Записки научных
семинаров ПОМИ
Том 423, 2014 г.

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INHERENTLY NON-FINITELY GENERATED VARIETIES OF APERIODIC MONOIDS WITH CENTRAL IDEMPOTENTS

Abstract. Let $\mathscr{A}$ denote the class of aperiodic monoids with central idempotents. A subvariety of $\mathscr{A}$ that is not contained in any finitely generated subvariety of $\mathscr{A}$ is said to be *inherently non-finitely generated*. A characterization of inherently non-finitely generated subvarieties of $\mathscr{A}$, based on identities that they cannot satisfy and monoids that they must contain, is given. It turns out that there exists a unique minimal inherently non-finitely generated subvariety of $\mathscr{A}$, the inclusion of which is both necessary and sufficient for a subvariety of $\mathscr{A}$ to be inherently non-finitely generated. Further, it is decidable in polynomial time if a finite set of identities defines an inherently non-finitely generated subvariety of $\mathscr{A}$.

§1. Introduction

Recall that a monoid is *aperiodic* if all its subgroups are trivial. The *index* of an aperiodic monoid is the least positive integer n for which the identity $x^{n+1} \approx x^n$ is satisfied by the monoid. The class $\mathscr{A}$ of aperiodic monoids with central idempotents constitutes an important source of examples in the study of the finite basis problem; see Jackson [2], Jackson and Sapir [4], Lee [5], Perkins [10], and Sapir [11]. For each $n \geqslant 1$, let $\mathbf{A}_n$ denote the variety of monoids from $\mathscr{A}$ of index at most n. The variety $\mathbf{A}_n$ is defined by the identities

$$x^{n+1} \approx x^n, \quad x^n y \approx y x^n \qquad (\blacktriangle_n)$$

and the inclusions $\mathbf{A}_1 \subset \mathbf{A}_2 \subset \cdots \subset \mathscr{A}$ hold and are proper. The class $\mathscr{A}$ is not a variety, but each of its subvarieties is contained in $\mathbf{A}_n$ for all sufficiently large n.

A finitely based, finitely generated variety that contains finitely many subvarieties is called a *Cross variety*. An *almost Cross variety* is a minimal non-Cross variety. By Zorn's lemma, each non-Cross variety contains

Key words and phrases: monoid, aperiodic monoid, central idempotent, variety, finitely generated, inherently non-finitely generated.

some almost Cross subvariety. Recent work of Jackson [3] and Lee [5, 6] has led to a complete description of Cross subvarieties of $\mathscr{A}$: there exist precisely three almost Cross subvarieties of $\mathscr{A}$, denoted by $\mathbf{J_1}$, $\mathbf{J_2}$, and $\mathbf{L}$, the exclusion of which is both necessary and sufficient for a subvariety of $\mathscr{A}$ to be Cross [9]. The varieties $\mathbf{J_1}$ and $\mathbf{J_2}$ are finitely generated [3] while the variety $\mathbf{L}$ is non-finitely generated [9]; the variety $\mathbf{L}$ is the subvariety of $\mathbf{A}_2$ defined by the identities

$$xyhxty \approx yxhxty, \quad xhxyty \approx xhyxty, \quad xhytxy \approx xhytyx$$

and it plays a crucial role in the present investigation.

Unless otherwise specified, all varieties in the present article are subvarieties of $\mathscr{A}$. A subvariety $\mathbf{V}$ of $\mathscr{A}$ that is not contained in any finitely generated subvariety of $\mathscr{A}$ is said to be *inherently non-finitely generated within* $\mathscr{A}$; since this article concentrates only on subvarieties of $\mathscr{A}$, it is unambiguous to refer to such a variety $\mathbf{V}$ simply as an *inherently non-finitely generated subvariety of* $\mathscr{A}$.[1] Although an inherently non-finitely generated subvariety of $\mathscr{A}$ is vacuously non-finitely generated, the converse is not true in general. A non-finitely generated subvariety of $\mathscr{A}$ that is not inherently non-finitely generated within $\mathscr{A}$ is exhibited in Section 6, and it is the first explicitly described example of its kind.

The present article is devoted to the description of inherently non-finitely generated subvarieties of $\mathscr{A}$. After developing some preliminary results in Section 2, some identities that are satisfied by subvarieties of $\mathscr{A}$ are introduced in Section 3. Section 4 is concerned with the investigation of the almost Cross variety $\mathbf{L}$, its subvarieties, and the identities it satisfies. In particular, the subvarieties of $\mathbf{L}$ are shown to constitute a countably infinite chain. Based on results from Sections 2–4, a characterization of inherently non-finitely generated subvarieties of $\mathscr{A}$ is established in Section 5; it includes identities that these varieties cannot satisfy and monoids that they must contain. It follows that the inclusion of the variety $\mathbf{L}$ is both necessary and sufficient for any subvariety of $\mathscr{A}$ to be inherently non-finitely generated within $\mathscr{A}$, whence $\mathbf{L}$ is the unique minimal inherently non-finitely generated subvariety of $\mathscr{A}$. A polynomial time algorithm is also presented that decides, given a finite set Σ of identities that defines

[1]Note that a subvariety of $\mathscr{A}$ that is inherently non-finitely generated within $\mathscr{A}$ may be contained in a finitely generated variety that is not a subvariety of $\mathscr{A}$. See Example 5.3.

a subvariety $\mathbf{V}$ of $\mathscr{A}$, if the variety $\mathbf{V}$ is inherently non-finitely generated within $\mathscr{A}$.

§2. Preliminaries

Let $\mathcal{X}$ be a countably infinite alphabet throughout. For any subset $\mathcal{Y}$ of $\mathcal{X}$, let $\mathcal{Y}^*$ denote the free monoid over $\mathcal{Y}$. Elements of $\mathcal{X}$ and $\mathcal{X}^*$ are called *letters* and *words*, respectively. An identity is written as $\mathbf{u} \approx \mathbf{v}$ where $\mathbf{u}$ and $\mathbf{v}$ are nonempty words; this identity is *nontrivial* if $\mathbf{u} \neq \mathbf{v}$. A monoid M *satisfies* an identity $\mathbf{u} \approx \mathbf{v}$ if, for any substitution φ from $\mathcal{X}$ into M, the elements $\mathbf{u}\varphi$ and $\mathbf{v}\varphi$ of M coincide. A class of monoids *satisfies* an identity if every monoid in the class satisfies the identity. The variety *defined* by a set Σ of identities is the class of monoids that satisfy all identities in Σ; in this case, Σ is a *basis* for the variety. A variety is *finitely based* if it possesses a finite basis.

