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LIOUVILLE THEOREM FOR 2D NAVIER–STOKES EQUATIONS IN HALF SPACE

Abstract. A Liouville type theorem for mild bounded ancient solutions to the Navier-Stokes system in a half plane has been proven provided that a certain scale invariant quantity is bounded.

Dedicated to the 80th jubilee of V. A. Solonnikov

§1. Motivation

In the paper, we deal with the so-called mild bounded ancient solutions to the 2D Navier–Stokes equations in half-space with the homogeneous Dirichlet boundary conditions. As it has been explained in [4, 5], and [6], such type of solutions appears as a result of re-scaling solutions to the Navier–Stokes equations around a possible singular point. If they are in a sense "trivial," then this point is not singular.

There are several interesting cases for which Liouville type theorems for ancient solutions to the Navier–Stokes equations turn out to be true. And their proofs are based on a reduction to a scalar equation with the further application of the strong maximum principle to it. For example, in 2D case, such a scalar equations is just the 2D vorticity equation. Unfortunately, this approach does not work in a half plane since non-slip boundary conditions in terms of the velocity does not implies the homogeneous Dirchlet boundary condition for the vorticity. However, there are some interesting results coming out from this approach, see paper [1] and reference in it.

In the paper, we exploit a different approach related to the long time behaviour of solutions to a conjugate system. It has been already used in the proof of the Liouville type theorem for the Stokes system in half-space, see the paper [3] and the paper [2] for another approach.

Let u be a mild bounded ancient solution to the Navier–Stokes equations in a half space, i.e., $u \in L_\infty(Q^+_-)$ ($|u| \leqslant 1$ a.e. in $Q^+_- = \{x \in \mathbb{R}^2_+,\, t < 0\}$,

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where $\mathbb{R}^2_+ = \{x = (x_1, x_2) \in \mathbb{R}^2 : \ x_2 > 0\}$) and there exists a scalar function p such that, for any $t < 0$, $p = p^1 + p^2$, where

$$\triangle p^1 = -\operatorname{div}\operatorname{div} u \otimes u \tag{1.1}$$

in Q^+_- with $p^1_{,2} = 0$ and $p^2(\cdot, t)$ is a harmonic function in $\mathbb{R}^2_+$ whose gradient obeys the inequality

$$|\nabla p^2(x,t)| \leqslant c \ln(2 + 1/x_2) \tag{1.2}$$

for $(x,t) \in Q^+_-$ and has the property

$$\sup_{x_1 \in \mathbb{R}} |\nabla p^2(x,t)| \to 0 \tag{1.3}$$

as $x_2 \to \infty$; u and p satisfy the classical Navier–Stokes system and boundary condition $u(x_1, 0, t) = 0$ in the following weak sense

$$\int\limits_{Q^+_-} \Big(u \cdot (\partial_t \varphi + \triangle \varphi) + u \otimes u : \nabla \varphi + p \operatorname{div} \varphi \Big) \, dx \, dt = 0 \tag{1.4}$$

for any $\varphi \in C^\infty_0(Q_-)$ with $\varphi(x_1, 0, t) = 0$ for $x_1 \in \mathbb{R}$ and

$$\int\limits_{Q^+_-} u \cdot \nabla q \, dx \, dt = 0 \tag{1.5}$$

for any $q \in C^\infty_0(Q_-)$.

Here, $Q_- = \mathbb{R}^2 \times \{t < 0\}$.

We are going to prove the following fact:

Theorem 1.1. *Let u be a mild bounded ancient solution to the Navier–Stokes equations in a half space. Assume in addition that*

$$u \in L_{2,\infty}(Q^+_-). \tag{1.6}$$

Then u is identically equal to zero.

Remark 1.2. Motivation for additional condition (1.6) is as follows. The norm of the space $L_{2,\infty}(Q^+_-)$ is invariant with respect to the Navier–Stokes scaling

$$v(x,t) \to \lambda u(\lambda x, \lambda^2 t).$$

So, if we study the smoothness of energy solutions in 2D, the corresponding norm stays bounded under scaling and a limiting procedure, leading to a mild bounded ancient solution, and thus condition (1.6) holds. For details, see [6].

