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SPECTRAL ESTIMATION PROBLEM IN INFINITE DIMENSIONAL SPACES

Abstract. We consider the generalized spectral estimation problem in infinite dimensional spaces. We solve this problem using the boundary control approach to inverse theory and provide an application to the initial boundary value problem for a hyperbolic system.

§1. Introduction

The classical spectral estimation problem consists of the recovery of the coefficients a_n, λ_k, $k=1,\dots,N$, $N\in\mathbb{N}$ of a signal

$$s(t)=\sum_{n=1}^{N} a_k e^{\lambda_k t},\quad t\geqslant 0$$

from the given observations $s(j)$, $j=0,\dots,2N-1$, where the coefficients a_k, λ_k may be arbitrary complex numbers. The literature describing variuos methods for solving the spectral estimation problem is very extensive: see for example the list of references in [6].

In papers [2,3] a new approach to this problem was proposed. In this approach the signal $s(t)$ was treated as a kernel of certain convolution operator corresponding to an input-output map for some linear discrete-time dynamical system. While the system realized from the input-output map is not unique, the coefficients a_n and λ_n can be determined uniquely using the non-selfadjoint version of the boundary control method [1].

Later on the infinite-dimensional version of this method has been developed in [6]. More precisely, the problem of the recovering the coefficients $a_k,\lambda_k\in\mathbb{C}$, $k\in\mathbf{N}$, of the given signal

$$s(t)=\sum_{k=1}^{\infty} a_k e^{\lambda_k t},\quad t\in(0,T),$$

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provided the sum converges in $L_2(0,T)$ was solved there. In the present paper we solve the so-called generalized spectral estimation problem. It is set up in the following way: to recover the coefficients $a_k(t)$, λ_k, $k \in \mathbb{N}$, of a signal

$$S(t)=\sum_{k=1}^{\infty} a_k(t) e^{\lambda_k t}, \quad t \in (0,T), \tag{1.1}$$

from the given data $S \in L_2(0,T)$. We assume that T ia a positive number, $\lambda_k \in \mathbb{C}$ and for each k, $a_k(t)=\sum\limits_{i=0}^{L_k} a_k^i t^i$ are polynomials of the order L_k with complex valued coefficients a_k^i.

In Sec. 2, we recover the unknown parameters λ_k. L_k, a_k^i; $i=0,\dots,L_k$, $k \in \mathbb{N}$, from $S(t)$, $t \in (0,T)$. In Sec. 3, as an application of the generalized spectral estimation, we consider the continuation problem of the inverse dynamical data in the identification problem for the first order hyperbolic system.

§2. Spectral estimation. The case of multiple poles

We consider the dynamical system in a complex Hilbert space H:

$$\dot{x}(t)=Ax(t)+bf(t), \quad t \in (0,T), \quad x(0)=0. \tag{2.1}$$

Here $b \in H$, $f \in L_2(0,T)$, and we assume that the spectrum of the operator A, $\{\lambda_k\}_{k=1}^{\infty}$ is not simple. We denote the algebraic multiplicity of λ_k by L_k, $k \in \mathbb{N}$, and assume also that the set of all root vectors $\{\phi_k^i\}$, $i=1,\dots,L_k$, $k \in \mathbb{N}$, forms a Riesz basis in H. Here the vectors from the chain $\{\phi_k^i\}_{i=1}^{L_k}$, $k \in \mathbb{N}$, satisfy the equations

$$(A-\lambda_k)\,\phi_k^1=0, \quad (A-\lambda_k)\,\phi_k^i=\phi_k^{i-1}, \quad 2 \leqslant i \leqslant L_k.$$

