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**D. Pauly**

# ON CONSTANTS IN MAXWELL INEQUALITIES FOR BOUNDED AND CONVEX DOMAINS

Abstract. For a bounded and convex domain $\Omega \subset \mathbb{R}^3$ we show that the Maxwell constants are bounded from below and above by Friedrichs' and Poincaré's constants of $\Omega$.

## §1. Introduction

Throughout this paper we fix a bounded and convex domain $\Omega \subset \mathbb{R}^3$. It is well known that, e.g., by Rellich's selection theorem using standard indirect arguments, the Poincaré[1] inequalities

$$\exists\, c_{\mathsf{p},0} > 0 \qquad \forall\, u \in \overset{\circ}{\mathsf{H}}{}^1 \qquad |u| \leqslant c_{\mathsf{p},0} |\nabla u|, \tag{1}$$

$$\exists\, c_{\mathsf{p}} > 0 \qquad \forall\, u \in \mathsf{H}^1 \cap \mathbb{R}^{\perp} \qquad |u| \leqslant c_{\mathsf{p}} |\nabla u| \tag{2}$$

hold. We assume to pick the best constants, i.e.,

$$\frac{1}{c_{\mathsf{p},0}} := \inf_{0 \neq u \in \overset{\circ}{\mathsf{H}}{}^1} \frac{|\nabla u|}{|u|}, \quad \frac{1}{c_{\mathsf{p}}} := \inf_{0 \neq u \in \mathsf{H}^1 \cap \mathbb{R}^{\perp}} \frac{|\nabla u|}{|u|}.$$

Then $c_{\mathsf{p},0}$ and $c_{\mathsf{p}}$ are the well known Friedrichs and Poincaré constants, respectively, which satisfy

$$0 < c_{\mathsf{p},0}^2 = \frac{1}{\lambda_1} < \frac{1}{\mu_2} = c_{\mathsf{p}}^2,$$

where $\lambda_1$ is the first Dirichlet and $\mu_2$ the second Neumann eigenvalue of the Laplacian. By $\langle\,\cdot\,,\,\cdot\,\rangle$ and $|\cdot|$ we denote the standard inner product and induced norm in $\mathsf{L}^2$ and we will write the usual $\mathsf{L}^2$-Sobolev spaces as $\mathsf{H}^1$ and $\overset{\circ}{\mathsf{H}}{}^1$, the latter is defined as the closure in $\mathsf{H}^1$ of smooth and compactly supported test functions. All spaces and norms are defined on $\Omega$. Moreover,

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[1]The estimate (1) is often called Friedrichs'/Steklov inequality as well.

we introduce the standard Sobolev spaces for the rotation and divergence by $\mathsf{R}$ and $\mathsf{D}$. More precisely,

$$\mathsf{R} := \{E \in \mathsf{L}^2 : \operatorname{rot} E \in \mathsf{L}^2\}, \quad \mathsf{D} := \{E \in \mathsf{L}^2 : \operatorname{div} E \in \mathsf{L}^2\}$$

hold, where rot = curl and div are to be understood in the usual distributional or weak sense. As before, we will denote the closures of test vector fields in the respective graph norms by $\overset{\circ}{\mathsf{R}}$ and $\overset{\circ}{\mathsf{D}}$. An index zero at the lower right corner of the latter spaces indicates a vanishing derivative, e.g.,

$$\mathsf{R}_0 := \{E \in \mathsf{R} : \operatorname{rot} E = 0\}, \quad \overset{\circ}{\mathsf{D}}_0 := \{E \in \overset{\circ}{\mathsf{D}} : \operatorname{div} E = 0\}.$$

As $\Omega$ is convex, it is especially simply connected and has got a connected boundary. Hence, the Neumann and Dirichlet fields of $\Omega$ vanish, i.e.,

$$\mathcal{H}_{\mathtt{N}} := \mathsf{R}_0 \cap \overset{\circ}{\mathsf{D}}_0 = \{0\} = \overset{\circ}{\mathsf{R}}_0 \cap \mathsf{D}_0 =: \mathcal{H}_{\mathtt{D}}.$$