Lemma 1.3. *Under assumptions of Theorem* 1.1*,*

$$\nabla u \in L_2(Q^+_-). \tag{1.7}$$

Proof. For fixed $A < 0$, we can construct $\widetilde{u}$ as a solution to the initial boundary value problem

$$\partial_t \widetilde{u} - \triangle \widetilde{u} + \nabla \widetilde{p}^2 = -\mathrm{divH}$$

in $\mathbb{R}^2_+ \times]A, 0[$, where $H = u \otimes u + p^1 \mathbb{I}$,

$$\widetilde{u}(x_1, 0, t) = 0,$$
$$\widetilde{u}(x, A) = u(x, A)$$

with the help of the Green function G and the kernel K introduced by Solonnikov in [7], i.e.,

$$\widetilde{u}(x,t) = \int\limits_{\mathbb{R}^2_+} G(x, y, t - A) u(y, A)\, dy + \int\limits_A^t \int\limits_{\mathbb{R}^2_+} K(x, y, t - \tau) F(y, \tau)\, dy\, d\tau.$$

For the further details, we refer the reader to the paper [6].

Let us describe the properties of $\widetilde{u}$. Our first observation is that

$$\mathrm{div}\, u \otimes u = u \cdot \nabla u \in L_{2,\infty}(Q^+_-)$$

since $u \in L_{2,\infty}(Q^+_-)$ and $\nabla u \in L_\infty(Q^+_-)$. The last fact has been proven in [6]. Hence,

$$\mathrm{divH} \in L_{2,\infty}(Q^+_-).$$

By the properties of the kernels G and K, such a solution $\widetilde{u}$ is bounded and satisfies the energy identity

$$\int\limits_{\mathbb{R}^2_+} |\widetilde{u}(x,t)|^2 dx + 2 \int\limits_A^t \int\limits_{\mathbb{R}^2_+} |\nabla \widetilde{u}(x,\tau)|^2\, dx\, d\tau$$
$$= \int\limits_{\mathbb{R}^2_+} |u(x,A)|^2\, dx + 2 \int\limits_A^t \int\limits_{\mathbb{R}^2_+} \mathrm{divH}(x,\tau) \cdot \widetilde{u}(x,\tau)\, dx\, d\tau$$

for all $A \leqslant t \leqslant 0$. In addition, we can state that for any $\delta > 0$,

$$\int\limits_{A+\delta}^0 \int\limits_{\mathbb{R}^2_+} |\nabla \widetilde{p}^2|^2\, dx\, dt < C(\delta, A) < \infty. \tag{1.8}$$

Our aim is to show that $u = \widetilde{u}$ in $\mathbb{R}^2_+ \times]A, 0[$. It is easy to see that, for any $R > 0$,

$$\|v(\cdot, t)\|_{2, B_+(R)} \to 0$$

as $t \to A$, where $v = u - \widetilde{u}$. This follows from the facts that u is continuous on the completion of the set $Q_+(R)$ for any $R > 0$, see details in [6], and that $\widetilde{u} \in C([A, 0]; L_2(\mathbb{R}^2_+))$.

The latter property allows us to show that v satisfies that the identity

$$\int\limits_A^0 \int\limits_{\mathbb{R}^2_+} (v \cdot \partial_t \varphi + v \cdot \triangle \varphi)\, dx\, dt = 0$$

for any $\varphi \in C_0^\infty(Q_-)$ such that $\varphi(x_1, 0, t) = 0$ for any $x_1 \in \mathbb{R}$ and any $t \in]-\infty, 0[$ and $\operatorname{div} \varphi = 0$ in Q_-^+. If we extend v by zero for $t < A$, this field will be bounded ancient solution to the Stokes system and therefore has the form $v = (v_1(x_2, t), 0)$, see [2] and [3]. The gradient of the corresponding pressure $p^2 - \widetilde{p}^2$ depends only on t. However, by (1.3) and by (1.8), this gradient must be zero. And the Liouville theorem for the heat equation in the half-space implies that $v = 0$.

Now, since $u = \widetilde{u}$, the energy identity implies

$$\int\limits_{\mathbb{R}^2_+} |u(x, 0)|^2 dx + 2 \int\limits_A^0 \int\limits_{\mathbb{R}^2_+} |\nabla u(x, \tau)|^2\, dx\, d\tau = \int\limits_{\mathbb{R}^2_+} |u(x, A)|^2\, dx$$

for any $A < 0$. This completes the proof of the lemma. □

Remark 1.4. In fact, we have proven that

$$\int\limits_A^0 \int\limits_{\mathbb{R}^2_+} |\nabla p^2|^2\, dx\, dt \leqslant c < \infty$$

for any $A < 0$.

