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ESTIMATES OF THE DISTANCE TO THE EXACT SOLUTION OF PARABOLIC PROBLEMS BASED ON LOCAL POINCARÉ TYPE INEQUALITIES

ABSTRACT. The goal of the paper is to derive two-sided bounds of the distance between the exact solution of the evolutionary reaction-diffusion problem with mixed Dirichlet–Robin boundary conditions and any function in the admissible energy space. The derivation is based upon special transformations of the integral identity, that defines the generalized solution. In order to obtain estimates with easily computable local constants we exploit classical Poincaré inequalities and Poincaré type inequalities for functions with zero mean boundary traces. The corresponding constants are estimated in [10] and [8]. Bounds of the distance to the exact solution contain only these constants associated with subdomains. It is proved that the bounds are equivalent to the energy norm of the error.

Dedicated to the 80th jubilee of V. A. Solonnikov

§1. PROBLEM STATEMENT

We consider the evolutionary reaction-diffusion problem: find $u = u(x,t)$ and $p = p(x,t)$ such that

$$u_t - \nabla \cdot p + \varrho^2 u = f \qquad \text{in} \quad Q_T := \Omega \times (0,T), \qquad (1.1)$$

$$p = A\nabla u \quad \text{in} \quad Q_T, \qquad (1.2)$$

$$u(\cdot, 0) = u_0 \qquad \text{in} \quad \Omega, \qquad (1.3)$$

$$u = 0 \qquad \text{on} \quad S_D, \qquad (1.4)$$

$$p \cdot n + \sigma^2 u = F \qquad \text{on} \quad S_R. \qquad (1.5)$$

Here, Ω is a bounded connected domain in $\mathbb{R}^d$ $(d \geqslant 2)$ with Lipchitz continuous boundary $\partial\Omega$, which consists of two measurable non-intersecting parts

Γ_D and $\Gamma_R \neq \varnothing$ associated with the Dirichlet and Robin boundary conditions, respectively, n denotes the unit normal vector to $\partial\Omega$ directed outwards, T is a finite positive number, $S := \partial\Omega \times]0, T[$ is the lateral surface of the space–time cylinder Q_T, $S_D := \Gamma_D \times]0, T[$, and $S_R := \Gamma_R \times]0, T[$. We assume that A is a symmetric matrix with coefficients in $L^\infty(\Omega)$, which for almost all $x \in \Omega$ satisfies the condition

$$\underline{\lambda}_A |\xi|^2 \leqslant A(x)\,\xi \cdot \xi \leqslant \overline{\lambda}_A |\xi|^2, \quad \forall \xi \in \mathbb{R}^d, \quad 0 < \underline{\lambda}_A \leqslant \overline{\lambda}_A < +\infty. \quad (1.6)$$

Here, $|\xi| := \sqrt{\xi \cdot \xi}$, and $\cdot$ stands for the scalar product in $\mathbb{R}^d$. Also, we assume that $f \in L^2(Q_T)$, $u_0 \in L^2(\Omega)$, $F \in L^2(S_R)$, and the coefficients $\varrho(x)$ and $\sigma(x)$ are uniformly bounded by the constants C_ϱ and C_σ in Q_T and S_T, respectively.

Throughout the paper, the norms of $L^2(\Omega)$ and $L^2(Q_T)$ are denoted by $\|\cdot\|_\Omega$ and $\|\cdot\|_{Q_T}$, respectively, $H^1_0(\Omega)$ is a subspace of $H^1(\Omega)$ containing the functions vanishing on the Dirichlet part of the boundary, and $H^1_0(Q_T) = V_0$. The generalized solution of (1.1)–(1.5) is a function $u \in V_0$ satisfying the integral identity

$$\int_\Omega \Big(u(x,T)\eta(x,T) - u(x,0)\eta(x,0) \Big)\,\mathrm{d}x - \int_{Q_T} u\eta_t\,\mathrm{d}x\mathrm{d}t$$

$$+ \int_{Q_T} A\nabla u \cdot \nabla\eta\,\mathrm{d}x\mathrm{d}t + \int_{Q_T} \varrho^2 u\eta\,\mathrm{d}x\mathrm{d}t + \int_{S_R} \sigma^2 u\eta\,\mathrm{d}s\mathrm{d}t$$

$$= \int_{Q_T} f\eta\,\mathrm{d}x\mathrm{d}t + \int_{S_R} F\eta\,\mathrm{d}s\mathrm{d}t, \quad \forall \eta \in V_0. \quad (1.7)$$

Due to known results (see, e.g., [3, 5, 6, 15]), the solution of (1.7) exists and is unique.

Assume that $v \in V_0$ is a function compared with u (e.g., it can be an approximation generated by some numerical method). Our goal is to deduce explicitly computable and realistic estimates of the error $e := u - v$

in terms of the measure

$$[e]^2_{(\nu,\theta,\zeta,\chi)} := \nu \int\limits_0^T \|\nabla e\|^2_{A,\Omega} \ \mathrm{d}t + \int\limits_0^T \|\theta e\|^2_\Omega \ \mathrm{d}t + \zeta \|e(\cdot,T)\|^2_\Omega$$

$$+ \chi \int\limits_0^T \|\sigma e\|^2_{\Gamma_R} \ \mathrm{d}t, \quad (1.8)$$

where $\|\nabla e\|^2_{A,\Omega} := \int_\Omega A \nabla e \cdot \nabla e \, \mathrm{d}x$, is the spatial error norm defined on Ω, and ν, θ, ζ, χ are positive weights. The first two terms present a measure equivalent to the natural energy norm, the third term represents the error at $t = T$, and the last one measures possible violations of the Robin boundary condition. The weights can be selected in order to balance different components of the error in desired proportions. In other words, the quantity (1.8) generates a collection of different error measures, which can be used for judging on the distance between u and v.

In this paper, we derive computable majorants of $[e]^2_{(\nu,\theta,\zeta,\chi)}$ by means of the method close to that has been originally suggested in [14] (see also Section 9.3 of the monograph [12]). They are derived by special transformations of the integral identity (1.7). However, in this paper we apply a somewhat new approach based on domain decomposition and local Poincaré type inequalities for functions with zero mean boundary traces (sharp constants in these inequalities has been recently found in [8]). As a result, we obtain fully computable estimates, which are applicable for problems with complicated geometry and non-trivial boundary conditions.

In Section 2, we deduce general two-sided bounds of the distance to the exact solution. The error majorant (Theorems 1) contains the constants $C_{\mathrm{F}\Omega}$ and $C_{\Gamma\mathrm{R}}$ in the Friedrichs and trace type inequalities

$$\|w\|_\Omega \leqslant C_{\mathrm{F}\Omega} \|\nabla w\|_\Omega, \qquad (1.9)$$

$$\|w\|_{\Gamma_R} \leqslant C_{\Gamma\mathrm{R}} \|\nabla w\|_\Omega, \qquad (1.10)$$

respectively, which are valid for the functions in $H_0^1(\Omega)$.

In general, finding global constants $C_{\mathrm{F}\Omega}$ and $C_{\Gamma\mathrm{R}}$ may be not an easy task (especially for geometrically complicated domains and mixed boundary conditions). A way to overcome these difficulties is suggested in Section 3, where we deduce new forms of majorants, which are based on decomposition of Ω into a collection of non-overlapping convex sub-domains

and local constants associated with these subdomains. Such constants are defined either by means of the Payne–Weinberger estimate [10] related to the Poincaré inequality for convex domains or due to the results of [8], where sharp constants for Poincaré type inequalities for functions with zero mean boundary traces has been found. Therefore, we obtain different forms of the respective error majorants, which involve only local constants and known functions. In Section 4, we prove that the majorants are equivalent to the distance to the exact solution measured either in terms of the measure (1.8) or in terms of a combined primal-dual energy norm.

§2. Estimates based on global constants

2.1. Majorant of $[e]_{(\nu,\theta,\zeta,\chi)}$**.** Let $v \in V_0$ be a function considered as an approximation of the exact solution u. We transform (1.7) and arrive at the relation

$$
\int\limits_0^T \|\nabla e\|^2_{A,\Omega}\, \mathrm{d}t + \int\limits_T^T \|\varrho e\|^2_{\Omega}\,\mathrm{d}t + \tfrac{1}{2}\|e(\cdot,T)\|^2_{\Omega}
$$
$$
+ \int\limits_0^T \|\sigma e\|^2_{\Gamma_R}\,\mathrm{d}t = \int\limits_{Q_T} \left(f - v_t - \varrho^2 v\right) e\, \mathrm{d}x\mathrm{d}t
$$
$$
- \int\limits_{Q_T} A\nabla v \cdot \nabla e\, \mathrm{d}x\mathrm{d}t + \int\limits_{S_R} (g - \sigma^2 v)e\, \mathrm{d}s\mathrm{d}t + \tfrac{1}{2}\|e(\cdot,0)\|^2_{\Omega}, \quad (2.1)
$$

which can be viewed as the basic error identity. In order to rearrange the right hand side of (2.1), we introduce a vector valued function $y \in Y_{\mathrm{div}}(Q_T)$, where $Y_{\mathrm{div}}(Q_T)$ denotes the space of vector valued functions $y \in L^2(\Omega, \mathbb{R}^d)$ such that $\mathrm{div}\, y \in L^2(\Omega)$ and $y \cdot n \in L^2(\Gamma_R)$ for almost all $t \in (0,T)$. We introduce the quantities

$$
\mathbf{r}_f(v,y) := f - v_t - \varrho^2 v + \mathrm{div}\, y, \tag{2.2}
$$

$$
\mathbf{r}_A(v,y) := y - A\nabla v, \tag{2.3}
$$

$$
\mathbf{r}_F(v,y) := F - \sigma^2 v - y \cdot n, \tag{2.4}
$$

which have clear meaning: they are *residuals* of (1.1), (1.2), and (1.5), respectively. The theorem below shows that certain norms of these quantities control the distance between u and v. Also, we define weighted residuals

$$
\mathbf{r}_{f,\mu}(v,y) := \mu\mathbf{r}_f \quad \text{and} \quad \mathbf{r}_{f,1-\mu}(v,y) := (1-\mu)\mathbf{r}_f. \tag{2.5}
$$

Here, $\mu(x,t)$ is a real-valued function taking values in $[0,1]$ (these weighted quantities are motivated later).