Refer to the monograph of Burris and Sankappanavar [1] for more information on varieties of algebras in general.

2.1. Rees quotients of $\mathcal{X}^*$. For any set $\mathcal{U}$ of words, let $\mathrm{S}(\mathcal{U})$ denote the Rees quotient monoid of $\mathcal{X}^*$ over the ideal of all words that are not factors of any word in $\mathcal{U}$. Equivalently, $\mathrm{S}(\mathcal{U})$ can be treated as the monoid that consists of every factor of every word in $\mathcal{U}$, together with a zero element 0, with binary operation $\cdot$ given by

$$\mathbf{u} \cdot \mathbf{v} = \begin{cases} \mathbf{uv} & \text{if } \mathbf{uv} \text{ is a factor of some word in } \mathcal{U}, \\ 0 & \text{otherwise.} \end{cases}$$

The empty factor, more conveniently written as 1, is the identity of the monoid $\mathrm{S}(\mathcal{U})$. If $\mathcal{U} = \{\mathbf{u}_1, \dots, \mathbf{u}_m\}$, then write $\mathrm{S}(\mathcal{U}) = \mathrm{S}(\mathbf{u}_1, \dots, \mathbf{u}_m)$.

Example 2.1 (Jackson [3, Section 5])**.** The almost Cross varieties $\mathbf{J_1}$ and $\mathbf{J_2}$ introduced in Section 1 are generated by the monoids $\mathrm{S}(xhxyty)$ and $\mathrm{S}(xhytxy, xyhxty)$, respectively. These varieties are non-finitely based.

A nonempty word $\mathbf{u}$ is an *isoterm* for a variety $\mathbf{V}$ if $\mathbf{V}$ does not satisfy any nontrivial identity of the form $\mathbf{u} \approx \mathbf{v}$.

Lemma 2.2 (Jackson [3, Lemma 3.3])**.** *For any set $\mathcal{U}$ of words and any variety $\mathbf{V}$, the monoid $\mathrm{S}(\mathcal{U})$ belongs to $\mathbf{V}$ if and only if every word in $\mathcal{U}$ is an isoterm for $\mathbf{V}$.*

2.2. The Straubing identities. A variety is *finitely generated* if it is generated by a single finite monoid. The *Straubing identities*

$$x \prod_{i=1}^{n-1} (h_i x) \approx x^n \prod_{i=1}^{n-1} h_i, \tag{$\bigstar_n$}$$

where $n \in \{2, 3, \dots\}$, play a significant role in the study of finitely generated subvarieties of $\mathscr{A}$.

Lemma 2.3 (Jackson and Sapir [4, Corollary 3.1])**.** *For each $n \geqslant 2$, the variety defined by the identities $\{\blacktriangle_n, \bigstar_n\}$ is finitely generated.*

Lemma 2.4 (Straubing [13])**.** *Let $\mathbf{V}$ be any subvariety of $\mathscr{A}$. If $\mathbf{V}$ is finitely generated, then $\mathbf{V}$ satisfies the identities $\{\blacktriangle_n, \bigstar_n\}$ for some $n \geqslant 2$.*

The converse of Lemma 2.4 does not hold in general since a subvariety of $\mathscr{A}$ that satisfies the identities $\{\blacktriangle_3, \bigstar_3\}$ is shown in Section 6 to be non-finitely generated.

§3. Rigid words and rigid identities

Results established in the present section are required in Sections 4 and 5, where all subvarieties of $\mathbf{L}$ and all inherently non-finitely generated subvarieties of $\mathscr{A}$ are described.

Define a *rigid word* to be the word

$$\mathbf{u} = x^{e_0} \prod_{i=1}^{m} (h_i x^{e_i})$$

where $m \geqslant 0$ and $e_0, \dots, e_m \geqslant 0$; the number m is the *level* of the word $\mathbf{u}$. Note that a rigid word of level 0 is of the form x^e. The rigid word $\mathbf{u}$ above is *square-free* if $e_0, \dots, e_m \leqslant 1$. A *rigid identity* is an identity that is formed by a pair of rigid words of the same level. Note that each Straubing identity $\bigstar_n$ is a rigid identity formed by rigid words of level $n - 1$.

Lemma 3.1. *Let $\mathbf{V}$ be any subvariety of $\mathscr{A}$ that satisfies a nontrivial rigid identity*

$$x^{e_0} \prod_{i=1}^{m} (h_i x^{e_i}) \approx x^{f_0} \prod_{i=1}^{m} (h_i x^{f_i}),$$

where at least one side of the identity is a square-free word. Suppose that at least one of the following conditions holds:

(a) $m = 0$;

(b) $(e_0, \ldots, e_m) = (0, \ldots, 0)$;
(c) $(f_0, \ldots, f_m) = (0, \ldots, 0)$.
Then $\mathbf{V}$ *is commutative.*

Proof. This lemma is routinely verified based on the assumption that the variety $\mathbf{V}$ satisfies the identities $\blacktriangle_n$ for some $n \geqslant 1$. □

Lemma 3.2. *The variety* $\mathbf{A}_n$ *satisfies the rigid identity*

$$x^{e_0} \prod_{i=1}^{m} (h_i x^{e_i}) \approx x^n \prod_{i=1}^{m} h_i \tag{3.1}$$

whenever $e_j \geqslant n$ *for some* $j \in \{0, \ldots, m\}$.

Proof. It is easily shown that the basis $\blacktriangle_n$ for $\mathbf{A}_n$ implies the identity (3.1) whenever $e_j \geqslant n$ for some $j \in \{0, \ldots, m\}$. □

Lemma 3.3. *Suppose that* $\mathbf{V}$ *is any subvariety of* $\mathscr{A}$ *that satisfies some nontrivial rigid identity* $\mathbf{u} \approx \mathbf{v}$ *where either* $\mathbf{u}$ *or* $\mathbf{v}$ *is square-free. Then* $\mathbf{V}$ *satisfies the Straubing identity* $\bigstar_k$ *for some* $k \geqslant 2$.