Given a tensor-valued function $F \in C_0^\infty(Q_-^+)$, let us consider the following initial boundary value problem:

$$\partial_t v + u \cdot \nabla v + \triangle v + \nabla q = \operatorname{div} F, \qquad \operatorname{div} v = 0 \tag{1.9}$$

in $Q_+ = \mathbb{R}^2_+ \times]-\infty, 0[$,

$$v(x_1, 0, t) = 0 \tag{1.10}$$

for any $x_1 \in \mathbb{R}$ and $t \leqslant 0$, and

$$v(x,0) = 0 \tag{1.11}$$

for $x \in \mathbb{R}^2_+$. Here, vector-valued field v and scalar function q are unknown.

Why we consider this system? At least formally, we have the following identity

$$\begin{aligned}\int\limits_{Q^+_-} u \cdot \operatorname{div} F \, dx\, dt &= \int\limits_{Q^+_-} u \cdot \Big(\partial_t v + u \cdot \nabla v + \triangle v + \nabla q\Big)\, dx\, dt \\ &= \int\limits_{Q^+_-} u \cdot \Big(\partial_t v + u \cdot \nabla v + \triangle v\Big)\, dx\, dt \\ &= \int\limits_{Q^+_-} \Big(- \partial_t u - \operatorname{div} u \otimes u + \triangle u\Big) \cdot v \, dx\, dt \\ &= \int\limits_{Q^+_-} \Big(- \partial_t u - \operatorname{div} u \otimes u + \triangle u - \nabla p\Big) \cdot v \, dx\, dt = 0.\end{aligned}$$

This would imply that u is the function of t only and thus, since u is a mild bounded ancient solution, u must be identically zero.

§2. Properties of solutions to dual problem

Proposition 2.1. *There exists a unique solution v to* (1.9), (1.10), *and* (1.11) *with the following properties:*

$$v \in L_{2,\infty}(Q^+_-), \quad \nabla v \in L_2(Q^+_-),$$

and, for all $T < 0$,

$$\partial_t p, \nabla^2 v, \nabla q \in L_2(\mathbb{R}^2_+ \times]T, 0[).$$

Proof. First of all, there exists a unique energy solution. This follows from the identity

$$\int\limits_{Q^+_-} (u \cdot \nabla v) \cdot v \, dx\, dt = 0$$

and from the inequality

$$\Big|-\int\limits_{Q^+_-}\operatorname{div}F\cdot v\,dx\,dt\Big|=\Big|\int\limits_{Q^+_-}F:\nabla v\,dx\,dt\Big|$$
$$\leqslant\left(\int\limits_{Q^+_-}|F|^2\,dx\,dt\right)^{\frac12}\left(\int\limits_{Q^+_-}|\nabla v|^2\,dx\,dt\right)^{\frac12}.$$

So, we can state that

$$v\in L_{2,\infty}(Q^+_-),\quad \nabla v\in L_2(Q^+_-). \tag{2.1}$$

The latter means that $u\cdot\nabla v\in L_2(Q^+_-)$. So, statements of Proposition 2.1 follows from the theory for Stokes system. □

