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ESTIMATES OF THE DISTANCE TO THE SET OF DIVERGENCE FREE FIELDS

Abstract. We are concerned with computable estimates of the distance to the set of divergence free fields, which are necessary for quantitative analysis of mathematical models of incompressible media (e.g., Stokes, Oseen, and Navier–Stokes problems). *The distance is measured in terms of L^q norm of the gradient with $q \in (1, +\infty)$. For $q = 2$, these estimates follow from* the so-called inf-sup condition (or Aziz–Babuška-Ladyzhenskaya–Solonnikov inequality) and require sharp estimates of the respective constant, which are known only for a very limited amount of cases. We suggest a way to bypass this difficulty and show that for a vide class of domains (and different boundary conditions) computable estimates of the distance to the set of divergence free field can be presented in the form, which uses inf-sup constants for certain basic problems. In the last section, these estimates are applied to problems in the theory of viscous incompressible fluids. They generate fully computable bounds of the distance to generalized solutions of the problems considered.

Dedicated to the 80th jubilee of V. A. Solonnikov

§1. Introduction

Let Ω be a bounded connected domain in $\mathbb{R}^d$ $(d \geqslant 2)$ with Lipschitz boundary Γ. We consider estimates of the distance between a function

$$v \in V := W_0^{1,q}(\Omega, \mathbb{R}^d) := \{v \in W^{1,q}(\Omega, \mathbb{R}^d) \mid v = 0 \text{ on } \Gamma\} \quad 1 < q < +\infty$$

and the space $S_0^{1,q}(\Omega, \mathbb{R}^d)$, which is the closure (with respect to the norm of V) of smooth divergence free fields having compact supports in Ω. Also, we consider similar estimates for vector valued functions vanishing only on a measurable part $\Gamma_D \subset \Gamma$ and the set of divergence free fields satisfying the same boundary condition.

Key words and phrases: Incompressible viscous fluids, inf-sup condition, distance to divergence free fields, Stokes, Oseen, and Navier–Stokes problems, computable bounds of the distance to the generalized solution.

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Throughout the paper, $\{f\}_\Omega$ denotes the mean value of f in Ω, $\|\cdot\|_\omega$ denotes the L^2 norm of a scalar or vector valued function over the set ω (if ω coincides with Ω, then the subindex is omitted).

If $q=2$, then an estimate of the distance between $v \in \overset{\circ}{W}{}^{1,2}(\Omega,\mathbb{R}^d)$ and the set of divergence free fields is based on the following principal result.

Theorem 1. *For any $f \in L^2(\Omega)$ satisfying the condition $\{f\}_\Omega = 0$, there exists a function $w_f \in \overset{\circ}{W}{}^{1,2}(\Omega,\mathbb{R}^d)$ such that*

$$\mathrm{div} w_f = f \quad \text{and} \quad \|\nabla w_f\| \leqslant \kappa_\Omega \|f\|, \tag{1.1}$$

where κ_Ω is a positive constant depending only on Ω.

We refer to [2,12] for the proof of Theorem 1, which has several important applications. It was used by O. A. Ladyzhenskaya and V. A. Solonnikov for proving existence of a generalized solution to the Stokes problem (see, e.g., [11,12]).

Theorem 1 implies the key relation in the mathematical theory of incompressible fluids (it is often called the Inf–Sup (or LBB) condition): there exists a positive constant c_Ω such that

$$\inf_{\substack{p\in L^2(\Omega)\\ \{p\}_\Omega=0,\ p\neq 0}} \sup_{\substack{w\in V_0\\ w\neq 0}} \frac{\int_\Omega p\,\mathrm{div} w\,dx}{\|p\|\,\|\nabla w\|} \geqslant c_\Omega. \tag{1.2}$$

In view of (1.1), the condition (1.2) holds with $c_\Omega = (\kappa_\Omega)^{-1}$.

Also, (1.2) can be justified by means of the Nečas inequality [14]:

$$\|p\|^2 \leqslant \|p\|^2_{-1,\Omega} + \sum_{i=1}^d \left\| \frac{\partial p}{\partial x_i} \right\|^2_{-1,\Omega} \qquad \forall p \in L^2(\Omega),$$

where $\|\zeta\|^2_{-1,\Omega} := \sup_{\eta\in H^1_0(\Omega)} (\zeta,\eta)/\|\eta\|_{H^1}$. For domains with Lipschitz boundaries a simple proof can be found in [3].

In [1] and [6], the LBB condition was introduced, proved, and used in order to justify the convergence of the so–called mixed approximation methods, in which a boundary-value problem is reduced to a saddle-point problem for a certain Lagrangian. Conditions analogous to (1.2) written for various pairs of finite dimensional spaces are often used for proving stability and convergence of numerical methods developed for viscous incompressible fluids (see, e.g., [13]). In [8], it was suggested a numerical method, which provides approximate values of c_Ω for certain classes of domains.

Theorem 1 can be extended to L^q spaces for $1 < q < +\infty$ (see [4, 5, 17, 18]).

Theorem 2. *Let $f \in L^q(\Omega)$. If $\{f\}_\Omega = 0$, then there exists*

$$v_f \in \overset{\circ}{W}{}^{1,q}_0(\Omega, \mathbb{R}^d)$$

such that

$$\mathrm{div} v_f = f \quad \text{and} \quad \|\nabla v_f\|_{q,\Omega} \leqslant \kappa_{\Omega,q} \|\mathrm{div} v_f\|_{q,\Omega}, \tag{1.3}$$

where $\kappa_{\Omega,q}$ ($\kappa_{\Omega,2} = \kappa_\Omega$) is a positive constant, which depends only on Ω.