Proof. By assumption, the variety $\mathbf{V}$ satisfies the identities $\blacktriangle_n$ for some $n \geqslant 2$ and

$$\mathbf{u} = x^{e_0} \prod_{i=1}^{m} (h_i x^{e_i}) \quad \text{and} \quad \mathbf{v} = x^{f_0} \prod_{i=1}^{m} (h_i x^{f_i})$$

for some $e_0, f_0, \ldots, e_m, f_m \geqslant 0$ with$(e_0, \ldots, e_m) \neq (f_0, \ldots, f_m)$. Further, it suffices to assume that $m \geqslant 1$ and $(e_0, \ldots, e_m), (f_0, \ldots, f_m) \neq (0, \ldots, 0)$, since otherwise the variety $\mathbf{V}$ is commutative by Lemma 3.1 and so satisfies the identity $\bigstar_2$.

Let $e = e_0 + \cdots + e_m$ and $f = f_0 + \cdots + f_m$. Without loss of generality, assume that one of the following cases holds:

(a) $\mathbf{u}$ is square-free and $\mathbf{v}$ is not square-free;
(b) $\mathbf{u}$ and $\mathbf{v}$ are both square-free with $0 < e \leqslant f$.

Then $e_0, \ldots, e_m \leqslant 1$ in both (a) and (b). Since $\mathbf{u}$ is a square-free rigid word and x occurs e times in $\mathbf{u}$, there exists an appropriate deletion φ_1 of the letters h_i such that

$$\mathbf{u}\varphi_1 = x \cdot h_{j_1} x \cdot h_{j_2} x \cdots h_{j_{e-1}} x,$$

where $1 \leqslant j_1 < \cdots < j_{e-1} \leqslant m$. Let φ_2 denote the substitution that renames the letters $h_{j_1}, \ldots, h_{j_{e-1}}$ by $h_1, \ldots, h_{e-1}$. Then

$$\mathbf{u}\varphi_1\varphi_2 = x\prod_{i=1}^{e-1}(h_i x)$$

is a square-free rigid word of level $e-1$.[2] Now perform the deletion φ_1 on $\mathbf{v}$ followed by the substitution φ_2 on $\mathbf{v}\varphi_1$ to obtain $\mathbf{v}\varphi_1\varphi_2$. It is clear that in case (a), the word $\mathbf{v}\varphi_1\varphi_2$ is a rigid word of level $e-1$ that is not square-free. In case (b), since the identity $\mathbf{u} \approx \mathbf{v}$ is nontrivial with $e \leqslant f$, the word $\mathbf{v}\varphi_1\varphi_2$ is also rigid and of level $e-1$ that is not square-free. Therefore in both cases, $\mathbf{v}\varphi_1\varphi_2 = \mathbf{p}x^r\mathbf{q}$ for some $r \geqslant 2$ and $\mathbf{p}, \mathbf{q} \in \mathcal{X}^*$, whence the identity $\mathbf{u} \approx \mathbf{v}$ implies the rigid identity

$$x\prod_{i=1}^{d}(h_i x) \approx \mathbf{p}x^r\mathbf{q} \tag{3.2}$$

where $d = e-1$. The identity (3.2) clearly implies a rigid identity of the form

$$x^r\prod_{i=1}^{d}(h_i x^r) \approx \mathbf{p}'x^{r^2}\mathbf{q}' \tag{3.3}$$

for some $\mathbf{p}', \mathbf{q}' \in \mathcal{X}^*$. Since

$$\begin{aligned} x\prod_{i=1}^{d^2+2d}(h_i x) &= \left(x\prod_{i=1}^{d}(h_i x)\right)h_{d+1}\left(x\prod_{i=d+2}^{2d+1}(h_i x)\right)h_{2d+2}\left(x\prod_{i=2d+3}^{3d+2}(h_i x)\right)\cdots \\ &\qquad \cdots h_{d^2+d}\left(x\prod_{i=d^2+d+1}^{d^2+2d}(h_i x)\right) \\ &\overset{(3.2)}{\approx} (\cdots x^r \cdots)h_{d+1}(\cdots x^r \cdots)h_{2d+2}(\cdots x^r \cdots)\cdots h_{d^2+d}(\cdots x^r \cdots) \\ &\overset{(3.3)}{\approx} \cdots x^{r^2} \cdots, \end{aligned}$$

the identity $\mathbf{u} \approx \mathbf{v}$ implies a rigid identity of the form (3.2) with r replaced by r^2. The same argument can be repeated sufficiently many times so that the identity $\mathbf{u} \approx \mathbf{v}$ implies a rigid identity of the form (3.2) with r replaced some number r^s that is greater than n. Therefore generality is not lost by

[2]For instance, if $\mathbf{u} = h_1xh_2h_3xh_4h_5h_6xh_7x$ where $e = 4$ and $m = 7$, then $\mathbf{u}\varphi_1 = xh_2xh_4xh_7x$ and $\mathbf{u}\varphi_1\varphi_2 = xh_1xh_2xh_3x$.

assuming that $r \geqslant n$ in (3.2) to begin with. Since $\mathbf{p}x^r\mathbf{q}$ is a rigid word of level d, it follows from Lemma 3.2 that

$$x^n \prod_{i=1}^{d} h_i \overset{(3.1)}{\approx} \mathbf{p}x^r\mathbf{q} \overset{(3.2)}{\approx} x \prod_{i=1}^{d}(h_i x).$$

The variety $\mathbf{V}$ thus satisfies the identity

$$x \prod_{i=1}^{d}(h_i x) \approx x^n \prod_{i=1}^{d} h_i. \tag{3.4}$$

If $d = n - 1$, then the identity (3.4) is $\bigstar_n$. If $d > n - 1$, then

$$x \prod_{i=1}^{d}(h_i x) \overset{(3.4)}{\approx} x^n \prod_{i=1}^{d} h_i \overset{\blacktriangle_n}{\approx} x^{d+1} \prod_{i=1}^{d} h_i$$

so that the variety $\mathbf{V}$ satisfies the identity $\bigstar_d$. If $d < n - 1$, then

$$x \prod_{i=1}^{n-1}(h_i x) \overset{(3.4)}{\approx} \bigg(x^n \prod_{i=1}^{d} h_i\bigg)\bigg(\prod_{i=d+1}^{n-1}(h_i x)\bigg) \overset{(3.1)}{\approx} x^n \prod_{i=1}^{n-1} h_i$$

so that the variety $\mathbf{V}$ satisfies the identity $\bigstar_n$. □

§4. The variety L

This section is concerned with the almost Cross variety $\mathbf{L}$. Recall from Section 1 that $\mathbf{L}$ is defined by the identities $\blacktriangle_2$ and

$$xyhxty \approx yxhxty, \quad xhxyty \approx xhyxty, \quad xhytxy \approx xhytyx. \tag{4.1}$$

Subsection 4.1 provides a complete description of all subvarieties of $\mathbf{L}$. For this purpose, the *reduced Straubing identities*

$$x \prod_{i=1}^{n-1}(h_i x) \approx x^2 \prod_{i=1}^{n-1} h_i, \tag{$\star_n$}$$

where $n \in \{2, 3, \dots\}$, are required. Define the set $\mathcal{W}_\infty = \{\mathbf{w}_2, \mathbf{w}_3, \dots\}$ where

$$\mathbf{w}_n = x \prod_{i=1}^{n-1}(h_i x)$$

is the word on the left side of the identity $\star_n$.