Along with (2.1), we consider the dynamical system for the adjoint operator:

$$\dot{y}(t)=A^*y(t)+dg(t), \quad t \in (0,T), \quad y(0)=0, \tag{2.2}$$

where $d \in H$, $g \in L_2(0,T)$. The spectrum of A^* is $\{\overline{\lambda}_k\}_{k=1}^{\infty}$ and the root vectors $\{\psi_k^i\}_{i=1}^{L_k}$, $i=1,\dots,L_k$, $k \in \mathbb{N}$, also form a Riesz basis in H and satisfy the equations

$$\left(A^*-\overline{\lambda}_k\right)\psi_k^{L_k}=0, \quad \left(A^*-\overline{\lambda}_k\right)\psi_k^i=\phi_k^{i+1}, \quad 1 \leqslant i \leqslant L_k-1.$$

Moreover, the root vectors of A and A^* are normalized in accordance with

$$\langle \phi_k^i, \psi_l^j \rangle = 0 \ \text{ if } \ k \neq l \ \text{ or } \ i \neq j;$$
$$\langle \phi_k^i, \psi_k^i \rangle = 1, \quad i = 1, \dots, L_k, \quad k \in \mathbb{N}.$$

We consider f and g as the inputs of the systems (2.1) and (2.2) and define the outputs z and w by the formulas

$$z(t) = \langle x(t), d \rangle, \quad w(t) = \langle y(t), b \rangle.$$

Suppose that the vector b has a representation $b = \sum\limits_{k=1}^{\infty} \sum\limits_{i=1}^{L_k} b_k^i \phi_k^i$. We look for the solution to (2.1) in the form

$$x(t) = \sum_{k=1}^{\infty} \sum_{i=1}^{L_k} c_k^i(t) \phi_k^i. \tag{2.3}$$

Plugging (2.3) into (2.1), multiplying by ψ_k^i, $i = 1, \dots, L_k$, $k \in \mathbb{N}$, we get the following equations for $c_k^i(t)$:

$$\dot{c}_k^{L_k}(t) = \lambda_k c_k^{L_k}(t) + b_k^{L_k} f(t), \quad c_k^{L_k}(0) = 0,$$
$$\dot{c}_k^i(t) = \lambda_k c_k^i(t) + c_k^{i+1}(t) + b_k^i f(t), \quad c_k^i(0) = 0, \quad i = 1, \dots, L_k - 1.$$

Solving the system of ODEs we find the coefficients $c_k^i(t)$:

$$c_k^{L_k}(t) = \int_0^t e^{\lambda_k (t-\tau)} b_k^{L_k} f(\tau)\, d\tau,$$

$$c_k^{L_k-1}(t) = \int_0^t e^{\lambda_k (t-\tau)} \left[(t-\tau) b_k^{L_k} + b_k^{L_k-1} \right] f(\tau)\, d\tau,$$

$$c_k^{L_k-2}(t) = \int_0^t e^{\lambda_k (t-\tau)} \left[\frac{(t-\tau)^2}{2} b_k^{L_k} + (t-\tau) b_k^{L_k-1} + b_k^{L_k-2} \right] f(\tau)\, d\tau,$$

$$c_k^1(t) = \int_0^t e^{\lambda_k (t-\tau)} \left[\frac{(t-\tau)^{L_k-1}}{(L_k-1)!} b_k^{L_k} + \ldots + (t-\tau) b_k^2 + b_k^1 \right] f(\tau)\, d\tau.$$

Similarly, we represent the vector d in the form $d=\sum_{k=1}^{\infty}\sum_{i=1}^{L_k} d_k^i\psi_k^i$. Then the output z can be written as

$$z(t)=\langle x(t),d\rangle=\sum_{k=1}^{\infty}\sum_{i=1}^{L_k} c_k^i(t)d_k^i=\int_0^t r(t-\tau)f(\tau)\,d\tau,$$

where $r(t)$ is defined as

$$r(t)=\sum_{k=1}^{\infty} e^{\lambda_k t}\Big[a_k^1+a_k^2 t+a_k^3\frac{t^2}{2}+\dots+a_k^{L_k-1}\frac{t^{L_k-2}}{(L_k-2)!}+a_k^{L_k}\frac{t^{L_k-1}}{(L_k-1)!}\Big]. \quad (2.4)$$

Here we introduced the notations

$$a_k^1=\sum_{i=1}^{L_k} b_k^i d_k^i,\quad a_k^2=\sum_{i=2}^{L_k} b_k^i d_k^{i-1},\quad a_k^3=\sum_{i=3}^{L_k} b_k^i d_k^{i-2},\dots\dots a_k^{L_k-1}=\sum_{i=L_k-1}^{L_k} b_k^i d_k^{i-(L_k-2)},\quad a_k^{L_k}=b_k^{L_k}d_k^1,\quad k\in\mathbb{N}. \quad (2.5)$$

It is important to notice that the *response function* $r(t)$ has the form of the series in (1.1).

