcal{H}^0(\Omega)$ in the space $L_2(\Omega)$ with the scalar product

$$\int_\Omega \mu \boldsymbol{H} \cdot \boldsymbol{h} dx.$$

By $\mathcal{H}^0(\Omega)$ we mean the closure of $\mathcal{H}^1(\Omega)$ in L_2 norm induced by this scalar product.

The characteristic property of $\mathcal{A}$ is

$$\int_\Omega \mu \mathcal{A}\boldsymbol{H} \cdot \boldsymbol{h} dx = \alpha^{-1} \int_{\mathcal{F}_1} \operatorname{rot} \boldsymbol{H} \cdot \operatorname{rot} \boldsymbol{h} dx, \quad \forall \boldsymbol{h}, \boldsymbol{H} \in \mathcal{H}^2,$$

which implies that $\mathcal{A}$ is a positive defined self-adjoint operator, the spectrum of $-\mathcal{A}$ consists of a countable number of real negative eigenvalues with the accumulation point at $-\infty$. This guarantees the weighted estimate (3.25) (see details in [5]).

§4. Nonlinear problem.

The goal of this section is to prove Theorem 2. We begin with proving existence of a generalized solution on the finite time interval $[0, T]$. We divide initial conditions in (2.2), (2.3) in two parts

$$\boldsymbol{u}_0 = \boldsymbol{u}_0'' + \boldsymbol{u}_0', \quad \rho_0 = \rho_0'' + \rho_0', \quad \boldsymbol{h}_0 = \boldsymbol{h}_0'' + \boldsymbol{h}_0',$$

where the functions $\boldsymbol{u}_0''$, ρ_0'', $\boldsymbol{h}_0''$ satisfy the same compatibility conditions as $\boldsymbol{u}_0$, ρ_0, $\boldsymbol{h}_0$ and have the order ε^2. More precisely, the conditions (1.6) imply

$$\int\limits_{S_1} \rho_0''(R_0y)dS = -\frac{1}{R_0}\int\limits_{S_1} \rho_0^2(R_0y)dS - \frac{1}{3R_0^2}\int\limits_{S_1} \rho_0^3(R_0y)dS,$$

$$\int\limits_{S_1} y_i\rho_0''(R_0y)dS = -\frac{3}{2R_0}\int\limits_{S_1} y_i\rho_0^2(R_0y)dS - \frac{1}{R_0^2}\int\limits_{S_1} y_i\rho_0^3(R_0y)dS$$
$$-\frac{1}{4R_0^3}\int\limits_{S_1} y_i\rho_0^4(R_0y)dS, \quad i=1,2,3,$$

Compatibility conditions in (2.2), (2.3) imply

$$\begin{aligned}
&\nabla\cdot\boldsymbol{u}_0''(x) = l_2(\boldsymbol{u}_0,\rho_0), \quad y\in\mathcal{F}_1,\\
&\nu\Pi_{S_{R_0}} S(\boldsymbol{u}_0'')\boldsymbol{N}(y) = \boldsymbol{l}_3(\boldsymbol{u}_0,\rho_0), \quad y\in S_{R_0},\\
&\operatorname{rot}\boldsymbol{h}_0'' = \operatorname{rot}\boldsymbol{l}_8(\boldsymbol{h}_0^{(2)},\rho_0), \quad y\in\mathcal{F}_2,\\
&\nabla\cdot\boldsymbol{h}_0'' = 0, \quad y\in\mathcal{F}_1\cup\mathcal{F}_2,\\
&[\boldsymbol{h}_{0\tau}''] = \boldsymbol{l}_9(\boldsymbol{h}_0,\rho_0), \quad [\mu\boldsymbol{h}_0''\cdot\boldsymbol{N}] = 0, \quad y\in S_{R_0},\\
&\boldsymbol{h}_0''\cdot\boldsymbol{n} = 0, \quad y\in\Sigma\cup S, \quad \operatorname{rot}_\tau\boldsymbol{h}_0'' = 0, \quad y\in\Sigma.
\end{aligned}$$

These functions satisfy the estimates

$$\|\rho_0''\|_{W_2^{2+l}(S_{R_0})} + \|\boldsymbol{u}_0''\|_{W_2^{1+l}(\mathcal{F}_1)} \leqslant c\big(\|\rho_0\|_{W_2^{2+l}(S_{R_0})} + \|\boldsymbol{u}_0\|_{W_2^{l+l}(\mathcal{F}_1)}\big)^2. \quad (4.1)$$

$$\sum_{i=1}^{2}\|\boldsymbol{h}_0''\|_{W_2^{1+l}(\mathcal{F}_i)} \leqslant c\Big(\sum_{i=1}^{2}\|\boldsymbol{h}_0^{(i)}\|_{W_2^{1+l}(\mathcal{F}_i)} + \|\rho_0\|_{W_2^{2+l}(S_{R_0})}\Big)^2. \quad (4.2)$$

We note that such functions can be indeed constructed due to inverse trace theorems (see [3,4]).

To simplify the presentation, we use the notation

$$Y(t) = \|\boldsymbol{u}(\cdot,t)\|_{W_2^{1+l}(\mathcal{F}_1)} + \|\rho(\cdot,t)\|_{W_2^{2+l}(S_{R_0})} + \sum_{i=1}^{2}\|\boldsymbol{h}(\cdot,t)\|_{W_2^{1+l}(\mathcal{F}_i)},$$

and denote by $Y'(t)$ and $Y''(t)$ the same expression defined by means of the functions $\boldsymbol{u}'$, ρ', $\boldsymbol{h}'$ and $\boldsymbol{u}''$, ρ'', $\boldsymbol{h}''$, respectively.

The functions $\boldsymbol{u}_0' = \boldsymbol{u}_0 - \boldsymbol{u}_0''$, $\rho_0' = \rho_0 - \rho_0''$, $\boldsymbol{h}_0' = \boldsymbol{h}_0 - \boldsymbol{h}_0''$ evidently satisfy compatibility conditions in corresponding to (2.2), (2.3) linear problem

(3.1), (3.23). This problem has a unique solution $\boldsymbol{u}^{'}, q^{'}, \rho^{'}, \boldsymbol{h}^{'}$. By (3.5), (3.19), (3.25), the following estimates

$$X_{(0,T)}\left(e^{at}\boldsymbol{u}^{'}, e^{at}q^{'}, e^{at}\rho^{'}, e^{at}\boldsymbol{h}^{'}\right) \leqslant c\left(Y^{'}(0) + \|e^{at}\boldsymbol{f}\|_{W_2^{l,l/2}(Q_T^1)}\right), \quad (4.3)$$

$$Y^{'}(t) \leqslant c_1 e^{-at}\left(Y^{'}(0) + \left(\int_0^t \|e^{a\tau}\boldsymbol{f}(\cdot,\tau)\|^2_{L_2(\mathcal{F}_1)} d\tau\right)^{1/2}\right) \quad (4.4)$$

hold with a certain $0 < a < b$.

