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THE YOGA OF COMMUTATORS: FURTHER APPLICATIONS

Abstract. In the present paper we describe some recent applications of localisation methods to the study of commutators in the groups of points of algebraic and algebraic-like groups, such as $\mathrm{GL}(n, R)$, Bak's unitary groups $\mathrm{GU}(2l, R, \Lambda)$ and Chevalley groups $G(\Phi, R)$. In particular, we announce multiple relative commutator formula and general multiple relative commutator formula, as well as results on the bounded width of relative commutators in elementary generators. We also state some of the intermediate results as well as some corollaries of these results. At the end of the paper we attach an updated list of unsolved problems in the field.

The present paper is a direct sequel of [39]. Its goal is to announce some major advances we achieved in 2010–2013, after the publication of [39], with the use of the new localisation methods described therein. Namely,

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Key words and phrases: Unitary groups, Chevalley groups, elementary subgroups, elementary generators, localisation, relative subgroups, conjugation calculus, commutator calculus, Noetherian reduction, Quillen–Suslin lemma, localisation-completion, commutator formulae, commutator width, nilpotency of K_1, nilpotent filtration.

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in [39] we described the methods themselves, and stated three typical applications to the study of commutators in algebraic-like groups:

- **relative standard commutator formulae** [45, 46, 50, 118, 120];
- **universal length bound for commutators**, [42, 86, 92, 94];
- **nilpotent structure of relative** K_1, [9, 11, 36, 43].

In the present paper we briefly outline some further fragments of commutator calculus and state three fresh applications of these methods to the study of commutators in algebraic-like groups:

- **multiple commutator formula** [47, 49, 50];
- **general multiple commutator formula** [41, 49];
- **relative commutator length**, [40, 42, 90].

In particular, this solves several of the problems stated in [39]. Moreover, we describe two further recent relative versions of localisation methods themselves:

- **relative Quillen–Suslin principle** [6];
- **relative localisation-completion** [41].

Also, we state some subsidiary results, which are not directly based on localisation methods, but rather are pure group theory. Nevertheless, we believe they are fun in themselves and could be useful in further studies of commutators in algebraic-like groups. Here are three typical such results for the relative elementary subgroups, which we discuss in the present paper:

- **generation of relative commutator subgroups**, [47–49, 51].
- **reduction of multiple commutators to double commutators** [40, 41, 47, 48];
- **relative splitting principle** [6].

At the end of the paper we attach a thoroughly updated and extended list of unsolved problems in the area.

This paper is based on our joint plenary talks at the following conferences:

- Advances in Group Theory and Applications (Porto Cesareo, Italy, June 2011, see [40]),
- USTC/PDMI Algebra and Dynamic System Workshop (Hefei, China, November 2011),

• Conference in Mathematics and its Applications (Bangkok, Thailand, December 2011),

• Nonstable classical algebraic K-theory (ICTP, Trieste, Italy, December 2012, 6 one-hour talks).

• 2nd Biennial International Conference in Group Theory (Doguš University, Istambul, Turkey, February 2013),

• Torsors, non-associative algebras and algebraic groups (Fields Institute, Toronto, Canada, June 2013),

• Classical algebraic K-theory (Tata Institute for Fundamental Research, Mumbai, India, July 2013, 8 one-hour talks);

on our talks at the following research seminars: Indian Statistical Insitute, Bangalore, India (February 2011), Institute of Research in Fundamental Sciences, Tehran, Iran (March 2011), Saint Petersburg Algebra Seminar (= Faddeev's Seminar, PDMI, April 2012), Moscow State Univ. (October 2012), Colloquium Univ. Mainz (November 2012), Univ. Milano II, Bicocca (November, 2012), Univ. Padova (December 2012), Univ. Autonoma di Madrid (January 2013), Univ. Waterloo (June 2013), Univ. Ottawa (June 2013);

and our *scheduled* invited talks at the following conferences:

• Group Theory Conference (Ischia, March 2012),

• 4th Group Theory Conference (Isfahan, March 2012),

• Polynomial Computer Algebra (St. Petersburg, April 2012).

§1. The groups

For reader's convenience we reproduce with minor changes the first section of [39]. As there, we consider algebraic-like or classical-like group functors G. Further, let $G(R)$ be the group of points of G over a ring R or a form ring (R, Λ). Observe, that groups of types other than A_l only exist over commutative rings. Typically, $G(R)$ is one of the following groups.

A. General linear group $\mathrm{GL}(n, R)$ of degree n over R.

Unlike [39], where many results were already known in this setting, *all* results we discuss in the present paper are already new even for the general linear group. Actually, at present our proofs of some of the more complicated results depend on deep external results, known only in this case.

For GL_n, we use notation from [35, 93, 121]. In this context the ring R does not have to be commutative. Of course, if it isn't, we still have to impose *some* commutativity conditions for our results to hold. One of our favourite classes are quasi-finite rings. Recall, that a ring R is called *almost commutative* – or, sometimes, *module finite* – if it is finitely generated as a module over its centre. *Quasi-finite* rings are direct limits of inductive systems of almost commutative rings. To avoid unnecessary repetions, in the sequel, speaking of ideals of an associative ring R, we always mean *two-sided* ideals of R.

The two other more general contexts, for which we already have some of these new results, or substantial parts thereof, are as follows.