§3. Main formula, integration by parts

For smooth function $\psi\in C_0^\infty(\mathbb{R}^2\times\mathbb{R})$, we have

$$\int\limits_{Q^+_-}u\cdot\psi\operatorname{div}F\,dx\,dt=\int\limits_{Q^+_-}u\cdot\psi\Big(\partial_t v+u\cdot\nabla v+\triangle v+\nabla q\Big)\,dx\,dt$$
$$=\int\limits_{Q^+_-}\Big(-u\cdot v\partial_t\psi-u\cdot vu\cdot\nabla\psi-u_iv_{i,j}\psi_{,j}+u_{i,j}v_i\psi_{,j}-qu\cdot\nabla\psi\Big)\,dx\,dt$$
$$-v\psi\cdot\Big(\partial_t u+u\cdot\nabla u-\triangle u\Big)\,dx\,dt=\int\limits_{Q^+_-}\Big(-u\cdot v\partial_t\psi-u\cdot vu\cdot\nabla\psi$$
$$-2u_iv_{i,j}\psi_{,j}+(u_{i,j}v_i+u_iv_{i,j})\psi_{,j}-qu\cdot\nabla\psi\Big)\,dx\,dt+\int\limits_{Q^+_-}v\psi\cdot\nabla p\,dx\,dt$$
$$=-\int\limits_{Q^+_-}\Big(u\cdot v\partial_t\psi+u\cdot vu\cdot\nabla\psi+2u_iv_{i,j}\psi_{,j}+u\cdot v\triangle\psi+(qu+pv)\cdot\nabla\psi\Big)\,dx\,dt.$$

We pick $\psi(x,t)=\chi(t)\varphi(x)$. Using simple arguments and smoothness of u and v, we can get rid of χ and have

$$J(T)=\int\limits_T^0\int\limits_{\mathbb{R}^2_+} u\cdot\varphi\operatorname{div}F\,dx\,dt=-\int\limits_{\mathbb{R}^2_+}\varphi(x)u(x,T)\cdot v(x,T)\,dx$$
$$+\int\limits_T^0\int\limits_{\mathbb{R}^2_+}\Big(u\cdot vu\cdot\nabla\varphi+2u_iv_{i,j}\varphi_{,j}+u\cdot v\triangle\varphi+(qu+pv)\cdot\nabla\varphi\Big)\,dx\,dt.$$

Fix a cut-off function $\varphi(x)=\xi(x/R)$, where $\xi\in C_0^\infty(\mathbb{R}^3)$ with the following properties: $0\leqslant\xi\leqslant1$, $\xi(x)=1$ if $|x|\leqslant1$, and $\xi(x)=0$ if $|x|\geqslant2$. Our aim is to show that

$$J_R=\int\limits_T^0\int\limits_{\mathbb{R}^2_+}\Big(u\cdot\widetilde{v}u\cdot\nabla\varphi+2u_i\widetilde{v}_{i,j}\varphi_{,j}+u\cdot\widetilde{v}\triangle\varphi+(\widetilde{q}u+p\widetilde{v})\cdot\nabla\varphi\Big)\,dx\,dt$$

tends to zero if $R\to\infty$.

We start with

$$\left|\int\limits_T^0\int\limits_{\mathbb{R}^2_+}2u_iv_{i,j}\varphi_{,j}\,dx\,dt\right|$$
$$\leqslant\frac{c}{R}\left(\int\limits_T^0\int\limits_{B_+(2R)}|u|^2\,dx\,dt\right)^{\frac12}\left(\int\limits_T^0\int\limits_{B_+(2R)\setminus B_+(R)}|\nabla v|^2\,dx\,dt\right)^{\frac12}$$
$$\leqslant c\sqrt{-T}\left(\int\limits_T^0\int\limits_{B_+(2R)\setminus B_+(R)}|\nabla v|^2\,dx\,dt\right)^{\frac12}\to0$$

as $R\to\infty$.

Next, since $|u|\leqslant1$, we have

$$\left|\int\limits_T^0\int\limits_{\mathbb{R}^2_+}u\cdot v\triangle\varphi\,dx\,dt\right|\leqslant\frac{c}{R^2}\left(\int\limits_T^0\int\limits_{B_+(2R)}|u|^2\,dx\,dt\right)^{\frac12}\left(\int\limits_T^0\int\limits_{\mathbb{R}^2_+}|v|^2\,dx\,dt\right)^{\frac12}$$
$$\leqslant c\frac{-T}{R}\|v\|_{2,\infty,Q^+_-}\to0$$

as $R \to \infty$.