The functions $\boldsymbol{u}^{''} = \boldsymbol{u} - \boldsymbol{u}^{'}$, $q^{''} = q - q^{'}$, $\rho^{''} = \rho - \rho^{'}$, $\boldsymbol{h}^{''} = \boldsymbol{h} - \boldsymbol{h}^{'}$ we find from the following nonlinear system of equations:

$$\begin{aligned}
&\boldsymbol{u}_t^{''} - \nu\nabla^2\boldsymbol{u}^{''} + \nabla q^{''} \\
&= \int_0^1 \nabla\boldsymbol{f}(y + s(\boldsymbol{N}^*(\rho^{'} + \rho^{''})^* + \chi\boldsymbol{\xi}), t)ds\left(\boldsymbol{N}^*(\rho^{'} + \rho^{''})^* + \chi\boldsymbol{\xi}\right) \\
&+ \boldsymbol{l}_1(\boldsymbol{u}' + \boldsymbol{u}^{''}, q' + q^{''}, \boldsymbol{h}' + \boldsymbol{h}^{''}, \rho' + \rho^{''}), \quad \text{in} \quad \mathcal{F}_1, \\
&\nabla\cdot\boldsymbol{u}^{''} = l_2(\boldsymbol{u}' + \boldsymbol{u}^{''}, \rho' + \rho^{''}), \quad \text{in} \quad \mathcal{F}_1, \\
&\mu_1\boldsymbol{h}_t^{''} + \alpha^{-1}\operatorname{rot}\operatorname{rot}\boldsymbol{h}^{''} = \boldsymbol{l}_7(\boldsymbol{h}' + \boldsymbol{h}^{''}, \boldsymbol{u}' + \boldsymbol{u}^{''}, \rho' + \rho^{''}), \quad \text{in} \quad \mathcal{F}_1, \\
&\nabla\cdot\boldsymbol{h}^{''} = 0, \quad in \quad \mathcal{F}_1, \\
&\operatorname{rot}\boldsymbol{h}^{''} = \operatorname{rot}\boldsymbol{l}_8(\boldsymbol{h}' + \boldsymbol{h}^{''}, \rho' + \rho^{''}), \quad \nabla\cdot\boldsymbol{h}^{''} = 0, \quad \text{in} \quad \mathcal{F}_2,
\end{aligned} \quad (4.5)$$

supplied with the boundary conditions

$$\begin{aligned}
&\nu\Pi_0 S(\boldsymbol{u}^{''})\boldsymbol{N} = \boldsymbol{l}_3(\boldsymbol{u}' + \boldsymbol{u}^{''}, \rho' + \rho^{''}), \quad \text{on} \quad S_{R_0}, \\
&- q^{''} + \nu\boldsymbol{N}\cdot S(\boldsymbol{u}^{''})\boldsymbol{N}(y) + \sigma B_0\rho^{''} \\
&= l_4(\boldsymbol{u}' + \boldsymbol{u}^{''}, \boldsymbol{h}^{'} + \boldsymbol{h}^{''}, \rho' + \rho^{''}) + l_5(\rho' + \rho^{''}), \quad \text{on} \quad S_{R_0}, \\
&\rho_t^{''} - \boldsymbol{u}^{''}\cdot\boldsymbol{N}(y) + |\Omega_0|^{-1}\int_{\mathcal{F}_1}\boldsymbol{u}^{''}dz\cdot\boldsymbol{N}(y) \\
&= l_6(\boldsymbol{u}' + \boldsymbol{u}^{''}, \rho' + \rho^{''}), \quad \text{on} \quad S_{R_0}, \quad \boldsymbol{u}^{''}(y,t)\big|_{y\in\Sigma} = 0, \\
&[\mu\boldsymbol{h}^{''}\cdot\boldsymbol{N}] = 0, \quad [\boldsymbol{h}_\tau^{''}] = \boldsymbol{l}_9(\boldsymbol{h}' + \boldsymbol{h}^{''}, \rho' + \rho^{''}), \quad \text{on} \quad S_{R_0}, \\
&\boldsymbol{h}^{''}(y,t)\cdot\boldsymbol{n}(y) = 0, \quad \text{on} \quad S\cup\Sigma, \quad \operatorname{rot}_\tau\boldsymbol{h}^{''} = 0, \quad \text{on} \quad \Sigma,
\end{aligned}$$

and initial conditions

$$\boldsymbol{u}''(y,0)=\boldsymbol{u}_0''(y),\quad y\in\mathcal{F}_1,\quad \boldsymbol{h}''(y,0)=\boldsymbol{h}_0''(y),\qquad y\in\mathcal{F}_1\cup\mathcal{F}_2,$$
$$\rho''(y,0)=\rho_0''(y),\quad y\in S_{R_0}.$$

We choose T sufficiently large, so that $c_1e^{-aT}<\frac{1}{4}$ (c_1 is the constant in (4.4)). Problem (4.5) can be solved for $t\in[0,T]$, provided ε is sufficiently small. We start with estimates of the nonlinear terms and the first term at the right-hand side of the first equation in (4.5).

Lemma 4.1. [3, 4, 7] *Let* $l\in(1/2,1)$,

$$\|\rho(\cdot,t)\|_{W_2^{2+l}(S_{R_0})}\leqslant\delta_1<1,\qquad \|\boldsymbol{u}(\cdot,t)\|_{W_2^{1+l}(\mathcal{F}_1)}\leqslant\delta_2<1,\qquad t<T, \tag{4.6}$$

the functions $\boldsymbol{u},q,\rho,\boldsymbol{h}$ *have a finite norm*

$$X_{(0,T)}\,(\boldsymbol{u},q,\rho,\boldsymbol{h})+\sup_{t<T}Y(t).$$

Then the sum of the norms of the nonlinear terms

$$\begin{aligned}
Z[(\boldsymbol{u},q,\rho,\boldsymbol{h})](T)&=\|\boldsymbol{l}_1\|_{W_2^{l,l/2}(Q_T^1)}+\|l_2\|_{W_2^{1+l,0}(Q_T^1)}+\sup_{t<T}\|l_2\|_{W_2^l(\mathcal{F}_1)}\\
&+\|\frac{\partial}{\partial t}\big((I-\hat{\mathcal{L}}^T)\boldsymbol{u}\big)\|_{W_2^{0,l/2}(Q_T^1)}+\|\boldsymbol{l}_3\|_{W_2^{l+1/2,l/2+1/4}(G_T)}\\
&+\|l_4\|_{W_2^{l+1/2,0}(G_T)}+\|l_4\|_{W_2^{l/2}(0,T,W_2^{1/2}(S_{R_0}))}\\
&+\|l_5\|_{W_2^{l+1/2,0}(G_T)}+\|l_5\|_{W_2^{l/2}(0,T,W_2^{1/2}(S_{R_0}))}\\
&+\|l_6\|_{W_2^{l+3/2,l/1+3/4}(G_T)}+\|\boldsymbol{l}_7\|_{W_2^{l,l/2}(Q_T^1)}+\|\operatorname{rot}\boldsymbol{l}_8\|_{W_2^{1+l,0}(Q_T^2)}\\
&+\sup_{t<T}\|\operatorname{rot}\boldsymbol{l}_8\|_{W_2^l(\mathcal{F}_2)}+\|\frac{\partial}{\partial t}\boldsymbol{l}_8\|_{W_2^{0,l/2}(Q_T^2)}\\
&+\|\boldsymbol{l}_9\|_{W_2^{l+3/2,0}(G_T)}+\sup_{t<T}\|\boldsymbol{l}_9\|_{W_2^{l+1/2}(S_{R_0})}\\
&+\sum_{i=1}^{2}\|\frac{\partial}{\partial t}\boldsymbol{A}^{(i)}(\rho,\boldsymbol{h})\|_{W_2^{0,l/2}(Q_T^i)}
\end{aligned}\tag{4.7}$$

satisfies the inequality

$$Z[(\boldsymbol{u},q,\rho,\boldsymbol{h})](T)\leqslant C_2(T)\Big[\Big(X_{(0,T)}+\sup_{t<T}Y(t)\Big)^2+\Big(X_{(0,T)}+\sup_{t<T}Y(t)\Big)^3\Big]. \tag{4.8}$$