Theorems 1 and 2 imply estimates of the distance between a vector function $v \in \overset{\circ}{W}{}^{1,q}_0(\Omega, \mathbb{R}^d)$ and the subspace $S^{1,q}_0(\Omega, \mathbb{R}^d) \subset \overset{\circ}{W}{}^{1,q}_0(\Omega, \mathbb{R}^d)$ containing divergence free functions if the distance is measured in terms of the quantity

$$d(v, S^{1,q}_0(\Omega, \mathbb{R}^d)) := \inf_{v_0 \in S^{1,q}_0(\Omega, \mathbb{R}^d)} \|\nabla(v - v_0)\|_{q,\Omega}.$$

Lemma 1. *For any $v \in \overset{\circ}{W}{}^{1,q}_0(\Omega, \mathbb{R}^d)$,*

$$d(v, S^{1,q}_0(\Omega, \mathbb{R}^d)) \leqslant \kappa_{\Omega,q} \|\mathrm{div} v\|_{q,\Omega}. \tag{1.4}$$

This result directly follows from Theorem 2 if we set $f = \mathrm{div} v$. Then, a function $v_f \in \overset{\circ}{W}{}^{1,q}_0(\Omega, \mathbb{R}^d)$ exists such that (1.3) holds. We set

$$v_0 := v - v_f \in S^{1,q}_0(\Omega)$$

and obtain

$$\|\nabla(v - v_0)\|_{q,\Omega} = \|\nabla v_f\|_{q,\Omega} \leqslant \kappa_{\Omega,q} \|\mathrm{div} v\|_{q,\Omega}.$$

Hence, the distance between $v \in \overset{\circ}{W}{}^{1,q}_0(\Omega, \mathbb{R}^d)$ and the set of divergence free fields is easily estimated from above provided that the constant $\kappa_{\Omega,q}$ (or a suitable upper bound of it) is known. Regrettably, the latter requirement generates a very difficult problem. Even for the most simple case $q = 2$ estimates of the constant are known only for a restricted amount of special (simple) domains (see, e.g., [7, 15, 16, 26]). In particular, for $d = 2$ it is known that the constant c_Ω can be expressed throughout the constant L in the inequality $\|u\|^2 \leqslant L\|v\|^2$, which holds for an analytic function $u + iv$ provided that $\{u\}_\Omega = 0$ (see [9]). It was shown (see [26]) that $c_\Omega = \frac{1}{\sqrt{1+L}} \leqslant \frac{1}{\sqrt{2}}$. For star shaped domains estimation of the constant L is based on simple geometrical properties of Ω and, in particular, leads to the conclusion that $c_\Omega = \frac{1}{\sqrt{2}}$ for the circle, $\sin\frac{\pi}{8} \leqslant c_\Omega \leqslant \sqrt{\frac{\pi-2}{2\pi}}$ for the square and $\sin\frac{\pi}{16} \leqslant c_\Omega \leqslant \sqrt{\frac{\pi-2\sqrt{2}}{2\pi}}$ for the isosceles right triangle. Analogous constants can be found analytically or computed numerically for certain basic three dimensional domains.

However, in general, the constants $\kappa_{\Omega,q}$ are unknown. Moreover, so far we do not know any method able to compute guaranteed and realistic bounds of these constants for arbitrary three dimensional Lipshitz domains or, at least, for polygonal 3D domains. This fact imposes the question, which often arises in quantitative analysis of incompressible media: *how to estimate the distance between a function $v \in W_0^{1,q}(\Omega, \mathbb{R}^d)$ and the set of divergence free fields for a sufficiently wide class of domains?* Moreover, it is necessary to answer the same question in the case where the functions are vanishing only on a part of the boundary. Below we show that estimates of the distance can be obtained provided that upper bounds of the respective constants associated with some basic (elementary) domains has been either obtained by analytic methods or precomputed.

In Section 2, we deduce estimates of the distance to the set of divergence free fields for functions vanishing on a part Γ_D of the boundary Γ and show that regardless of a particular form of Γ_D the corresponding estimate holds with the same constant as for $\Gamma_D = \Gamma$ provided that the function has zero mean divergence (this result generalizes Lemma 6.2.1 in [22]). After that, a more sophisticated estimate is derived, which provides an upper bound of the distance to the set of divergence free fields without this zero mean condition. Section 3 presents two advanced forms of Theorem 1 in which the mean value condition is imposed for a collection of subdomains (these subdomains cover Ω and may be overlapping or non-overlapping). The respective estimates of the distance follow from these results. It can be useful for polygonal domains, which can be decomposed into simplicial and polyhedral subdomains (cells). If the constants $\kappa_{\Omega,q}$ for these cells are known, then the distance to the set of divergence free fields is easy to estimate. Finally, in Section 4 we discuss applications of these results to a posteriori estimates for problems in the theory of viscous incompressible fluids.