B. Bak's unitary groups $\mathrm{GU}(2n, A, \Lambda)$, over a form ring (A, Λ).

The notation we use for these groups, their subgroups and elements are mostly standard. As in [14], in the case of hyperbolic unitary groups we number columns and rows of matrices as follows: $1, \dots, n, -n, \dots, -1$. Recall, that in this setting A is a [not necessarily commutative] ring with involution $\bar{\ } : A \longrightarrow A$, Λ is the form parameter. To somewhat simplify matters, we usually assume that A is module finite over a commutative ring R. In general, Λ is not an R-module. Thus, R has to be replaced by its subring R_0, generated by $\xi\overline{\xi}$, for $\xi \in R$. One can find all necessary background and many further references in [14, 35–37, 44, 59, 80–82]. On the other hand, all technical details are discussed in [42, 45, 47, 49] where one can find precise statements of auxiliary results and conclusive proofs.

Another favourite setting in our papers is that of Chevalley groups, see [1–3, 78, 91, 113, 117] for basic definitions and many further references.

C. Chevalley groups $G(\Phi, R)$ of type Φ over R. Chevalley groups are indeed *algebraic*, and the ground rings are *commutative* in this case, which usually makes life easier. The universal localisation works smoothly here, and we illustrate it in this example.

Together with the algebraic-like group $G(R)$ we consider the following subgroups.

• First of all, the elementary group $E(R)$, generated by elementary unipotents.

∘ In the linear case, the elementary generators are elementary [linear] transvections $t_{ij}(\xi)$, $1 \leqslant i \neq j \leqslant n$, $\xi \in R$.

∘ In the unitary case, the elementary generators are elementary unitary transvections $T_{ij}(\xi)$, $1 \leqslant i \neq j \leqslant -1$, $\xi \in A$. In the even hyperbolic case they come in two modifications. They can be short root type, $i \neq \pm j$, when the parameter ξ can be any element of A. On the other hand, for the long root type $i = -j$ and the parameter ξ must belong to [something defined in terms of] the form parameter Λ.

∘ Finally, for Chevalley groups, the elementary generators are the elementary root unipotents $x_\alpha(\xi)$ for a root $\alpha \in \Phi$ and a ring element $\xi \in R$.

Further, let $I \trianglelefteq R$ be an ideal of R. We also consider the following relative subgroups. (For unitary groups, see [14] for precise definitions).

• The elementary group $E(I)$ of level I, generated by elementary unipotents of level I.

• The relative elementary group $E(R, I) = E(I)^{E(R)}$ of level I.

• The principal congruence subgroups $G(R, I)$ of level I, the kernel of reduction homomorphism $\rho_I : G(R) \longrightarrow G(R/I)$.

• The full congruence subgroups $C(R, I)$ of level I, the inverse image of the centre of $G(R/I)$ with respect to ρ_I.

Recall the usual notation for these groups in the above contexts A–C.

$G(R)$	$\mathrm{GL}(n, R)$	$\mathrm{GU}(n, R, \Lambda)$	$G(\Phi, R)$
$E(R)$	$E(n, R)$	$\mathrm{EU}(n, R, \Lambda)$	$E(\Phi, R)$
$E(I)$	$E(n, I)$	$\mathrm{FU}(n, I, \Gamma)$	$E(\Phi, I)$
$E(R, I)$	$E(n, R, I)$	$\mathrm{EU}(n, I, \Gamma)$	$E(\Phi, R, I)$
$G(R, I)$	$\mathrm{GL}(n, R, I)$	$\mathrm{GU}(n, I, \Gamma)$	$G(\Phi, R, I)$
$C(R, I)$	$C(n, R, I)$	$\mathrm{CU}(n, I, \Gamma)$	$C(\Phi, R, I)$

There are two more general contexts, where Quillen–Suslin localisation method has been successfully used by Victor Petrov, Anastasia Stavrova, and Alexander Luzgarev, [73, 80–83, 90].

D. Isotropic reductive groups $G(R)$,

E. Odd unitary groups $U(V, q)$.

We are positive that one could obtain results similar to the ones stated in [39] and in the present paper also in these contexts. However, at present even at the absolute level similar results are somewhat problematic, but several mathematicians world-wide, including ourselves, Ravi Rao, Rabeya Basu, Anastasia Stavrova, and others are presently working towards it.

§ 2. Double commutator formula

The starting point for the main results of the present paper is the [double] relative standard commutator formula. In fact, all of them can be viewed as generalisations of this formula, complements to it, or are directly based thereon. This formula and some of its refinements were already stated in the first installment of this work, see [39, §8]. To motivate what follows, we recall it, adducing somewhat more details.

Theorem 1A. *Let A be a quasi-finite ring, $n \geqslant 3$. Then for any two ideals $I, J \trianglelefteq A$ one has*

$$[E(n, A, I), \mathrm{GL}(n, A, J)] = [E(n, A, I), E(n, A, J)].$$

Theorem 1B. *Let $n \geqslant 3$, R be a commutative ring, (A, Λ) be a form ring such that A is a quasi-finite R-algebra. Further, let (I, Γ) and (J, Δ) be two form ideals of the form ring (A, Λ). Then*

$$[\mathrm{EU}(2n, I, \Gamma), \mathrm{GU}(2n, J, \Delta)] = [\mathrm{EU}(2n, I, \Gamma), \mathrm{EU}(2n, J, \Delta)].$$

Actually, in the *commutative* case the principal congruence subgroup in the left hand side of the equalities can be replaced by the full congruence subgroup. In other words, when R is commutative, one has

$$[E(n, R, I), C(n, R, J)] = [E(n, R, I), E(n, R, J)].$$

Similarly, when A is commutative, one has

$$[\mathrm{EU}(2n, I, \Gamma), \mathrm{CU}(2n, J, \Delta)] = [\mathrm{EU}(2n, I, \Gamma), \mathrm{EU}(2n, J, \Delta)].$$

On the other hand, it is easy to construct non-commutative counter-examples to these stronger assertions, see [75].

Since Chevalley groups of types other than A_l are only defined over commutative rings, we can state the next result with the full congruence subgroup right from the outset. The following result is Theorem 3 of [46].