Lemma 4.2 *Let all the assumptions of Lemma* 4.1 *be fulfilled,* $\chi(y)$ *be a smooth function with uniformly bounded derivatives, and*

$$\|\nabla \boldsymbol{f}\|_{W_2^{l,l/2}(\Omega\times(0,T))} \leqslant \varepsilon, \qquad \|D^2\boldsymbol{f}\|_{L_2(\Omega\times(0,T))} \leqslant \varepsilon. \tag{4.9}$$

Then, the function

$$\mathcal{K}(N^*\rho^* + \chi\xi) = \int\limits_0^1 \nabla \boldsymbol{f}(y + s(\boldsymbol{N}^*\rho^* + \chi\boldsymbol{\xi}), t)ds\,(\boldsymbol{N}^*\rho^* + \chi\boldsymbol{\xi}),$$

where

$$\boldsymbol{\xi}(t) = \frac{1}{|\Omega_0|}\int\limits_0^t\int\limits_{\mathcal{F}_1} \boldsymbol{u}(y,\tau)|L|dyd\tau.$$

is subject to the following estimate:

$$\begin{aligned}\|\mathcal{K}(N^*\rho^* + \boldsymbol{\chi}\xi)\|_{W_2^{l,l/2}(Q_T^1)} &\leqslant C_3(T)\Big(\varepsilon X_{(0,T)} + \varepsilon X^2_{(0,T)} + \varepsilon \sup_{t<T} Y(t)\\ &\quad + \sup_{t<T}|\boldsymbol{\xi}(t)|\|\nabla \boldsymbol{f}\|_{W_2^{l,l/2}(\Omega\times(0,T))}\Big).\end{aligned} \tag{4.10}$$

Proof. In our case $l \in (1/2, 1)$, consequently $l < n/2$, and we estimate the product of two functions as follows (see (4.6) in [4]):

$$\|uv\|_{W_2^l(\mathcal{F}_1)} \leqslant c\|u\|_{W_2^l(\mathcal{F}_1)}\|v\|_{W_2^{3/2+\eta}(\mathcal{F}_1)}, \quad \eta > 0.$$

By this result we obtain

$$\begin{aligned}\|\mathcal{K}(N^*\rho^* + \boldsymbol{\chi}\xi)\|_{W_2^{l,0}(Q_T^1)} &\leqslant c\sup_{t<T}\|\boldsymbol{N}^*\rho^*(\cdot,t) + \chi\boldsymbol{\xi}(t)\|_{W_2^{3/2+\eta}(\mathcal{F}_1)}\\ &\quad\times \Big\|\int\limits_0^1 \nabla \boldsymbol{f}(y + s(\boldsymbol{N}^*\rho^* + \chi\boldsymbol{\xi}), t)ds\Big\|_{W_2^{l,0}(Q_T^1)}\\ &\leqslant c\left(\sup_{t<T}\|\rho(\cdot,t)\|_{W_2^{2+l}(S_{R_0})} + \sup_{t<T}|\boldsymbol{\xi}(t)|\right)\|\nabla \boldsymbol{f}\|_{W_2^{l,0}(\Omega\times(0,T))}.\end{aligned} \tag{4.11}$$

Similarly, $l/2 < 1/2$, and we have

$$\|\mathcal{K}(N^*\rho^* + \chi\xi)\|_{W_2^{0,l/2}(Q_T^1)} \leqslant c\|\int_0^1 \nabla \boldsymbol{f}(y + s(\boldsymbol{N}^*\rho^* + \chi\boldsymbol{\xi}), t)ds\|_{W_2^{0,l/2}(Q_T^1)} \times \|\boldsymbol{N}^*\rho^* + \chi\boldsymbol{\xi}\|_{W_2^{0,1/2+\eta}(Q_T^1)}, \quad \eta > 0. \quad (4.12)$$

We can set $\eta = 1/2$, and use the relation

$$\boldsymbol{\xi}^{'}(t) = \frac{1}{|\Omega_0|}\int_{\mathcal{F}_1} \boldsymbol{u}(y,t)|L|dy.$$

Due to the assumptions (4.6), Jacobian L is uniformly bounded, the cut-off function χ also has uniformly bounded derivatives. Consequently, we obtain

$$\|\boldsymbol{N}^*\rho^* + \chi\boldsymbol{\xi}\|_{W_2^{0,1}(Q_T^1)} \leqslant c\left(\|\rho^*\|_{W_2^{0,1}(Q_T^1)} + \left(\int_0^T \|\boldsymbol{u}(\cdot,\tau)\|^2_{L_2(\mathcal{F}_1)}d\tau\right)^{1/2}\right) \leqslant cX_{(0,T)}. \quad (4.13)$$

The first norm in the right-hand side of (4.12) is small due to the assumptions (4.9), indeed,

$$\|\int_0^1 \nabla \boldsymbol{f}(y + s(\boldsymbol{N}^*\rho^* + \chi\boldsymbol{\xi}), t)ds\|_{W_2^{0,l/2}(Q_T^1)} \leqslant c\left(\|\nabla\boldsymbol{f}\|_{W_2^{0,l/2}(\Omega\times(0,T))} + \|D^2\boldsymbol{f}\|_{L_2(\Omega\times(0,T))}\|\boldsymbol{N}^*\rho^* + \chi\boldsymbol{\xi}\|_{W_2^{0,1}(Q_T^1)}\right) \leqslant c\varepsilon\left(1 + X_{(0,T)}\right). \quad (4.14)$$

Inequalities (4.11), (4.13), (4.14) imply (4.10). □

Theorem 5. *Let all the assumptions of Theorem* 2 *be fulfilled. The functions* $\boldsymbol{u}^{'}$, $q^{'}$, $\rho^{'}$, $\boldsymbol{h}^{'}$ *are subject to* (4.3), (4.4). *For a given* $T > 0$, *there exists such* $\varepsilon > 0$ *that if the given functions satisfy conditions* (2.4), (2.5) *with this* ε*, then problem* (4.5) *is uniquely solvable on the time interval* $[0,T]$ *and the solution satisfies the estimate*

$$\begin{aligned} &X_{(0,T)}\left(\boldsymbol{u}'',q'',\rho'',\boldsymbol{h}''\right)+\sup_{t<T}Y''(t) \\ &\leqslant c_2(T)\varepsilon\left(Y(0)+\|\boldsymbol{f}\|_{W_2^{l,l/2}(Q_T^1)}+\|\nabla\boldsymbol{f}\|_{W_2^{l,l/2}(\Omega\times(0,T))}\right). \end{aligned} \tag{4.15}$$

Proof. To prove this result, we apply the successive approximations method.