Theorem 1. *For any $v \in V_0$, $y \in Y_{\mathrm{div}}(Q_T)$, $\delta \in (0,2]$, and real-valued function $\gamma(t) \geqslant \frac{1}{2}$, we have the estimate*

$$[e]^2_{(\nu,\theta,1,2)} \leqslant \overline{\mathrm{M}}^2_{\mathrm{I}}(v,y;\delta,\gamma,\mu) := \|e(\cdot,0)\|^2_\Omega + \int_0^T \Big(\alpha_1(t)\|\mathbf{r}_A(v,y)\|^2_{A^{-1},\Omega}$$

$$+ \gamma \left\| \frac{1}{\varrho}\mathbf{r}_{f,\mu}(v,y) \right\|^2_\Omega + \alpha_2(t)\frac{C^2_{\mathrm{F}\Omega}}{\underline{\lambda}_A}\|\mathbf{r}_{f,1-\mu}(v,y)\|^2_\Omega$$

$$+ \alpha_3(t)\frac{C^2_{\Gamma\mathrm{R}}}{\underline{\lambda}_A}\|\mathbf{r}_F(v,y)\|^2_{\Gamma_R} \Big) \, \mathrm{d}t, \quad (2.6)$$

where $\nu = 2 - \delta$, $\theta(x,t) = \varrho(x)\left(2 - \frac{1}{\gamma(t)}\right)^{\frac{1}{2}}$, and $\alpha_1(t)$, $\alpha_2(t)$, $\alpha_3(t)$ are arbitrary positive functions satisfying the relation

$$\frac{1}{\alpha_1(t)} + \frac{1}{\alpha_2(t)} + \frac{1}{\alpha_3(t)} = \delta. \qquad (2.7)$$

Proof. We transform the right-hand side of (2.1) by means of the integral identity

$$\int_{Q_T} \mathrm{div}\, ye \, \mathrm{d}x\mathrm{d}t + \int_{Q_T} y \cdot \nabla e \, \mathrm{d}x\mathrm{d}t = \int_{S_R} y \cdot ne \, \mathrm{d}s\mathrm{d}t, \qquad (2.8)$$

which yields

$$\int_0^T \|\nabla e\|^2_{A,\Omega} \, \mathrm{d}t + \int_0^T \|\varrho e\|^2_\Omega \, \mathrm{d}t + \frac{1}{2}\|e(\cdot,T)\|^2_\Omega + \int_0^T \|\sigma e\|^2_{\Gamma_R} \, \mathrm{d}t$$

$$= \mathscr{I}_f + \mathscr{I}_A + \mathscr{I}_F + \frac{1}{2}\|e(x,0)\|^2_\Omega, \quad (2.9)$$

where

$$\mathscr{I}_f := \int_{Q_T} \mathbf{r}_f(v,y)e \, \mathrm{d}x\mathrm{d}t, \qquad (2.10)$$

$$\mathscr{I}_A := \int_{Q_T} \mathbf{r}_A(v,y) \cdot \nabla e \, \mathrm{d}x\mathrm{d}t, \qquad (2.11)$$

$$\mathscr{I}_F := \int_{S_R} \mathbf{r}_F(v,y)e \, \mathrm{d}s\mathrm{d}t. \qquad (2.12)$$

It is easy to see that

$$|\mathscr{I}_A| \leqslant \int\limits_0^T \|\mathbf{r}_A\|_{A^{-1},\Omega} \, \|\nabla e\|_{A,\Omega} \, \mathrm{d}t \tag{2.13}$$

and (cf. (1.10))

$$|\mathscr{I}_F| \leqslant \int\limits_0^T \|\mathbf{r}_F\|_{\Gamma_R} \, \|e\|_{\Gamma_R} \, \mathrm{d}t \leqslant \int\limits_0^T \|\mathbf{r}_F\|_{\Gamma_R} \, \frac{C_{\Gamma R}}{\sqrt{\underline{\lambda}_A}} \|\nabla e\|_{A,\Omega} \, \mathrm{d}t. \tag{2.14}$$

In order to estimate the term $\mathscr{I}_f$, we apply the same method as in [13] and introduce a function $\mu(x,t)$, which takes values in $[0,1]$. The idea behind is to split the integral into two parts, which will be later subject to different parts of the error norm. If the function ϱ has very different values and may be close to zero, then the resulting estimate is much more accurate. We can select μ such that large factors of the type $\frac{1}{\varrho}$ (if ϱ is small), arising in the estimate are compensated. Hence, we obtain

$$|\mathscr{I}_f| \leqslant \int\limits_0^T \left(\left\| \tfrac{\mu}{\varrho}\mathbf{r}_f \right\|_\Omega \|\varrho e\|_\Omega + \frac{C_{F\Omega}}{\sqrt{\underline{\lambda}_A}} \|(1-\mu)\mathbf{r}_f\|_\Omega \|\nabla e\|_{A,\Omega} \right) \mathrm{d}t. \tag{2.15}$$

Combining (2.13)–(2.15), we find that

$$\begin{aligned}
\int\limits_0^T \|\nabla e\|_{A,\Omega}^2 \, \mathrm{d}t + \int\limits_0^T \|\varrho e\|_\Omega^2 \, \mathrm{d}t &+ \tfrac{1}{2}\|e(\cdot,T)\|_\Omega^2 + \int\limits_0^T \|\sigma e\|_{\Gamma_R}^2 \, \mathrm{d}t \\
\leqslant \tfrac{1}{2}\|e(x,0)\|_\Omega^2 &+ \int\limits_0^T \left(\left\| \tfrac{\mu}{\varrho}\mathbf{r}_f \right\|_\Omega \|\varrho e\|_\Omega + \frac{C_{F\Omega}}{\sqrt{\underline{\lambda}_A}} \|(1-\mu)\mathbf{r}_f\|_\Omega \|\nabla e\|_{A,\Omega} \right. \\
&\left. + \|\mathbf{r}_A\|_{A^{-1},\Omega} \|\nabla e\|_{A,\Omega} + \frac{C_{\Gamma R}}{\sqrt{\underline{\lambda}_A}} \|\mathbf{r}_F\|_{\Gamma_R} \|\nabla e\|_{A,\Omega} \right) \mathrm{d}t.
\end{aligned} \tag{2.16}$$

The second term on the right-hand side of (2.16) can be estimated by the Young–Fenchel inequality

$$\int\limits_0^T \left\| \tfrac{\mu}{\varrho}\mathbf{r}_f \right\|_\Omega \|\varrho e\|_\Omega \, \mathrm{d}t \leqslant \int\limits_0^T \left(\tfrac{\gamma(t)}{2} \left\| \tfrac{\mu}{\varrho}\mathbf{r}_f \right\|_\Omega^2 + \tfrac{1}{2\gamma(t)} \|\varrho e\|_\Omega^2 \right) \mathrm{d}t, \tag{2.17}$$

where $\gamma(t)$ is an arbitrary real-valued function taking values in $\left[\frac{1}{2}, +\infty\right[$. Analogously,

$$\int_0^T \frac{C_{\mathrm{F}\Omega}}{\sqrt{\underline{\lambda}_A}} \left\|(1-\mu)\mathbf{r}_f\right\|_\Omega \|\nabla e\|_{A,\Omega} \, \mathrm{d}t$$

$$\leqslant \frac{1}{2} \int_0^T \left(\alpha_1(t)\frac{C_{\mathrm{F}\Omega}^2}{\underline{\lambda}_A}\left\|(1-\mu)\mathbf{r}_f\right\|_\Omega^2 + \frac{1}{\alpha_1(t)}\|\nabla e\|_{A,\Omega}^2\right) \, \mathrm{d}t,$$

$$\int_0^T \left\|\mathbf{r}_A\right\|_{A^{-1},\Omega}\|\nabla e\|_{A,\Omega} \, \mathrm{d}t$$

$$\leqslant \frac{1}{2} \int_0^T \left(\alpha_2(t)\left\|\mathbf{r}_A\right\|_{A^{-1},\Omega}^2 + \frac{1}{\alpha_2(t)}\|\nabla e\|_{A,\Omega}^2\right) \, \mathrm{d}t,$$

and

$$\int_0^T \left\|\mathbf{r}_F\right\|_{\Gamma_R} \frac{C_{\mathrm{TR}}}{\sqrt{\underline{\lambda}_A}}\|\nabla e\|_{A,\Omega} \, \mathrm{d}t$$

$$\leqslant \frac{1}{2} \int_0^T \left(\alpha_3(t)\frac{C_{\mathrm{TR}}^2}{\underline{\lambda}_A}\left\|\mathbf{r}_F\right\|_{\Gamma_R}^2 + \frac{1}{\alpha_3(t)}\|\nabla e\|_{A,\Omega}^2\right) \, \mathrm{d}t. \quad (2.18)$$

Here, $\alpha_1(t)$, $\alpha_2(t)$, and $\alpha_3(t)$ are positive functions satisfying (2.7). Then, the estimate (2.6) follows from (2.17)–(2.18). $\qquad\square$

Remark 1. The function y can be viewed as an approximation of the exact flux $A\nabla u$. If it is defined (e.g., by means of some reconstruction of a numerical solution v), then the functions

$$r_1(t) := \left\|\mathbf{r}_A(v, y)\right\|_{A^{-1},\Omega}, \qquad\qquad (2.19)$$

$$r_2(t) := \frac{C_{\mathrm{F}\Omega}}{\sqrt{\underline{\lambda}_A}}\left\|\mathbf{r}_{f,1-\mu}(v, y)\right\|_\Omega, \qquad\qquad (2.20)$$

$$r_3(t) := \frac{C_{\mathrm{TR}}}{\sqrt{\underline{\lambda}_A}}\left\|\mathbf{r}_F(v, y)\right\|_{\Gamma_R} \qquad\qquad (2.21)$$

are known. In this case, the majorant $\overline{\mathrm{M}}_{\mathrm{I}}^2(v, y; \delta, \gamma, \mu)$ can be minimized with respect to $\alpha_1(t)$, $\alpha_2(t)$, $\alpha_3(t)$. The optimal functions $\alpha_i^*(t)$ $(i = 1, 2, 3)$

can be easily found by the method of Lagrangian multipliers and are defined by the relation

$$\alpha_i^*(t) = \frac{\sum\limits_{i=1}^{3} r_i(t)}{\delta\, r_i(t)}. \tag{2.22}$$

However, if we wish to minimize the majorant with respect to y, then it is more advantageous to keep the quadratic structure of (2.6). In this case, we can apply iterative minimization procedures similar to those used in [7, 12] and some other publications cited therein.

Remark 2. The majorant $\overline{\mathrm{M}}_{\mathrm{I}}^2(v, y; \delta, \gamma, \mu)$ has a clear structure. The first term contains the error in the initial condition and vanishes if the function v exactly satisfies it. Other terms are formed by norms of the residuals $\mathbf{r}_A$, $\mathbf{r}_f$, and $\mathbf{r}_F$ and weight factors formed by the global constants $C_{\mathrm{F}\Omega}$ and $C_{\Gamma\mathrm{R}}$ related to Ω. Since v satisfies the boundary condition on Γ_D, the majorant vanishes if and only if all the residuals are equal to zero, i.e., if and only if v coincides with u and y coincides with $A\nabla u$.