§2. Estimates for functions vanishing on a part of the boundary

Assume that Γ consists of two measurable non-intersecting parts Γ_D and Γ_N, $\mathrm{meas}_{d-1}\Gamma_D > 0$, and

$$v \in W^{1,q}_{0,\Gamma_D}(\Omega, \mathbb{R}^d) := \{v \in W^{1,q}(\Omega, \mathbb{R}^d) \mid v = 0 \text{ on } \Gamma_D\}.$$

We define the set

$$K_{\mu,\Gamma_D}(\Omega) := \left\{ w \in W^{1,q}_{0,\Gamma_D}(\Omega, \mathbb{R}^d) \mid \int\limits_\Omega \mathrm{div} w \, dx = \mu \in \mathbb{R} \right\}.$$

Our goal is to find an upper bound of

$$d(v, S^{1,q}_{0,\Gamma_D}(\Omega, \mathbb{R}^d)) := \inf_{v_0 \in S^{1,q}_{0,\Gamma_D}(\Omega,\mathbb{R}^d)} \|\nabla(v - v_0)\|_{q,\Omega}, \tag{2.1}$$

where

$$S^{1,q}_{0,\Gamma_D}(\Omega, \mathbb{R}^d) = \left\{ v \in W^{1,q}_{0,\Gamma_D}(\Omega, \mathbb{R}^d) \mid \mathrm{div} v = 0 \right\},$$

and to show that the estimate (1.4) holds for the functions vanishing only on Γ_D with the same constant $\kappa_{\Omega,q}$ as in (1.4).

Lemma 2. *Let $v \in K_{0,\Gamma_D}(\Omega)$. Then,*

$$d(v, S^{1,q}_{0,\Gamma_D}(\Omega, \mathbb{R}^d)) \leqslant \kappa_{\Omega,q} \|\mathrm{div} v\|_{q,\Omega}. \tag{2.2}$$

Indeed, the function $f = \mathrm{div} v$ has zero mean, so that Theorem 2 guarantees existence of $v_f \in \overset{\circ}{W}{}^{1,q}_0(\Omega, \mathbb{R}^d)$ such that (1.3) holds. Since

$$v_0 := v - v_f \in S^{1,q}_{0,\Gamma_D}(\Omega, \mathbb{R}^d),$$

we arrive at (2.2).

Lemma 2 generalizes results presented in [24] and Chapter 6 of [22], where it was shown that the condition $\{\mathrm{div} v\}_\Omega = 0$ allows to apply the usual LBB constant for problems with boundary conditions.

Now we consider estimates of the distance, which use the same constant $\kappa_{\Omega,q}$ and hold without the condition $\int\limits_\Omega \mathrm{div} v \, dx = 0$.

We begin with the most interesting case $q = 2$ and first of all deduce an upper bound of the quantity

$$\inf_{\widetilde{v} \in K_{0,\Gamma_D}(\Omega)} \|\nabla(\widetilde{v} - v)\|. \tag{2.3}$$

Since any function $\widetilde{v} \in K_{0,\Gamma_D}(\Omega)$ can be represented in the form $\widetilde{v} = v - \widetilde{w}$, where $\widetilde{w} \in K_{\mu,\Gamma_D}(\Omega)$ and $\mu = \int\limits_\Omega \operatorname{div} v\, dx$, this task leads to the auxiliary variational problem

$$\inf_{\widetilde{w} \in K_{\mu,\Gamma_D}(\Omega)} J(\widetilde{w}), \quad J(\widetilde{w}) := \frac{1}{2}\|\nabla \widetilde{w}\|^2, \tag{2.4}$$

which is equivalent to the minimax problem

$$\inf_{w \in W^{1,2}_{0,\Gamma_D}(\Omega,\mathbb{R}^d)} \sup_{\lambda \in \mathbb{R}} \left\{ \frac{1}{2}\|\nabla w\|^2 + \lambda \left(\int\limits_\Omega \operatorname{div} w dx - \mu \right) \right\}.$$

The corresponding dual problem is generated by the functional

$$G(\lambda) = \inf_{w \in W^{1,2}_{0,\Gamma_D}(\Omega,\mathbb{R}^d)} \left\{ \frac{1}{2}\|\nabla w\|^2 + \lambda \int\limits_\Omega \operatorname{div} w dx \right\} - \lambda\mu, \tag{2.5}$$

which contains a well posed convex minimization problem. Let u_* denote the minimizer of this problem for $\lambda = 1$. It meets the integral identity

$$\int\limits_\Omega \nabla u_* : \nabla w\, dx + \int\limits_{\Gamma_N} n \cdot w\, ds = 0 \quad \forall w \in W^{1,2}_{0,\Gamma_D}(\Omega,\mathbb{R}^d) \tag{2.6}$$

and solves the problem

$$\begin{aligned} \Delta u_* &= 0 && \text{in } \Omega, \\ u_* &= 0 && \text{on } \Gamma_D, \\ \nabla u_*\, n + n &= 0 && \text{on } \Gamma_N. \end{aligned}$$

It is easy to see that

$$\|\nabla u_*\|^2 + \int\limits_\Omega \operatorname{div} u_*\, dx = 0 \tag{2.7}$$

and λu_* is the minimizer of the problem (2.5).

We obtain

$$G(\lambda) = \frac{1}{2}\lambda^2\|\nabla u_*\|^2 + \lambda\left(\lambda \int\limits_\Omega \operatorname{div} u_*\, dx - \mu\right) = -\frac{1}{2}\lambda^2\|\nabla u_*\|^2 - \lambda\mu.$$

Therefore, $\sup\limits_{\lambda} G(\lambda)$ is attained at $\lambda = \lambda_* := -\frac{\mu}{\|\nabla u_*\|^2}$. By (2.6) we conclude that $\|\nabla u_*\| \neq 0$ so that λ_* is a finite real number and

$$G(\lambda_*) = \frac{1}{2}\frac{\mu^2}{\|\nabla u_*\|^2}.$$

Note that

$$\lambda_* \int\limits_{\Omega} \mathrm{div} u_* \, dx = \mu. \tag{2.8}$$

Hence, $\lambda_* u_* \in K_{\mu,\Gamma_D}(\Omega)$. Since

$$J(\lambda_* u_*) = \frac{1}{2}\|\nabla \lambda_* u_*\|^2 = \frac{1}{2}\frac{\mu^2}{\|\nabla u_*\|^2}$$

we see that the values of the primal and dual functionals associated with the auxiliary problem coincide and, therefore, $\lambda_* u_*$ is the minimizer of the auxiliary problem (2.4).