For the first approximation, we take $\boldsymbol{u}_1''$ and ρ_1'', which satisfy the initial conditions

$$\boldsymbol{u}_1''\big|_{t=0}=\boldsymbol{u}_0'', \qquad \rho_1''\big|_{t=0}=\rho_0'',$$

and the inequalities

$$\|\boldsymbol{u}_1''\|_{W_2^{2+l,1+l/2}(Q_T^1)}+\sup_{t<T}\|\boldsymbol{u}_1''(\cdot,t)\|_{W_2^{1+l}(\mathcal{F}_1)}\leqslant c\|\boldsymbol{u}_0''\|_{W_2^{1+l}(\mathcal{F}_1)}, \tag{4.16}$$

$$\begin{aligned} &\|\rho_1''\|_{W_2^{5/2+l,0}(G_T)}+\|\rho_1''\|_{W_2^{l/2}(0,T,W_2^{5/2}(S_{R_0}))}+\|\rho_{1t}''\|_{W_2^{l+3/2,l/2+3/4}(G_T)} \\ &\quad+\sup_{t<T}\|\rho_1''(\cdot,t)\|_{W_2^{2+l}(S_{R_0})}\leqslant c\|\rho_0''\|_{W_2^{2+l}(S_{R_0})}. \end{aligned}$$

The existence of extensions of initial data with such regularity properties follows from inverse trace theorems in Sobolev–Slobodetskii spaces and Proposition 4.1 in [11]. We put $q_1''=0$.

For $\boldsymbol{h}_1''$, we take a divergence free vector field such that

$$\boldsymbol{h}_1''\big|_{t=0}=\boldsymbol{h}_0'', \qquad y\in\mathcal{F}_1\cup\mathcal{F}_2,$$

and the estimate

$$\begin{aligned} &\sum_{i=1}^{2}\left(\|(\boldsymbol{h}_1'')^{(i)}\|_{W_2^{2+l,1+l/2}(Q_T^i)}+\sup_{t<T}\|\boldsymbol{h}_1''(\cdot,t)\|_{W_2^{1+l}(\mathcal{F}_i)}\right) \\ &\qquad\leqslant c\sum_{i=1}^{2}\|\boldsymbol{h}_0''\|_{W_2^{1+l}(\mathcal{F}_i)} \end{aligned} \tag{4.17}$$

holds (construction of this vector field is presented in [4]). Due to (4.1), (4.2), (4.16), (4.17), we have

$$X_{(0,T)}\left(\boldsymbol{u}_1'',q_1'',\rho_1'',\boldsymbol{h}_1''\right)+\sup_{t<T}Y_1''(t)\leqslant C_1(T)\varepsilon Y(0). \tag{4.18}$$

For $\rho(y,t)$, $y \in S_{R_0}$, we define a linear extension operator E with the following properties:

$$\operatorname{supp} E\rho \subset \Omega, \qquad \frac{\partial E\rho}{\partial n}\Big|_{S_{R_0}} = 0,$$

$$\|E\rho(\cdot,t)\|_{W_2^{r+1/2}(\Omega)} \leqslant c\|\rho\|_{W_2^r(S_{R_0})} \quad r \in (0, l+5/2],$$

$$\|\tfrac{\partial}{\partial t}E\rho(\cdot,t)\|_{W_2^{r+1/2}(\Omega)} \leqslant c\|\rho_t\|_{W_2^r(S_{R_0})} \quad r \in (0, l+3/2].$$

We set $\rho_1^{''*}(y,t) = E\rho_1^{''}$, $\rho^{'*}(y,t) = E\rho^{'}$.

Approximations $\boldsymbol{u}_{m+1}^{''}, q_{m+1}^{''}, \rho_{m+1}^{''}, \boldsymbol{h}_{m+1}^{''}$ for $m \geqslant 1$ can be found step by step from the following linear system of equations

$$\begin{aligned}
&\frac{\partial}{\partial t}\boldsymbol{u}_{m+1}^{''} - \nu\nabla^2\boldsymbol{u}_{m+1}^{''} + \nabla q_{m+1}^{''} = \mathcal{K}\big(\boldsymbol{N}^*(\rho' + \rho_m'')^* + \chi\xi_m\big) + \boldsymbol{l}_1^{(m)},\\
&\nabla\cdot\boldsymbol{u}_{m+1}^{''} = l_2^{(m)}, \quad \text{in} \quad \mathcal{F}_1,\\
&\mu_1\frac{\partial}{\partial t}\boldsymbol{h}_{m+1}^{''} + \alpha^{-1}\operatorname{rot}\operatorname{rot}\boldsymbol{h}_{m+1}^{''} = \boldsymbol{l}_7^{(m)},\\
&\nabla\cdot\boldsymbol{h}_{m+1}^{''} = 0, \quad in \quad \mathcal{F}_1,\\
&\operatorname{rot}\boldsymbol{h}_{m+1}^{''} = \operatorname{rot}\boldsymbol{l}_8^{(m)}, \quad \nabla\cdot\boldsymbol{h}_{m+1}^{''} = 0, \quad in \quad \mathcal{F}_2,
\end{aligned} \tag{4.19}$$

supplied with the following boundary conditions

$$\begin{aligned}
&\nu\Pi_0 S(\boldsymbol{u}_{m+1}^{''})\boldsymbol{N} = \boldsymbol{l}_3^{(m)}, \quad on \quad S_{R_0},\\
&-q_{m+1}^{''} + \nu\boldsymbol{N}\cdot S(\boldsymbol{u}_{m+1}^{''})\boldsymbol{N}(y) + \sigma B_0\rho_{m+1}^{''} = l_4^{(m)} + l_5^{(m)}, \quad on \quad S_{R_0},\\
&\frac{\partial}{\partial t}\rho_{m+1}^{''} - \boldsymbol{u}_{m+1}^{''}\cdot\boldsymbol{N}(y) + |\Omega_0|^{-1}\int_{\mathcal{F}_1}\boldsymbol{u}_{m+1}^{''}dz\cdot\boldsymbol{N}(y) = l_6^{(m)}, \quad on \quad S_{R_0},\\
&[\mu\boldsymbol{h}_{m+1}^{''}\cdot\boldsymbol{N}] = 0, \quad [\boldsymbol{h}_{m+1,\tau}^{''}] = \boldsymbol{l}_9^{(m)}, \quad \text{on} \quad S_{R_0},\\
&\boldsymbol{h}_{m+1}^{''}(y,t)\cdot\boldsymbol{n}(y) = 0 \text{ on } S\cup\Sigma, \quad \operatorname{rot}_\tau\boldsymbol{h}_{m+1}^{''} = 0 \text{ on } \Sigma, \quad \boldsymbol{u}_{m+1}^{''}\big|_\Sigma = 0,
\end{aligned}$$

and initial conditions

$$\begin{aligned}
&\boldsymbol{u}_{m+1}^{''}(y,0) = \boldsymbol{u}_0^{''}(y), \quad y \in \mathcal{F}_1, \quad \boldsymbol{h}_{m+1}^{''}(y,0) = \boldsymbol{h}_0^{''}(y), \ in\ \mathcal{F}_1\cup\mathcal{F}_2,\\
&\rho_{m+1}^{''}(y,0) = \rho_0^{''}(y), \quad on \quad S_{R_0}.
\end{aligned}$$