Looking for the solution of (2.2) in the form

$$y(t)=\sum_{k=1}^{\infty}\sum_{i=1}^{L_k} h_k^i(t)\psi_k^i,$$

we derive the following system of ODEs for $h_k^i(t)$, $i=1,\dots,L_k$, $k\in\mathbb{N}$:

$$\dot h_k^1(t)=\overline{\lambda}_k h_k^1(t)+d_k^1 g(t),\quad h_k^1(0)=0,$$
$$\dot h_k^i(t)=\overline{\lambda}_k h_k^i(t)+h_k^{i-1}(t)+d_k^i g(t),\quad h_k^i(0)=0,\quad i=2,\dots,L_k.$$

Solving this system we obtain the coefficients $h_k^i(t)$:

$$h_k^1(t)=\int_0^t e^{\overline{\lambda}_k(t-\tau)}d_k^1 g(\tau)\,d\tau,$$

$$d_k^2(t) = \int_0^t e^{\overline{\lambda}_k(t-\tau)} \left[(t-\tau)d_k^1 + d_k^2\right] g(\tau)\, d\tau,$$

$$h_k^3(t) = \int_0^t e^{\overline{\lambda}_k(t-\tau)} \left[\frac{(t-\tau)^2}{2} d_k^1 + (t-\tau)d_k^2 + d_k^3\right] g(\tau)\, d\tau,$$

$$h_k^{L_k}(t) = \int_0^t e^{\overline{\lambda}_k(t-\tau)} \left[\frac{(t-\tau)^{L_k-1}}{(L_k-1)!} d_k^1 + \ldots + (t-\tau)d_k^{L_k-1} + d_k^{L_k}\right] g(\tau)\, d\tau,$$

The output of the system (2.2) is given by

$$w(t) = \langle\, y(t), b\rangle = \sum_{k=1}^{\infty}\sum_{i=1}^{L_k} h_k^i(t) b_k^i = \int_0^t \overline{r(t-\tau)} g(\tau)\, d\tau.$$

We introduce now the *connecting operator* $C^T : L_2(0,T) \mapsto L_2(0,T)$ defined through its bilinear form by the formula:

$$\langle\, C^T f, g\rangle = \langle\, x(T), y(T)\rangle.$$

Lemma 1. *The connecting operator* C^T *has a representation* $(C^T f)(t) = (Rf)(2T-t)$, *or*

$$(C^T f)(t) = \int_0^T r(2T - t - \tau) f(\tau)\, d\tau.$$

Proof. We introduce the function $\chi(s,t) := (x(s), y(t))_H$. It is straightforward to check that for $s, t > 0$, this function satisfies the equation

$$\chi_t(s,t) - \chi_s(s,t) = (r * f)(s) g(t) - (r * g)(t) f(s)$$

with the boundary conditions $\chi(0,t) = \chi(s,0) = 0$. This initial boundary value problem can be solved explicitly. Since $x(T)$ and $y(T)$ are independent of the value of $f(t)$ and $g(t)$ for $t > T$, we may put $f(t) = g(t) = 0$, if $t > T$, when compute $(C^T f, g)_H$. Taking this into account, we obtain:

$$\langle\, C^T f, g\rangle = \chi(T,T) = \int_0^T \int_0^{2T-\gamma} r(2T - \gamma - \tau) f(\tau) g(\gamma)\, d\tau\, d\gamma,$$

and therefore,

$$(C^T f)(t) = \int_0^{2T-t} r(2T-t-\tau) f(\tau)\, d\tau = \int_0^T r(2T-t-\tau) f(\tau)\, d\tau. \quad (2.6)$$