Subsection 4.2 demonstrates that it is decidable in polynomial time if an arbitrarily given identity is satisfied by $\mathbf{L}$.

4.1. Subvarieties of L. For any set Σ of identities, let $\mathbf{L}\Sigma$ denote the subvariety of $\mathbf{L}$ defined by Σ. For any set $\mathcal{U}$ of words, let $\mathbf{S}(\mathcal{U})$ denote the variety generated by the monoid $\mathrm{S}(\mathcal{U})$. Let $\mathbf{0}$ denote the variety of trivial monoids.

Proposition 4.1. *The subvarieties of* $\mathbf{L}$ *constitute the chain*

$$\mathbf{0} \subset \mathbf{S}(\varnothing) \subset \mathbf{S}(x) \subset \mathbf{S}(xy) \subset \mathbf{S}(\mathbf{w}_2) \subset \mathbf{S}(\mathbf{w}_3) \subset \cdots \subset \mathbf{S}(\mathcal{W}_\infty) = \mathbf{L}. \quad (4.2)$$

The proof of Proposition 4.1 is given at the end of the subsection.

Lemma 4.2. *Let* $e_0, \ldots, e_m \geqslant 0$ *and* $\ell \geqslant 2$ *be such that* $\ell \leqslant e_0 + \cdots + e_m$. *Then the identities* $\{\blacktriangle_2, \star_\ell\}$ *imply the identity*

$$x^{e_0} \prod_{i=1}^{m} (h_i x^{e_i}) \approx x^2 \prod_{i=1}^{m} h_i.$$

Consequently, the following inclusions hold:

$$\mathbf{L}\{\star_2\} \subseteq \mathbf{L}\{\star_3\} \subseteq \cdots \subseteq \mathbf{L}. \quad (4.3)$$

Proof. Let $e = e_0 + \cdots + e_m$. Then $e_0, \ldots, e_m \geqslant 0$ and $\ell \leqslant e$ imply that

$$x^{e_0} \prod_{i=1}^{m} (h_i x^{e_i}) = \mathbf{q}_0 \Bigg(x \prod_{i=1}^{\ell-1} (\mathbf{q}_i x) \Bigg) \Bigg(\prod_{i=\ell}^{e-1} (\mathbf{q}_i x) \Bigg) \mathbf{q}_e$$

for some $\mathbf{q}_0, \ldots, \mathbf{q}_e \in \mathcal{X}^*$ such that $\mathbf{q}_0 \cdots \mathbf{q}_e = h_1 \cdots h_m$. Hence

$$x^{e_0} \prod_{i=1}^{m} (h_i x^{e_i}) \overset{\star_\ell}{\approx} \mathbf{q}_0 \Bigg(x^2 \prod_{i=1}^{\ell-1} \mathbf{q}_i \Bigg) \Bigg(\prod_{i=\ell}^{e-1} (\mathbf{q}_i x) \Bigg) \mathbf{q}_e \overset{(3.1)}{\approx} x^2 \prod_{i=0}^{e} \mathbf{q}_i = x^2 \prod_{i=1}^{m} h_i,$$

where the second deduction holds by Lemma 3.2 with $n = 2$. □

Lemma 4.3. *The variety* $\mathbf{L}$ *satisfies a nontrivial rigid identity* $\mathbf{u} \approx \mathbf{v}$ *if and only if both of the rigid words* $\mathbf{u}$ *and* $\mathbf{v}$ *are not square-free.*

Proof. This is easily verified by Lemma 3.2 and has been performed in Lee [9, Lemma 13]. □

Lemma 4.4 (Lee [7, Proposition 4.1])**.** *Let* $\mathbf{V}$ *be any variety that satisfies the identities* (4.1). *Then each noncommutative subvariety of* $\mathbf{V}$ *is defined by the identities* (4.1) *together with some set of rigid identities.*

Lemma 4.5. *The noncommutative subvarieties of* $\mathbf{L}$ *are precisely the varieties in the chain* (4.3).

Proof. Let $\mathbf{V}$ be any noncommutative proper subvariety of $\mathbf{L}$. Then the variety $\mathbf{V}$ is Cross because $\mathbf{L}$ is almost Cross. Since the variety $\mathbf{L}$ satisfies the identities (4.1), it follows from Lemma 4.4 that $\mathbf{V} = \mathbf{L}\Sigma$ for some set Σ of rigid identities that are not satisfied by $\mathbf{L}$. By Lemma 2.4, the variety $\mathbf{V}$ satisfies the identity $\bigstar_k$ for some $k \geqslant 2$. The identities $\blacktriangle_2$ and $\bigstar_k$ clearly imply the identity $\star_k$ so that the variety $\mathbf{V}$ satisfies $\star_k$. Let ℓ be the least possible integer for which the identity $\star_\ell$ is satisfied by $\mathbf{V}$.

Let $\mathbf{u} \approx \mathbf{v}$ be any identity from Σ. Then

$$\mathbf{u} = x^{e_0} \prod_{i=1}^{m} (h_i x^{e_i}) \quad \text{and} \quad \mathbf{v} = x^{f_0} \prod_{i=1}^{m} (h_i x^{f_i})$$

for some $e_0, f_0, \ldots, e_m, f_m \geqslant 0$ with $(e_0, \ldots, e_m) \neq (f_0, \ldots, f_m)$. Further, it suffices to assume that $m \geqslant 1$ and $(e_0, \ldots, e_m), (f_0, \ldots, f_m) \neq (0, \ldots, 0)$, since otherwise, the variety $\mathbf{V}$ is commutative by Lemma 3.1, contradicting the assumption.