By the Maxwell compactness properties, see [5, 13, 6, 14, 7, 12], i.e., the compactness of the two embeddings

$$\overset{\circ}{\mathsf{R}} \cap \mathsf{D} \hookrightarrow \mathsf{L}^2, \quad \mathsf{R} \cap \overset{\circ}{\mathsf{D}} \hookrightarrow \mathsf{L}^2,$$

(and again by a standard indirect argument) the Maxwell inequalities

$$\exists\, c_{\mathtt{m},\mathtt{t}} > 0 \qquad \forall\, E \in \overset{\circ}{\mathsf{R}} \cap \mathsf{D} \qquad |E| \leqslant c_{\mathtt{m},\mathtt{t}}\big(|\operatorname{rot} E|^2 + |\operatorname{div} E|^2\big)^{1/2}, \tag{3}$$

$$\exists\, c_{\mathtt{m},\mathtt{n}} > 0 \qquad \forall\, H \in \mathsf{R} \cap \overset{\circ}{\mathsf{D}} \qquad |H| \leqslant c_{\mathtt{m},\mathtt{n}}\big(|\operatorname{rot} H|^2 + |\operatorname{div} H|^2\big)^{1/2} \tag{4}$$

hold. Again, we assume that we have chosen the best constants, i.e.,

$$\frac{1}{c_{\mathtt{m},\mathtt{t}}^2} := \inf_{0 \neq E \in \overset{\circ}{\mathsf{R}} \cap \mathsf{D}} \frac{|\operatorname{rot} E|^2 + |\operatorname{div} E|^2}{|E|^2},$$

$$\frac{1}{c_{\mathtt{m},\mathtt{n}}^2} := \inf_{0 \neq H \in \mathsf{R} \cap \overset{\circ}{\mathsf{D}}} \frac{|\operatorname{rot} H|^2 + |\operatorname{div} H|^2}{|H|^2}.$$

The notation $c_{\mathtt{m},\mathtt{t}}$ and $c_{\mathtt{m},\mathtt{n}}$ should indicate the homogeneous tangential and normal boundary condition, respectively. To the best of the author's knowledge, general bounds for the Maxwell constants $c_{\mathtt{m},\mathtt{t}}$ and $c_{\mathtt{m},\mathtt{n}}$ are missing. On the other hand, at least estimates for $c_{\mathtt{m},\mathtt{t}}$ and $c_{\mathtt{m},\mathtt{n}}$ from above are very important from the point of view of applications, such as preconditioning or a priori and a posteriori error estimation for numerical methods, see e.g. [10, 8].

In the paper at hand, we will prove

$$c_{\mathtt{p},0} \leqslant c_{\mathtt{m},\mathtt{t}} \leqslant c_{\mathtt{m},\mathtt{n}} = c_{\mathtt{p}} \leqslant \frac{\operatorname{diam}(\Omega)}{\pi}. \tag{5}$$

We note that (5) is already well known in two dimensions, where even

$$c_{\mathtt{p},0} < c_{\mathtt{m},\mathtt{t}} = c_{\mathtt{m},\mathtt{n}} = c_{\mathtt{p}} \leqslant \frac{\operatorname{diam}(\Omega)}{\pi}$$

holds[2], see Appendix, but new in three dimensions. Furthermore, the last inequality in (5) has been proved in the famous paper of Payne and Weinberger [9], where also the optimality of the estimate was shown.

## §2. Results and Proofs

We start with an inequality for irrotational fields.

**Lemma 1.** *For all* $E \in \nabla\mathring{\mathsf{H}}^1 \cap \mathsf{D}$ *and all* $H \in \nabla\mathsf{H}^1 \cap \mathring{\mathsf{D}}$

$$|E| \leqslant c_{\mathtt{p},0}|\operatorname{div} E|, \quad |H| \leqslant c_{\mathtt{p}}|\operatorname{div} H|.$$

**Proof.** Let $\varphi \in \mathring{\mathsf{H}}^1$ with $E = \nabla\varphi$. By (1) we get

$$\begin{aligned}|E|^2 &= \langle E, \nabla\varphi\rangle = -\langle \operatorname{div} E, \varphi\rangle \\ &\leqslant |\operatorname{div} E||\varphi| \leqslant c_{\mathtt{p},0}|\operatorname{div} E||\nabla\varphi| = c_{\mathtt{p},0}|\operatorname{div} E||E|.\end{aligned}$$