We set in (2.3) $\widetilde{v} = v_* := v - \lambda_* u_*$ and find that

$$\inf_{\widetilde{v} \in K_{0,\Gamma_D}(\Omega)} \|\nabla(\widetilde{v} - v)\| = \frac{1}{\|\nabla u_*\|} \left| \int\limits_{\Omega} \mathrm{div} v \, dx \right|. \tag{2.9}$$

Now

$$\inf_{v_0 \in S^{1,2}_{0,\Gamma_D}(\Omega,\mathbb{R}^d)} \|\nabla(v - v_0)\| \leqslant \inf_{v_0 \in S^{1,2}_{0,\Gamma_D}(\Omega,\mathbb{R}^d)} \|\nabla(v_* - v_0)\| + \|\lambda_* \nabla u_*\|$$

$$\leqslant \kappa_\Omega \|\mathrm{div} v - \lambda_* \, \mathrm{div} u_*\| + \frac{1}{\|\nabla u_*\|} \left| \int\limits_{\Omega} \mathrm{div} v \, dx \right|. \tag{2.10}$$

In view of (2.8), we arrive at the following result.

Lemma 3. *For any* $v \in W^{1,2}_{0,\Gamma_D}(\Omega, \mathbb{R}^d)$,

$$d(v, S^{1,2}_{0,\Gamma_D}(\Omega,\mathbb{R}^d)) \leqslant$$

$$\frac{\kappa_\Omega}{|\{\mathrm{div} u_*\}_\Omega|} \| \{\mathrm{div} u_*\}_\Omega \mathrm{div} v - \mathrm{div} u_* \{\mathrm{div} v\}_\Omega \| + \frac{1}{\|\nabla u_*\|} \left| \int\limits_{\Omega} \mathrm{div} v \, dx \right|. \tag{2.11}$$

It is easy to see that this estimate converts into (2.2) if $\{\mathrm{div} v\}_\Omega = 0$.

Corollary 1. (2.10)*implies a somewhat different estimate:*

$$d(v, S^{1,2}_{0,\Gamma_D}(\Omega, \mathbb{R}^d)) \leqslant \kappa_\Omega \|\mathrm{div} v\| + C_* \left| \int\limits_\Omega \mathrm{div} v \, dx \right|, \tag{2.12}$$

where

$$C_* = \frac{1}{\|\nabla u_*\|} \left(\kappa_\Omega \frac{\|\mathrm{div} u_*\|}{\|\nabla u_*\|} + 1 \right).$$

□

A similar estimate can be derived for $q \in (1, +\infty)$. Let u_* be the minimizer of the problem

$$\inf_{w \in W^{1,q}_{0,\Gamma_D}(\Omega)} \left\{ \frac{1}{q} \|\nabla w\|^q_{q,\Omega} + \lambda \int\limits_\Omega \mathrm{div} w dx \right\}, \tag{2.13}$$

which meets the integral identity

$$\int\limits_\Omega \Big(| \nabla u_* |^{q-2} \nabla u_* : \nabla w + \mathrm{div} w \Big) \, dx = 0 \qquad \forall w \in W^{1,q}_{0,\Gamma_D}(\Omega).$$

Then,

$$\|\nabla u_*\|^q_{q,\Omega} + \int\limits_\Omega \mathrm{div} u_* \, dx = 0.$$

Set $v_* = v - \lambda_* u_*$, where

$$\lambda_* = \frac{\int_\Omega \mathrm{div} v \, dx}{\int_\Omega \mathrm{div} u_* \, dx} = -\frac{\int_\Omega \mathrm{div} v \, dx}{\|\nabla u_*\|^q_{q,\Omega}}.$$

We obtain

$$\begin{aligned}
&\inf_{v_0 \in S^{1,q}_{0,\Gamma_D}(\Omega,\mathbb{R}^d)} \|\nabla(v - v_0)\|_{q,\Omega} \\
&\qquad\leqslant \inf_{v_0 \in S^{1,q}_{0,\Gamma_D}(\Omega,\mathbb{R}^d)} \|\nabla(v_* - v_0)\|_{q,\Omega} + \|\lambda_* \nabla u_*\|_{q,\Omega} \\
&\qquad\leqslant \kappa_{\Omega,q}\|\mathrm{div} v - \lambda_* \,\mathrm{div} u_*\|_{q,\Omega} + \frac{1}{\|\nabla u_*\|^{q-1}_{q,\Omega}} \left| \int\limits_\Omega \mathrm{div} v \, dx \right| \\
&\qquad\leqslant \kappa_{\Omega,q}\|\mathrm{div} v\|_{q,\Omega} + C_{*,q} \left| \int\limits_\Omega \mathrm{div} v \, dx \right|, \qquad (2.14)
\end{aligned}$$

where

$$C_{*,q} = \frac{1}{\|\nabla u_*\|^{q-1}_{q,\Omega}} \left(\kappa_{\Omega,q} \frac{\|\mathrm{div} u_*\|_{q,\Omega}}{\|\nabla u_*\|_{q,\Omega}} + 1 \right).$$