Theorem 1C. *Let Φ be a reduced irreducible root system, $rk(\Phi) \geqslant 2$. Further, let R be a commutative ring, and $I, J \trianglelefteq R$ be two ideals of R. In the cases $\Phi = \mathrm{C}_2, \mathrm{G}_2$ assume that R does not have residue fields $\mathbb{F}_2$ of 2 elements and in the case $\Phi = \mathrm{C}_l$, $l \geqslant 2$, assume additionally that any $c \in R$ is contained in the ideal $c^2R + 2cR$. Then*

$$[E(\Phi, R, I), C(\Phi, R, J)] = [E(\Phi, R, I), E(\Phi, R, J)].$$

Actually, relative standard commutator formulas can be proven by localisation, as in [45,46,50], and this is precisely the proof which most generalisations are based upon. Otherwise, they can be reduced to the *absolute* standard commutator formulas by level calculations, as in [45,46,120,127]. Of course, the usual proofs of the absolute commutator formulas themselves in this generality involve some forms of localisation, at least in the non-commutative case.

Before proceeding to higher generalisations, we dwell a bit more on the structure and generation of the relative commutator subgroups $[E(R,I),E(R,J)]$ that appear in these theorems. These results are essentially elementary, sheer abstract or algebraic group theory, and do not use localisation. But they are useful and amusing, and serve to motivate, prove or amplify our main theorems.

§3. Levels of relative commutator subgroups

We start with an elementary observation that under some mild restrictions $[E(R,I),E(R,J)]/E(R,I\circ J)$ lives inside $K_1(R,I\circ J)$. In other words,

$$E(R,I\circ J)\leqslant [E(R,I),E(R,J)]\leqslant [E(R,I),G(R,J)] \leqslant [G(R,I),G(R,J)]\leqslant G(R,I\circ J).$$

Here, $I\circ J$ denotes the symmetrised product of ideals/form ideals I and J, which in the commutative case coincides with their usual product. By definition, the symmetrised product of ideals equals $I\circ J=IJ+JI$. It is somewhat trickier to come up with a definition of the symmetrised product of form ideals, which will be recalled below.

These level results were known for some time, but they are fundamental for what follows. For $\mathrm{GL}(n,A)$ the level was first computed in the works by Hyman Bass, Alec Mason and Wilson Stothers [15], [74] – [77], the proofs are also reproduced in [118].

Theorem 2A. *Let A be any associative ring, $n\geqslant 3$, and let $I,J\trianglelefteq A$ be two two-sided ideals of A. Then*

$$E(n,A,I\circ J)\leqslant [E(n,A,I),E(n,A,J)]\leqslant [E(n,R,A),\mathrm{GL}(n,R,B)] \leqslant [\mathrm{GL}(n,A,I),\mathrm{GL}(n,A,J)]\leqslant \mathrm{GL}(n,A,I\circ J).$$

For unitary group, it is *highly* non-trivial even to correctly *state* the analogue of this result, especially in the non-commutative case. In [45] the symmetrised product of two ideals was defined as

$$(I,\Gamma)\circ(J,\Delta)=(IJ+JI,\Gamma_{\min}(IJ+JI)+{}^J\Gamma+{}^I\Delta),$$

where

$$\Gamma_{\min}(I)=\{\xi-\lambda\overline{\xi}\mid \xi\in I\}+\langle \xi\alpha\overline{\xi}\mid \xi\in I,\alpha\in\Lambda\rangle$$

and

$${}^J\Gamma=\langle \xi\Gamma\overline{\xi}\mid \xi\in J\rangle,\quad {}^I\Delta=\langle \xi\Delta\overline{\xi}\mid \xi\in I\rangle.$$

Recall, that [33, 34, 36] and [37] operated with the usual product of form ideals, $(I,\Gamma)(J,\Delta)=(IJ,\Gamma\Delta)$, where by definition $\Gamma\Delta=\Gamma_{\min}(IJ)+{}^J\Gamma+{}^I\Delta$. Clearly, with this notation one indeed has

$$(I,\Gamma)\circ(J,\Delta)=(I,\Gamma)(J,\Delta)+(J,\Delta)(I,\Gamma).$$

For unitary groups, the level was calculated by Günter Habdank [33,34] and then by the first author [36,37], see also [45,48].

Theorem 2B. *Let (I,Γ) and (J,Δ) be two form ideals of a form ring (A,Λ). Then*

$$\begin{aligned}\mathrm{EU}(2n,(I,\Gamma)\circ(J,\Delta))&\leqslant[\mathrm{FU}(2n,I,\Gamma),\mathrm{FU}(2n,J,\Delta)]\\&\leqslant[\mathrm{EU}(2n,I,\Gamma),\mathrm{EU}(2n,J,\Delta)]\leqslant\mathrm{GU}(2n,(I,\Gamma)\circ(J,\Delta)).\end{aligned}$$

Finally, for Chevalley groups [a weaker form of] the corresponding result was first officially stated by You Hong [127, Theorem 1], see also [46, Lemmas 17 and 19]. In the present form, the result was only stated in the very recent paper [48], as a prerequisite to the proof of Theorems 4C and 5C.

Theorem 2C. *Let $\mathrm{rk}(\Phi)\geqslant 2$. In the cases $\Phi=\mathrm{C}_2,\mathrm{G}_2$ assume that R does not have residue fields $\mathbb{F}_2$ of 2 elements and in the case $\Phi=\mathrm{C}_l$, $l\geqslant 2$, assume additionally that any $c\in R$ is contained in the ideal c^2R+2cR.*

Then for any two ideals I and J of the ring R one has the following inclusion

$$\begin{aligned}E(\Phi,R,IJ)&\leqslant[E(\Phi,I),E(\Phi,J)]\leqslant[E(\Phi,R,I),E(\Phi,R,J)]\\&\leqslant[E(\Phi,R,I),G(\Phi,R,J)]\leqslant[G(\Phi,R,I),C(\Phi,R,J)]\leqslant G(\Phi,R,IJ).\end{aligned}$$

For groups of rank 2, these additional assumptions are indeed necessary. It is classically known that when the ground ring R has residue fields of 2 elements, the groups of types C_2 and G_2 are not perfect. Thus, the left-most inclusion fails even at the absolute level, when $I=J=R$.