Let $\varphi \in \mathsf{H}^1$ with $H = \nabla\varphi$ and $\varphi \perp \mathbb{R}$. Since $H \in \mathring{\mathsf{D}}$ and by (2), we obtain

$$\begin{aligned}|H|^2 &= \langle H, \nabla\varphi\rangle = -\langle \operatorname{div} H, \varphi\rangle \\ &\leqslant |\operatorname{div} H||\varphi| \leqslant c_{\mathtt{p}}|\operatorname{div} H||\nabla\varphi| = c_{\mathtt{p}}|\operatorname{div} H||H|. \qquad \square\end{aligned}$$

**Remark 2.** *Lemma 1 extends to arbitrary Lipschitz domains* $\Omega \subset \mathbb{R}^N$, $N \in \mathbb{N}$.

As usual in the theory of Maxwell's equations, we need another crucial tool, the Helmholtz decompositions of vector fields into irrotational and solenoidal vector fields. For convex domains, these decompositions are very simple. We have

$$\mathsf{L}^2 = \nabla\mathring{\mathsf{H}}^1 \oplus \operatorname{rot}\mathsf{R}, \quad \mathsf{L}^2 = \nabla\mathsf{H}^1 \oplus \operatorname{rot}\mathring{\mathsf{R}}, \tag{6}$$

[2]In 2D, the equality $c_{\mathtt{m},\mathtt{t}} = c_{\mathtt{m},\mathtt{n}} = c_{\mathtt{p}}$ holds even for general Lipschitz domains, see Appendix.

where $\oplus$ denotes the orthogonal sum in $\mathsf{L}^2$. We note

$$\overset{\circ}{\mathsf{R}}_0 = \nabla\overset{\circ}{\mathsf{H}}{}^1, \quad \mathsf{R}_0 = \nabla\mathsf{H}^1, \quad \mathsf{D}_0 = \operatorname{rot}\mathsf{R}, \quad \overset{\circ}{\mathsf{D}}_0 = \operatorname{rot}\overset{\circ}{\mathsf{R}}.$$

Moreover, with

$$\overset{\circ}{\mathcal{R}} := \overset{\circ}{\mathsf{R}} \cap \operatorname{rot}\mathsf{R}, \quad \mathcal{R} := \mathsf{R} \cap \operatorname{rot}\overset{\circ}{\mathsf{R}}$$

we have

$$\overset{\circ}{\mathsf{R}} = \nabla\overset{\circ}{\mathsf{H}}{}^1 \oplus \overset{\circ}{\mathcal{R}}, \quad \mathsf{R} = \nabla\mathsf{H}^1 \oplus \mathcal{R} \tag{7}$$

and see

$$\operatorname{rot}\overset{\circ}{\mathsf{R}} = \operatorname{rot}\overset{\circ}{\mathcal{R}}, \quad \operatorname{rot}\mathsf{R} = \operatorname{rot}\mathcal{R}.$$

We note that all occurring spaces of range-type are closed subspaces of $\mathsf{L}^2$, which follows immediately by the estimates (1)-(4). More details about the Helmholtz decompositions can be found e.g. in [5].

To get similar inequalities for solenoidal vector fields as in Lemma 1 we need a crucial lemma from [1, Theorem 2.17], see also [11, 4, 3, 2] for related partial results.

**Lemma 3.** *Let $E$ belong to $\overset{\circ}{\mathsf{R}} \cap \mathsf{D}$ or $\mathsf{R} \cap \overset{\circ}{\mathsf{D}}$. Then $E \in \mathsf{H}^1$ and*

$$|\nabla E|^2 \leqslant |\operatorname{rot} E|^2 + |\operatorname{div} E|^2. \tag{8}$$

We emphasize that for $E \in \overset{\circ}{\mathsf{H}}{}^1$ and any domain $\Omega \subset \mathbb{R}^3$

$$|\nabla E|^2 = |\operatorname{rot} E|^2 + |\operatorname{div} E|^2 \tag{9}$$

holds since $-\Delta = \operatorname{rot}\operatorname{rot} - \nabla\operatorname{div}$ . This formula is no longer valid if $E$ has just the tangential or normal boundary condition but for convex domains the inequality (8) remains true.