2.2. Minorant of $[e]_{(\nu,\theta,\varsigma,\chi)}$. Computable minorants of the deviations from the exact solution of partial differential equations provide useful information, which allows us to judge on the quality of error majorants. For elliptic variational problems, a minorant can be derived fairly easily by means of variational arguments (see [9]). Another derivation method (which does not exploit variational arguments) was suggested in [12]. Below, we apply this method to the considered class of parabolic problems and deduce computable minorants of the distance to the exact solution.

Theorem 2. *Let* $v \in V_0$ *and* $\eta \in H_0^1(Q_T)$, *then the following estimate holds:*

$$\underline{\mathrm{M}}^2(v) := \sup_{\eta \in H_0^1(Q_T)} \left\{ \sum_{i=1}^{5} G_i(\eta, v, \kappa_i) + G_0(\eta, f, F, u_0) \right\} \leqslant [e]^2_{(\underline{\nu},\underline{\theta},\underline{\varsigma},\underline{\chi})} \tag{2.23}$$

where

$$G_1(v, \eta, \kappa_1) := \int\limits_{Q_T} \left(-\nabla\eta \cdot A\nabla v - \tfrac{1}{2\kappa_1} A\nabla\eta \cdot \nabla\eta \right) \mathrm{d}x\mathrm{d}t,$$

$$G_2(v, \eta, \kappa_2) := \int\limits_{Q_T} \left(\eta_t v - \tfrac{1}{2\kappa_2} |\eta_t|^2 \right) \mathrm{d}x\mathrm{d}t,$$

$$G_3(v, \eta, \kappa_3) := \int\limits_{Q_T} \varrho^2 \left(-v\eta - \tfrac{1}{2\kappa_3} |\eta|^2 \right) \mathrm{d}x\mathrm{d}t,$$

$$G_4(v, \eta, \kappa_4) := \int\limits_{\Omega} \left(-v(x, T)\eta(x, T) - \tfrac{1}{2\kappa_4} |\eta(x, T)|^2 \right) \mathrm{d}x,$$

$$G_5(v, \eta, \kappa_5) := \int\limits_{S_R} \sigma^2 \left(-v\eta - \tfrac{1}{2\kappa_5} |\eta|^2 \right) \mathrm{d}s\mathrm{d}t, \qquad (2.24)$$

$$G_0(\eta, f, F, u_0) := \int\limits_{Q_T} f\eta \, \mathrm{d}x\mathrm{d}t + \int\limits_{S_R} F\eta \, \mathrm{d}s\mathrm{d}t + \int\limits_{\Omega} u_0\eta(\cdot, 0) \, \mathrm{d}x, \qquad (2.25)$$

and $\underline{\nu} = \frac{\kappa_1}{2}$, $\underline{\theta}(x) = \left(\frac{1}{2} \left(\kappa_2 + \kappa_3\varrho(x)^2 \right) \right)^{\frac{1}{2}}$, $\underline{\zeta} = \frac{\kappa_4}{2}$, $\underline{\chi} = \frac{\kappa_5}{2}$, *and* κ_i *(i = 1, ..., 5) are arbitrary positive numbers.*

Proof. Consider the functional

$$\mathcal{M}(e) := \sup_{\eta \in H_0^1(Q_T)} \left\{ \int\limits_{Q_T} \left(\nabla\eta \cdot A\nabla e - \tfrac{1}{2\kappa_1} A\nabla\eta \cdot \nabla\eta - \eta_t e \right.\right.$$

$$\left. - \tfrac{1}{2\kappa_2} |\eta_t|^2 + \varrho^2 \left(e\eta - \tfrac{1}{2\kappa_3} |\eta|^2 \right) \right) \mathrm{d}x\mathrm{d}t$$

$$+ \int\limits_{\Omega} \left(e(x, T)\eta(x, T) - \tfrac{1}{2\kappa_4} |\eta(x, T)|^2 \right) \mathrm{d}x + \int\limits_{S_R} \sigma \left(e\eta - \tfrac{1}{2\kappa_5} |\eta|^2 \right) \mathrm{d}s\mathrm{d}t \left. \right\}.$$

It is not difficult to see that for any $\eta \in \nabla_0$

$$\int_{Q_T} \left(\nabla \eta \cdot A \nabla e - \tfrac{1}{2\kappa_1} A \nabla \eta \cdot \nabla \eta \right) \mathrm{d}x\mathrm{d}t \leqslant \frac{\kappa_1}{2} \int_0^T \|\nabla e\|_A^2 \, \mathrm{d}t, \qquad (2.26)$$

$$\int_{Q_T} \left(-\eta_t e - \tfrac{1}{2\kappa_2} |\eta_t|^2 \right) \mathrm{d}x\mathrm{d}t \leqslant \frac{\kappa_2}{2} \int_0^T \|e\|_\Omega^2 \, \mathrm{d}t, \qquad (2.27)$$

$$\int_{Q_T} \varrho^2 \left(e\eta - \tfrac{1}{2\kappa_3} |\eta|^2 \right) \mathrm{d}x\mathrm{d}t \leqslant \frac{\kappa_3}{2} \int_0^T \|\varrho e\|_\Omega^2 \, \mathrm{d}t, \qquad (2.28)$$

$$\int_\Omega \left(e(x,T)\eta(x,T) - \tfrac{1}{2\kappa_4} |\eta(x,T)|^2 \right) \mathrm{d}x \leqslant \frac{\kappa_4}{2} \|e(x,T)\|_\Omega^2, \qquad (2.29)$$

$$\int_{S_R} \sigma \left(e\eta - \tfrac{1}{2\kappa_5} |\eta|^2 \right) \mathrm{d}s\mathrm{d}t \leqslant \frac{\kappa_5}{2} \int_0^T \|\sigma^{\frac{1}{2}} e\|_{\Gamma_R}^2 \, \mathrm{d}t. \qquad (2.30)$$

Hence, we find that

$$\mathcal{M}(e) \leqslant [e]_{(\underline{\nu},\underline{\theta},\underline{\varsigma},\underline{\chi})}^2. \qquad (2.31)$$

By means of (1.7) we rewrite $\mathcal{M}(e)$ in the form

$$\mathcal{M}(e) = \sup_{\eta \in H_0^1(Q_T)} \left\{ \sum_{i=1}^5 G_i(\eta, v, \kappa_i) + G_0(\eta, f, F, u_0) \right\},$$

and arrive at (2.23). $\qquad\qquad \square$

Remark 3. $\underline{\mathrm{M}}(v)$ vanishes if and only if v coincides with u.

§3. ESTIMATES BASED ON DOMAIN DECOMPOSITION AND LOCAL CONSTANTS

3.1. Estimates of constants in local Poincaré type inequalities.
The majorant defined in Theorem 1 contains global constants $C_{\mathrm{F}\Omega}$ and $C_{\Gamma\mathrm{R}}$. In general, finding these constants may be not an easy task (which is equivalent to deriving a guaranteed lower bound of the minimal eigenvalue for the respective differential operator). Below, we suggest the method, which allows us to overcome this difficulty. The key idea is to decompose

Ω (which may have a complicated structure) into a collection of simple sub-domains and derive such an estimate of the distance to the exact solution that uses only local constants associated with subdomains (a close method for elliptic problems is considered in [15]). We note that for the minorant $\underline{M}$ such a procedure is not required because it does not contain constants arising in functional inequalities.

Assume that

$$\overline{\Omega} := \bigcup_{\Omega_i \subset \mathcal{O}_\Omega} \overline{\Omega}_i, \quad \Omega_i \cap \Omega_j = \varnothing, \quad i \neq j, \ i,j = 1, \ldots, N, \qquad (3.1)$$

where Ω_i are convex subdomains with Lipschitz boundaries, and $\mathcal{O}_\Omega$ is the partition formed by Ω_i (in practice $\{\Omega_i\}_{i=1}^N$ are typically presented by simplicial or polyhedral cells). Henceforth, we use the notation $\Gamma_{ij} = \overline{\Omega}_i \cap \overline{\Omega}_j$, $\Gamma_{Di} = \overline{\Omega}_i \cap \Gamma_D$, and $\Gamma_{Ri} = \overline{\Omega}_i \cap \Gamma_R$.

For any Ω_i we have the classical Poincaré inequality [11]

$$\|w\|_{\Omega_i} \leqslant C_{P\Omega_i} \|\nabla w\|_{\Omega_i}, \qquad (3.2)$$

which holds for any function

$$w \in \widetilde{H}^1(\Omega_i) := \left\{ w \in H^1(\Omega_i) \big| \{w\}_{\Omega_i} = 0 \right\},$$

where $\{w\}_{\Omega_i} := \frac{1}{\Omega_i} \int_{\Omega_i} w \, \mathrm{d}x$. Due to [10], we know that $C_{P\Omega_i} \leqslant \frac{\mathrm{diam}\,\Omega_i}{\pi}$. This estimate of the Poincaré constant admits various generalizations (see, e.g., [1, 2]; similar estimates for spaces of vector-valued functions are considered in [4]).

Poincaré type estimates also hold for functions having zero mean traces on the boundary. Let

$$\widetilde{H}^1(\Omega_i, \mathcal{T}) := \left\{ w \in H^1(\Omega_i) \big| \{w\}_{\mathcal{T}} = 0 \right\}, \qquad (3.3)$$

where $\mathcal{T}$ is a measurable part of the boundary $\partial\Omega_i$, which coincides with Γ_{ij} or Γ_{Ri} (we assume that $\mathcal{T}$ has a positive measure). For any $w \in \widetilde{H}^1(\Omega_i, \mathcal{T})$, we have the estimate

$$\|w\|_{\mathcal{T}} \leqslant C_{\mathcal{T}\Omega_i} \|\nabla w\|_{\Omega_i}. \qquad (3.4)$$

Sharp values of $C_{\mathcal{T}\Omega_i}$ are found in [8] for some classes of domains. For our subsequent analysis, we need results related to the cases, where Ω_i is either a triangle or a quadrilateral and $\mathcal{T}$ is one side of it. We can extend these results to the case of $d = 3$ and Ω_i presented by parallelepiped (or

domains obtained by affine transformations of parallelepiped). Below, for the convenience of the reader, we recall some of these results.