In the above relations, $\boldsymbol{\xi}_m(t) := \frac{1}{|\Omega_0|}\int\limits_0^t\int\limits_{\mathcal{F}_1}\left(\boldsymbol{u}^{'} + \boldsymbol{u}_m^{''}\right)|L|dyd\tau$,

$$l_j^{(m)} = l_j\left(\boldsymbol{u}^{'} + \boldsymbol{u}_m^{''}, q^{'} + q_m^{''}, \boldsymbol{h}^{'} + \boldsymbol{h}_m^{''}, \rho^{'} + \rho_m^{''}\right).$$

Now, we are aimed to prove (4.15) for all the approximations.

The problem (4.19) is uniquely solvable due to known results for linear problems (see [3, 6, 7]). Moreover,

$$\begin{aligned} X''_{m+1}(0,T) + \sup_{t<T} Y''_{m+1}(t) \\ \leqslant C_4(T)\Big(Z(\boldsymbol{u}' + \boldsymbol{u}''_m, q' + q''_m, \rho' + \rho''_m, \boldsymbol{h}' + \boldsymbol{h}''_m) \\ + \|\mathcal{K}\big(\boldsymbol{N}^*(\rho' + \rho''_m)^* + \chi\xi_m\big)\|_{W_2^{l,l/2}(Q_T^1)} + Y''(0)\Big). \quad (4.20) \end{aligned}$$

Let for the m-th approximation conditions (4.6) be fulfilled and Jacobian L is uniformly bounded. We apply Lemmas 4.1, 4.2 and the estimate

$$\begin{aligned} \sup_{t<T} |\boldsymbol{\xi}_m(t)| \leqslant \tfrac{1}{|\Omega_0|} \int_0^T d\tau \int_{\mathcal{F}_1} \Big(|\boldsymbol{u}'(y,\tau)| + |\boldsymbol{u}''_m(y,\tau)|\Big)\, |L| dy \\ \leqslant C_5(T) \Big(\|\boldsymbol{u}'\|_{L_2(Q_T^1)} + \|\boldsymbol{u}''_m\|_{L_2(Q_T^1)}\Big) \\ \leqslant C_5(T) \Big(X'(0,T) + X''_m(0,T)\Big) \quad (4.21) \end{aligned}$$

to the right-hand side of (4.20). This gives

$$\begin{aligned} X''_{m+1}(0,T) + \sup_{t<T} Y''_{m+1}(t) \\ \leqslant C_6(T)\Bigg(\Big(X'(0,T) + \sup_{t<T} Y'(t)\Big)^2 + \Big(X''_m(0,T) + \sup_{t<T} Y''_m(t)\Big)^2 \\ + \Big(X'(0,T) + \sup_{t<T} Y'(t)\Big)^3 + \Big(X''_m(0,T) + \sup_{t<T} Y''_m(t)\Big)^3 \Bigg) \\ + C_3(T)C_4(T)\Big(\varepsilon(X'(0,T) + \sup_{t<T} Y'(t)) + \varepsilon(X''_m(0,T) + \sup_{t<T} Y''_m(t)) \\ + C_5(T)(X'(0,T) + X''_m(0,T))\|\nabla \boldsymbol{f}\|_{W_2^{l,l/2}(\Omega\times(0,T))}\Big) \\ + C_4(T)Y''(0). \quad (4.22) \end{aligned}$$

By (4.3), (4.4) with $a = 0$ we obtain

$$X'(0,T) + \sup_{t<T} Y'(t) \leqslant C_7 \Big(Y(0) + \|\boldsymbol{f}\|_{W_2^{l,l/2}(Q_T^1)}\Big).$$

For the first approximation we have (4.18). Also, we have the smallness assumptions (2.4), (2.5) for the given functions. Hence, for the second approximation, (4.22) implies

$$\begin{gathered} X_2''(0,T) + \sup_{t<T} Y_2''(t) \\ \leqslant C_8(T)\varepsilon\left(Y(0) + \|\boldsymbol{f}\|_{W_2^{l,l/2}(Q_T^1)} + \|\nabla \boldsymbol{f}\|_{W_2^{l,l/2}(\Omega\times(0,T))}\right) \\ + C_9(T)\varepsilon^2\left(Y(0) + \|\boldsymbol{f}\|_{W_2^{l,l/2}(Q_T^1)} + \|\nabla \boldsymbol{f}\|_{W_2^{l,l/2}(\Omega\times(0,T))}\right), \\ C_8 = C_6C_7 + C_3C_4C_7(1+C_5) + C_4. \end{gathered} \tag{4.23}$$

We choose ε in such a way that the right-hand side of (4.23) is less or equal

$$2C_8(T)\varepsilon\left(Y(0)+F\right), \qquad F = \|\boldsymbol{f}\|_{W_2^{l,l/2}(Q_T^1)} + \|\nabla \boldsymbol{f}\|_{W_2^{l,l/2}(\Omega\times(0,T))}.$$

Let the estimate

$$X_m''[0,T] + \sup_{t<T} Y_m''(t) \leqslant 2C_8(T)\varepsilon\left(Y(0)+F\right) \tag{4.24}$$

holds for m-th approximation. For $(m+1)$-th approximation, (4.22) gives

$$\begin{gathered} X_{m+1}''(0,T) + \sup_{t<T} Y_{m+1}''(t) \leqslant C_8(T)\varepsilon\left(Y(0)+F\right) \\ +\Big(4C_8^2\varepsilon^3 + 8C_8^3\varepsilon^5 + C_6C_7\varepsilon^2 + 2C_3C_4C_8(C_5+1)\varepsilon^2\Big)(Y(0)+F) \end{gathered} \tag{4.25}$$

Hence, for the $(m+1)$-th approximation the estimate (4.24) follows from (4.25), provided that ε is sufficiently small. We conclude that if (4.6) hold for the mth approximation, then the same conditions hold for the $(m+1)$th approximation.