**Lemma 4.** *For all vector fields $E$ in $\overset{\circ}{\mathsf{R}} \cap \operatorname{rot}\mathsf{R}$ or $\mathsf{R} \cap \operatorname{rot}\overset{\circ}{\mathsf{R}}$*

$$|E| \leqslant c_{\mathrm{p}} |\operatorname{rot} E|.$$

**Proof.** Let $E \in \operatorname{rot}\mathsf{R} = \operatorname{rot}\mathcal{R}$ and $\Phi \in \mathcal{R}$ with $\operatorname{rot}\Phi = E$. Then $\Phi \in \mathsf{H}^1$ by Lemma 3 since $\mathcal{R} = \mathsf{R} \cap \overset{\circ}{\mathsf{D}}_0$. Moreover, $\Phi = \operatorname{rot}\Psi$ can be represented by some $\Psi \in \overset{\circ}{\mathsf{R}}$. Hence, for any constant vector $a \in \mathbb{R}^3$ we have $\langle \Phi, a \rangle = 0$. Thus, $\Phi$ belongs to $\mathsf{H}^1 \cap (\mathbb{R}^3)^{\perp}$. Then, for $E \in \overset{\circ}{\mathsf{R}} \cap \operatorname{rot}\mathsf{R}$ and by Lemma 3 we get

$$|E|^2 = \langle E, \operatorname{rot}\Phi\rangle = \langle \operatorname{rot}E, \Phi\rangle$$
$$\leqslant |\operatorname{rot}E||\Phi| \leqslant c_{\mathtt{p}}|\operatorname{rot}E||\nabla\Phi| \leqslant c_{\mathtt{p}}|\operatorname{rot}E||\underbrace{\operatorname{rot}\Phi}_{=E}|.$$

If $E \in \mathsf{R}\cap\operatorname{rot}\mathring{\mathsf{R}}$, there exists $\Phi \in \mathring{\mathsf{R}}$ with $\operatorname{rot}\Phi = E$. As before, by Lemma 3 we see $E \in \mathsf{H}^1 \cap (\mathbb{R}^3)^\perp$ and $|E| \leqslant c_{\mathtt{p}}|\nabla E| \leqslant c_{\mathtt{p}}|\operatorname{rot}E|$, which completes the proof. □

**Theorem 5.** *For all vector fields* $E \in \mathring{\mathsf{R}} \cap \mathsf{D}$ *and* $H \in \mathsf{R} \cap \mathring{\mathsf{D}}$

$$|E|^2 \leqslant c_{\mathtt{p},0}^2|\operatorname{div}E|^2 + c_{\mathtt{p}}^2|\operatorname{rot}E|^2, \quad |H|^2 \leqslant c_{\mathtt{p}}^2|\operatorname{div}H|^2 + c_{\mathtt{p}}^2|\operatorname{rot}H|^2$$

*hold, i.e.,* $c_{\mathtt{m},\mathtt{t}}, c_{\mathtt{m},\mathtt{n}} \leqslant c_{\mathtt{p}}$. *Moreover,* $c_{\mathtt{p},0} \leqslant c_{\mathtt{m},\mathtt{t}} \leqslant c_{\mathtt{m},\mathtt{n}} = c_{\mathtt{p}} \leqslant \operatorname{diam}(\Omega)/\pi$.

**Proof.** By the Helmholtz decomposition (6) we have

$$\mathring{\mathsf{R}} \cap \mathsf{D} \ni E = E_\nabla + E_{\mathrm{rot}} \in \nabla\mathring{\mathsf{H}}^1 \oplus \operatorname{rot}\mathsf{R}$$

with $E_\nabla \in \nabla\mathring{\mathsf{H}}^1 \cap \mathsf{D}$ and $E_{\mathrm{rot}} \in \mathring{\mathsf{R}} \cap \operatorname{rot}\mathsf{R}$ and $\operatorname{div}E_\nabla = \operatorname{div}E$ as well as $\operatorname{rot}E_{\mathrm{rot}} = \operatorname{rot}E$. By Lemma 1 and Lemma 4 and orthogonality we obtain

$$|E|^2 = |E_\nabla|^2 + |E_{\mathrm{rot}}|^2 \leqslant c_{\mathtt{p},0}^2|\operatorname{div}E|^2 + c_{\mathtt{p}}^2|\operatorname{rot}E|^2.$$