To explain the relevance of the second assumption, we should distinguish the ideal I^2, generated by the products ab, where $a, b \in I$, from the ideal $I^{\boxed{2}}$, generated by a^2, where $a \in I$. Clearly, when $2 \in R^*$ these ideals coincide, but this case is trivial *anyway*. The second assumption for C_2 is not visible at the absolute level. But without that assumption the upper and lower levels of the relative commutator subgroup $[E(\Phi, R, I), E(\Phi, R, J)]$ do not coincide, so that the left-most inclusion in the above lemma should be replaced by

$$E(\Phi, R, IJ, I^{\boxed{2}}J + 2IJ + IJ^{\boxed{2}}) \leqslant [E(\Phi, R, I), E(\Phi, R, J)].$$

Here, $E(\Phi, R, I, J)$ is the elementary subgroup corresponding to an admissible pair (I, J) in the sense of Abe, where I is an ideal of R, expressing the short root level (= upper level), whereas a Jordan ideal J, expressing the long root level (= lower level), plays the role of a form parameter. Not to complicate things any further, in the sequel we *always* impose these additional restrictions on R, when $\Phi = \mathrm{C}_2, \mathrm{G}_2$. These two cases, especially that of the group $\mathrm{Sp}(4, R)$, require separate analysis anyway, [23], [24].

§4. Relative commutator subgroups are not elementary

In view of Theorem 2, it is natural to ask, whether the commutators of relative elementary subgroups are themselves elementary of the corresponding level, in other words, whether

$$[E(R, I), E(R, J)] = E(R, I \circ J)?$$

This is known to be the case in many important classical situations, for instance, at the absolute level, where $I = R$ or $J = R$. In fact, this equality holds under much weaker assumptions. Specifically, it is easy verified when the ideals I and J are comaximal, $I + J = R$. Let us reproduce the current versions of Theorem 4 of [39], whose proofs in all cases can be now found in print: for GL_n it is [120], Theorem 5; for unitary groups it is [45, Theorem 3]; finally, for Chevalley groups it is [46, Theorem 3].

Theorem 3A. *Let A be a quasi-finite ring, $n \geqslant 3$. Then for any two comaximal ideals $I, J \trianglelefteq A$, $I + J = A$, one has*

$$[E(n, A, I), E(n, A, J)] = E(n, A, I \circ J).$$

Theorem 3B. *Let $n \geqslant 3$, and (A, Λ) be an arbitrary form ring for which absolute standard commutator formulae are satisfied. Then for any two* *comaximal form ideals* (I,Γ) *and* (J,Δ) *of the form ring* (A,Λ), $I+J=A$, *one has the following equality*

$$[\mathrm{EU}(2n,I,\Gamma),\mathrm{EU}(2n,J,\Delta)]=\mathrm{EU}(2n,IJ+JI,{}^J\Gamma+{}^I\Delta+\Gamma_{\min}(IJ+JI)).$$

Theorem 3C. *Let* Φ *be a reduced irreducible root system,* $rk(\Phi)\geqslant 2$. *Further, let* R *be a commutative ring, and* $I,J\trianglelefteq R$ *be two ideals of* R. *In the cases* $\Phi=\mathrm{C}_2,\mathrm{G}_2$ *assume that* R *does not have residue fields* $\mathbb{F}_2$ *of 2 elements. Then for any two comaximal ideals* $I,J\trianglelefteq R$, $I+J=R$, *one has the following equality*

$$[E(\Phi,R,I),E(\Phi,R,J)]=E(\Phi,R,IJ).$$

Observe, that unlike Theorem 1C, in Theorem 3C the extra assumption on R for type $\Phi=\mathrm{C}_l$, $l\geqslant 2$, turned out to be redundant (due to more accurate level calculations in terms of admissible pairs).

But the relative commutator subgroup $[E(R,I),E(R,J)]$ cannot be always elementary, in general. Let us reproduce from [74,76] one such example based on the calculation of relative K_1-functors for Dedekind rings of arithmetic type by Hyman Bass, John Milnor and Jean-Pierre Serre [16]. We do not make any attempt to recall the explicit formula for

$$\mathrm{SK}_1(n,R,I)=\mathrm{SL}(n,R,I)/E(n,R,I)$$

in the general case.

Instead, we cite the explicit answer for the *first* non-trivial case of Gaussian integers $R=\mathbb{Z}[i]$. Consider the prime ideal $\mathfrak{p}=(1+i)R$. Then for any $n\geqslant 3$ and any ideal $I\trianglelefteq R$ one has

$$\mathrm{SK}_1(n,R,I)=\mathrm{SK}_1(n,R,\mathfrak{p}^s),\qquad s=\mathrm{ord}_{\mathfrak{p}}(I).$$

On the other hand,

$$SK_1(n,R,\mathfrak{p}^s)=\begin{cases}1, & s\leqslant 3,\\ 2, & s=4,5,\\ 4, & s\geqslant 6.\end{cases}$$

Now a straightforward calculation shows that

$$\begin{aligned}E(n,\mathbb{Z}[i],\mathfrak{p}^6)&<[E(n,\mathbb{Z}[i],\mathfrak{p}^3),E(n,\mathbb{Z}[i],\mathfrak{p}^3)]\\ &=[\mathrm{SL}(n,\mathbb{Z}[i],\mathfrak{p}^3),\mathrm{SL}(n,\mathbb{Z}[i],\mathfrak{p}^3)]<\mathrm{SL}(n,\mathbb{Z}[i],\mathfrak{p}^6),\end{aligned}$$

where *both* inclusions are strict. In fact, both indices are equal to 2.