The third term is estimated as follows (by boundedness of u):

$$\begin{aligned}
&\left| \int_T^0 \int_{\mathbb{R}^2_+} u \cdot vu \cdot \nabla\varphi \, dx \, dt \right| \\
&\quad \leqslant \frac{c}{R} \left(\int_T^0 \int_{B_+(2R)} |u|^4 \, dx \, dt \right)^{\frac{1}{2}} \left(\int_T^0 \int_{B_+(2R)\setminus B_+(R)} |v|^2 \, dx \, dt \right)^{\frac{1}{2}} \\
&\quad \leqslant \frac{c}{R} \left(\int_T^0 \int_{B_+(2R)} |u|^2 \, dx \, dt \right)^{\frac{1}{2}} \left(\int_T^0 \int_{B_+(2R)\setminus B_+(R)} |v|^2 \, dx \, dt \right)^{\frac{1}{2}} \\
&\quad \leqslant \frac{c}{R} \left(\int_T^0 \int_{\mathbb{R}^2_+} |u|^2 \, dx \, dt \right)^{\frac{1}{2}} \left(\int_T^0 \int_{\mathbb{R}^2_+} |v|^2 \, dx \, dt \right)^{\frac{1}{2}} \\
&\quad \leqslant c \frac{-T}{R} \|u\|_{2,\infty,Q^+_-} \|u\|_{2,\infty,Q^+_-} \to 0
\end{aligned}$$

as $R \to \infty$.

The first term containing the pressure is estimated as follows. We have

$$\int_T^0 \int_{\mathbb{R}^2_+} pv \cdot \nabla\varphi \, dx \, dt = \int_T^0 \int_{\mathbb{R}^2_+} p_R v \cdot \nabla\varphi \, dx \, dt,$$

where

$$p_R = p^1_R + p^2_R$$

with $p^1_R = p^1 - [p^1]_{B_+(2R)}$ and $p^2_R = p^2 - [p^2]_{B_+(2R)}$. By the assumptions, after even extension, the function p^1 belongs to $L_\infty(-\infty, 0; \mathrm{BMO})$ and thus

$$\frac{1}{R^2} \int_{B_+(2R)} |p^1_R(x,t)|^2 \, dx \leqslant c$$

for all $t \leqslant 0$. As to p^2_R, we use Poincaré inequality

$$\frac{1}{R^2} \int_{B_+(2R)} |p^2_R(x,t)|^2 \, dx \leqslant \int_{B_+(2R)} |\nabla p^2(x,t)|^2 \, dx \leqslant \int_{\mathbb{R}^2_+} |\nabla p^2(x,t)|^2 \, dx.$$

So, by Lemma 1.3 and by the Lebesgue theorem about dominated convergence,

$$\begin{aligned}\left|\int_T^0\int_{\mathbb{R}^2_+} pv\cdot\nabla\varphi\,dx\,dt\right| &\leqslant c\int_T^0 d\tau\left(\int_{B_+(2R)\setminus B_+(R)}|v(x,\tau)|^2\,dx\right)^{\frac12}\\ &\quad+\frac{c}{R}\left(R^2\int_T^0\int_{\mathbb{R}^2_+}|\nabla p^2|^2\,dx\,dt\right)^{\frac12}\left(\int_T^0\int_{B_+(2R)\setminus B_+(R)}|v|^2\,dx\,dt\right)^{\frac12}\to 0\end{aligned}$$

as $R\to\infty$.

The last term is treated with the help of Poincaré inequality in the same way as p_R^2. Indeed,

$$\begin{aligned}\left|\int_T^0\int_{\mathbb{R}^2_+} qu\cdot\nabla\varphi\,dx\,dt\right| &= \left|\int_T^0\int_{\mathbb{R}^2_+}(q-[q]_{B_+(2R)})u\cdot\nabla\varphi\,dx\,dt\right|\\ &\leqslant\frac{c}{R}\left(R^2\int_T^0\int_{B_+(2R)}|\nabla q|^2\,dx\,dt\right)^{\frac12}\left(\int_T^0\int_{B_+(2R)\setminus B_+(R)}|u|^2\,dx\,dt\right)^{\frac12}.\end{aligned}$$

The right-hand side of the latter inequality tends to zero as $R\to\infty$ by the assumption that $u\in L_{2,\infty}(Q^+_-)$.

So, finally, we have

$$\begin{aligned}\int_T^0\int_{\mathbb{R}^2_+} u\cdot\operatorname{div}F\,dx\,dt &= -\lim_{R\to\infty}\int_{\mathbb{R}^2_+}\varphi(x)u(x,T)\cdot v(x,T)\,dx\\ &= -\int_{\mathbb{R}^2_+}u(x,T)\cdot v(x,T)\,dx.\end{aligned}$$

Now, our aim to see what happens if $T \to -\infty$.