1. If $d = 2$, Ω_i is the right quadrilateral $\Pi_2 := (0, h_1) \times (0, h_2)$, and $\mathcal{T}$ is the face $x_1 = 0$, then

$$C_{\mathcal{T}\Pi_2} = \left(\frac{\pi}{h_2} \tanh \left(\frac{\pi h_1}{h_2} \right) \right)^{\frac{1}{2}}. \qquad (3.5)$$

Analogously, if $d = 3$, $\Pi_3 := (0, h_1) \times (0, h_2) \times (0, h_3)$, and $\mathcal{T}$ is again the face defied by the condition $x_1 = 0$, then

$$C_{\mathcal{T}\Pi_3} = \left(\frac{\pi}{h_+} \tanh \left(\frac{\pi h_1}{h_+} \right) \right)^{\frac{1}{2}}, \quad h_+ = \max \{ h_2, h_3 \}. \qquad (3.6)$$

2. If $d = 2$, Ω_i is the triangle $\overline{T} := \mathrm{conv}\Big\{ (0,0), (0,h), (h,0) \Big\}$, and $\mathcal{T}$ is the leg defined by the condition $x_1 = 0$, then $C_{\mathcal{T}T} = \left(\frac{h}{\sigma_1} \right)^{\frac{1}{2}}$, where $\sigma_1 = \zeta_1 \tanh(\zeta_1)$, and ζ_1 is the unique root of the equation $\tan(z) + \tanh(z) = 0$ in $(0, \pi)$.

3. Also, we may use another result of [8] related to the case, where functions have zero mean values on the hypotenuse of the isosceles right triangle T with legs h. In this case, $C_{\mathcal{T}T} = \left(\frac{h}{2} \right)^{\frac{1}{2}}$.

By means of 2 and 3 and standard affine transformation of the coordinates, we can obtain estimates of $C_{\mathcal{T}T}$ for any non-degenerate triangle.

Proposition 3.1. *Let T be the triangle with the nodes*

$$\Big\{ (0,0), (h_1, 0), (h_2 \cos \alpha, h_2 \sin \alpha) \Big\}$$

and

$$\mathcal{T} := \Big\{ x_1 \in [0, h_1]; \;\; x_2 = 0 \Big\}.$$

Then, for any $v \in H^1(T)$ with zero mean trace on $\mathcal{T}$ we have the estimate

$$\|v\|_{\mathcal{T}} \leqslant C_{\mathcal{T}T} h_1^{\frac{1}{2}} \|\nabla v\|_T, \quad C_{\mathcal{T}T} = \widehat{C}_{\mathcal{T}T} \widehat{C}(\rho, \alpha), \qquad (3.7)$$

where

$$\widehat{C}(\rho, \alpha) = \left(\frac{\mu(\rho)}{\rho \sin \alpha} \right)^{\frac{1}{2}}, \quad \mu(\rho) = \frac{1}{2} \left(1 + \rho^2 + \left(1 + \rho^4 + 2 \cos(2\alpha) \rho^2 \right)^{\frac{1}{2}} \right), \;\; \rho = \frac{h_2}{h_1}, \tag{3.8}$$

and $\widehat{C}_{\mathcal{T}T}$ is the corresponding constant for the basic right triangle.

Remark 4. *It is clear that for the inequality (3.4), we have a certain monotonicity property, which allows us to easily estimate the constant $C_{\Gamma\Omega}$. Namely, if Ω_1 and Ω_2 have a common part Γ and $\Omega_1 \subset \Omega_2$, then*

$$\|w\| \leqslant C_{\Omega_1}\|\nabla w\|_{\Omega_1} \leqslant C_{\Omega_1}\|\nabla w\|_{\Omega_2}. \tag{3.9}$$

Therefore, $C_{\Gamma\Omega_2} \leqslant C_{\Gamma\Omega_1}$.

3.2. The first estimate. Let the sub-domains be collected into two different sets

$$\overline{\Omega}_{\mathrm{P}} := \bigcup_{\Omega_l \subset \mathcal{O}_{\mathrm{P}}} \overline{\Omega}_l, \quad \mathcal{O}_{\mathrm{P}} := \left\{ \Omega_l \subset \mathcal{O}_\Omega \mid \varrho|_{\Omega_l} \geqslant \mathrm{P}, \; l = 1, \dots, N_{\mathrm{P}} \right\} \quad \text{and}$$
$$\tag{3.10}$$
$$\overline{\Omega}_0 := \bigcup_{\Omega_k \subset \mathcal{O}_0} \overline{\Omega}_k, \quad \mathcal{O}_0 := \left\{ \Omega_k \subset \mathcal{O}_\Omega \mid \varrho|_{\Omega_k} < \mathrm{P}, \; k = 1, \dots, N_0 \right\},$$
$$\tag{3.11}$$

which contain regions with relatively large and small reaction, respectfully. For the sub-domains in $\mathcal{O}_0$, we impose an additional condition, namely,

$$\left\{ \mathbf{r}_{f,1-\mu}(v, y) \right\}_{\Omega_k \subset \mathcal{O}_0} = 0, \quad \text{for a.a.} \quad t \in [0, T]. \tag{3.12}$$

Since y is in our disposal, then selecting it in such a way that the mean value condition (3.12) holds is technically not difficult.

We impose similar local type conditions on Γ_R, which is decomposed into $\Gamma_{Rj} = \partial\Omega_j \cap \Gamma_R$, $j = 1, \dots, M$, $M \leqslant N$.

Assume that

$$\left\{ \mathbf{r}_F(v, y) \right\}_{\Gamma_{Rj} \subset \mathcal{S}_R} = 0, \quad \text{for a.a.} \quad t \in [0, T], \tag{3.13}$$

holds. Here, $\mathcal{S}_R$ denotes a collection of non-overlapping faces Γ_{Rj}.

Using the idea of Proposition 3.1, we deduce another form of the error majorant, which involves constants in Poincaré type inequalities. Henceforth, we use the following quantities based on localized residuals

$$R_{\mathcal{O}_P,\{\cdot\}}(t) := \sum_{\Omega_l \subset \mathcal{O}_P} \frac{|\Omega_l|}{P^2} \Big\{ \mathbf{r}_{f,1-\mu}(v,y) \Big\}_{\Omega_l}^2, \tag{3.14}$$

$$R_{\mathcal{O}_P,\|\cdot\|}(t) := \sum_{\Omega_l \subset \mathcal{O}_P} \frac{C_{P\Omega_l}^2}{\lambda_A} \|\mathbf{r}_{f,1-\mu}(v,y)\|_{\Omega_l}^2, \tag{3.15}$$

$$R_{\mathcal{O}_0}(t) := \sum_{\Omega_k \subset \mathcal{O}_0} \frac{C_{P\Omega_k}^2}{\lambda_A} \|\mathbf{r}_{f,1-\mu}(v,y)\|_{\Omega_k}^2, \tag{3.16}$$

$$R_{\mathcal{S}_R}(t) := \sum_{\Gamma_{Rj} \subset \mathcal{S}_R} \frac{C_{\Gamma\Omega_j}^2}{\lambda_A} \|\mathbf{r}_F(v,y)\|_{\Gamma_{Rj}}^2. \tag{3.17}$$

Theorem 3. (i) *Assume that* (3.12) *and* (3.13) *hold, then for any* $v \in V_0$ *and* $y \in Y_{\mathrm{div}}(Q_T)$, $\delta \in (0,2]$, $\rho_2(t) \geqslant 1$, $\rho_1(t) \geqslant 1$, *we have the estimate*

$$[e]_{(\nu,\theta,1,2)}^2 \leqslant \overline{\mathrm{M}}_{\mathrm{I,N}}^2(v,y;\delta,\rho_1,\rho_2,\mu)$$

$$:= \int_0^T \Big(\rho_1 \Big\| \tfrac{1}{\varrho} \mathbf{r}_{f,\mu}(v,y) \Big\|_\Omega^2 + \rho_2 R_{\mathcal{O}_P,\{\cdot\}}(t)$$

$$+ \alpha_1(t)\|\mathbf{r}_A(v,y)\|_{A^{-1},\Omega}^2 + \alpha_2(t)\Big(R_{\mathcal{O}_P,\|\cdot\|}(t)$$

$$+ R_{\mathcal{O}_0}(t) \Big) + \alpha_3(t) R_{\mathcal{S}_R}(t) \Big) \mathrm{d}t, \quad (3.18)$$

where $\mathbf{r}_f(v,y)$, $\mathbf{r}_{f,1-\mu}(v,y)$ *and* $\mathbf{r}_{f,\mu}(v,y)$, $\mathbf{r}_A(v,y)$, $\mathbf{r}_F(v,y)$ *are defined in* (2.2), (2.5), (2.3), *and* (2.4), *respectively,* $\nu = 2 - \delta$,

$$\theta(x) = \varrho(x) \left(2 - \frac{1}{\rho_1(t)} - \frac{1}{\rho_2(t)} \right)^{\frac{1}{2}}$$

are positive weights, $\mu(x,t)$ *is a real-valued function taking values in* $[0,1]$, *the reaction function* $\varrho(x) > 0$, $\alpha_1(t)$, $\alpha_2(t)$, $\alpha_3(t)$ *are positive scalar-valued functions satisfying the relation* (2.7).

(ii) *For any* $\delta \in (0,2]$, $\rho_1(t) \geqslant 1$, $\rho_2(t) \geqslant 1$, *and a real-valued function* $\mu(x,t)$ *taking values in* $[0,1]$, *the upper bound of the variation problem*

generated by the majorant

$$\inf_{\substack{v \in H_0^1(Q_T) \\ y \in Y_{\mathrm{div}}(Q_T)}} \overline{\mathrm{M}}_{\mathrm{I,N}}^2(v, y; \delta, \rho_1, \rho_2, \mu) \tag{3.19}$$

is zero, and it is attained if and only if $v = u$ and $y = A\nabla u$.