This, and many further examples of arithmetic and algebro-geometric nature show that in general the relative commutator subgroup

$$[E(n, A, I), E(n, A, J)]$$

is *strictly larger* than the relative elementary subgroup $E(n, A, I \circ J)$.

In particular, it follows that in general

$$[E(n, A, I), E(n, A, J)] \neq [E(n, A, K), E(n, A, L)]$$

for two pairs of ideals (I, J) and (K, L), such that $I \circ J = K \circ L$. In fact, this already follows from the previous example, for pairs (I, J) and $(K, L) = (I \circ J, A)$, but it is easy to construct many further examples, much fancier than that.

Summarising the above, we can conclude that in general the double relative commutator subgroups do not reduce to relative elementary subgroups, and reveal some new layers of the internal structure of $\mathrm{K}_1(R, I)$.

Amazingly, and this is one of the main new results we wish to report, all higher multiple commutator subgroups reduce to *double* commutator subgroups. In other words, forming successive commutators of relative elementary subgroups *never* results in anything new inside $K_1(R, K)$, apart from the groups

$$[E(R, I), E(R, J)]/E(R, K) \leqslant K_1(R, K),$$

for some other ideals I and J, such that $I \circ J = K$.

§5. Generators of relative commutator subgroups as normal subgroups

Here, we describe generators of relative commutator subgroups $[E(R, I), E(R, J)]$ as normal subgroups of $E(R)$. These results are elementary algebraic group theory, but they are an essential complement to Theorem 1, an important tool in the proof of multiple commutator formula, and the starting point for results on relative commutator width.

By Theorem 2 the relative commutator subgroup $[E(R, I), E(R, J)]$ contains the elementary subgroup $E(R, I \circ J)$. In particular, it contains the generators of that group. However, we know that in general the subgroup $[E(R, I), E(R, J)]$ may be strictly larger, than $E(R, I \circ J)$. Thus, we have to produce the missing generators. As in the case of the relative elementary subgroups $E(R, I)$ themselves, these generators will sit in the fundamental SL_2's and are in fact commutators of *some* elementary generators of $E(\Phi, R, I)$ and $E(\Phi, R, J)$.

Unlike Theorem 1 itself, whose proof crucially depends on localisation, this is a purely group theoretic result that holds over *arbitrary* associative/form/commutative rings. For the general linear group the following result is Lemma 12 of [51].

Theorem 4A. *Let A be an associative ring with 1 and let I, J be two two-sided ideals of A. Then as a normal subgroup of $E(n,A)$, $n\geqslant 3$, the mixed commutator subgroup $[E(n,A,I),E(n,A,J)]$ is generated by the elements of the form*

- $[t_{ji}(\xi),{}^{t_{ij}(\eta)}t_{ji}(\zeta)]$,
- $[t_{ji}(\xi),t_{ij}(\zeta)]$,
- $t_{ij}(\xi\zeta)$ *and* $t_{ij}(\zeta\xi)$,

where $1\leqslant i\neq j\leqslant n$, $\xi\in I$, $\zeta\in J$, $\eta\in A$.

A similar result for unitary groups, Theorem 9 of [47], is somewhat more technical. To somewhat shorten the next statement, we describe conditions on the generators in the form $T_{ji}(\xi)\in \mathrm{EU}(2n,I,\Gamma)$. Recall [14,36,44] that this means that $\xi\in I$, for $i\neq\pm j$, and $\xi\in\Gamma$, for $i=-j$.

Theorem 4B. *Let (A,Λ) be a form ring and (I,Γ), (J,Δ) be two form ideals of (A,Λ). Then as a normal subgroup of $\mathrm{EU}(2n,R,\Lambda)$, $n\geqslant 3$, the mixed commutator subgroup $[\mathrm{EU}(2n,I,\Gamma),\mathrm{EU}(2n,J,\Delta)]$ is generated by the elements of the form*

- $[T_{ji}(\xi),{}^{T_{ij}(\eta)}T_{ji}(\zeta)]$,
- $[T_{ji}(\xi),T_{ij}(\zeta)]$,
- $T_{ij}(\xi\zeta)$ *and* $T_{ij}(\zeta\xi)$,

where $T_{ji}(\xi)\in\mathrm{EU}(2n,I,\Gamma)$, $T_{ji}(\zeta)\in\mathrm{EU}(2n,J,\Delta)$, $T_{ij}(\eta)\in\mathrm{EU}(2n,A,\Lambda)$, *and* $T_{ij}(\theta)\in\mathrm{EU}\,(2n,(I,\Gamma)\circ(J,\Delta))$.

The proof for Chevalley groups is similar, with some additional complications in the rank 2 case. The following result is [48, Theorem 2].

Theorem 4C. *Let $\mathrm{rk}(\Phi)\geqslant 2$ and let I, J be two ideals of a commutative ring R. In the cases $\Phi=\mathrm{C}_2,\mathrm{G}_2$ assume that R does not have residue fields $\mathbb{F}_2$ of 2 elements and in the case $\Phi=\mathrm{C}_l$, $l\geqslant 2$, assume additionally that any $c\in R$ is contained in the ideal c^2R+2cR.*

Then as a normal subgroup of the elementary Chevalley group $E(\Phi,R)$ the mixed commutator subgroup $[E(\Phi,R,I),E(\Phi,R,J)]$ is generated by the elements of the form

- $[x_\alpha(\xi), {}^{x_{-\alpha}(\eta)}x_\alpha(\zeta)]$,
- $[x_\alpha(\xi), x_{-\alpha}(\zeta)]$,
- $x_\alpha(\xi\zeta)$,

where $\alpha \in \Phi$, $\xi \in I$, $\zeta \in J$, $\eta \in R$.