□

Next, we demonstrate how to find λ_k, L_k and a_k^i, $i = 1, \dots, L_k$, $k \in \mathbb{N}$, given the function $r(t)$ in the form (1.1). To do that we use the ideas of the boundary control method, more precisely, the possibility to extract the spectral data from the dynamical data (see [7,8]). We assume that the system (2.1) is spectrally controllable in time T. This means that, for any $i \in \{1, \dots, L_k\}$ and any $k \in \mathbb{N}$, there exists $\{f_k^i\} \in H_0^1(0,T)$, such that $x^{f_k^i}(T) = \phi_k^i$. By the definition of $\{f_k^i\}$,

$$\dot{x}^{f_k^1}(T) = Ax^{f_k^1}(T) + bf_k^1(T) = A\phi_k^1 = \lambda_k \phi_k^1 = \lambda_k x^{f_k^1}(T), \quad k \in \mathbb{N}, \quad (2.7)$$

$$\dot{x}^{f_k^i}(T) = A\phi_k^i = \lambda_k \phi_k^i + \phi_k^{i-1} = \lambda_k x^{f_k^i}(T) + x^{f_k^{i-1}}(T), \quad i = 2, \dots, L_k, \quad k \in \mathbb{N}. \quad (2.8)$$

The definition of the operator C^T and equations (2.7) imply that for any $g \in L_2(0,T)$ one has

$$\langle C^T \dot{f}_k^1, g \rangle = \langle x^{\dot{f}_k^1}(T), y^g(T) \rangle = \langle \dot{x}^{f_k^1}(T), y^g(T) \rangle = \langle \lambda_k x^{f_k^1}(T), y^g(T) \rangle = \langle \lambda_k C^T f_k^1, g \rangle, \quad k \in \mathbb{N}.$$

Similarly, making use of (2.8) for $k = 1, \dots \infty$, $2 \leqslant i \leqslant L_k$, we obtain

$$\langle C^T \dot{f}_k^i, g \rangle = \langle x^{\dot{f}_k^i}(T), y^g(T) \rangle = \langle \dot{x}^{f_k^i}(T), y^g(T) \rangle = \langle \lambda_k x^{f_k^i}(T) + x^{f_k^{i-1}}(T), y^g(T) \rangle = \langle \lambda_k C^T f_k^i + C^T f_k^{i-1}, g \rangle.$$

Using (2.6), one gets the following integral eigenvalue equations for finding λ_k and f_k^i, $1 \leqslant i \leqslant L_k$, $k \in \mathbb{N}$:

$$\int_0^T r(2T-t-\tau) \dot{f}_k^1(\tau) - \lambda_k r(2T-t-\tau) f_k^1(\tau)\, d\tau = 0,$$

$$\int_0^T r(2T-t-\tau) \dot{f}_k^i(\tau) - \lambda_k r(2T-t-\tau) f_k^i(\tau) - r(2T-t-\tau) f_k^{i-1}(\tau)\, d\tau = 0.$$

Integrating by parts we finally have:

$$\int_0^T \dot r(2T-t-\tau)f_k^1(\tau) - \lambda_k r(2T-t-\tau)f_k^1(\tau)\,d\tau = 0,$$

$$\int_0^T \dot r(2T-t-\tau)f_k^i(\tau) - \lambda_k r(2T-t-\tau)f_k^i(\tau) - r(2T-t-\tau)f_k^{i-1}(\tau)\,d\tau = 0.$$

This leads to the following conclusion: the set λ_k, f_k^i, $i=1,\dots,L_k$, $k\in\mathbb{N}$, are eigenvalues and root vectors of the following generalized eigenvalue problem in $L_2(0,T)$:

$$\int_0^T \dot r(2T-t-\tau)f(\tau) - \lambda r(2T-t-\tau)f(\tau)\,d\tau = 0. \tag{2.9}$$