The convergence of the sequence $(\boldsymbol{u}_m'', q_m'', \rho_m'', \boldsymbol{h}_m'')$ follows from the estimates of the differences

$$\boldsymbol{k}_{m+1} = \boldsymbol{h}_{m+1}'' - \boldsymbol{h}_m'', \quad \boldsymbol{w}_{m+1} = \boldsymbol{u}_{m+1}'' - \boldsymbol{u}_m'',$$
$$r_{m+1} = \rho_{m+1}'' - \rho_m'', \quad p_{m+1} = q_{m+1}'' - q_m''.$$

Differences of the nonlinear terms can be estimated by the same technic as the nonlinear terms (Lemma 4.1, Lemma 4.2). Precisely, if all the assumptions of Lemma 4.1 be fulfilled, and (4.6) holds for $m \in \mathbf{N}$, then

$$\begin{aligned} &Z_T[(\boldsymbol{u}' + \boldsymbol{u}_m'', q' + q_m'', \boldsymbol{h}' + \boldsymbol{h}_m'', \rho' + \rho_m'') \\ &- (\boldsymbol{u}' + \boldsymbol{u}_{m-1}'', q' + q_{m-1}'', \boldsymbol{h}' + \boldsymbol{h}_{m-1}'', \rho' + \rho_{m-1}'')] \\ &= c\theta_1(\varepsilon,T) X\left(\boldsymbol{w}_m, p_m, \boldsymbol{k}_m, r_m\right), \end{aligned} \tag{4.26}$$

where the function $\theta_1(\varepsilon,T)$ is small for small ε.

If all the assumptions of Lemma 4.2 are fulfilled, and (4.6) holds for $m \in \mathbf{N}$, then

$$\big\|\mathcal{K}(N^*(\rho'+\rho''_m)^*+\chi\xi_m)-\mathcal{K}(N^*(\rho'+\rho''_{m-1})^*+\chi\xi_{m-1}\big\|_{W_2^{l,l/2}(Q_T^1)} \leqslant c\theta_2(\varepsilon,T)X\left(\boldsymbol{w}_m,p_m,\boldsymbol{k}_m,r_m\right), \quad (4.27)$$

where the function $\theta_2(\varepsilon,T)$ is small for small ε.

With the help of (4.26), (4.27), we deduce

$$X_{(0,T)}[\boldsymbol{w}_{m+1},p_{m+1},\boldsymbol{k}_{m+1},r_{m+1}] \leqslant c\theta_3(\varepsilon,T)X_{(0,T)}[\boldsymbol{w}_m,p_m,\boldsymbol{k}_m,r_m]. \quad (4.28)$$

For sufficiently small $\theta_3(\varepsilon,T)$, the estimate (4.28) guarantees the convergence of the sequence $(\boldsymbol{u}_m^{''},q_m^{''},\rho_m^{''},\boldsymbol{h}_m^{''})$ to the solution of problem (4.5). Passing to the limit in (4.24), we arrive at (4.15). Uniqueness of the solution follows from the above estimates applied to the difference of two solutions of (4.5). □

Taking a sum of solution $\boldsymbol{u}^{'},q^{'},\rho^{'},\boldsymbol{h}^{'}$ to linear problem (3.1), (3.23) with initial data $\boldsymbol{u}_0^{'},\rho_0^{'},\boldsymbol{h}_0^{'}$ and solution $\boldsymbol{u}^{''},q^{''},\rho^{''},\boldsymbol{h}^{''}$ to problem (4.5), we obtain a solution to problem (2.2), (2.3) on time interval $[0,T]$. Remind, that the value T was fixed in such a way that the factor c_1e^{-aT} in (4.4) is not grater then $\frac{1}{4}$. Now we choose such ε that $c_2(T)\varepsilon$ in (4.15) is also not grater then $\frac{1}{4}$. In consequence of (4.4), (4.15), solution to problem (2.2), (2.3) satisfies the estimate

$$Y(T) \leqslant \frac{1}{2}Y(0)+\frac{1}{4}\Big(\|\boldsymbol{f}\|_{W_2^{l,l/2}(Q_T^1)}+\|\nabla\boldsymbol{f}\|_{W_2^{l,l/2}(\Omega\times(0,T))} + \Big(\int_0^T \|e^{a\tau}\boldsymbol{f}(\cdot,\tau)\|^2_{L_2(\mathcal{F}_1)}d\tau\Big)^{1/2}\Big). \quad (4.29)$$

We prove the existence result in an infinite time interval step by step. Let we have proved existence of a solution to problem (2.2), (2.3) on time interval $[0,kT]$. Let $|\boldsymbol{\xi}(t)|$ is uniformly bounded for $t\in[0,kT]$, and the estimates

$$
\begin{aligned}
Y(iT) \leqslant \frac{1}{2} Y((i-1)T) \\
+ \frac{1}{4}\left(F[i] + \left(\int\limits_{(i-1)T}^{iT} \|e^{a(\tau-(i-1)T)} \boldsymbol{f}(\cdot,\tau)\|^2_{L_2(\mathcal{F}_1)} d\tau \right)^{1/2} \right),
\end{aligned}
\tag{4.30}
$$

where

$$
F[i] = \|\boldsymbol{f}\|_{W_2^{l,l/2}(\mathcal{F}_1\times((i-1)T,iT))} + \|\nabla \boldsymbol{f}\|_{W_2^{l,l/2}(\Omega\times((i-1)T,iT))}
$$

are valid for $i = 1,\dots,k$. On time interval $[(i-1)T, iT]$, the solution can be decomposed in two parts: $\boldsymbol{u} = \boldsymbol{u}^{'} + \boldsymbol{u}^{''}$, $q = q^{'} + q^{''}$, $\rho = \rho^{'} + \rho^{''}$, $\boldsymbol{h} = \boldsymbol{h}^{'} + \boldsymbol{h}^{''}$, satisfying the following estimates

$$
X_{[(i-1)T,iT]}\left(\boldsymbol{u}^{''}, q^{''}, \rho^{''}, \boldsymbol{h}^{''}\right) \leqslant \frac{1}{4}\left(Y((i-1)T) + F[i]\right), \tag{4.31}
$$

$$
\begin{aligned}
X_{[(i-1)T,iT]}\left(e^{a(t-(i-1)T)}\boldsymbol{u}^{'}, e^{a(t-(i-1)T)}q^{'}, e^{a(t-(i-1)T)}\rho^{'}, e^{a(t-(i-1)T)}\boldsymbol{h}^{'}\right) \\
\leqslant c\left(Y((i-1)T) + \|e^{a(t-(i-1)T)}\boldsymbol{f}\|_{W_2^{l,l/2}(\mathcal{F}_1\times((i-1)T,iT))}\right), \quad a < b.
\end{aligned}
\tag{4.32}
$$

Now we prove existence in time interval $[kT, (k+1)T]$. We consider $\boldsymbol{u}_{kT} = \boldsymbol{u}(\cdot, kT)$, $\rho_{kT} = \rho(\cdot, kT)$, $\boldsymbol{h}_{kT} = \boldsymbol{h}(\cdot, kT)$, as initial data at $t = kT$ and repeat the above scheme on $[kT, (k+1)T]$. Due to the conservation of the volume, the first of conditions (1.6) holds for $\rho(y, kT)$. The barycenter is located at the point $\xi(kT)$, which not necessary coincides with the origin. We have

$$
\int\limits_{\Omega_{kT}} x_i dx = \xi_i(kT)\frac{4}{3}\pi R_0^3 = \xi_i(kT)\int\limits_{\Omega_{kT}} dx,
$$

i.e. $\int\limits_{\Omega_{kT}} (x_i - \xi_i(kT))\, dx = 0$, $i = 1, 2, 3$. We pass to the spherical coordinates with the center at the point $\boldsymbol{\xi}(kT)$ and see that the linear part of the second condition (1.6) for $\rho(y, kT)$ has the same form as for ρ_0, precisely, $\int\limits_{S_1} y_i \rho(R_0 y, kT) dS = 0$. Consequently, we can use all the results of section 3 on the time interval $[kT, (k+1)T]$.