Similarly we have

$$\mathsf{R} \cap \mathring{\mathsf{D}} \ni H = H_\nabla + H_{\mathrm{rot}} \in \nabla\mathsf{H}^1 \oplus \operatorname{rot}\mathring{\mathsf{R}}$$

with $H_\nabla \in \nabla\mathsf{H}^1 \cap \mathring{\mathsf{D}}$ and $H_{\mathrm{rot}} \in \mathsf{R} \cap \operatorname{rot}\mathring{\mathsf{R}}$ and $\operatorname{div}H_\nabla = \operatorname{div}H$ and $\operatorname{rot}H_{\mathrm{rot}} = \operatorname{rot}H$. As before,

$$|H|^2 = |H_\nabla|^2 + |H_{\mathrm{rot}}|^2 \leqslant c_{\mathtt{p}}^2|\operatorname{div}H|^2 + c_{\mathtt{p}}^2|\operatorname{rot}H|^2.$$

This shows the upper bounds. For the lower bounds, let $\lambda_1$ be the first Dirichlet eigenvalue of the negative Laplacian $-\Delta$, i.e.,

$$\frac{1}{c_{\mathtt{p},0}^2} = \lambda_1 = \inf_{0\neq\varphi\in\mathring{\mathsf{H}}^1} \frac{|\nabla\varphi|^2}{|\varphi|^2},$$

and let $u \in \mathring{\mathsf{H}}^1$ be an eigenfunction to $\lambda_1$. Note that $u$ satisfies

$$\forall\,\varphi \in \mathring{\mathsf{H}}^1 \quad \langle\nabla u, \nabla\varphi\rangle = \lambda_1\langle u, \varphi\rangle.$$

Then $0 \neq E := \nabla u \in \nabla\mathring{\mathsf{H}}^1 \cap \mathsf{D} = \mathring{\mathsf{R}}_0 \cap \mathsf{D}$ and $-\operatorname{div} E = -\operatorname{div}\nabla u = \lambda_1 u$. By (3) and (1) we have

$$|E| \leqslant c_{\mathtt{m},\mathtt{t}}|\operatorname{div} E| = c_{\mathtt{m},\mathtt{t}}\lambda_1|u| \leqslant c_{\mathtt{m},\mathtt{t}}\lambda_1 c_{\mathtt{p},0}|\nabla u| = \frac{c_{\mathtt{m},\mathtt{t}}}{c_{\mathtt{p},0}}|E|,$$

yielding $c_{\mathtt{p},0} \leqslant c_{\mathtt{m},\mathtt{t}}$. Now, let $\mu_2$ be the second Neumann eigenvalue of the negative Laplacian $-\Delta$, i.e.,

$$\frac{1}{c_{\mathtt{p}}^2} = \mu_2 = \inf_{0\neq\varphi\in\mathsf{H}^1\cap\mathbb{R}^\perp} \frac{|\nabla\varphi|^2}{|\varphi|^2},$$

and let $u \in \mathsf{H}^1 \cap \mathbb{R}^\perp$ be an eigenfunction to $\mu_2$. Note that $u$ satisfies

$$\forall\,\varphi \in \mathsf{H}^1 \cap \mathbb{R}^\perp \quad \langle\nabla u, \nabla\varphi\rangle = \mu_2\langle u, \varphi\rangle$$

and that this relation holds even for all $\varphi \in \mathsf{H}^1$. Then $0 \neq H := \nabla u$ belongs to $\nabla\mathsf{H}^1 \cap \mathring{\mathsf{D}} = \mathsf{R}_0 \cap \mathring{\mathsf{D}}$ and satisfies $-\operatorname{div} H = -\operatorname{div}\nabla u = \mu_2 u$. By (4) and (2) we have

$$|H| \leqslant c_{\mathtt{m},\mathtt{n}}|\operatorname{div} H| = c_{\mathtt{m},\mathtt{n}}\mu_2|u| \leqslant c_{\mathtt{m},\mathtt{n}}\mu_2 c_{\mathtt{p}}|\nabla u| = \frac{c_{\mathtt{m},\mathtt{n}}}{c_{\mathtt{p}}}|H|,$$

yielding $c_{\mathtt{p}} \leqslant c_{\mathtt{m},\mathtt{n}}$ and completing the proof. □