Actually, the proof of this result in [48] replaces most of explicit fiddling with the Chevalley commutator formula and commutator identites, by a reference to some obvious properties of parabolic subgroups, which makes it *considerably* less computational, than the proofs of Theorem 2A and Theorem 2B in [47, 51].

Let us sketch this proof, using this occasion to introduce requisite notation. First of all, observe that these elements indeed belong to the relative commutator subgroups $[E(R,I), E(R,J)]$ by Theorem 2. Next, recall that the elementary generators of the elementary groups $E(R,I)$ themselves are classically known, and look as follows:

- $z_{ji}(\xi,\eta) = t_{ij}(\eta)t_{ji}(\xi)t_{ij}(-\eta)$, for GL_n, see [110],
- $Z_{ji}(\xi,\eta) = T_{ij}(\eta)T_{ji}(\xi)T_{ij}(-\eta)$, for unitary groups, see [14],
- $z_\alpha(\xi,\eta) = x_{-\alpha}(\eta)x_\alpha(\xi)x_{-\alpha}(-\eta)$, for Chevalley groups [3, 91, 102, 106].

Observe, that these generators are precisely the second factors of the first type of generators in the above theorems, and we use this shorthand notation in the sequel.

Let us focus on Theorem 3C. The usual commutator identities imply that as a normal subgroup $[E(\Phi,R,I), E(\Phi,R,J)]$ is generated by the commutators of the form $[z_\alpha(\xi,\eta), z_\beta(\zeta,\theta)]$. Since we are working up to elementary conjugation, we can replace these generators by $[x_\alpha(\xi), {}^{x_{-\alpha}(-\eta)}z_\beta(\zeta,\theta)]$. Since the groups $E(\Phi,R,J)$ are normal in $E(\Phi,R)$, the conjugates ${}^{x_{-\alpha}(-\eta)}z_\beta(\zeta,\theta)$ can be again expressed as products of elementary generators. Once more applying commutator identities, we see that as a normal subgroup $[E(\Phi,R,I), E(\Phi,R,J)]$ is generated by the commutators $[x_\alpha(\xi), z_\beta(\zeta,\theta)]$. At this point, we are left with three options:

- $\alpha = \beta$, and we get the first type of generators,
- $\alpha = -\beta$, and we get the second type of generators, up to conjugation,
- $\alpha \neq \pm\beta$. If α and β are strictly orthogonal, then $[x_\alpha(\xi), z_\beta(\zeta,\theta)] = e$.

Thus, we can assume that α and β generate an irreducible root system of rank 2, and fiddle with the Chevalley commutator formula therein. Alternatively, we can choose an order such that β is fundamental, whereas α is positive. Then $[x_\alpha(\xi), z_\beta(\zeta,\theta)]$ sits inside the unipotent radical U_β of the minimal (=rank 1) standard parabolic subgroup P_β. On the other hand, by Theorem 2C it sits inside $G(\Phi, R, IJ)$. Clearly, $U_\beta \cap G(\Phi, R, IJ) \leqslant E(\Phi, IJ)$. Thus, in this last case $[x_\alpha(\xi), z_\beta(\zeta, \theta)]$ is a product of generators of the third type.

§6. Generators of relative commutator subgroups as groups

However, to state results on relative commutator width we need generators of relative commutator subgroups $[E(R, I), E(R, J)]$ *as groups*, rather than just their generators as normal subgroups of $E(R)$.

Again, this *seems* to be a purely group theoretic result that should be valid over *arbitrary* associative/form/commutative rings. However, the easy proofs we have at this time, only work in the situations, where the standard commutator formula holds.

Theorem 5A. *Let A be a quasi-finite ring, $n \geqslant 3$, and let I, J be two ideals of A. Then the mixed commutator subgroup $[E(n, A, I), E(n, A, J)]$ is generated as a group by the elements of the form*

- $[t_{ji}(\xi), z_{ji}(\zeta, \eta)]$,
- $[t_{ji}(\xi), t_{ij}(\zeta)]$,
- $z_{ij}(\xi\zeta, \eta)$ *or* $z_{ij}(\zeta\xi, \eta)$,

where in all cases $\xi \in I$, $\zeta \in J$, $\eta, \theta \in A$.

Theorem 5B. *Let $n \geqslant 3$, R be a commutative ring, (A, Λ) be a form ring such that A is a quasi-finite R-algebra. Further, let (I, Γ) and (J, Δ) be two form ideals of the form ring (A, Λ).*

Then the mixed commutator subgroup $[\mathrm{EU}(2n, I, \Gamma), \mathrm{EU}(2n, J, \Delta)]$ is generated as a group by the elements of the form

- $[T_{ji}(\xi), Z_{ji}(\zeta, \eta)]$,
- $[T_{ji}(\xi), T_{ij}(\zeta)]$,
- $Z_{ij}(\theta, \eta)$,

where $T_{ji}(\xi) \in \mathrm{EU}(2n, I, \Gamma)$, while $T_{ij}(\zeta), Z_{ji}(\zeta, \eta) \in \mathrm{EU}(2n, J, \Delta)$, and $Z_{ij}(\theta, \eta) \in \mathrm{EU}\,(2n, (I, \Gamma) \circ (J, \Delta))$.