Remark 1. The constant C_* depends on the solution u_* of the auxiliary boundary value problem (2.6) (or problem (2.13)). In general, this function is unknown. It can be replaced by a finite dimensional approximation $u_{*,h}$, which solves the problem (for the case $q = 2$)

$$\int\limits_\Omega (\nabla u_{*,h} : \nabla w_h + \mathrm{div} w_h) \, dx = 0 \quad \forall w \in K^h_{0,\Gamma_D}(\Omega),$$

where K^h_{0,Γ_D} is a certain finite dimensional subspace of $K_{0,\Gamma_D}(\Omega)$. Then, repeating above arguments, we find that

$$\begin{aligned}
&\inf_{\widetilde{v} \in K_{0,\Gamma_D}(\Omega)} \|\nabla(\widetilde{v} - v)\| \\
&\qquad\leqslant \|\nabla(v - u_{*,h})\| = \frac{1}{\|\nabla u_{*,h}\|} \left| \int\limits_\Omega \mathrm{div} v \, dx \right| \qquad (2.15)
\end{aligned}$$

and (2.12) holds with the fully computable constant

$$C_{*,h} = \frac{1}{\|\nabla u_{*,h}\|} \left(\kappa_\Omega \frac{\|\mathrm{div} u_{*,h}\|}{\|\nabla u_{*,h}\|} + 1 \right).$$

By applying known argumentation of the approximation theory one can prove that $u_{*,h}$ tends to u_* provided that standard regularity assumptions on the structure of subspaces $K_{0,\Gamma_D}(\Omega)$ are satisfied. Then, $C_{*,h}$ tends to C_*.

§3. Estimates based upon decomposition of Ω

First, we prove a modified version of Theorem 2, which is adapted to the case where Ω is divided into a collection of non-overlapping Lipschitz subdomains Ω_i, $i = 1, 2, \ldots N$.

Theorem 3. *Let $f \in L^q(\Omega)$. If f satisfies the conditions $\{f\}_{\Omega_i} = 0$ for $i = 1, 2, \ldots, N$, then there exists $v_f \in \overset{\circ}{W}{}^{1,q}(\Omega, \mathbb{R}^d)$ such that*

$$\mathrm{div} v_f = f \quad \text{and} \quad \|\nabla v_f\|^q_{q,\Omega} \leqslant \sum_{i=1}^N \kappa^q_{\Omega_i,q} \|f\|^q_{q,\Omega}, \tag{3.1}$$

where $\kappa_{\Omega_i,q}$ are positive constants associated with subdomains Ω_i.

Proof. In view of Theorem 2, for any Ω_i there exists $v_{f,i} \in \overset{\circ}{W}{}^{1,q}(\Omega_i, \mathbb{R}^d)$ such that

$$\mathrm{div} v_{f,i} = f \text{ in } \Omega_i \quad \text{and} \quad \|\nabla v_{f,i}\|_{q,\Omega_i} \leqslant \kappa_{\Omega_i,q} \|f\|_{q,\Omega_i}. \tag{3.2}$$

We define v_f as the function, which is equal to $v_{f,i}$ in Ω_i. Since $v_{f,i}$ vanishes on $\partial\Omega_i$, the function v_f belongs to $\overset{\circ}{W}{}^{1,q}(\Omega_i, \mathbb{R}^d)$. By raising (3.2) to the power q and summing over i, we arrive at (3.1). $\square$

Theorem 3 implies an estimate of the distance to the set of divergence free fields, which instead of one global constant operates with the constants $\kappa_{\Omega i}$ (for $q = 2$, this estimate has been earlier established in [23]).

Lemma 4. *Let $v \in W^{1,q}_{0,\Gamma_D}(\Omega, \mathbb{R}^d)$ and*

$$\{\mathrm{div} v\}_{\Omega_i} = 0 \qquad i = 1, 2, \ldots, N. \tag{3.3}$$

Then,

$$d(v, S^{1,q}_{0,\Gamma_D}(\Omega)) \leqslant \left(\sum_{i=1}^N \kappa^q_{\Omega_i,q} \|\mathrm{div} v\|^q_{\Omega_i} \right)^{1/q}. \tag{3.4}$$

Proof. We set $f = \mathrm{div} v$. Then, there exists $v_f \in \overset{\circ}{W}{}^{1,q}(\Omega, \mathbb{R}^d)$ satisfying (3.1). The function $v_0 = v - v_f$ is divergence free and

$$\|\nabla(v - v_0)\|^q_{q,\Omega} = \|\nabla v_f\|^q_{q,\Omega} \leqslant \sum_{i=1}^N \kappa^q_{q,\Omega_i} \|\mathrm{div} v\|^q_{q,\Omega_i},$$

which implies (3.4). $\square$

Now we present another version of the estimate based on domain decomposition. It requires values of constants κ for a collection of overlapping Lipschitz subdomains D_k, $k = 1, 2, \ldots, m$, which consist of smaller subdomains Ω_i. Assume that $q = 2$,

$$\overline{\Omega} = \bigcup_{k=1}^{m} \overline{D}_k = \bigcup_{i=1}^{n} \overline{\Omega}_i, \quad \Omega_i \cap \Omega_j = \varnothing \text{ if } i \neq j \tag{3.5}$$

and

$$\text{for any } \Omega_i \text{ there exists } D_k \text{ such that } \Omega_i \subset D_k. \tag{3.6}$$