**Remark 6.** *Looking at the proof, the lower bounds $c_{\mathtt{p},0} \leqslant c_{\mathtt{m},\mathtt{t}}$ as well as $c_{\mathtt{p}} \leqslant c_{\mathtt{m},\mathtt{n}}$ remain true in more general situations, i.e., for bounded Lipschitz[3] domains $\Omega \subset \mathbb{R}^N$.*

## Appendix §A. The 2D Case

In 2D there are two rotations $\operatorname{rot} = \operatorname{div} R$ and $\vec{\operatorname{rot}} = \nabla^\perp = R\nabla$, a scalar and vector valued one. The scalar valued one is just the divergence div after a 90°-rotation

$$R := \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

and the vector valued one is actually the gradient $\nabla$ followed by the same rotation $R$. Hence, applying the Poincaré estimates to the potentials generated by the Helmholtz decompositions yields immediately the desired estimates. Of course, this special trick works just in 2D.

More precisely, let $\Omega \subset \mathbb{R}^2$ be a Lipschitz domain. Then Lemma 1 holds by Remark 2. Moreover, even a stronger version of Lemma 4 is true.

[3]The Lipschitz assumption can also be weakened. It is sufficient that $\Omega$ admits the Maxwell compactness properties.

**Lemma 7.** *For all vector fields* $E \in \overset{\circ}{\mathsf{R}} \cap \vec{\operatorname{rot}}\, \mathsf{H}^1$ *and* $H \in \mathsf{R} \cap \vec{\operatorname{rot}}\, \overset{\circ}{\mathsf{H}}{}^1$

$$|E| \leqslant c_{\mathtt{p}} |\operatorname{rot} E|, \quad |H| \leqslant c_{\mathtt{p},0} |\operatorname{rot} H|.$$

This follows immediately from Lemma 1 by the following arguments.

**Proof.** Let $E \in \overset{\circ}{\mathsf{R}} \cap \vec{\operatorname{rot}}\, \mathsf{H}^1 = \overset{\circ}{\mathsf{R}} \cap R\nabla \mathsf{H}^1$. Then $H := RE \in \overset{\circ}{\mathsf{D}} \cap \nabla \mathsf{H}^1$. By Lemma 1 we get

$$|E| = |H| \leqslant c_{\mathtt{p}} |\operatorname{div} H| = c_{\mathtt{p}} |\operatorname{rot} E|.$$

If $H \in \mathsf{R} \cap \vec{\operatorname{rot}}\, \overset{\circ}{\mathsf{H}}{}^1 = \mathsf{R} \cap R\nabla \overset{\circ}{\mathsf{H}}{}^1$, then $E := RH \in \mathsf{D} \cap \nabla \overset{\circ}{\mathsf{H}}{}^1$. By Lemma 1 we obtain

$$|H| = |E| \leqslant c_{\mathtt{p},0} |\operatorname{div} E| = c_{\mathtt{p},0} |\operatorname{rot} H|.$$

□

We note that in 2D the Helmholtz decompositions read

$$\mathsf{L}^2 = \nabla \overset{\circ}{\mathsf{H}}{}^1 \oplus \mathcal{H}_{\mathtt{D}} \oplus \vec{\operatorname{rot}}\, \mathsf{H}^1, \quad \mathsf{L}^2 = \nabla \mathsf{H}^1 \oplus \mathcal{H}_{\mathtt{N}} \oplus \vec{\operatorname{rot}}\, \overset{\circ}{\mathsf{H}}{}^1,$$

where due to the possibly non-trivial topology (We do not assume $\Omega$ to be convex.) non-vanishing Dirichlet or Neumann fields may exist.

**Theorem 8.** *For all vector fields* $E \in \overset{\circ}{\mathsf{R}} \cap \mathsf{D} \cap \mathcal{H}_{\mathtt{D}}^{\perp}$ *and* $H \in \mathsf{R} \cap \overset{\circ}{\mathsf{D}} \cap \mathcal{H}_{\mathtt{N}}^{\perp}$

$$|E|^2 \leqslant c_{\mathtt{p},0}^2 |\operatorname{div} E|^2 + c_{\mathtt{p}}^2 |\operatorname{rot} E|^2, \quad |H|^2 \leqslant c_{\mathtt{p}}^2 |\operatorname{div} H|^2 + c_{\mathtt{p},0}^2 |\operatorname{rot} H|^2$$

*hold, i.e.,* $c_{\mathtt{m},\mathtt{t}}, c_{\mathtt{m},\mathtt{n}} \leqslant c_{\mathtt{p}}$. *Moreover, even* $c_{\mathtt{p},0} < c_{\mathtt{m},\mathtt{t}} = c_{\mathtt{m},\mathtt{n}} = c_{\mathtt{p}}$.