Using the same arguments we can deduce that $\overline{\lambda}_k$, g_k^i, $k=1,\dots\infty$, $i=1,\dots,L_k$ are eigenvalues and root vectors of the eigenvalue problem

$$\int_0^T \overline{\dot r(2T-t-\tau)}g(\tau) - \lambda\overline{r(2T-t-\tau)}g(\tau)\,d\tau = 0. \tag{2.10}$$

We notice that solving (2.9) and (2.10) yields eigenvalues λ_k, their multiplicities L_k, $k\in\mathbb{N}$, and non-normalized functions f_k^i and g_k^i for which $x^{f_k^i}(T)=\alpha_k^i\phi_k^i$, $y^{g_k^i}(T)=\beta_k^i\psi_k^i$, with some (unknown) constants α_k^i, β_k^i.

Now we describe the algorithm of recovering $a_k^1,\dots a_k^{L_k}$, $k\in\mathbb{N}$ (see the representation (2.4)). We normalize the solutions to (2.9), (2.10) by the rule

$$\langle C^T\widetilde{f}_k^i, \widetilde{g}_k^i\rangle = 1. \tag{2.11}$$

So if $x^{f_k^i}(T)=\phi_k^i$ and $y^{g_k^i}(T)=\psi_k^i$, then $x^{\widetilde{f}_k^i}(T)=\alpha_k^i\phi_k^i$ and $y^{\widetilde{g}_k^i}(T)=\frac{1}{\alpha_k^i}\psi_k^i$. In the case we define

$$\widetilde{b}_k^i = \langle y^{\widetilde{g}_k^i}(T), b\rangle = \int_0^T \overline{r}(T-\tau)\widetilde{g}_k^i(\tau)\,d\tau, \tag{2.12}$$

$$\widetilde{d}_k^i = \langle x^{\widetilde{f}_k^i}(T), d\rangle = \int_0^T r(T-\tau)\widetilde{f}_k^i(\tau)\,d\tau, \tag{2.13}$$

then (see (2.5))

$$a_k^1 = \sum_{i=1}^{L_k} \widetilde{b}_k^i \widetilde{d}_k^i. \tag{2.14}$$

Denote by ∂ and I the operator of differentiation and unitary operator. Bearing in mind (2.9), which we rewrite as $C^T\left(\partial - \lambda_k I\right) f_k^i = C^T f_k^{i-1}$, we evaluate

$$\langle C^T\left(\partial - \lambda_k I\right) \widetilde{f}_k^i, \widetilde{g}_k^{i-1}\rangle = \alpha_k^i \langle C^T f_k^{i-1}, \widetilde{g}_k^{i-1}\rangle = \frac{\alpha_k^i}{\alpha_k^{i-1}}.$$

So, normalizing the solutions to (2.9), (2.10) by the rule

$$\langle C^T\left(\partial - \lambda_k I\right) \widehat{f}_k^i, \widehat{g}_k^{i-1}\rangle = 1,$$

we can define

$$\widehat{b}_k^i = \int_0^T \overline{r}(T-\tau)\widehat{g}_k^i(\tau)\, d\tau, \tag{2.15}$$

$$\widehat{d}_k^i = \int_0^T r(T-\tau)\widehat{f}_k^i(\tau)\, d\tau. \tag{2.16}$$

and compute $a_k^2 = \sum\limits_{i=2}^{L_k} \widehat{b}_k^i \widehat{d}_k^{i-1}$, cf. (2.5).