Theorem 5C. *Let $\mathrm{rk}(\Phi) \geqslant 2$ and let I, J be two ideals of a commutative ring R. In the cases $\Phi = \mathrm{C}_2, \mathrm{G}_2$ assume that R does not have residue fields $\mathbb{F}_2$ of 2 elements and in the case $\Phi = \mathrm{C}_l$, $l \geqslant 2$, assume additionally that any $c \in R$ is contained in the ideal $c^2R + 2cR$.*

Then the mixed commutator subgroup $[E(\Phi,R,I),E(\Phi,R,J)]$ *is generated as a group by the elements of the form*

- $[x_\alpha(\xi),z_\alpha(\zeta,\eta)]$,
- $[z_\alpha(\xi),z_{-\alpha}(\zeta)]$,
- $z_\alpha(\xi\zeta,\eta)$, *where* $\alpha\in\Phi$, $\xi\in I$, $\zeta\in J$, $\eta,\in R$.

Let us sketch the proof of Theorem 5C. From this proof, it will be clear, why a similar slick argument does not prove Theorem 5A and Theorem 5B for arbitrary associative rings or arbitrary form rings.

The set described in this theorem contains the set described in Theorem 2C, which already generates $[E(\Phi,R,I),E(\Phi,R,J)]$ as a normal subgroup of $E(\Phi,R)$. Therefore, it suffices to show that elementary conjugates of the above generators are themselves products of such generators. Let g be one of these generators and let $h\in E(\Phi,R)$. By Theorem 2C, one has $g\in G(\Phi,R,IJ)$. Now the [absolute] standard commutator formula implies that

$$[h,g]\in[G(\Phi,R,IJ),E(\Phi,R)]=E(\Phi,R,IJ).$$

Being an element $E(\Phi,R,IJ)$, the commutator $[h,g]$ is a product of some elementary generators $z_\alpha(\xi\zeta,\eta)$, where $\alpha\in\Phi$, $\xi\in I$, $\zeta\in J$, $\eta\in R$. Thus, any conjugate $hgh^{-1}=[h,g]g$ is a product of some generators of the third type and the generator g itself.

In fact, mostly this argument relied on *elementary* calculations, such as the one needed to prove Theorem 2C and Theorem 3C. But at one instance we had to invoke a special case of Theorem 1C, the [absolute] standard commutator formula. This last result is not elementary, and certainly it does not hold over arbitrary associative rings. There are explicit counterexamples to the standard commutator formula in this generality, the first of them by Victor Gerasimov [31].

It seems incongruous that [what appears to be] a pure group theoretic result should depend on commutativity conditions. This poses the followimg problem.

Problem 1. *Find elementary proofs of Theorems* 5A *and* 5B *that work over arbitrary associative rings/form rings.*

By juggling with commutator identities, we succeeded in proving a slightly weaker version of Theorem 5A, with a somewhat larger set of generators, all of them still sitting inside fundamental GL_2's. However, a straightforward calculation, based on induction on the length of the conjugating element, is so long and appalling, that it strongly discoraged us from any attempt to prove the technically much fancier Theorem 5B for arbitrary form rings along these lines.

A closer look at the generators in Theorems 5A–5C shows that all of them in fact belong already to $[E(\Phi, I), E(\Phi, R, J)]$. By symmetry, we may switch the role of factors. In particular, this means that Theorems 5A–5C imply the following curious corollaries.

Corollary 1A. *Let A be a quasi-finite ring, $n \geqslant 3$, and let I, J be two ideals of R. Then one has*

$$[E(n, I), E(n, R, J)] = [E(n, R, I), E(n, J)] = [E(n, R, I), E(n, R, J)].$$

Corollary 1B. *Let $n \geqslant 3$, R be a commutative ring, (A, Λ) be a form ring such that A is a quasi-finite R-algebra. Further, let (I, Γ) and (J, Δ) be two form ideals of the form ring (A, Λ). Then one has*

$$\begin{aligned}[\mathrm{FU}(2n, I, \Gamma), \mathrm{EU}(n, J, \Delta)] &= [\mathrm{EU}(2n, I, \Gamma), \mathrm{FU}(n, J, \Delta)] \\ &= [\mathrm{EU}(2n, I, \Gamma), \mathrm{EU}(n, J, \Delta)].\end{aligned}$$

Corollary 1C. *Let $\mathrm{rk}(\Phi) \geqslant 2$ and let I, J be two ideals of a commutative ring R. In the cases $\Phi = \mathrm{C}_2, \mathrm{G}_2$ assume that R does not have residue fields $\mathbb{F}_2$ of 2 elements and in the case $\Phi = \mathrm{C}_l$, $l \geqslant 2$, assume additionally that any $c \in R$ is contained in the ideal $c^2R + 2cR$. Then one has*

$$[E(\Phi, I), E(\Phi, R, J)] = [E(\Phi, R, I), E(\Phi, J)] = [E(\Phi, R, I), E(\Phi, R, J)].$$

§7. Higher commutators

Once we understand double commutators, it is natural to consider higher commutators of relative elementary subgroups and congruence subgroups. Let G be a group and $H_1, \dots, H_m \leqslant G$ be its subgroups. There are many ways to form a higher commutator of these groups, depending on where we put the brackets. Thus, for three subgroups $F, H, K \leqslant G$ one can form two triple commutators $[[F, H], K]$ and $[F, [H, K]]$. For four subgroups $F, H, K, L \leqslant G$ one can form 5 such commutators

$$[[[F, H], K], L], \qquad [[F, [H, K]], L], \qquad [[F, H], [K, L]], \qquad [F, [H, [K, L]]], \qquad [F, [[H, K], L]].$$

As is well known, for m subgroups one can form c_{m-1}, where c_m is the m-th Catalan number.