To repeat the above scheme on time interval $[kT,(k+1)T]$, we again separate the data at $t = kT$ in two parts

$$\boldsymbol{u}_{kT} = \boldsymbol{u}''_{kT} + \boldsymbol{u}'_{kT}, \quad \rho_{kT} = \rho''_{kT} + \rho'_{kT}, \quad \boldsymbol{h}_{kT} = \boldsymbol{h}''_{kT} + \boldsymbol{h}'_{kT},$$

where the functions $\boldsymbol{u}''_{kT}$, ρ''_{kT}, $\boldsymbol{h}''_{kT}$ satisfy the same compatibility conditions as $\boldsymbol{u}_{kT}$, ρ_{kT}, $\boldsymbol{h}_{kT}$ in nonlinear problem (2.2), (2.3) and have the order ε^2. The solution $\boldsymbol{u}', q', \rho', \boldsymbol{h}'$ to linear problem (3.1), (3.23) with initial data $\boldsymbol{u}'_{kT}, \rho'_{kT}, \boldsymbol{h}'_{kT}$ satisfies (4.3), (4.4) on time interval $[kT,(k+1)T]$. It gives

$$Y'((k+1)T) \leqslant \frac{1}{4}\left(Y'(kT) + \left(\int\limits_{kT}^{(k+1)T} \|e^{a(\tau-kT)}\boldsymbol{f}(\cdot,\tau)\|^2_{L_2(\mathcal{F}_1)} d\tau\right)^{1/2}\right) \tag{4.33}$$

and (4.32) for $i = k+1$.

To apply Theorem 5 on time interval $[kT,(k+1)T]$, we have to take care of the term

$$\sup_{kT<t<(k+1)T} |\boldsymbol{\xi}(t)|.$$

It is clear that $\boldsymbol{\xi}(t) - \boldsymbol{\xi}(kT)$ is estimated in the same way as in (4.21) by $\|\boldsymbol{u}\|_{L_2(\mathcal{F}_1\times(kT,(k+1)T))}$, and it remains to estimate $|\xi(kT)|$.

Lemma 4.3 *Let inequalities* (4.30), (4.31), (4.32) *are valid for* $i = 1,\dots,k$. *Then*

$$|\boldsymbol{\xi}(kT)| \leqslant C\varepsilon, \tag{4.34}$$

where the constant C is independent of kT and ε.

Proof. We use (4.30) for $i = 1,\dots,k.$, and deduce

$$Y(kT) \leqslant \frac{1}{2^k}Y(0) + \sum_{i=1}^{k} \frac{1}{2^{k-i+2}}\left(F[i] + \left(\int\limits_{(i-1)T}^{iT} \|e^{a(\tau-(i-1)T)}\boldsymbol{f}(\cdot,\tau)\|^2_{L_2(\mathcal{F}_1)}\right)^{1/2}\right). \tag{4.35}$$

Under the assumptions of Theorem 2, the norms $\|e^{at}\boldsymbol{f}\|_{W_2^{l,l/2}\Omega\times(0,+\infty)}$, $\|e^{at}\nabla\boldsymbol{f}\|_{W_2^{l,l/2}\Omega\times(0,+\infty)}$, $a\leqslant b$ are bounded. It is clear that

$$F[i]\leqslant e^{-(i-1)aT}\Big(\|e^{at}\boldsymbol{f}\|_{W_2^{l,l/2}(\Omega\times((i-1)T,iT))} + \|e^{at}\nabla\boldsymbol{f}\|_{W_2^{l,l/2}(\Omega\times((i-1)T,iT))}\Big).$$

Similarly,

$$\Big(\int\limits_{(i-1)T}^{iT}\|e^{a(\tau-(i-1)T)}\boldsymbol{f}(\cdot,\tau)\|^2_{L_2(\mathcal{F}_1)}\Big)^{1/2}\leqslant e^{-(i-1)aT}\|e^{at}\boldsymbol{f}\|_{L_2(\mathcal{F}_1)\times((i-1)T,iT)}.$$

As a result, (4.35) leads to

$$\begin{aligned}Y(kT)&\leqslant\frac{1}{2^k}Y(0)\\&+\frac{1}{(\min\{2,e^{aT}\})^k}\Big(\|e^{at}\boldsymbol{f}\|_{W_2^{l,l/2}(\Omega\times(0,kT))}+\|e^{at}\nabla\boldsymbol{f}\|_{W_2^{l,l/2}(\Omega\times(0,kT))}\Big)\\&\leqslant\frac{1}{(\min\{2,e^{aT}\})^k}\Big(Y(0)+\|e^{at}\boldsymbol{f}\|_{W_2^{l,l/2}(\Omega\times(0,+\infty))}\\&\qquad+\|e^{at}\nabla\boldsymbol{f}\|_{W_2^{l,l/2}(\Omega\times(0,+\infty))}\Big).\end{aligned}\tag{4.36}$$

Note, that (4.36) is valid for every $k_1=1,2,\dots,k$. In particular, for $t\in[0,kT]$

$$\begin{aligned}&\|\boldsymbol{u}(\cdot,t)\|_{L_2(\mathcal{F}_1)}\\&\leqslant ce^{-\alpha t}\Big(Y(0)+\|e^{at}\boldsymbol{f}\|_{W_2^{l,l/2}(\Omega\times[0,+\infty))}+\|e^{at}\nabla\boldsymbol{f}\|_{W_2^{l,l/2}(\Omega\times[0,+\infty))}\Big)\\&\leqslant 3ce^{-\alpha t}\varepsilon,\end{aligned}$$

with a certain $\alpha>0$. In consequence of (4.31), (4.32), Jacobian L is uniformly bounded for $t\in[0,kT]$. Using this fact and the Holder inequality, we obtain

$$|\boldsymbol{\xi}(kT)| = \Big|\int\limits_0^{kT} dt \int\limits_{\Omega_{1t}} \boldsymbol{v}(x,t)dx\Big| \leqslant \int_0^{kT} dt \int_{\mathcal{F}_1} |\boldsymbol{u}(y,t)||L|dy$$

$$\leqslant c\int\limits_0^{kT} \|\boldsymbol{u}(\cdot,t)\|_{L_2(\mathcal{F}_1)}dt \leqslant c_1 \int\limits_0^{+\infty} \varepsilon e^{-\alpha t}dt \leqslant C\varepsilon, \tag{4.37}$$

with the constant C independent of kT and ε. □

Now, we can repeat the proof of Theorem 5 on time interval $[kT, (k+1)T]$, replacing everywhere $Y(0)$ by $Y(kT)$ and the first term at the right-hand side of (4.23) by

$$\left(C_8(T) + C\right)\varepsilon\left(Y(kT) + F[k+1]\right),$$

where C is the constant from (4.37). By Theorem 5, solution to problem (4.5) exists for $t \in [kT, (k+1)T]$ and satisfies the estimate

$$X_{[kT,(k+1)T]}\left(\boldsymbol{u}^{''}, q^{''}, \rho^{''}, \boldsymbol{h}^{''}\right) + \sup_{kT<t<(k+1)T} Y^{''}(t) \leqslant \tilde{c}_2(T)\varepsilon\left(Y(kT) + F[k+1]\right). \tag{4.38}$$

Due to the fact that the constant C in (4.37) is independent of k, we can be sure that the constant $\tilde{c}_2(T)$ is also independent of k. Starting from $k = 1$, we fix so small ε that $\tilde{c}_2(T)\varepsilon < \frac{1}{4}$. Hence, we choose ε when we construct solution on $[T, 2T]$. For $k \geqslant 2$ we can repeat the proof with the same ε.