Notice that since C^T commutes with the differentiation, we have for $l < i$: $\left[C^T\left(\partial - \lambda_k I\right)\right]^l f_k^i = C^T f_k^{i-l}$. Then

$$\left\langle \left[C^T\left(\partial - \lambda_k I\right)\right]^l \widetilde{f}_k^i, \widetilde{g}_k^{i-l}\right\rangle = \alpha_k^i \langle C^T f_k^{i-l}, \widetilde{g}_k^{i-l}\rangle = \frac{\alpha_k^i}{\alpha_k^{i-l}}.$$

Again, normalizing the solutions to (2.9), (2.10) (for $i > l$) by the rule

$$\left\langle \left[C^T\left(\partial - \lambda_k I\right)\right]^l \widehat{f}_k^i, \widehat{g}_k^{i-l}\right\rangle = 1, \tag{2.17}$$

we define $\widehat{b}_k^i$, $\widehat{d}_k^i$ by (2.12), (2.13) and evaluate

$$a_k^l = \sum_{i=l}^{L_k} \widehat{b}_k^i \widehat{d}_k^{i-l}. \tag{2.18}$$

We conclude this section with the algorithm for solving the spectral estimation problem: suppose that we are given with the function $r \in L_2(0, 2T)$

of the form (2.4) and the family $\bigcup_{k=1}^{\infty}\{e^{\lambda_k t},\dots,t^{L_k-1}e^{\lambda_k t}\}$ is minimal in $L_2(0,T)$. Then to recover λ_k, L_k and coefficients of polynomials, one should follow the

Algorithm

a) solve generalized eigenvalue problems (2.9), (2.10) to find λ_k, L_k and non-normalized controls.
b) Normalize $\widetilde{f}_k^i, \widetilde{g}_k^i$ by (2.11), define $\widetilde{b}_k^i$, $\widetilde{d}_k^i$ by (2.12), (2.13) to recover a_k^1 by (2.14)(see (2.4), (2.5))
c) Normalize $\widehat{f}_k^i, \widehat{g}_k^i$ by (2.17), define $\widetilde{b}_k^i$, $\widehat{d}_k^i$ by (2.15), (2.16) to recover a_k^l by (2.18)(see (2.4), (2.5))

§3. Continuation of the inverse data for the first order hyperbolic system

We consider the initial boundary value problem

$$\frac{\partial}{\partial t}\begin{pmatrix}u\\v\end{pmatrix}-\frac{\partial}{\partial x}\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}u\\v\end{pmatrix}-\begin{pmatrix}p_{11}&p_{12}\\p_{21}&p_{22}\end{pmatrix}\begin{pmatrix}u\\v\end{pmatrix}=0,\qquad 0\leqslant x\leqslant 1,\quad t>0, \tag{3.1}$$

$$u(0,t)=u(1,t)=0,\quad t>0, \tag{3.2}$$

$$\begin{pmatrix}u(x,0)\\v(x,0)\end{pmatrix}=\begin{pmatrix}d_1(x)\\d_2(x)\end{pmatrix},\quad 0\leqslant x\leqslant 1. \tag{3.3}$$

Here $p_{ij}\in C^1([0,1];\mathbb{C})$ and $d_1,d_2\in L_2(0,1;\mathbb{C})$. We fix some $T>0$ and define $R(t):=\{v(0,t),v(1,t)\}$, $0\leqslant t\leqslant T$. The problem of the recovering unknown potential p_{ij} and initial state $c_{1,2}$ has been considered in [9, 10], where the authors established the uniqueness result for large enough T. The inverse problem by one measurement for the one-dimensional Schrödinger equation has been considered in [5], and the procedure of the recovering the potential and the initial state has been proposed. Here we focus on the problem of the continuation of the inverse data: we assume that $R(t)$ is known on the interval $(0,T)$ and recover it on the whole real axis.

We introduce the notations $B=\begin{pmatrix}0&1\\1&0\end{pmatrix}$, $P=\begin{pmatrix}p_{11}&p_{12}\\p_{21}&p_{22}\end{pmatrix}$, $D=\begin{pmatrix}d_1\\d_2\end{pmatrix}$, and the operator A acting by the rule

$$A\varphi=\left(B\frac{d}{dx}+P\right)\varphi,\quad 0\leqslant x\leqslant 1$$

with the domain

$$D(A)=\left\{\varphi=\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix}\in H_1(0,1;\mathbb{C}^2)\,|\,\varphi_1(0)=\varphi_1(1)=0\right\}.$$