The identity $\mathbf{u} \approx \mathbf{v}$ is not satisfied by $\mathbf{L}$ so that by Lemma 4.3, either $\mathbf{u}$ or $\mathbf{v}$ is square-free. Let $e = e_0 + \cdots + e_m$ and $f = f_0 + \cdots + f_m$. Without loss of generality, assume that one of the following cases holds:

(a) $\mathbf{u}$ is square-free and $\mathbf{v}$ is not square-free;
(b) $\mathbf{u}$ and $\mathbf{v}$ are both square-free with $0 < e \leqslant f$.

Following the arguments in the proof of Lemma 3.3, the identity $\mathbf{u} \approx \mathbf{v}$ implies the rigid identity

$$x \prod_{i=1}^{e-1} (h_i x) \approx \mathbf{p} x^r \mathbf{q} \tag{4.4}$$

for some $\mathbf{p}, \mathbf{q} \in \mathcal{X}^*$ and $r \geqslant 2$. Since $\mathbf{V}$ is a subvariety of $\mathbf{A}_2$, it follows that

$$x \prod_{i=1}^{e-1} (h_i x) \overset{(4.4)}{\approx} \mathbf{p} x^r \mathbf{q} \overset{(3.1)}{\approx} x^2 \prod_{i=1}^{e-1} h_i$$

by Lemma 3.2 with $n = 2$, whence $\mathbf{V}$ satisfies the identity $\star_e$. The minimality of ℓ implies that $\ell \leqslant e$. In case (a), since $f_j \geqslant 2$ for some j, the deductions

$$\mathbf{u} = x^{e_0} \prod_{i=1}^{m} (h_i x^{e_i}) \overset{\blacktriangle_2, \star_\ell}{\approx} x^2 \prod_{i=1}^{m} h_i \overset{(3.1)}{\approx} x^{f_0} \prod_{i=1}^{m} (h_i x^{f_i}) = \mathbf{v}$$

hold, respectively, by Lemma 4.2 and Lemma 3.2 with $n = 2$. In case (b), the deductions

$$\mathbf{u} = x^{e_0} \prod_{i=1}^{m} (h_i x^{e_i}) \overset{\blacktriangle_2, \star_\ell}{\approx} x^2 \prod_{i=1}^{m} h_i \overset{\blacktriangle_2, \star_\ell}{\approx} x^{f_0} \prod_{i=1}^{m} (h_i x^{f_i}) = \mathbf{v}$$

hold by Lemma 4.2. In both cases, the identities $\{\blacktriangle_2, \star_\ell\}$ imply the identity $\mathbf{u} \approx \mathbf{v}$. Since the identity $\mathbf{u} \approx \mathbf{v}$ from Σ is arbitrarily chosen,

$$\mathbf{L}\{\star_\ell\} = \mathbf{L}\Sigma = \mathbf{V}. \qquad \square$$

Proof of Proposition 4.1. It is easily shown that the variety $\mathbf{S}(\varnothing)$ of semilattice monoids and the variety $\mathbf{S}(x)$ are the only nontrivial commutative subvarieties of $\mathbf{L}$. By Lemma 4.5, the subvarieties of $\mathbf{L}$ constitute the chain

$$\mathbf{0} \subset \mathbf{S}(\varnothing) \subset \mathbf{S}(x) \subset \mathbf{L}\{\star_2\} \subseteq \mathbf{L}\{\star_3\} \subseteq \cdots \subseteq \mathbf{L}.$$

It is known that $\mathbf{S}(xy) = \mathbf{L}\{\star_2\}$ [3, Lemma 4.5]. For each $n \geqslant 2$, it is routinely shown that the monoid $\mathrm{S}(\mathbf{w}_n)$ satisfies the identities $\{\blacktriangle_2, (4.1), \star_{n+1}\}$ but does not satisfy the identity $\star_n$, whence

$$\mathbf{S}(\mathbf{w}_n) = \mathbf{L}\{\star_{n+1}\} \neq \mathbf{L}\{\star_n\} \quad \text{and} \quad \mathbf{S}(\mathcal{W}_\infty) = \mathbf{L} \neq \mathbf{L}\{\star_n\}.$$

Consequently, the subvarieties of $\mathbf{L}$ constitute the chain (4.2). $\square$

4.2. Identities satisfied by L. The *content* of a word $\mathbf{u}$, denoted by $\mathsf{con}(\mathbf{u})$, is the set of letters occurring in $\mathbf{u}$. A letter of a word $\mathbf{u}$ is *simple* if it occurs exactly once in $\mathbf{u}$; otherwise, it is *non-simple* in $\mathbf{u}$.

Suppose that the simple letters of a word $\mathbf{u}$ are $h_1, \ldots, h_m$ when listed in order of first occurrence, and that the distinct non-simple letters of $\mathbf{u}$ are $x_1, \ldots, x_r$ when listed in alphabetical order. Then the word $\mathbf{u}$ is in *canonical form* if

$$\mathbf{u} = \mathbf{u}'\mathbf{u}_0 \prod_{i=1}^{m} (h_i \mathbf{u}_i) \qquad (4.5)$$

where

(CF1) $\mathbf{u}' = x_1^{e_1} \cdots x_r^{e_r}$ for some $e_1, \ldots, e_r \in \{0, 2\}$;
(CF2) $\mathbf{u}_0, \ldots, \mathbf{u}_m \in \{x_1^{f_1} \cdots x_r^{f_r} \mid f_1, \ldots, f_r \in \{0, 1\}\}$;
(CF3) $\mathsf{con}(\mathbf{u}') \cap \mathsf{con}(\mathbf{u}_0 \cdots \mathbf{u}_m) = \varnothing$.

Note that if the word $\mathbf{u}$ in (4.5) contains only simple letters, then $\mathbf{u} = \prod_{i=1}^{m} h_i$; if it contains only non-simple letters, then $\mathbf{u} = \mathbf{u}' = x_1^2 \cdots x_r^2$.

Lemma 4.6. *For any word* $\mathbf{u}$, *there exists some word* $\widehat{\mathbf{u}}$ *in canonical form such that the identities* $\{\blacktriangle_2, (4.1)\}$ *imply the identity* $\mathbf{u} \approx \widehat{\mathbf{u}}$.

Proof. It suffices to convert the word $\mathbf{u}$, using the identities $\{\blacktriangle_2, (4.1)\}$, into a word in canonical form. Without loss of generality, assume that the simple letters of $\mathbf{u}$ are $h_1, \dots, h_m$ when listed in order of first occurrence, and that the distinct non-simple letters of $\mathbf{u}$ are $x_1, \dots, x_r$ when listed in alphabetical order. Then

$$\mathbf{u} = \mathbf{u}_0 \prod_{i=1}^{m} (h_i \mathbf{u}_i)$$

for some $\mathbf{u}_0, \dots, \mathbf{u}_m \in \{x_1, \dots, x_r\}^*$.