Proof. We consider (2.9) and estimate $\mathscr{I}_A$ and $\mathscr{I}_F$ analogously to the proof of Theorem 1. The term $\mathscr{I}_f$ is decomposed as follows:

$$\begin{aligned}
\mathscr{I}_f &= \int_0^T \left(\int_\Omega \mathbf{r}_{f,\mu} e \, dx + \int_\Omega \mathbf{r}_{f,1-\mu} e \, dx \right) dt \\
&= \int_0^T \left(\int_\Omega \mathbf{r}_{f,\mu} e \, dx + \int_{\Omega_P} \mathbf{r}_{f,1-\mu} e \, dx + \int_{\Omega_0} \mathbf{r}_{f,1-\mu} e \, dx \right) dt \\
&\qquad\qquad\qquad = \mathscr{I}_{f,\mu} + \mathscr{I}_{f,1-\mu}^{\mathcal{O}_P} + \mathscr{I}_{f,1-\mu}^{\mathcal{O}_0}. \tag{3.20}
\end{aligned}$$

Each term on the right-hand side of (3.20) is estimated by different methods. We use the Hölder inequality, to estimate $\mathscr{I}_{f,\mu}$. If (3.12) holds, the term $\mathscr{I}_{f,1-\mu}^{\mathcal{O}_0}$ can be estimated by (3.2)

$$|\mathscr{I}_{f,1-\mu}^{\mathcal{O}_0}| \leqslant \int_0^T R_{\mathcal{O}_0}^{\frac{1}{2}} \|\nabla e\|_{A,\Omega_0} \, dt. \tag{3.21}$$

After the representation

$$\mathscr{I}_{f,1-\mu}^{\mathcal{O}_P} = \int_0^T \left(\sum_{\Omega_l \subset \mathcal{O}_P} \int_{\Omega_l} \widetilde{\mathbf{r}}_{f,1-\mu} e \, dx + \sum_{\Omega_l \subset \mathcal{O}_P} \left\{ \mathbf{r}_{f,1-\mu} \right\}_{\Omega_l} \int_{\Omega_l} e \, dx \right) dt, \tag{3.22}$$

we estimate this term by the inequalities

$$\left| \int\limits_0^T \sum_{\Omega_l \subset \mathcal{O}_{\mathrm{P}}} \int\limits_{\Omega_l} \widetilde{\mathbf{r}}_{f,1-\mu} e \, \mathrm{d}x \, \mathrm{d}t \right| \leqslant \int\limits_0^T R_{\mathcal{O}_{\mathrm{P}},\|\cdot\|}^{\frac{1}{2}} \|\nabla e\|_{A,\Omega_{\mathrm{P}}} \, \mathrm{d}t, \qquad (3.23)$$

$$\left| \int\limits_0^T \sum_{\Omega_l \subset \mathcal{O}_{\mathrm{P}}} \left\{ \mathbf{r}_{f,1-\mu} \right\}_{\Omega_l} \int\limits_{\Omega_l} e \, \mathrm{d}x \, \mathrm{d}t \right| \leqslant \int\limits_0^T \sum_{\Omega_l \subset \mathcal{O}_{\mathrm{P}}} \frac{|\Omega_l|^{\frac{1}{2}}}{\mathrm{P}} \left\{ \mathbf{r}_{f,1-\mu} \right\}_{\Omega_l} \|\varrho e\|_{\Omega_l} \, \mathrm{d}t$$

$$\leqslant \int\limits_0^T R_{\mathcal{O}_{\mathrm{P}},\{\cdot\}}^{\frac{1}{2}} \|\varrho e\|_{\Omega} \, \mathrm{d}t. \qquad (3.24)$$

By means of the Minkowski inequality, the sum of right-hand sides of (3.21) and (3.23) is estimated as follows:

$$\int\limits_0^T R_{\mathcal{O}_0}^{\frac{1}{2}} \|\nabla e\|_{A,\Omega_0} \, \mathrm{d}t + \int\limits_0^T R_{\mathcal{O}_{\mathrm{P}},\|\cdot\|}^{\frac{1}{2}} \|\nabla e\|_{A,\Omega_{\mathrm{P}}} \, \mathrm{d}t$$

$$\leqslant \int\limits_0^T \left(R_{\mathcal{O}_0} + R_{\mathcal{O}_{\mathrm{P}},\|\cdot\|} \right)^{\frac{1}{2}} \|\nabla e\|_{A,\Omega} \, \mathrm{d}t. \quad (3.25)$$

We recall (3.13) and apply (3.4) to obtain

$$\mathscr{I}_F = \int\limits_0^T \sum_{\Gamma_{Rj} \subset \mathcal{S}_R} \int\limits_{\Gamma_{Rj}} \widetilde{\mathbf{r}}_F e \, \mathrm{d}s \, \mathrm{d}t \leqslant \int\limits_0^T R_{\mathcal{S}_R}^{\frac{1}{2}} \|\nabla e\|_{A,\Omega} \, \mathrm{d}t. \qquad (3.26)$$

In view of the Young–Fenchel inequalities, we have

$$\int\limits_0^T \|\mathbf{r}_A\|_{A^{-1},\Omega} \|\nabla e\|_{A,\Omega} \, \mathrm{d}t \leqslant \tfrac{1}{2} \int\limits_0^T \left(\alpha_1(t) \|\mathbf{r}_A\|_{A^{-1},\Omega}^2 + \tfrac{1}{\alpha_1(t)} \|\nabla e\|_{A,\Omega}^2 \right) \mathrm{d}t,$$

$$(3.27)$$

$$\int\limits_0^T \left\| \tfrac{1}{\varrho} \mathbf{r}_{f,\mu} \right\|_{\Omega} \|\varrho e\|_{\Omega} \, \mathrm{d}t \leqslant \tfrac{1}{2} \int\limits_0^T \left(\rho_1(t) \left\| \tfrac{1}{\varrho} \mathbf{r}_{f,\mu} \right\|_{\Omega}^2 + \tfrac{1}{\rho_1(t)} \|\varrho e\|_{\Omega}^2 \right) \mathrm{d}t,$$

$$(3.28)$$

$$\int_0^T R_{\mathcal{O}_P,\{\cdot\}}^{\frac{1}{2}} \|\varrho e\|_\Omega \, \mathrm{d}t \leqslant \frac{1}{2} \int_0^T \left(\rho_2(t) R_{\mathcal{O}_P,\{\cdot\}} + \frac{1}{\rho_2(t)} \|\varrho e\|_\Omega^2 \right) \mathrm{d}t, \quad (3.29)$$

$$\int_0^T R_{\mathcal{S}_R}^{\frac{1}{2}} \|\nabla e\|_{A,\Omega} \, \mathrm{d}t \leqslant \frac{1}{2} \int_0^T \left(\alpha_3(t) R_{\mathcal{S}_R}(t) + \frac{1}{\alpha_3(t)} \|\nabla e\|_{A,\Omega}^2 \right) \mathrm{d}t,$$
$$(3.30)$$

and

$$\int_0^T \left(R_{\mathcal{O}_P,\|\cdot\|} + R_{\mathcal{O}_0} \right)^{\frac{1}{2}} \|\nabla e\|_{A,\Omega} \, \mathrm{d}t$$

$$\leqslant \frac{1}{2} \int_0^T \left(\alpha_2(t) \left(R_{\mathcal{O}_P,\|\cdot\|} + R_{\mathcal{O}_0} \right) + \frac{1}{\alpha_2(t)} \|\nabla e\|_{A,\Omega}^2 \right) \mathrm{d}t. \quad (3.31)$$

By combining (3.27)–(3.31), we arrive at (3.18).

(ii) Existence of the pair $(v, y) \in H_0^1(Q_T) \times Y_{\mathrm{div}}^*(Q_T)$ minimizing the functional $\overline{\mathrm{M}}_{\mathrm{I,N}}^2(v, y; \delta, \rho_1, \rho_2, \mu)$ can be proven straightforwardly. Indeed, let $v = u$ and $y = A\nabla u$. Since $\mathrm{div}\,(A\nabla u) \in L^2(Q_T)$, we see that $y \in Y_{\mathrm{div}}^*(Q_T)$. In this case (cf. (1.1)–(1.5)),

$$\begin{aligned}
e(x, 0) &= (u - v)(x, 0) = u_0(x) - v(x, 0) = 0, \\
\mathbf{r}_f(u, A\nabla u) &= f - u_t - \varrho^2 u + \mathrm{div}\, A\nabla u = 0, \\
\mathbf{r}_A(u, A\nabla u) &= A\nabla u - A\nabla u = 0, \\
\mathbf{r}_F(u, A\nabla u) &= F - \sigma^2 v - A\nabla u \cdot n = 0,
\end{aligned}$$

Thus, we see that $\overline{\mathrm{M}}_{\mathrm{I,N}} = 0$. Since the majorant is nonnegative, the functions u and $A\nabla u$ minimize it.

Assume that $\overline{\mathrm{M}}_{\mathrm{I,N}}^2 = 0$. Then, the following relations hold:

$$y = A\nabla v \quad \text{a.a.} \quad (x, t) \in Q_T, \quad\quad\quad\quad\quad (3.32)$$

$$f - v_t - \varrho^2 v + \mathrm{div}\, y = 0 \quad \text{a.a.} \quad (x, t) \in \Omega_i \times (0, T), \quad \Omega_i \subset \mathcal{O}_\Omega, \quad (3.33)$$

$$v(\cdot, 0) = u_0 \quad \text{a.a.} \quad x \in \Omega, \quad\quad\quad\quad\quad (3.34)$$

$$v = 0 \quad \text{a.a.} \quad (x, t) \in S_D, \quad\quad\quad\quad\quad (3.35)$$

$$y \cdot n + \sigma^2 v = F \quad \text{a.a.} \quad (x, t) \in \Gamma_{Rj} \times (0, T), \quad \Gamma_{Rj} \subset \mathcal{S}_R. \quad (3.36)$$

From (3.33)–(3.36), it follows that for any $\eta \in H_0^1(Q_T)$

$$\int\limits_0^T \sum_{\Omega_i \subset \mathcal{O}_\Omega} \int\limits_{\Omega_i} \left((f - v_t - \varrho^2 v)\eta - y \cdot \nabla\eta \right) \mathrm{d}x + \int\limits_0^T \sum_{\Gamma_{Rj} \subset \mathcal{S}_R} \int\limits_{\Gamma_{Rj}} F\eta \, \mathrm{d}s \mathrm{d}t = 0,$$

or, equally,

$$\int\limits_{Q_T} \left((f - v_t - \varrho^2 v)\eta - y \cdot \nabla\eta \right) \mathrm{d}x + \int\limits_{S_R} F\eta \, \mathrm{d}s \mathrm{d}t = 0, \quad \forall \eta \in H_0^1(Q_T). \quad (3.37)$$

In view of (3.32), the identity (3.37) is equivalent to (1.7), whence it follows that $v = u$ and $y = A\nabla u$.

We conclude that the exact lower bound of $\overline{\mathrm{M}}_{\mathrm{I,N}}^2$ is equal to zero and it is attained only on the pair (v, y), which presents the exact solution of (1.1)–(1.5) and the respective flux. $\qquad\square$

3.3. Equivalence of $\overline{\mathrm{M}}_{\mathrm{I,N}}$ and the primal–dual error norm. Now, we are aimed to show that the majorant is equivalent to the error measure in terms of a combined (primal-dual) norm. This fact justifies the majorant as an adequate tool of error control.