Theorem 4. *Let $f \in H^2(\Omega)$ be such that*

$$\{f\}_{\Omega_i} = 0, \quad i = 1, 2, \ldots, n \tag{3.7}$$

and decomposition of Ω satisfies (3.5) *and* (3.6). *Then, there exists a function $v_f \in H^1_0(\Omega, \mathbb{R}^d)$ such that*

$$\mathrm{div} v = f \qquad \text{in } \Omega \tag{3.8}$$

and

$$\|\nabla v\|_\Omega \ \leqslant \ \sum_{i=1}^{n} \kappa_i \|f\|_{\Omega_i} \tag{3.9}$$

where

$$\kappa_i = \inf_{k=1,\ldots,m} \rho_k, \quad \rho_k = \begin{cases} \kappa_{D_k} & \text{if } \Omega_i \subset D_k, \\ +\infty & \text{if } \Omega_i \not\subset D_k. \end{cases} \tag{3.10}$$

Proof. Define

$$f_i(x) = \begin{cases} f & \text{if } x \in \Omega_i, \\ 0 & \text{if } x \notin \Omega_i. \end{cases}$$

Let M_i denote the number k of D_k containing Ω_i with minimal κ_{D_k}. Since $\{f_i\}_{D_{M_i}} = 0$, we can find $v^{(i,M_i)}$ such that $v^{(i,M_i)} = 0$ on ∂D_{M_i},

$$\mathrm{div} v^{(i,M_i)} = f_i \qquad \text{in } D_{M_i} \tag{3.11}$$

and

$$\|\nabla v^{(i,M_i)}\|_{D_k} \ \leqslant \ \kappa_i \|f_i\|_{D_{M_i}} = \kappa_i \|f\|_{\Omega_i}. \tag{3.12}$$

We extend $v^{(i,M_i)}$ by zero to $\Omega \setminus D_{M_i}$ and find that (3.11) holds for the whole Ω. Moreover,

$$\|\nabla v^{(i,M_i)}\|_\Omega \ \leqslant \ \kappa_i \|f\|_{\Omega_i}. \tag{3.13}$$

Set $v_f = \sum_{i=1}^{n} v^{(i,M_i)} \in H^1_0(\Omega, \mathbb{R}^d)$. Then $\mathrm{div} v_f = f$. Since

$$\|\nabla v_f\|_\Omega \leqslant \sum_{i=1}^{n} \|\nabla v^{(i,M_i)}\|_{D_{M_i}} \leqslant \sum_{i=1}^{n} \kappa_i \|f_i\|_{D_{M_i}} = \sum_{i=1}^{n} \kappa_i \|f_i\|_{\Omega_i},$$

we arrive at (3.9). □

Corollary 2. *Assume that the constants κ_{D_k} are known. Then Theorem 4 yields computable estimate of the distance to the set of divergence free fields for any $v \in H^1(\Omega, \mathbb{R}^d)$ such that v vanishes on Γ_D and $\{\mathrm{div} v\}_{\Omega_i} = 0$, $i = 1, 2, \ldots, n$.*

Set $f = \mathrm{div} v$. There exists $v_f \in H^1(\Omega, \mathbb{R}^d)$ vanishing on the boundary such that (3.8) and (3.9) hold. Then $v_0 = v - v_f$ is a divergence free field such that $v_0 = 0$ on Γ_D. We have

$$d(v, S^{1,2}_{0,\Gamma_D}(\Omega)) \leqslant \|\nabla(v - v_0)\|_\Omega \leqslant \|\nabla v_f\|_\Omega \leqslant \sum_{i=1}^{n} \kappa_i \|\mathrm{div} v\|_{\Omega_i}. \quad (3.14)$$

A quite similar estimate for $d(v, S^{1,q}_{0,\Gamma_D}(\Omega))$ with $q \in (1, +\infty)$ and

$$\kappa_i = \inf_{k=1,\ldots,m} \rho_k, \quad \rho_k = \begin{cases} \kappa_{D_k,q} & \text{if } \Omega_i \subset D_k, \\ +\infty & \text{if } \Omega_i \not\subset D_k, \end{cases}$$

is obviously true and can be justified by repeating the above arguments.

Remark 2. Lemma 4 answers the question stated in the introduction. It says that if Ω is decomposed into a set of "simple" subdomains (for which the constants $\kappa_{\Omega,q}$ are known), then an upper bound of the distance is easy to compute provided that mean values of the divergence in each subdomain are zero.

It should be noted that satisfaction of a certain amount of integral conditions (3.3) can be performed without essential difficulties unlike the methods based on constructing a sufficiently wide subspace of divergence free functions and computing the estimate directly (especially in the three dimensional case). Indeed, if v does not satisfy (3.3), then the corresponding correction can be done be changing N parameters in the representation of this function. Since

$$\int_{\Omega_i} \mathrm{div} v \, dx = \int_{\partial\Omega_i} v \cdot n_i \, ds \qquad i = 1, 2, \ldots, N,$$

where n_i is the outward normal to the boundary $\partial\Omega_i$, changing the parameters should be done such that all the boundary integrals vanish. If N is not very large, then this requirement do not lead to essential difficulties (especially if v is presented by edge based approximations such as, e.g., Raviart–Thomas elements).