(I) For each $i \in \{0, \dots, m\}$, since the letters of $\mathbf{u}_i$ are non-simple in $\mathbf{u}$, they can be alphabetically ordered within $\mathbf{u}_i$ by the identities (4.1). Hence each $\mathbf{u}_i$ can be converted to a word of the form $x_1^{f_1} \cdots x_r^{f_r}$ with $f_1, \dots, f_r \geqslant 0$.

(II) For each $j \in \{1, \dots, r\}$, if a square x_j^2 occurs as a factor in some of $\mathbf{u}_0, \dots, \mathbf{u}_m$, then the identities $\blacktriangle_2$ can be used to gather every x_j in $\mathbf{u}$ to the left. This forms the prefix $\mathbf{u}' = x_1^{e_1} \cdots x_r^{e_r}$ with $e_1, \dots, e_r \in \{0, 2, 3, \dots\}$ such that (CF3) is satisfied. Further, (CF2) is satisfied since all squares are removed from $\mathbf{u}_0, \dots, \mathbf{u}_m$.

(III) If an exponent e_j in $\mathbf{u}'$ is 3 or greater, then apply the identity $x^3 \approx x^2$ from $\blacktriangle_2$ to reduce e_j to 2. Hence (CF1) is satisfied. □

Lemma 4.7. *Given any identity* $\mathbf{u} \approx \mathbf{v}$, *there exists a polynomial time algorithm that decides if the variety* $\mathbf{L}$ *satisfies the identity* $\mathbf{u} \approx \mathbf{v}$.

Proof. By Lemma 4.6, there exist words $\widehat{\mathbf{u}}$ and $\widehat{\mathbf{v}}$ in canonical form such that the identities $\{\blacktriangle_2, (4.1)\}$ imply the identities $\mathbf{u} \approx \widehat{\mathbf{u}}$ and $\mathbf{v} \approx \widehat{\mathbf{v}}$. Hence the variety $\mathbf{L}$ satisfies the identity $\mathbf{u} \approx \mathbf{v}$ if and only if it satisfies the identity $\widehat{\mathbf{u}} \approx \widehat{\mathbf{v}}$. By Lemma 2.2 and Proposition 4.1, the words $\{x, x^2\} \cup \mathcal{W}_\infty$ are all isoterms for $\mathbf{L}$. It is then routinely shown that the variety $\mathbf{L}$ satisfies the identity $\widehat{\mathbf{u}} \approx \widehat{\mathbf{v}}$ if and only if the words $\widehat{\mathbf{u}}$ and $\widehat{\mathbf{v}}$ are identical.

Steps (I)–(III) in the proof of Lemma 4.6 provide a polynomial time algorithm that converts the words $\mathbf{u}$ and $\mathbf{v}$, using the identities $\{\blacktriangle_2, (4.1)\}$, into the words $\widehat{\mathbf{u}}$ and $\widehat{\mathbf{v}}$ in canonical form. □

§5. Main results

Theorem 5.1. *The following statements on any subvariety* $\mathbf{V}$ *of* $\mathscr{A}$ *are equlvalent*:

(a) $\mathbf{V}$ *is inherently non-finitely generated within* $\mathscr{A}$;

(b) *for any* $n \geqslant 2$, *the Straubing identity*

$$\bigstar_n\colon\ x\prod_{i=1}^{n-1}(h_i x) \approx x^n \prod_{i=1}^{n-1} h_i$$

is not satisfied by $\mathbf{V}$;

(c) *for any* $n \geqslant 2$, *the word*

$$\mathbf{w}_n = x\prod_{i=1}^{n-1}(h_i x)$$

is an isoterm for $\mathbf{V}$;

(d) *for any* $n \geqslant 2$, *the monoid* $\mathrm{S}(\mathbf{w}_n)$ *belongs to* $\mathbf{V}$;

(e) *the almost Cross variety* $\mathbf{L}$ *is a subvariety of* $\mathbf{V}$.

Proof. (a) ⇒ (b). Suppose that for some $n \geqslant 2$, the variety $\mathbf{V}$ satisfies the identity $\bigstar_n$. Then $\mathbf{V}$ satisfies the identities $\{\blacktriangle_k, \bigstar_k\}$ for all sufficiently large k. By Lemma 2.3, the variety defined by $\{\blacktriangle_k, \bigstar_k\}$ is a finitely generated subvariety of $\mathbf{A}_k$. Therefore $\mathbf{V}$ is a subvariety of $\mathbf{A}_k$ and so is not inherently non-finitely generated within $\mathscr{A}$.

(b) ⇒ (c). Suppose that for some $n \geqslant 2$, the word $\mathbf{w}_n$ is not an isoterm for $\mathbf{V}$. Then the variety $\mathbf{V}$ satisfies some nontrivial identity

$$x\prod_{i=1}^{n-1}(h_i x) \approx \mathbf{v}.$$

Case 1. The following conditions hold:

- $\mathsf{con}(\mathbf{v}) = \{x, h_1, \ldots, h_{n-1}\}$;
- $h_1, \ldots, h_{n-1}$ are simple in $\mathbf{v}$;
- for any i, the letter h_i occurs before h_{i+1} in $\mathbf{v}$.

Then

$$\mathbf{v} = x^{e_0}\prod_{i=1}^{n-1}(h_i x^{e_i})$$

for some $e_0, \ldots, e_n \geqslant 0$ with $(e_0, \ldots, e_n) \neq (1, \ldots, 1)$. By Lemma 3.3, the variety $\mathbf{V}$ satisfies the identity $\bigstar_k$ for some $k \geqslant 2$.

CASE 2. Any one of the three conditions in Case 1 fails. Then it is straightforwardly shown that the variety $\mathbf{V}$ is either commutative or idempotent, whence it satisfies the identity $\bigstar_2$.

(c) $\Leftrightarrow$ (d) $\Leftrightarrow$ (e). These follow from Lemma 2.2 and Proposition 4.1.