Consider the solution of (1.1)–(1.5) as a pair $(u, p) \in V_0 \times Y_{\mathrm{div}}(Q_T)$. In order to measure the deviation of the approximation $(v, y) \in V_0 \times Y_{\mathrm{div}}(Q_T)$ from (u, p), we use the following form of combined primal-dual norm

$$\| [(u, p) - (v, y)] \|_{(\check\nu, \check\theta, \check\zeta, \check\kappa, \check\chi, \check\vartheta, \check\varpi)}^2$$

$$:= \check\nu \int\limits_0^T \|\nabla u - v\|_{A,\Omega}^2 \, \mathrm{d}t + \check\kappa \int\limits_0^T \|\varrho(u - v)\|_\Omega^2 \, \mathrm{d}t + \check\chi \|(u - v)(\cdot, T)\|_\Omega^2$$

$$+ \check\theta \int\limits_0^T \|y - p\|_{A^{-1},\Omega}^2 \, \mathrm{d}t + \check\zeta \int\limits_0^T \|\mathrm{div}\,(p - y) - (u - v)_t\|_\Omega^2 \, \mathrm{d}t$$

$$+ \check\vartheta \int\limits_0^T \|\sigma(u - v)\|_{\Gamma_R}^2 \, \mathrm{d}t + \check\varpi \int\limits_0^T \|(p - y) \cdot n\|_{\Gamma_R}^2 \, \mathrm{d}t. \quad (3.38)$$

It is easy to see that the first three terms of (3.38) present an energy norm of the error in the primal variable, the forth can be viewed as an error associated with the flux. The fifth term is generated by both errors in primal

and dual variables. The last two terms are related to errors in boundary conditions. For simplicity, further (3.38) is used as $\|[(u,p)-(v,y)]\|^2$.

From Theorem 1 (with α_1, α_2, $\alpha_3 = $ const, $\mu = 0$, and exactly satisfied initial condition $u_0 = v(\cdot, 0)$), the estimate can be written in the form

$$
(2-\delta)\int_0^T \|\nabla e\|_{A,\Omega}^2 + \left(2 - \tfrac{1}{\gamma}\right)\int_0^T \|\varrho e\|_\Omega^2 + \|e(x,T)\|_\Omega^2
$$

$$
+ 2\int_0^T \|\sigma e\|_{\Gamma_R}^2 \, \mathrm{d}t \leqslant \overline{\mathrm{M}}_{\mathrm{I,N}}^2(v,y) := \gamma \int_0^T R_{\mathcal{O}_{\mathrm{P}},\{\cdot\}} \, \mathrm{d}t
$$

$$
+ \alpha_1 \int_0^T \|\mathbf{r}_A\|_{A^{-1},\Omega}^2 \, \mathrm{d}t + \alpha_2 \int_0^T \left(R_{\mathcal{O}_{\mathrm{P}},\|\cdot\|} + R_{\mathcal{O}_0}\right) \mathrm{d}t + \alpha_3 \int_0^T R_{\mathcal{S}_R} \, \mathrm{d}t. \quad (3.39)
$$

Set

$$
C_{\Omega\mathrm{P}} := \max_{\Omega_l \subset \mathcal{O}_{\mathrm{P}}} \left\{ \frac{|\Omega_l|}{\mathrm{P}^2} \right\}, \qquad\qquad \overline{C}_{\mathrm{P}\Omega} := \max_{\Omega_i \subset \mathcal{O}_\Omega} \left\{ C_{\mathrm{P}\Omega i} \right\}, \qquad (3.40)
$$

$$
\overline{C}_{\Gamma\Omega} := \max_{\Gamma_{Rj} \subset \mathcal{S}_R} \left\{ C_{\Gamma\Omega j} \right\}, \qquad C_{\varrho,\alpha} := \max\{\gamma, \alpha_2\}, \qquad (3.41)
$$

then

$$
\overline{\mathrm{M}}_{\mathrm{I,N}}^2 \leqslant \|e(\cdot, T)\|_\Omega^2 + \alpha_1 \int_0^T \|y - A\nabla v\|_{A^{-1},\Omega}^2 \, \mathrm{d}t
$$

$$
+ \frac{C_{\gamma,\alpha}}{\underline{\lambda}_A} \max\left\{ \overline{C}_{\mathrm{P}\Omega}^2, C_{\Omega\mathrm{P}} \right\} \int_0^T \|f - v_t + \mathrm{div}\, y - \varrho^2 v\|_\Omega^2 \, \mathrm{d}t
$$

$$
+ \alpha_3 \frac{\overline{C}_{\Gamma\Omega}^2}{\underline{\lambda}_A} \int_0^T \|F - \sigma^2 v - y \cdot n\|_{\Gamma_R}^2 \, \mathrm{d}t. \quad (3.42)
$$

For further simplification, let

$$
C_{\max} := \frac{C_{\gamma,\alpha}}{\underline{\lambda}_A} \max\left\{ \overline{C}_{\mathrm{P}\Omega}^2, C_{\Omega\mathrm{P}} \right\}, \qquad C_{\alpha_3 \Gamma} := \alpha_3 \frac{\overline{C}_{\Gamma\Omega}^2}{\underline{\lambda}_A}. \qquad (3.43)
$$

By means of (1.1) and (1.5), the right-hand side of (3.42) can be decomposed as follows:

$$\overline{M}_{I,N}^2 \leqslant \|e(\cdot,T)\|_{\Omega}^2 + \alpha_1 \left(\int_0^T \|y-p\|_{A^{-1},\Omega}^2 \, dt + \int_0^T \|\nabla(u-v)\|_{A,\Omega}^2 \, dt \right)$$

$$+ C_{\max} \Big(\int_0^T \|\mathrm{div}\,(y-p) + (u-v)_t\|_{\Omega}^2 \, dt + \int_0^T \left\|\varrho^2(u-v)\right\|_{\Omega}^2 \, dt \Big)$$

$$+ C_{\alpha_3\Gamma} \left(\int_0^T \left\|\sigma^2(u-v)\right\|_{\Gamma_R}^2 \, dt + \int_0^T \|(p-y)\cdot n\|_{\Gamma_R}^2 \, dt \right).$$

We recall that ϱ and σ are uniformly bounded by the constants C_ϱ and C_σ, respectively, and estimate the right-hand side of the latter inequality as

$$\overline{M}_{I,N}^2 \leqslant \|e(\cdot,T)\|_{\Omega}^2 + \alpha_1 \left(\int_0^T \|y-p\|_{A^{-1},\Omega}^2 \, dt + \int_0^T \|\nabla(u-v)\|_{A,\Omega}^2 \, dt \right)$$

$$+ C_{\max} \left(\int_0^T \|\mathrm{div}\,(y-p) + (u-v)_t\|_{\Omega}^2 \, dt + C_\varrho^2 \int_0^T \|\varrho(u-v)\|_{\Omega_P}^2 \, dt \right)$$

$$+ C_{\alpha_3\Gamma} \left(C_\sigma^2 \int_0^T \|\sigma(u-v)\|_{\Gamma_R}^2 \, dt + \int_0^T \|(p-y)\cdot n\|_{\Gamma_R}^2 \, dt \right)$$

$$=: \left\| [(u,p)-(v,y)] \right\|_{(\check{\nu},\check{\theta},\check{\zeta},\check{\kappa},\check{\chi},\check{\vartheta},\check{\varpi})}^2. \quad (3.44)$$

Here, on the right-hand side we have the error measured in terms of (3.38) with positive weights

$$\check{\nu} = \check{\theta} = \alpha_1, \quad \check{\zeta} = C_{\max}, \quad \check{\kappa} = C_\varrho^2 C_{\max},$$
$$\check{\chi} = 1, \quad \check{\vartheta} = C_\sigma^2 C_{\alpha_3\Gamma}, \quad \check{\varpi} = C_{\alpha_3\Gamma}. \quad (3.45)$$

Next, we combine four terms related to the energy error norm of the primal variable on the right-hand side of (3.44) and estimate it by using (3.39). The rest of the terms related to dual component can be estimated by the

technique used above. Therefore, we obtain

$$\|[(u,p)-(v,y)]\|^2 \leqslant C_{\mathrm{ER}}\overline{\mathrm{M}}_{\mathrm{I,N}}^2$$

$$+ \alpha_1 \left(\int\limits_0^T \|y - A\nabla v\|_{A^{-1},\Omega}^2 \, \mathrm{d}t + \int\limits_0^T \|\nabla(u-v)\|_{A,\Omega}^2 \, \mathrm{d}t \right)$$

$$+ C_{\max}\left(\int\limits_0^T \|f + \operatorname{div} y - v_t - \varrho^2 v\|_\Omega^2 \, \mathrm{d}t + C_\varrho^2 \int\limits_0^T \|\varrho(u-v)\|_\Omega^2 \, \mathrm{d}t \right)$$

$$+ C_{\alpha_3\Gamma}\left(\int\limits_0^T \|F - \sigma^2 v - y\cdot n\|_{\Gamma_R}^2 \, \mathrm{d}t + C_\sigma^2 \int\limits_0^T \|\sigma(v-u)\|_{\Gamma_R}^2 \, \mathrm{d}t \right), \quad (3.46)$$

where

$$C_{\mathrm{ER}} = \max\left\{ \frac{\alpha_1}{(2-\delta)}, \quad \frac{\gamma}{(2\gamma-1)}C_\varrho^2 C_{\max}, \quad 1, \quad \tfrac{1}{2}C_\sigma^2 C_{\alpha_3\Gamma} \right\}. \qquad (3.47)$$

By using constants

$$\underline{C}_{\mathrm{P\Omega}} := \min_{\Omega_i \subset \mathcal{O}_\Omega}\{C_{\mathrm{P\Omega}\,i}\}, \quad \text{and} \quad \widetilde{C}_{\Gamma\Omega} := \frac{\overline{C}_{\Gamma\Omega}^2}{\underline{C}_{\Gamma\Omega}^2},$$

where

$$\underline{C}_{\Gamma\Omega} := \min_{\Gamma_{Rj}\subset\mathcal{S}_R}\{C_{\Gamma\Omega\,j}\},$$

we rewrite the right-hand side of (3.46) and obtain the following result:

$$\|[(u,p)-(v,y)]\|^2 \leqslant \alpha_1 \int\limits_0^T \|\nabla(u-v)\|_{A,\Omega}^2 \, \mathrm{d}t$$

$$+ C_\varrho^2 C_{\max} \int\limits_0^T \|\varrho(u-v)\|_\Omega^2 \, \mathrm{d}t + \|e(\cdot,T)\|_\Omega^2$$

$$+ C_\sigma^2 C_{\alpha_3\Gamma} \int\limits_0^T \|\sigma(v-u)\|_{\Gamma_R}^2 \, \mathrm{d}t + C_{\mathrm{ER}}\overline{\mathrm{M}}_{\mathrm{I,N}}^2 + \alpha_1 \int\limits_0^T \|y - A\nabla v\|_{A^{-1},\Omega}^2 \, \mathrm{d}t$$

$$+ \frac{C_{\max}}{\underline{C}_{\mathrm{P\Omega}}^2}\left(\int\limits_0^T R_{\mathcal{O}_{\mathrm{P}},\|\cdot\|} \, \mathrm{d}t + \int\limits_0^T R_{\mathcal{O}_0} \, \mathrm{d}t \right) + \alpha_3 \widetilde{C}_{\Gamma\Omega} \int\limits_0^T R_{\mathcal{S}_R} \, \mathrm{d}t. \quad (3.48)$$