The adjoint operator

$$A^*\psi=\left(-B\frac{d}{dx}+P^T\right)\psi,\quad 0\leqslant x\leqslant 1,$$

has the domain

$$D(A^*)=\left\{\psi=\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}\in H^1(0,1;\mathbb{C}^2)\,|\,\psi_1(0)=\psi_1(1)=0\right\}.$$

The spectrum of the operator A has the following structure (see [9, 10]): $\sigma(A)=\Sigma_1\cup\Sigma_2$, where $\Sigma_1\cap\Sigma_2=\varnothing$ and there exists $N_1\in\mathbb{N}$ such that

1) Σ_1 consists of $2N_1-1$ eigenvalues including algebraical multiplicities;
2) Σ_2 consists of infinite number of eigenvalues of multiplicity one;
3) root vectors of A form a Riesz basis in $L_2(0,1;\mathbb{C}^2)$.

Let m denote the algebraical multiplicity of eigenvalue λ, and we introduce the notations $e_1=\binom{1}{0}$,

$$\Sigma_1=\left\{\lambda^i\in\sigma(A),\quad m_i\geqslant 2,\quad 1\leqslant i\leqslant N\right\},$$
$$\Sigma_2=\left\{\lambda_n\in\sigma(A),\quad \lambda_n\ \text{ is simple },\ n\in\mathbb{Z}\right\}.$$

The root vectors are introduced by the following way:

$$\left(A-\lambda^i\right)\phi_1^i=0,\quad \left(A-\lambda^i\right)\phi_j^i=\phi_{j-1}^i,\quad 2\leqslant j\leqslant m_i,$$
$$\phi_j^i(0)=e_1,\quad \phi_j^i\in D(A),\quad 1\leqslant j\leqslant m_i.$$

For the adjoint operator the following equalities are valid:

$$\left(A^*-\overline{\lambda}^i\right)\psi_{m_i}^i=0,\quad \left(A^*-\overline{\lambda}^i\right)\psi_j^i=\psi_{j+1}^i,\quad 1\leqslant j\leqslant m_i-1,$$
$$\psi_j^i(0)=e_1,\quad \psi_j^i\in D(A^*),\quad 1\leqslant j\leqslant m_i.$$

For the simple eigenvalues we have:

$$(A-\lambda_n)\,\phi_n=0,\quad \left(A^*-\overline{\lambda}_n\right)\psi_n=0,$$
$$\phi_n(0)=\psi_n(0)=e_1,\quad \phi_n\in D(A),\quad \psi_n\in D(A^*).$$

Moreover, the following biorthigonality conditions hold:

$$\left(\phi_j^i,\psi_n\right)=0,\quad \left(\phi_n,\psi_j^i\right)=0,\quad (\phi_k,\psi_n)=0,$$
$$\left(\phi_j^i,\psi_l^k\right)=0\ \text{ if }\ i\neq k\ \text{ or }\ j\neq l.$$

Then we set

$$\rho_j^i = \left(\phi_j^i, \psi_j^i\right), \quad i = i \ldots, N, \quad j = 1, \ldots, m_i,$$
$$\rho_n = (\phi_n, \psi_n), \quad n \in \mathbb{Z},$$

and introduce the spectral data:

$$S(P) = \left\{\lambda^i, m_i, \rho_j^i\right\}_{1 \leqslant i \leqslant N}^{1 \leqslant j \leqslant m_i} \cup \left\{\lambda_n, \rho_n\right\}_{n \in \mathbb{Z}}$$

We represent the initial state as the series:

$$D = \sum_{i=1}^{N} \sum_{j=1}^{m_i} d_j^i \phi_j^i(x) + \sum_{n \in \mathbb{Z}} d_n \phi_n(x). \tag{3.4}$$

We are looking for the solution to (3.1)–(3.3) in the form

$$\begin{pmatrix} u \\ v \end{pmatrix}(x,t) = \sum_{i=1}^{N} \sum_{j=1}^{m_i} c_j^i(t) \phi_j^i(x) + \sum_{n \in \mathbb{Z}} c_n(t) \phi_n(x).$$