Moreover, we can deduce fully computable estimates of the distance, which are valid without the conditions or (3.3). Indeed, let $\mu_i = \int\limits_{\Omega_i} \operatorname{div} v \, dx$ and $w \in W^{1,q}_{0,\Gamma_D}(\Omega)$ be a "correction function" such that

$$\int\limits_{\Omega_i} \operatorname{div} w \, dx = \mu_i \qquad \text{for } i = 1, 2, \dots, N.$$

Then

$$d(v, S^{1,q}_{0,\Gamma_D}(\Omega)) \leqslant d(v - w, S^{1,q}_{0,\Gamma_D}(\Omega)) + \|\nabla w\|_{q,\Omega}$$

and (3.4) yields a simple estimate

$$d(v, S^{1,q}_{0,\Gamma_D}(\Omega)) \leqslant \left(\sum_{i=1}^{N} \kappa^q_{\Omega_i} \|\operatorname{div}(v - w)\|^q_{\Omega_i} \right)^{1/q} + \|\nabla w\|_{q,\Omega}. \tag{3.15}$$

This estimate provides an upper bound of the distance to the set of divergence free fields for any $w \in W^{1,q}_{0,\Gamma_D}(\Omega)$. In order to obtain the best estimate, w should be selected in such a way that the right hand side of (3.15) be minimal. For this purpose, we should use a generalized version of the method exposed in Lemma 3.

§4. Estimates of the distance to the exact solutions of boundary value problems

Finally, we consider applications of the above derived estimates to quantitative analysis of mathematical models arising in the theory of viscous incompressible fluids. For the sake of simplicity, we consider only stationary models with the Dirichl'et boundary conditions (i.e., $u = u_0$ on Γ, where u_0 is a given divergence free vector function). For this class of problems, a generalized solution u is defined as a divergence free field satisfying the integral identity

$$\int\limits_{\Omega} (\nu \nabla u - \eta(u)) : \nabla w \, dx = \int\limits_{\Omega} f \cdot w \, dx \qquad \forall w \in S^{1,2}_0(\Omega, \mathbb{R}^d), \tag{4.1}$$

where ν is a positive constant (viscosity), $\eta(u) = 0$ for the Stokes problem, $\eta(u) = \mathrm{a} \otimes u$ for the Oseen problem (where a is a certain bounded divergence free vector function), and $\eta(u) = u \otimes u$ for the Navier–Stokes problem (in the latter case, we assume that u is a certain weak solution).

Let v be a solenoidal vector function satisfying the Dirichlét boundary conditions, which we view as an approximation of u. In order to get an estimate of the distance between v and u we rewrite (4.1) as follows:

$$\int\limits_{\Omega} (\nu\nabla(u-v) + \eta(v) - \eta(u)) : \nabla w \, dx = \mathcal{L}_v(w), \tag{4.2}$$

where

$$\mathcal{L}_v(w) = \int\limits_{\Omega} (f \cdot w - \nu\nabla v : \nabla w + \eta(v) : \nabla w) \, dx$$

is the residual functional associated with v. This relation yields the general error identity

$$\sup_{w \in S_0^{1,2}(\Omega,\mathbb{R}^d)} \frac{\int\limits_{\Omega} (\nu\nabla(u-v) + \eta(v) - \eta(u)) : \nabla w \, dx}{\|\nabla w\|} =: \underline{\Re(u,v) = |\, \mathcal{L}_v \,|}. \tag{4.3}$$

Here, $\Re(u,v)$ is a measure of the distance between u and v. It is easy to see that the measure is symmetric and satisfies the triangle inequality. Since $|\, \mathcal{L}_v \,|$ is the norm of the residual functional (which contains all the available information concerning the quality of the approximate solution), the identity (4.3) shows that $\Re(u,v)$ is the accuracy measure to be used for this class of problems (see also [23]).

For the Stokes problem, $\Re(u,v) = \nu\|\nabla(u-v)\|$.

For the Oseen problem, we have

$$\int\limits_{\Omega} (\mathrm{a} \otimes w) : \nabla w \, dx = -\frac{1}{2} \int\limits_{\Omega} \mathrm{a} \cdot \nabla(|w|^2) \, dx = 0. \tag{4.4}$$

Therefore,

$$\sup_{w \in S_0^{1,2}(\Omega,\mathbb{R}^d)} \frac{\int\limits_{\Omega} (\nu\nabla(u-v) : \nabla w - (\mathrm{a} \otimes (u-v) : \nabla w) \, dx}{\|\nabla w\|} \geqslant \nu\|\nabla(u-v)\|$$

and $\Re(u,v) \geqslant \nu\|\nabla(u-v)\|$.

In general, such a simple bound does not take place for the Navier–Stokes problem. We can only prove (see [21]) that $\Re(u,v)$ is bounded from below by $\mu\|\nabla(u-v)\|$ (where μ is a positive multiplier) provided that ∇v is sufficiently small.

The residual functional can be decomposed into two physically meaningful parts by means of known methods using suitable integration by parts relations (see [22]). Let $q \in L^2(\Omega)$ and

$$\tau \in H(\Omega, \mathrm{Div}) := \left\{\tau \in L^2(\Omega, \mathbb{M}^{2\times 2}) \mid \mathrm{Div}\tau \in L^2(\Omega, \mathbb{R}^2)\right\}.$$

Then,

$$\mathcal{L}_v(w) = \int\limits_\Omega (f - \mathrm{Div}\tau)\cdot w\,dx + \int\limits_\Omega (\nu\nabla v + \eta(v) - \tau - q\mathbb{I}) : \nabla w\,dx. \tag{4.5}$$

Hence, we find that

$$|\mathcal{L}_v| \leqslant \|\nu\nabla v + \eta(v) - \tau - q\mathbb{I}\| + C_{F\Omega}\|f - \mathrm{Div}\tau\|, \tag{4.6}$$

where $C_{F\Omega}$ is a constant in the Friedrich's type inequality

$$\|v\| \leqslant C_{F\Omega}\|\nabla v\| \qquad \forall v \in W_0^{1,2}(\Omega, \mathbb{R}^d).$$

Now, (4.3) and (4.6) yield an upper bound of $\Re(u,v)$ for any $v \in S_0^{1,2}(\Omega, \mathbb{R}^d)$.