§4. $t \to -\infty$

We shall show that

$$\|v(\cdot,t)\|_{2,\mathbb{R}^2_+} \to 0. \tag{4.1}$$

as $t \to -\infty$.

Indeed, we also know

$$\int\limits_{-\infty}^{t}\int\limits_{\mathbb{R}^2_+} |\nabla v|^2\,dx\,d\tau \to 0 \tag{4.2}$$

as $t \to -\infty$. By Ladyzhenskaya's inequality,

$$v \in L_4(Q^+_-)$$

and thus

$$\int\limits_{-\infty}^{t}\int\limits_{\mathbb{R}^2_+} |v|^4\,dx\,d\tau \to 0 \tag{4.3}$$

as $t \to -\infty$.

Now, for sufficiently large $-t_0$, we have

$$v = v^1 + v^2,$$

where

$$\partial_t v^1 + \triangle v^1 + \nabla q^1 = 0, \quad \operatorname{div} v^1 = 0$$

in $R^2_+ \times]-\infty, t_0[$,

$$v^1(x_1, 0, t) = 0$$

for any $x_1 \in \mathbb{R}$ and for any $t \leqslant t_0$, and

$$v^1(x, t_0) = v(x, t_0)$$

for any $x \in \mathbb{R}^2_+$.

As to v^2, it satisfies

$$\partial_t v^2 + \triangle v^2 + \nabla q^2 = -\operatorname{div} v \otimes u, \quad \operatorname{div} v^2 = 0$$

in $R^2_+ \times]-\infty, t_0[$,

$$v^2(x_1, 0, t) = 0$$

for any $x_1 \in \mathbb{R}$ and for any $t \leqslant t_0$, and

$$v^1(x, t_0) = 0$$

for any $x \in \mathbb{R}^2_+$.

Then, it is well known that

$$\|v^1(\cdot,t)\|_{2,\mathbb{R}^2_+} \to 0 \tag{4.4}$$

as $t \to -\infty$. On the other hand, by the energy inequality,

$$\begin{aligned}
&\frac{1}{2}\big\|v^2(\cdot,t)\big\|^2_{2,\mathbb{R}^2_+} + \int\limits_t^{t_0}\int\limits_{\mathbb{R}^2_+} |\nabla v^2|^2\,dx\,d\tau \\
&\quad = \int\limits_t^{t_0}\int\limits_{\mathbb{R}^2_+} v^2\cdot \operatorname{div} v\otimes u\,dx\,d\tau = -\int\limits_t^{t_0}\int\limits_{\mathbb{R}^2_+} v\otimes u : \nabla v^2\,dx\,d\tau \\
&\quad \leqslant \left(\int\limits_t^{t_0}\int\limits_{\mathbb{R}^2_+} |\nabla v^2|^2\,dx\,d\tau\right)^{\frac{1}{2}} \left(\int\limits_t^{t_0}\int\limits_{\mathbb{R}^2_+} |u|^4\,dx\,d\tau\right)^{\frac{1}{4}} \left(\int\limits_t^{t_0}\int\limits_{\mathbb{R}^2_+} |v|^4\,dx\,d\tau\right)^{\frac{1}{4}}
\end{aligned}$$

for $t < t_0$. For the same reason as for v, we have

$$\int\limits_{\infty}^{0}\int\limits_{\mathbb{R}^2_+} |u|^4\,dx\,d\tau \leqslant c \tag{4.5}$$

and thus, by the Cauchy inequality,

$$\|v^2(\cdot,t)\|^2_{2,\mathbb{R}^2_+} \leqslant c\left(\int\limits_t^{t_0}\int\limits_{\mathbb{R}^2_+} |v|^4\,dx\,d\tau\right)^{\frac{1}{2}} \leqslant c\left(\int\limits_{-\infty}^{t_0}\int\limits_{\mathbb{R}^2_+} |v|^4\,dx\,d\tau\right)^{\frac{1}{2}} \tag{4.6}$$

for all $t < t_0$.

It is not so difficult to deduce (4.1) from (4.2), (4.4), and (4.6).

The only assumption we really need is (4.5) and it is true if $u \in L_{2,\infty}(Q^+_-)$ and $\nabla u \in L_2(Q^+_-)$. The latter follows from Ladyzhenskaya's inequality.

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