**Proof.** Following the proof of Theorem 5 yields for $E \in \overset{\circ}{\mathsf{R}} \cap \mathsf{D} \cap \mathcal{H}_{\mathtt{D}}^{\perp}$ by the Helmholtz decomposition

$$E = E_{\nabla} + E_{\mathrm{rot}} \in \nabla \overset{\circ}{\mathsf{H}}{}^1 \oplus \vec{\operatorname{rot}}\, \mathsf{H}^1$$

with $E_{\nabla} \in \nabla \overset{\circ}{\mathsf{H}}{}^1 \cap \mathsf{D}$ and $E_{\mathrm{rot}} \in \overset{\circ}{\mathsf{R}} \cap \vec{\operatorname{rot}}\, \mathsf{H}^1$ and $\operatorname{div} E_{\nabla} = \operatorname{div} E$ and $\operatorname{rot} E_{\mathrm{rot}} = \operatorname{rot} E$. Hence, by Lemma 1 and Lemma 7 and orthogonality we obtain

$$|E|^2 = |E_{\nabla}|^2 + |E_{\mathrm{rot}}|^2 \leqslant c_{\mathtt{p},0}^2 |\operatorname{div} E|^2 + c_{\mathtt{p}}^2 |\operatorname{rot} E|^2,$$

and the estimate for $H$ follows analogously. For the lower bounds, we look again at the second Neumann eigenvalue $\mu_2 = 1/c_{\mathtt{p}}^2$ of $-\Delta$ and a corresponding eigenfunction $u \in \mathsf{H}^1 \cap \mathbb{R}^{\perp}$ with $\nabla u \in \overset{\circ}{\mathsf{D}}$ and $-\Delta u = \mu_2 u$. Then, as before, $0 \neq H := \nabla u$ belongs to $\nabla \mathsf{H}^1 \cap \overset{\circ}{\mathsf{D}} = \mathsf{R}_0 \cap \overset{\circ}{\mathsf{D}} \cap \mathcal{H}_{\mathtt{N}}^{\perp}$ with

$-\operatorname{div} H = -\operatorname{div} \nabla u = \mu_2 u$. By the definition of $c_{\mathtt{m},\mathtt{n}}$ and (2) (for non-convex $\Omega$) we have

$$|H| \leqslant c_{\mathtt{m},\mathtt{n}} |\operatorname{div} H| = c_{\mathtt{m},\mathtt{n}} \mu_2 |u| \leqslant c_{\mathtt{m},\mathtt{n}} \mu_2 c_{\mathtt{p}} |\nabla u| = \frac{c_{\mathtt{m},\mathtt{n}}}{c_{\mathtt{p}}} |H|,$$

yielding $c_{\mathtt{p}} \leqslant c_{\mathtt{m},\mathtt{n}}$. On the other hand, $E := RH \in \mathsf{D}_0 \cap \mathring{\mathsf{R}} \cap \mathcal{H}_{\mathtt{D}}^{\perp}$ and

$$|E| \leqslant c_{\mathtt{m},\mathtt{t}} |\operatorname{rot} E| = c_{\mathtt{m},\mathtt{t}} |\operatorname{div} H| = c_{\mathtt{m},\mathtt{t}} \mu_2 |u| \leqslant c_{\mathtt{m},\mathtt{t}} \mu_2 c_{\mathtt{p}} |\nabla u| = \frac{c_{\mathtt{m},\mathtt{t}}}{c_{\mathtt{p}}} |E|,$$

showing $c_{\mathtt{p}} \leqslant c_{\mathtt{m},\mathtt{t}}$. □

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Fakultät für Mathematik,
Universität Duisburg-Essen,
Campus Essen, Germany
*E-mail*: dirk.pauly@uni-due.de

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