Using the method of moments we can derive the system of ODe's for c_j^i, $i \in \{1, \ldots, N\}$, $j \in \{1, \ldots, m_i\}$; c_n, $n \in \mathbb{Z}$ solving which we obtain

$$c_j^i(t) = e^{\lambda_i t} \left[d_j^i + d_{j+1}^i t + d_{j+2}^i \frac{t^2}{2} + \cdots + d_{m_i}^i \frac{t^{m_i - j}}{(m_i - j)!} \right],$$
$$c_n(t) = d_n e^{\lambda_n t}.$$

Notice that the response $\{v(0,t), v(1,t)\}$ has a form depicted in (1.1):

$$v(0,t) = \sum_{i=1}^{N} e^{\lambda_i t} a_i^0(t) + \sum_{n \in \mathbb{Z}} e^{\lambda_n t} d_n \left(\phi_n(0)\right)_2, \tag{3.5}$$

$$v(1,t) = \sum_{i=1}^{N} e^{\lambda_i t} a_i^1(t) + \sum_{n \in \mathbb{Z}} e^{\lambda_n t} d_n \left(\phi_n(1)\right)_2, \tag{3.6}$$

where the coefficients of $a_i^0(t) = \sum_{k=0}^{m_i - 1} \alpha_k^i t^k$ are given by

$$\alpha_0^i = \sum_{l=1}^{m_i} d_l^i \left(\phi_l^i(0)\right)_2, \quad \alpha_1^i = \sum_{l=2}^{m_i} d_l^i \left(\phi_{l-1}^i(0)\right)_2,$$
$$\alpha_2^i = \frac{1}{2}\sum_{l=3}^{m_i} d_l^i \left(\phi_{l-2}^i(0)\right)_2, \ldots, \alpha_k^i = \frac{1}{(k-1)!}\sum_{l=k+1}^{m_i} d_l^i \left(\phi_{l-k}^i(0)\right)_2, \ldots$$
$$\alpha_{m_i-1}^i = \frac{1}{(m_i-1)!} d_{m_i}^i \left(\phi_l^i(0)\right)_2.$$

The coefficients $a_i^1(t)$, $i = 1, \ldots, N$ are defined by the similar formulaes.

We introduce the following

Definition 1. *The state* $D \in L_2\left((0,1);\mathbb{C}^2\right)$ *is generic if all the Fourier coefficients in the expansion* (3.4) *are not equal to zero.*

We assume below that the initial state D is generic. The meaning of this restriction is clear – if the initial state is not generic, say $d_k = 0$ for some $k \in \mathbb{Z}$, the response (3.5), (3.6) does not contain any information on λ_k.

We introduce the notation $U := \binom{u}{v}$ and consider the dynamical system with the boundary control $f \in L_2(\mathbb{R}_+)$

$$\begin{aligned} U_t - AU &= 0, & 0 \leqslant x \leqslant 1, \quad & t > 0, \\ u(0,t) &= f(t), & u(1,t) = 0, \quad & t > 0, \\ U(x,0) &= 0. \end{aligned}$$

It is not difficult to show that this system is exactly controllable in time $T \geq 2$. This implies (see [4]) that the family $\bigcup_{i=1}^{N} \{e^{\lambda_i t}, \ldots, t^{m_i-1}e^{\lambda_i t}\} \cup \{e^{i\lambda_n t}\}_{n\in\mathbb{Z}}$ forms a Riesz basis in a closure of its linear span in $L_2((0,T);\mathbb{C})$. Because of this and the fact that each component of the response $\{v(0,t), v(1,t)\}$ has the form of (1.1), we can apply the method from the previous section and recover λ^i, m_i, coefficients of polynomials $a_i^{0,1}(t)$ $i = 1, \ldots, N$, λ_n, $n \in \mathbb{Z}$. The latter allows one to extend the inverse data $R(t)$ to all values of $t \in \mathbb{R}$ by formulas (3.5), (3.6). This is important for solving the identification problem, see [10].

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