(e) $\Rightarrow$ (a). Suppose that the variety $\mathbf{V}$ is not inherently non-finitely generated within $\mathscr{A}$. Then it follows from Lemma 2.4 that $\mathbf{V}$ satisfies the identities $\{\blacktriangle_n, \bigstar_n\}$ for some $n \geqslant 2$. But by Lemma 4.3, the variety $\mathbf{L}$ does not satisfy the identity $\bigstar_n$ and so cannot be a subvariety of $\mathbf{V}$. □

Corollary 5.2. *The almost Cross variety* $\mathbf{L}$ *is the unique minimal inherently non-finitely generated subvariety of* $\mathscr{A}$.

The following example demonstrates that subvarieties of $\mathscr{A}$ that are inherently non-finitely generated within $\mathscr{A}$ need not be inherently non-finitely generated within the class $\mathscr{M}$ of all monoids. (Another explicit example can be found in Lee [8, Proposition 6.9].)

Example 5.3. Let $\mathbf{B_2^1}$ denote the variety generated by the Brandt monoid

$$B_2^1 = \langle a, b \mid a^2 = b^2 = 0,\ aba = a,\ bab = b\rangle \cup \{1\}$$

of order six. Then $\mathbf{L}$ is a subvariety of $\mathbf{B_2^1}$, but $\mathbf{B_2^1}$ is not a subvariety of $\mathscr{A}$.

Proof. The idempotent ab of B_2^1 is not central since $ab \cdot a \neq a \cdot ab$. Hence $\mathbf{B_2^1}$ is not a subvariety of $\mathscr{A}$. It is routinely verified that for each $n \geqslant 2$, the word $\mathbf{w}_n$ is an isoterm for the variety $\mathbf{B_2^1}$ so that by Lemma 2.2, the monoid $\mathrm{S}(\mathbf{w}_n)$ belongs to $\mathbf{B_2^1}$. It follows from Proposition 4.1 that $\mathbf{L}$ is a subvariety of $\mathbf{B_2^1}$. □

Presently, the variety of idempotent monoids is the only known example of a variety of monoids that is minimal with respect to being inherently non-finitely generated within $\mathscr{M}$ [12].

Theorem 5.4. *Suppose that* Σ *is any finite set of identities that defines a subvariety* $\mathbf{V}$ *of* $\mathscr{A}$. *Then there exists a polynomial time algorithm that decides if* $\mathbf{V}$ *is inherently non-finitely generated within* $\mathscr{A}$.

Proof. By assumption, the variety $\mathbf{V}$ is a subvariety of $\mathbf{A}_n$ for some $n \geqslant 1$. Hence generality is not lost by assuming that Σ contains the identities $\blacktriangle_n$. By Lemma 4.7, there exists a polynomial time algorithm that decides if the variety $\mathbf{L}$ satisfies the identities in Σ. The result now follows from Theorem 5.1. □

§6. A non-finitely generated subvariety of $\mathbf{A}_2$

Let $\mathbf{N}$ denote the variety defined by the identities $\blacktriangle_2$, $\bigstar_3$, and

$$xhytxy \approx xhytyx, \tag{6.1}$$

$$xyhxty \approx yxhxty. \tag{6.2}$$

The main aim of the present section is to show that the variety $\mathbf{N}$ is non-finitely generated. But since the variety $\mathbf{N}$ satisfies the identities $\{\blacktriangle_3, \bigstar_3\}$, it follows from Lemma 2.3 that $\mathbf{N}$ is not inherently non-finitely generated within $\mathscr{A}$.

Lemma 6.1. *The variety* $\mathbf{N}$ *satisfies the identities*

$$x^2hxtx \approx xhx^2tx \approx xhxtx^2 \approx xhxtx, \tag{6.3}$$

$$xh^2yxty \approx x^2h^2yx^2ty, \tag{6.4}$$

$$xhyxt^2y \approx xhy^2xt^2y^2. \tag{6.5}$$

Proof. It is easily shown that the variety $\mathbf{N}$ satisfies the identities (6.3). Since

$$xh^2yxty \overset{\blacktriangle_2}{\approx} h^2xyxty \overset{(6.2)}{\approx} h^2yx^2ty \overset{\blacktriangle_2}{\approx} h^2yx^4ty \overset{\blacktriangle_2}{\approx} x^2h^2yx^2ty,$$

$$xhyxt^2y \overset{\blacktriangle_2}{\approx} xhyxyt^2 \overset{(6.1)}{\approx} xhy^2xt^2 \overset{\blacktriangle_2}{\approx} xhy^4xt^2 \overset{\blacktriangle_2}{\approx} xhy^2xt^2y^2,$$

the variety $\mathbf{N}$ satisfies the identities (6.4) and (6.5). □

Lemma 6.2. *For each* $n \geqslant 2$, *the variety* $\mathbf{N}$ *does not satisfy the identity*

$$\mathbf{z}_n \approx \mathbf{z}'_n,$$

where

$$\mathbf{z}_n = x_0h\left(\prod_{i=0}^{n}(x_{i+1}x_i)\right)tx_{n+1} = x_0h \cdot x_1x_0 \cdot x_2x_1 \cdots x_{n+1}x_n \cdot tx_{n+1},$$

$$\mathbf{z}'_n = x_0hx_0\left(\prod_{i=1}^{n}x_i^2\right)x_{n+1}tx_{n+1} = x_0hx_0 \cdot x_1^2x_2^2 \cdots x_n^2 \cdot x_{n+1}tx_{n+1}.$$

Proof. First observe that

(a) any letter occurs at most twice in the word $\mathbf{z}_n$;

(b) the word $\mathbf{z}_n$ does not contain any factor of the form $\mathbf{x}^2$;

(c) the word $\mathbf{z}_n$ does not contain any factor of the form $\mathbf{xhytxy}$ or $\mathbf{xyhxty}$, where $\mathbf{x}, \mathbf{y}, \mathbf{h}, \mathbf{t} \in \mathscr{X}^*$ with $\mathbf{x}, \mathbf{y} \neq \varnothing$.

It is then easily seen that it is impossible to convert the word $\mathbf{z}_n$ into a different word by applying only the identities $\blacktriangle_2$, $\bigstar_3$, (6.1), and (6.2). It follows that the variety $\mathbf{N}$ does not satisfy the identity $\mathbf{z}_n \approx \mathbf{z}'_n$. □

Lemma 6.3. *Any finite monoid in the variety* $\mathbf{N}$ *satisfies the identity* $\mathbf{z}_n \approx \mathbf{z}'_n$ *for all sufficient large* $n \geqslant 2$.