Finally, the terms related to the error norm of primal component on the right-hand side of (3.48) can be estimated by the majorant (3.39):

$$
\left\|[(u,p)-(v,y)]\right\|^2 \leqslant \int_0^T \Bigg(\alpha_1 \left(2C_{\mathrm{ER}}+1\right)\|y-A\nabla v\|^2_{A^{-1},\Omega} + 2C_{\mathrm{ER}}\gamma R_{\mathcal{O}_{\mathrm{P}},\{\cdot\}}
$$

$$
+ \alpha_2 \left(2C_{\mathrm{ER}} + \tfrac{C_{\max}}{\alpha_2 \underline{C}^2_{\mathrm{P}\Omega}}\right)\left(R_{\mathcal{O}_{\mathrm{P}},\|\cdot\|} + R_{\mathcal{O}_0}\right)
$$

$$
+ \alpha_3 \left(2C_{\mathrm{ER}} + \widetilde{C}_{\Gamma\Omega}\right) R_{\mathcal{S}_R} \Bigg) \mathrm{d}t \leqslant C_{\mathrm{MAJ}}\overline{\mathrm{M}}^2_{\mathrm{I,N}},
$$

where

$$
C_{\mathrm{MAJ}} = \max\left\{2C_{\mathrm{ER}}+1,\quad 2C_{\mathrm{ER}},\quad 2C_{\mathrm{ER}}+\tfrac{C_{\max}}{\alpha_2\underline{C}^2_{\mathrm{P}\Omega}},\quad 2C_{\mathrm{ER}}+\widetilde{C}_{\Gamma\Omega}\right\}.
$$
(3.49)

Therefore, we obtain the double inequality

$$
\overline{\mathrm{M}}^2_{\mathrm{I,N}} \leqslant \left\|[(u,p)-(v,y)]\right\|^2_{(\check{\nu},\check{\theta},\check{\zeta},\check{\kappa},\check{\chi},\check{\vartheta})} \leqslant C_{\mathrm{MAJ}}\overline{\mathrm{M}}^2_{\mathrm{I,N}}, \tag{3.50}
$$

which shows that the majorant introduced in Theorem 3 is equivalent to a certain form of combined (primal-dual) error norm. In other words, $\overline{\mathrm{M}}^2_{\mathrm{I,N}}$ (which contains only known functions and parameters) adequately reflects the distance between $(v,y) \in V_0 \times Y_{\mathrm{div}}(Q_T)$ and the exact solution (u,p). In particular, this means that if (u_h, p_h) is the sequence of approximations computed on a certain set of meshes $\mathcal{F}_h$, which converges to (u,p) with the rate h^α, then the values of the majorant tend to zero with the same rate.

3.4. The second estimate. Now, we deduce another estimate, which is in general sharper than (3.18), but contains an additional free function $w \in H_0^1(Q_T)$. The corresponding residuals of (1.2), (1.1), and (1.5) are presented as

$$
\mathbf{r}_f(v,y,w) := f - (v+w)_t - \varrho^2(v-w) + \mathrm{div}\, y, \tag{3.51}
$$

$$
\mathbf{r}_{f,\mu}(v,y,w) := \mu\mathbf{r}_f(v,y,w), \tag{3.52}
$$

$$
\mathbf{r}_{f,1-\mu}(v,y,w) := (1-\mu)\mathbf{r}_f(v,y,w), \tag{3.53}
$$

$$
\mathbf{r}_A(v,y,w) := y - A\nabla(v-w), \tag{3.54}
$$

$$
\mathbf{r}_F(v,y,w) := F - \sigma^2(v-w) - y\cdot n, \tag{3.55}
$$

respectively. On collections $\mathcal{O}_0$ and $\mathcal{S}_R$, we impose the mean conditions similar to (3.12) and (3.13), namely,

$$\left\{ \mathbf{r}_{f,1-\mu}(v,y,w) \right\}_{\Omega_k \subset \mathcal{O}_0} = 0, \quad \text{for a.a.} \quad t \in [0,T], \qquad (3.56)$$

and

$$\left\{ \mathbf{r}_F(v,y,w) \right\}_{\Gamma_{R_j} \subset \mathcal{S}_R} = 0, \quad \text{for a.a.} \quad t \in [0,T]. \qquad (3.57)$$

Correspondingly, the complexes $R_{\mathcal{O}_\mathrm{P},\{\mathbf{r}_{f,1-\mu}\}}(t)$, $R_{\mathcal{O}_\mathrm{P},\|\mathbf{r}_{f,1-\mu}\|}(t)$, $R_{\mathcal{O}_0}(t)$, $R_{\mathcal{S}_R}(t)$ are defined analogously (3.14)–(3.17) and depend on residuals (3.51)–(3.55) related to free functions v, y, and w.

Theorem 4. (i) *Assume that conditions* (3.56) *and* (3.57) *are satisfied. Then, for any* $v \in V_0$, $w \in H_0^1(Q_T)$ *and* $y \in Y_{\mathrm{div}}(Q_T)$, $\delta \in (0,2]$, $\epsilon \geqslant 1$, $\rho_2(t) \geqslant 1$, $\rho_1(t) \geqslant 1$, *the error has the following estimate:*

$$[e]^2_{(\nu,\theta,\zeta,2)} \leqslant \overline{\mathrm{M}}^2_{\mathrm{II,N}}(v,y,w;\delta,\epsilon,\rho_1,\rho_2,\mu)$$
$$:= \epsilon\|w(x,T)\|^2_\Omega + 2L(v,w) + l(v,w)$$
$$+ \int_0^T \Bigg(\rho_1(t)\left\|\tfrac{1}{\varrho}\mathbf{r}_{f,\mu}(v,y,w)\right\|^2_{\Omega_\mathrm{P}} + \rho_2(t)R_{\mathcal{O}_\mathrm{P},\{\cdot\}}(t) + \alpha_1(t)\|\mathbf{r}_A(v,y,w)\|^2_{A^{-1},\Omega}$$
$$+ \alpha_2(t)\big(R_{\mathcal{O}_\mathrm{P},\|\cdot\|}(t) + R_{\mathcal{O}_0}(t)\big) + \alpha(t)R_{\mathcal{S}_R}(t) \Bigg)\mathrm{d}t, \quad (3.58)$$

$$L(v,w) := \int_{Q_T} \left(v_t w + A\nabla v \cdot \nabla w + \varrho^2 vw - fw \right) \mathrm{d}x\mathrm{d}t - \int_{S_R} (F - \sigma^2 v)w\,\mathrm{d}s\mathrm{d}t,$$
$$(3.59)$$

$$l(v,w) := \int_\Omega |v(x,0) - u_0(x)|^2 - 2w(x,0)\big(u_0(x) - v(0,x)\big)\,\mathrm{d}x, \quad (3.60)$$

$\nu = 2 - \delta$, $\theta(x,t) = \varrho(x)\left(2 - \frac{1}{\rho_1(t)} - \frac{1}{\rho_2(t)}\right)^{\frac{1}{2}}$, $\zeta = 1 - \frac{1}{\epsilon}$ *are positive parameters,* $\mu(x,t) \in [0,1]$ *is real-valued function,* $\alpha_1(t)$, $\alpha_2(t)$, $\alpha_3(t)$ *are positive functions satisfying* (2.7).

(ii) *For any $\delta \in (0, 2]$, $\rho_1(t) \geqslant \frac{1}{\left(2 - \frac{1}{\rho_2}\right)}$, $\rho_2 \geqslant 1$, $\epsilon \geqslant 1$, and $\mu \in [0, 1]$, the lower bound of the variation problem generated by the majorant*

$$\inf_{\substack{v \in V_0 \\ w \in H_0^1(Q_T) \\ y \in Y_{\mathrm{div}}(Q_T)}} \overline{\mathrm{M}}_{\mathrm{II,N}}^2(v, y, w; \delta, \epsilon, \rho_1, \rho_2, \mu) \tag{3.61}$$

is zero, and it is attained if and only if $v = u$, $y = A\nabla u$, and $w = 0$.

Proof. (i) We rewrite the right-hand side of (2.1) by inserting functions $w \in H_0^1(Q_T)$ and $y \in Y_{\mathrm{div}}(Q_T)$, which implies the following relation

$$\int_0^T \|\nabla e\|_{A,\Omega}^2 \, \mathrm{d}t + \int_0^T \|\varrho e\|_\Omega^2 \, \mathrm{d}t + \tfrac{1}{2}\|e(\cdot, T)\|_\Omega^2 + \int_0^T \|\sigma e\|_{\Gamma_R}^2 \, \mathrm{d}t$$

$$= \int_\Omega e(x, T) w(x, T) \, \mathrm{d}x + \int_\Omega \left(\tfrac{1}{2}e^2(x, 0) - e(x, 0) w(x, 0)\right) \, \mathrm{d}x$$

$$+ \mathscr{I}_1 + \mathscr{I}_2 + \mathscr{I}_3 + L(v, w), \quad (3.62)$$

where $\mathscr{I}_f$, $\mathscr{I}_A$, $\mathscr{I}_F$ are quite analogous to (2.10)–(2.12) and depend on (3.51)–(3.55). As in the proof of Theorem 3, the term $\mathscr{I}_f$ can be represented as

$$\mathscr{I}_f = \mathscr{I}_{f,\mu} + \mathscr{I}_{f,1-\mu}^{\mathcal{O}_{\mathrm{P}}} + \mathscr{I}_{f,1-\mu}^{\mathcal{O}_0}, \tag{3.63}$$

where each of the summands is estimated as follows:

$$|\mathscr{I}_{f,\mu}| \leqslant \int_0^T \left\|\tfrac{1}{\varrho}\mathbf{r}_{f,\mu}\right\|_\Omega \|\varrho e\|_\Omega \, \mathrm{d}t, \tag{3.64}$$

$$|\mathscr{I}_{f,1-\mu}^{\mathcal{O}_{\mathrm{P}}}| \leqslant \int_0^T R_{\mathcal{O}_{\mathrm{P}},\|\cdot\|}^{\frac{1}{2}} \|\nabla e\|_{A,\Omega_{\mathrm{P}}} \, \mathrm{d}t + \int_0^T R_{\mathcal{O}_{\mathrm{P}},\{\cdot\}}^{\frac{1}{2}} \|\varrho e\|_\Omega \, \mathrm{d}t, \tag{3.65}$$

$$|\mathscr{I}_{f,1-\mu}^{\mathcal{O}_0}| \leqslant \int_0^T R_{\mathcal{O}_0}^{\frac{1}{2}} \|\nabla e\|_{A,\Omega_0} \, \mathrm{d}t. \tag{3.66}$$