Taking a sum of solutions to problem (4.5) with initial data $\boldsymbol{u}^{''}_{kT}$, $\rho^{''}_{kT}$, $\boldsymbol{h}^{''}_{kT}$ and to linear problems (3.1), (3.23) with initial data $\boldsymbol{u}^{'}_{kT}$, $\rho^{'}_{kT}$, $\boldsymbol{h}^{'}_{kT}$, we obtain a solution to problem (2.2), (2.3) on time interval $[kT, (k+1)T]$. Inequalities (4.30) − (4.32) for this solution are valid ($i = k+1$). We repeat the above scheme for any $k \in \boldsymbol{N}$ and step by step obtain a solution to problem (2.2), (2.3) on an infinite time interval $[0, +\infty)$.

Now, we pass to estimate (2.6). By (4.32) and (4.36), we have

$$X_{[(i-1)T,iT]}\Big(e^{a(t-(i-1)T)}\boldsymbol{u}', e^{a(t-(i-1)T)}q', e^{a(t-(i-1)T)}\rho', e^{a(t-(i-1)T)}\boldsymbol{h}'\Big)$$
$$\leqslant c\frac{1}{(\min\{2,e^{aT}\})^{i-1}}\Big(Y(0)+2\|e^{at}\boldsymbol{f}\|_{W_2^{l,l/2}(\Omega\times(0,+\infty))}$$
$$+\|e^{at}\nabla\boldsymbol{f}\|_{W_2^{l,l/2}(\Omega\times(0,+\infty))}\Big),\ i\in\mathbb{N}, \quad (4.39)$$

here the constant c is independent of i. We estimate the right-hand side of (4.31) by the same reasonings as in proving (4.36), which gives

$$X_{[(i-1)T,iT]}\Big(\boldsymbol{u}'', q'', \rho'', \boldsymbol{h}''\Big)$$
$$\leqslant \frac{1}{\min\{2,e^{aT}\}^{i}}\Big(Y(0)+\|e^{at}\boldsymbol{f}\|_{W_2^{l,l/2}(\Omega\times(0,+\infty))}$$
$$+\|e^{at}\nabla\boldsymbol{f}\|_{W_2^{l,l/2}(\Omega\times(0,+\infty))}\Big),\ i\in\mathbb{N}. \quad (4.40)$$

Estimates (4.39) and (4.40) imply (2.6), provided that $e^{aT}<2$.

Theorem 1 follows from Theorem 2. We find the position of the free boundary for any $t>0$ by the formula

$$\Gamma_t=\{x=y+\boldsymbol{N}(y)\rho(y,t)+\boldsymbol{\xi}(t),\quad y\in S_{R_0}\},$$

make the inverse coordinate transform and obtain a solution $\boldsymbol{v}$, p, $\boldsymbol{H}$ to the free boundary problem (1.1) – (1.5).

Due to (2.6), we conclude that Jacobian of the mapping (2.1) is uniformly bounded for any $t>0$, and exponential decay of the solution in terms of Sobolev norms takes place for $t\to+\infty$. By the same reasonings as in (4.37), we have

$$|\boldsymbol{\xi}(+\infty)|\leqslant\int_0^{+\infty}dt\int_{\Omega_{1t}}|\boldsymbol{v}(x,t)|dx$$
$$\leqslant C\int\limits_0^{+\infty}\|\boldsymbol{u}(\cdot,t)\|_{L_2(\mathcal{F}_1)}dt\leqslant C\int\limits_0^{+\infty}\varepsilon e^{-\alpha t}dt\leqslant C^*\varepsilon. \quad (4.41)$$

It means that $|\boldsymbol{\xi}(t)|$ is uniformly bounded for any $t>0$. To be sure that the free boundary do not intersect the fixed parts of the boundary, we have assumed that at the initial moment of time $\operatorname{dist}\{\Gamma_0,\Sigma\}>3d_0$, $\operatorname{dist}\{\Gamma_0,S\}>3d_0$, $d_0>(C^*+1)\varepsilon$ (see assumptions of Theorem 1).

Remark 1. Theorem 1 implies exponential stability of the trivial rest state with zero velocity and zero magnetic field.

Remark 2. The same scheme can be applied to the free boundary problem described the motion of a finite isolated mass of a viscous incompressible fluid in vacuum when the external force is acting on the fluid, but there is no magnetic field.

References

1. O. A. Ladyzhenskaya and V. A. Solonnikov, *Solution of some non-stationary problems of magnetohydrodynamics for a viscous incompressible fluid.* — Trudy mat. Inst. Steklov **59** (1960), 115–173.
2. O. A. Ladyzhenskaya and V. A. Solonnikov, *The linearization principle and invariant manifolds for problems of magnetohydrodynamics.* — Zap. Nauchn. Sem. POMI **38**, (1973), 46–93.
3. M. Padula and V. A. Solonnikov, *On the free boundary problem of magnethydrodynamics.* — Zap. Nauchn. Sem. POMI **385**, (2010), 135–186.
4. V.A. Solonnikov, *On the stability of uniformly rotating liquid in a weak magnetic field.* — Problemy Mat. Anal. **57**, (2011), 165–191.
5. S. Mosconi and V.A. Solonnikov, *On a problem of magnetohydrodinamics in a multi-connected domain.* — Nonlinear analysis, **74** N2 (2010), 462–478.
6. V. A. Solonnikov, *Free boundary problems of magnetohydrodynamics in multi-connected domains.* — Interfaces and Free Boundaries **14** (2012), 569–602.
7. V. A. Solonnikov and E. V. Frolova *Solvability of a free boundary problem of magnetohydrodynamics in an infinite time interval.* — Zap. Nauchn. Sem. POMI **410**, (2013), 131–167.
8. M. Padula and V. A. Solonnikov, *On stability of equlilbrium figures of a uniformly rotating liquid drop in n-dimentional space.* - RIMS Kokyuroku Bessatsu, B1, (2007).
9. M. Padula and V. A. Solonnikov, *On Raileigh-Teylor stability.* — Annali dell University di Ferrara, 45, 2000.
10. V. A. Solonnikov, *On the stability of uniformly rotating viscous incompressible self-gravitating fluid.* — Beijing, April 2009.
11. V.A. Solonnikov, *On the linear problem arising in the study of a free boundary problem for the Navier–Stokes equations.* — Algebra i Analiz, **22**, no. 6, (2010), 235–269.

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