Proof. Let M be any finite monoid in the variety $\mathbf{N}$ and fix any $n > |M|$. Suppose that φ is any substitution into the monoid M. Then it is shown in the following that $\mathbf{z}_n\varphi = \mathbf{z}'_n\varphi$ in M. Consequently, the monoid M satisfies the identity $\mathbf{z}_n \approx \mathbf{z}'_n$.

For notational brevity, write $x\varphi = \widehat{x}$. Since $n > |M|$, the list $\widehat{x}_1, \ldots, \widehat{x}_n$ of elements from M must contain some repetition, say $\widehat{x}_i = \widehat{x}_j$ with $1 \leqslant i < j \leqslant n$.

Case 1. $1 < i < j \leqslant n$. Note that the letter x_i occurs twice in the word $\mathbf{z}_n$. Since $\widehat{x}_i = \widehat{x}_j$, the element $\widehat{x}_i$ occurs at least thrice in the product $\mathbf{z}_n\varphi$, whence the identities (6.3) can be applied to replace any $\widehat{x}_i$ in $\mathbf{z}_n\varphi$ by $\widehat{x}_i^2$:

$$\begin{aligned}\mathbf{z}_n\varphi &= \cdots \widehat{x}_{i-1}\widehat{x}_{i-2} \cdot \widehat{x}_i\widehat{x}_{i-1} \cdot \widehat{x}_{i+1}\widehat{x}_i \cdot \widehat{x}_{i+2}\widehat{x}_{i+1} \cdots \\ &\overset{(6.3)}{=} \cdots \widehat{x}_{i-1}\widehat{x}_{i-2} \cdot \widehat{x}_i^2\widehat{x}_{i-1} \cdot \widehat{x}_{i+1}\widehat{x}_i^2 \cdot \widehat{x}_{i+2}\widehat{x}_{i+1} \cdots .\end{aligned}$$

Then the identity (6.4) can be applied to replace $\widehat{x}_{i+1}$ by $\widehat{x}_{i+1}^2$, and the identity (6.5) can be applied to replace $\widehat{x}_{i-1}$ by $\widehat{x}_{i-1}^2$:

$$\begin{aligned}&\cdots \widehat{x}_{i-1}\widehat{x}_{i-2} \cdot \widehat{x}_i^2\widehat{x}_{i-1} \cdot \widehat{x}_{i+1}\widehat{x}_i^2 \cdot \widehat{x}_{i+2}\widehat{x}_{i+1} \cdots \\ \overset{(6.4)}{=} &\cdots \widehat{x}_{i-1}\widehat{x}_{i-2} \cdot \widehat{x}_i^2\widehat{x}_{i-1} \cdot \widehat{x}_{i+1}^2\widehat{x}_i^2 \cdot \widehat{x}_{i+2}\widehat{x}_{i+1}^2 \cdots \\ \overset{(6.5)}{=} &\cdots \widehat{x}_{i-1}^2\widehat{x}_{i-2} \cdot \widehat{x}_i^2\widehat{x}_{i-1}^2 \cdot \widehat{x}_{i+1}^2\widehat{x}_i^2 \cdot \widehat{x}_{i+2}\widehat{x}_{i+1}^2 \cdots .\end{aligned}$$

This procedure can be repeated until $\widehat{x}_1, \ldots, \widehat{x}_n$ are replaced by $\widehat{x}_1^2, \ldots, \widehat{x}_n^2$. Hence

$$\begin{aligned}\mathbf{z}_n\varphi &= \widehat{x}_0\widehat{h} \cdot \widehat{x}_1^2\widehat{x}_0 \cdot \widehat{x}_2^2\widehat{x}_1^2 \cdot \widehat{x}_3^2\widehat{x}_2^2 \cdots \widehat{x}_n^2\widehat{x}_{n-1}^2 \cdot \widehat{x}_{n+1}\widehat{x}_n^2 \cdot \widehat{t}\widehat{x}_{n+1} \\ &\overset{\blacktriangle_2}{=} \widehat{x}_0\widehat{h}\widehat{x}_0 \cdot \widehat{x}_1^2 \cdots \widehat{x}_n^2 \cdot \widehat{x}_{n+1}\widehat{t}\widehat{x}_{n+1} = \mathbf{z}'_n\varphi.\end{aligned}$$

Case 2. $1 = i < j \leqslant n$. Note that the letter x_1 occurs twice in the word $\mathbf{z}_n$. Since $\widehat{x}_1 = \widehat{x}_j$, the element $\widehat{x}_1$ occurs at least thrice in the product $\mathbf{z}_n\varphi$, whence the identities (6.3) can be applied to replace any $\widehat{x}_1$ in $\mathbf{z}_n\varphi$ by $\widehat{x}_1^2$:

$$\mathbf{z}_n\varphi = \widehat{x}_0\widehat{h} \cdot \widehat{x}_1\widehat{x}_0 \cdot \widehat{x}_2\widehat{x}_1 \cdot \widehat{x}_3\widehat{x}_2 \cdots \overset{(6.3)}{=} \widehat{x}_0\widehat{h} \cdot \widehat{x}_1^2\widehat{x}_0 \cdot \widehat{x}_2\widehat{x}_1^2 \cdot \widehat{x}_3\widehat{x}_2 \cdots .$$

The identity (6.4) can then be applied to replace $\widehat{x}_2$ by $\widehat{x}_2^2$:

$$\widehat{x}_0\widehat{h}\cdot\widehat{x}_1^2\widehat{x}_0\cdot\widehat{x}_2\widehat{x}_1^2\cdot\widehat{x}_3\widehat{x}_2\cdots\overset{(6.4)}{=}\widehat{x}_0\widehat{h}\cdot\widehat{x}_1^2\widehat{x}_0\cdot\widehat{x}_2^2\widehat{x}_1^2\cdot\widehat{x}_3\widehat{x}_2^2\cdots.$$

This procedure can be repeated until $\widehat{x}_1,\ldots,\widehat{x}_n$ are replaced by $\widehat{x}_1^2,\ldots,\widehat{x}_n^2$. The equality $\mathbf{z}_n\varphi=\mathbf{z}'_n\varphi$ is then deduced in the same manner as in Case 1. □

Theorem 6.4. *The variety* $\mathbf{N}$ *is non-finitely generated.*

Proof. If the variety $\mathbf{N}$ is finitely generated, then by Lemma 6.3, it satisfies the identity $\mathbf{z}_n\approx\mathbf{z}'_n$ for some $n\geqslant 1$. But this contradicts Lemma 6.2. □

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Поступило October 3, 2013