The term $\mathscr{I}_A$ is estimated by the Hölder inequality, and $\mathscr{I}_F$ is bounded analogously to (3.26). By applying Yong's inequalities, we obtain the estimates

$$\int_\Omega e(x,T)w(x,T)\,\mathrm{d}x\mathrm{d}t \leqslant \tfrac{1}{2}\left(\tfrac{1}{\epsilon}\|e(\cdot,T)\|_\Omega^2 + \epsilon\|w(\cdot,T)\|_\Omega^2\right), \qquad (3.67)$$

$$\int_0^T \left\|\tfrac{1}{\varrho}\mathbf{r}_{f,\mu}\right\|_\Omega \|\varrho e\|_\Omega \,\mathrm{d}t \leqslant \tfrac{1}{2}\int_0^T \left(\rho_1\left\|\tfrac{1}{\varrho}\mathbf{r}_{f,\mu}\right\|_\Omega^2 + \tfrac{1}{\rho_1}\|\varrho e\|_\Omega^2\right)\mathrm{d}t, \qquad (3.68)$$

$$\int_0^T R_{\mathcal{O}_\mathrm{P},\{\cdot\}}^{\frac{1}{2}}\|\varrho e\|_\Omega \,\mathrm{d}t \leqslant \tfrac{1}{2}\int_0^T \left(\rho_2 R_{\mathcal{O}_\mathrm{P},\{\cdot\}} + \tfrac{1}{\rho_2}\|\varrho e\|_\Omega^2\right)\mathrm{d}t, \qquad (3.69)$$

$$\int_0^T \|\mathbf{r}_A\|_{A^{-1},\Omega}\|\nabla e\|_{A,\Omega} \,\mathrm{d}t \leqslant \tfrac{1}{2}\int_0^T \left(\alpha_1(t)\|\mathbf{r}_A\|_{A^{-1},\Omega}^2 + \tfrac{1}{\alpha_1(t)}\|\nabla e\|_{A,\Omega}^2\right)\mathrm{d}t,$$
$$(3.70)$$

$$\int_0^T R_{\mathcal{S}_R}^{\frac{1}{2}}\|\nabla e\|_{A,\Omega} \,\mathrm{d}t \leqslant \tfrac{1}{2}\int_0^T \left(\alpha_4(t)R_{\mathcal{S}_R} + \tfrac{1}{\alpha_4(t)}\|\nabla e\|_{A,\Omega}^2\right)\mathrm{d}t, \quad (3.71)$$

and

$$\int_0^T \left(R_{\mathcal{O}_\mathrm{P},\|\cdot\|} + R_{\mathcal{O}_0}\right)^{\frac{1}{2}}\|\nabla e\|_{A,\Omega} \,\mathrm{d}t$$

$$\leqslant \tfrac{1}{2}\int_0^T \left(\alpha_3(t)\left(R_{\mathcal{O}_\mathrm{P},\|\cdot\|} + R_{\mathcal{O}_0}\right) + \tfrac{1}{\alpha_3(t)}\|\nabla e\|_{A,\Omega}^2\right)\mathrm{d}t. \quad (3.72)$$

By combining (3.67)–(3.71), we obtain the required estimate (3.58).

(ii) The proof is similar to the proof of (ii) in Theorem 3. $\square$

3.5. Equivalence of $\overline{\mathrm{M}}_{\mathrm{II,N}}$ and $[e]_{(\nu,\theta,\zeta,\chi)}$. Finally, we prove that $\overline{\mathrm{M}}_{\mathrm{II,N}}^2$ is equivalent to the error measure (1.8). For this purpose, we estimate (3.58) from above and show that this upper bound is equivalent to the

error norm. Henceforth, we assume that $\mu = 0$ (this is done for the sake of simplicity only), $y = A\nabla u \in Y_{\mathrm{div}}(Q_T)$, and $w = e$, then

$$\mathbf{r}_f(v, A\nabla u, e) = 2\varrho^2 e, \quad \mathbf{r}_A(v, A\nabla u, e) = 2A\nabla e, \quad \mathbf{r}_F(v, A\nabla u, e) = 2\sigma^2 e.$$

The functional (3.59) can be represented as follows:

$$
\begin{aligned}
L(v, e) &= \int\limits_{Q_T} \left(v_t e + A\nabla v \cdot \nabla e + \varrho^2 v e - f e \right) \mathrm{d}x\mathrm{d}t - \int\limits_{S_R} (F - \sigma^2 v) e\, \mathrm{d}s\mathrm{d}t \\
&= \int\limits_{Q_T} \left(u_t e + A\nabla u \cdot \nabla e + \varrho^2 u e - f e \right) \mathrm{d}x\mathrm{d}t + \int\limits_{S_R} (F - \sigma^2 u) e\, \mathrm{d}s\mathrm{d}t \\
&\quad - \int\limits_{Q_T} \left(A\nabla e \cdot \nabla e + e_t e + \varrho^2 e^2 \right) \mathrm{d}x\mathrm{d}t - \int\limits_{S_R} \sigma^2 e^2\, \mathrm{d}s\mathrm{d}t. \quad (3.73)
\end{aligned}
$$

In view of (1.7), the first two terms in the right-hand side of (3.73) vanishes, and we find that

$$L(v, e) = - \int\limits_{Q_T} \left(A\nabla e \cdot \nabla e + e_t e + \varrho^2 e^2 \right) \mathrm{d}x\mathrm{d}t - \int\limits_{S_R} \sigma^2 e^2\, \mathrm{d}s\mathrm{d}t. \quad (3.74)$$

Next,

$$
\begin{aligned}
l(v, e) &= \int\limits_{\Omega} \left(|v(x, 0) - u_0(x)|^2 - 2e(x, 0)\big(u_0(x) - v(0, x)\big) \right)\, \mathrm{d}x \\
&= -\|e(x, 0)\|_{\Omega}^2. \quad (3.75)
\end{aligned}
$$

By means of differentiation by part and (3.75), we obtain the estimate

$$
\begin{aligned}
\overline{\mathrm{M}}_{\mathrm{II,N}}^2 &\leqslant (4\alpha_2 - 2) \int\limits_0^T \|\nabla e\|_{A,\Omega}^2\, \mathrm{d}t + 4\alpha_1 \int\limits_0^T \sum_{\Omega_l \subset \mathcal{O}_{\mathrm{P}}} \frac{C_{\mathrm{P}\Omega_l}^2}{\underline{\lambda}_A} \|\varrho e\|_{\Omega_l}^2\, \mathrm{d}t \\
&\quad - 2 \int\limits_0^T \|\varrho e\|_{\Omega}^2\, \mathrm{d}t + 4\alpha_3 \int\limits_0^T \sum_{\Gamma_{Rj} \subset \mathcal{S}_R} \frac{C_{\Gamma\Omega_j}^2}{\underline{\lambda}_A} \|\sigma e\|_{\Gamma_{Rj_m}}^2\, \mathrm{d}t \\
&\quad - 2 \int\limits_0^T \|\sigma e\|_{\Gamma_R}^2\, \mathrm{d}t + \epsilon \|e(\cdot, T)\|_{\Omega}^2 - 2 \int\limits_{Q_T} e_t e\, \mathrm{d}x\mathrm{d}t - \|e(x, 0)\|_{\Omega}^2
\end{aligned}
$$

$$\leqslant 2\left(2\alpha_2 - 1\right) \int\limits_0^T \|\nabla e\|_{A,\Omega}^2 \, \mathrm{d}t + 2\left(2\alpha_1 \frac{\overline{C}_{\mathrm{P}\Omega}^2}{\underline{\lambda}_A} - 1\right) \int\limits_0^T \|\varrho e\|_\Omega^2$$

$$+ 2\left(2\alpha_3 \frac{\overline{C}_{\Gamma\Omega}^2}{\underline{\lambda}_A} - 1\right) \int\limits_0^T \|\sigma e\|_{\Gamma_R}^2 + (\epsilon - 1)\|e(\cdot, T)\|_\Omega^2, \quad (3.76)$$

where $\overline{C}_{\mathrm{P}\Omega} = \max\limits_{\Omega_l \subset \mathcal{O}_{\mathrm{P}}} \{C_{\mathrm{P}\Omega l}\}$, $\overline{C}_{\Gamma\Omega} = \max\limits_{\Gamma_{Rj} \subset R} \{C_{\Gamma\Omega j}\}$. Therefore, for any $v \in V_0$ we arrive at double inequality

$$[e]_{(\nu,\theta,\zeta,\chi)}^2 \leqslant \overline{\mathrm{M}}_{\mathrm{II,N}}^2 \leqslant [e]_{(\nu',\theta',\zeta',\chi')}^2 \leqslant \mathcal{K}[e]_{(\nu,\theta,\zeta,\chi)}^2, \quad (3.77)$$

with parameters

$$\nu' = 2\left(2\alpha_2 - 1\right), \quad \theta' = \varrho\left(2\left(2\alpha_1 \frac{\overline{C}_{\mathrm{P}\Omega}^2}{\underline{\lambda}_A} - 1\right)\right)^{\frac{1}{2}},$$

$$\zeta' = \epsilon - 1, \quad \chi' = 2\left(2\alpha_3 \frac{\overline{C}_{\Gamma\Omega}^2}{\underline{\lambda}_A} - 1\right),$$

$$\nu = 2 - \delta, \quad \theta = \varrho\left(2 - \frac{1}{\gamma}\right)^{\frac{1}{2}}, \quad \zeta = 1 - \frac{1}{\epsilon}, \quad \chi = 2,$$

and

$$\mathcal{K} = \max\left\{ \frac{2(2\alpha_2 - 1)}{2 - \delta}, \quad 2\left(\frac{2\alpha_1 \frac{\overline{C}_{\mathrm{P}\Omega}^2}{\underline{\lambda}_A} - 1}{2 - \frac{1}{\gamma}}\right)^{\frac{1}{2}}, \quad \epsilon, \quad 2\alpha_3 \frac{\overline{C}_{\Gamma\Omega}^2}{\underline{\lambda}_A} - 1 \right\}.$$

The relation (3.77) shows that $\overline{\mathrm{M}}_{\mathrm{II,N}}^2$ is equivalent to the error measure (1.8).

Therefore, we obtain fully computable error majorants (presented in Theorems 3 and 4), which generate fully computable and realistic estimates of the distance to exact solution.

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