Using results of Sections 2 and 3, we can extend this estimate to functions, which do not satisfy the divergence free condition.

Let $v \in W_0^{1,2}(\Omega, \mathbb{R}^d)$ but $v \notin S_0^{1,2}(\Omega, \mathbb{R}^d)$. Note that

$$\Re(u,v) \leqslant \Re(u,v_0) + \Upsilon(v - v_0), \tag{4.7}$$

where v_0 is an arbitrary function in $S_0^{1,2}(\Omega, \mathbb{R}^d)$ and Υ is a nonnegative functional defined by the relation $\Upsilon(v-v_0) := \nu\|\nabla(v_0-v)\| + \|\eta(v)-\eta(v_0)\|$. In view of (4.6), we find that

$$\Re(u,v) \leqslant \|\nu\nabla v + \eta(v) - \tau - q\mathbb{I}\| + C_{F\Omega}\|f - \mathrm{Div}\tau\| \\ + 2 \inf_{v_0 \in S_0^{1,2}(\Omega,\mathbb{R}^d)} \Upsilon(v - v_0). \tag{4.8}$$

For the Stokes problem $\Upsilon(v - v_0) = \nu\|\nabla(v - v_0)\|$ and we obtain

$$\nu\|\nabla(u-v)\| \leqslant \|\tau + \widetilde{p}\mathbb{I} - \nu\nabla v\| + \\ + C_{F\Omega}\|\mathrm{Div}\tau + f\| + 2\nu d(v, S_0^{1,2}(\Omega, \mathbb{R}^d)), \tag{4.9}$$

where q is an approximation of the pressure p and τ is an approximation of the stress $\sigma = \nu\nabla u - p\mathbb{I}$.

If the constant κ_Ω is known, then $d(v, S^{1,2}(\Omega, \mathbb{R}^d) \leqslant \kappa_\Omega \|\mathrm{div} v\|$ and we obtain a fully computable upper bound of the error (cf. [19, 20, 22]). If the constant κ_Ω is unknown, then we can split Ω and represent it as a union of "simple" non-overlapping domains Ω_i for which the respective constants κ_{Ω_i} are known. Let v satisfy the conditions

$$\int\limits_{\Omega_i} \mathrm{div} v \, dx = 0, \qquad i = 1, 2, \ldots, N.$$

Then,

$$\nu\|\nabla(u-v)\| \leqslant \|\tau+\widetilde{p}\mathbb{I}-\nu\nabla v\| + C_{F\Omega}\|\mathrm{Div}\tau+f\| + 2\nu\left(\sum_{i=1}^{N}\kappa^2_{\Omega,i}\|\mathrm{div} v\|^2_{\Omega_i}\right)^{1/2}. \tag{4.10}$$

Analogously, for the Oseen problem

$$\inf_{v_0\in S_0^{1,2}(\Omega,\mathbb{R}^d)} \Upsilon(v-v_0) \leqslant C_{\mathrm{Os}}\, d(v, S^{1,2}(\Omega,\mathbb{R}^d)) \leqslant C_{\mathrm{Os}}\kappa_\Omega\|\mathrm{div} v\|, \tag{4.11}$$

where $C_{\mathrm{Os}} = (\nu + \|\mathrm{a}\|_{\infty,\Omega} C_{F\Omega})$. If the constant κ_Ω is unknown, then instead of (4.11) we can use the estimate

$$\inf_{v_0\in S_0^{1,2}(\Omega,\mathbb{R}^d)} \Upsilon(v-v_0) \leqslant C_{\mathrm{Os}}\left(\sum_{i=1}^{N}\kappa^2_{\Omega,i}\|\mathrm{div} v\|^2_{\Omega_i}\right)^{1/2}, \tag{4.12}$$

which together with (4.8) yields an error majorant.

For the Navier–Stokes problem, we need more sophisticated estimates. First, we note that $\|\eta(v)-\eta(v_0)\|^2 \leqslant \int\limits_\Omega (2|v|^2|v-v_0|^2 + |v-v_0|^4)\, dx$, which due to embedding of H^1 to L^4 yields the estimate

$$\begin{aligned}\inf_{v_0\in S_0^{1,2}(\Omega,\mathbb{R}^d)} & \Upsilon(v-v_0) \\ &\leqslant (\nu + \mu(\Omega)(2\|v\|^2_{4,\Omega} + \mu^2(\Omega) d^2(v, S^{1,2}(\Omega, R^d)))^{1/2}) d(v, S^{1,2}(\Omega,\mathbb{R}^d))\end{aligned} \tag{4.13}$$

where $\mu(\Omega)$ is the constant in the inequality $\|w\|_{4,\Omega} \leqslant \mu(\Omega)\|\nabla w\|_\Omega$ for functions vanishing on the boundary. Then, (4.8) yields the corresponding error majorant, in which the term related to the distance to the set of divergence free field is either estimated by a single global constant of by means of a collection of local constants $\kappa_{\Omega,i}$.

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