Usually, we write $[H_1, H_2, \dots, H_m]$ for the *left-normed* commutator, defined inductively by

$$[H_1, \dots, H_{m-1}, H_m] = [[H_1, \dots, H_{m-1}], H_m].$$

To stress that we consider any commutator of these subgroups, with an arbitrary placement of brackets, we write $[\![H_1, H_2, \dots, H_m]\!]$. Thus, for instance, $[\![F, H, K, L]\!]$ refers to any of the five arrangements above.

Actually, a specific arrangment of brackets usually does not play major role in our results – and in fact any role whatsoever over commutative rings! – apart from one important attribute. Namely, what will matter a lot is the position of the outermost pairs of inner brackets. Namely, every higher commutator subgroup $[\![H_1, H_2, \dots, H_m]\!]$ can be uniquely written as

$$[\![H_1, H_2, \dots, H_m]\!] = [[\![H_1, \dots, H_h]\!], [\![H_{h+1}, \dots, H_m]\!]],$$

for some $h = 1, \dots, m-1$. This h will be called the cut point of our multiple commutator. Thus, among the quadruple commutators $[\![F, H, K, L]\!]$, two arrangements, $[[[F, H], K], L]$ and $[[F, [H, K]], L]$, cut at 3; one, $[[F, H], [K, L]]$, cuts at 2; and the remaining two, $[F, [H, [K, L]]]$ $[F, [[H, K], L]]$, cut at 1.

Now, let I_i, $i = 1, \dots, m$, be ideals of the ring R. Our ultimate objective is to compute the commutator subgroups of congruence subgroups

$$[\![G(R, I_1), G(R, I_2), \dots, G(R, I_m)]\!],$$

but that is a *highly* strenuous enterprise. So far[1], we have done it for the case $G = \mathrm{SL}_n$, provided that $m > \delta(R)$.

In the next section we embark on the [somewhat easier] calculation of higher commutators of relative elementary subgroups

$$[\![E(R, I_1), E(R, I_2), \dots, E(R, I_m)]\!].$$

Even this turns out to be a rather non-trivial task. In fact, we do not see any other way to do that, but to prove a higher analogue of the standard commutator formula, viz.

$$[\![E(R, I_1), G(R, I_2), \dots, G(R, I_m)]\!] = [\![E(R, I_1), E(R, I_2), \dots, E(R, I_m)]\!].$$

This multiple commutator formula will be discussed in the next two sections. Unlike the *general* multiple commutator formula in which we are ultimately interested, and which only works for finite-dimensional rings, this weaker formula holds over arbitrary quasi-finite/commutative rings.

[1]In the revised version of [92] this is done for all simply connected Chevally groups.

Amazingly, the resulting *multiple* commutator subgroups will always coincide with some *double* relative commutator subgroups, depending not on the ideals I_i themselves, but only on two symmetrised products of these ideals. Since the symmetrised product of ideals is not associative, some traces of the initial arrangment will still be visible in these symmetrised products. However, for commutative rings the symmetrised product becomes the usual product of ideals, which is associative, so that the result will not depend on the arrangment itself either, but only on its cut point. We discuss these results in §§9,10.

§8. Multiple commutator formula

The following theorem is the main result of the paper [51] by the first and the fourth authors. Initially, it was conceived as *part* of the answer to a problem proposed by the second and the third author [118, 120]. As a matter of fact, it turned out to be of significant independent interest. The proof of the following result in [51] is based on a further enhancement of relative localisation which we outline in the next section.

Theorem 6A. *Let $n \geqslant 3$, let A be a quasi-finite ring with 1 and let $I_i \trianglelefteq A$, $i = 1, \dots, m$, be ideals of A. Then one has*

$$\llbracket E(n,R,I_1), \mathrm{GL}(n,R,I_2), \dots, \mathrm{GL}(n,R,I_m)\rrbracket = \llbracket E(n,R,I_1), E(n,R,I_2), \dots, E(n,R,I_m)\rrbracket.$$

In this theorem the arrangement of brackets on the left hand side may be arbitrary. But it is essential that the placement of brackets on the right hand side coincides with that on the left hand side. Without this assumption the equality may fail dramatically, even if all factors are elementary, as we shall see in § 10. Of course, the same observation applies to the theorems below.

For unitary groups, similar result is established in [47], by essentially the same method. However, as one could expect, the necessary calculations are tangibly more complicated and require a completely different level of technical strain.

Theorem 6B. *Let $n \geqslant 3$ and let (A, Λ) be a form ring such that A is a quasi-finite R-algebra over a commutative ring R. Further, let (I_i, Γ_i),*

$i = 1, \dots, m$, be form ideals of (A, Λ). Then

$$\begin{aligned}&[\![\mathrm{EU}(2n, I_1, \Gamma_1), \mathrm{GU}(2n, I_2, \Gamma_2), \dots, \mathrm{GU}(2n, I_m, \Gamma_m)]\!]\\ &\qquad = [\![\mathrm{EU}(2n, I_1, \Gamma_1), \mathrm{EU}(2n, I_2, \Gamma_2), \dots, \mathrm{EU}(2n, I_m, \Gamma_m)]\!].\end{aligned}$$

Finally, let us pass to Chevalley groups. We believe that at this point we possess two independent proofs of the following result. One of them, by the first, the third and the fourth authors, is conventional, and involves an further elaboration of the relative commutator calculus in the style of [46]. Another one, by the second author, is somewhat shorter, and employs his method of universal localisation [92][2].

Theorem 6C. *Let $\mathrm{rk}(\Phi) \geqslant 2$ and let $I_i \trianglelefteq R$, $i = 1, \dots, m$, be ideals of a commutative ring R. In the cases $\Phi = \mathrm{C}_2, \mathrm{G}_2$ assume that R does not have residue fields $\mathbb{F}_2$ of 2 elements and in the case $\Phi = \mathrm{C}_l$, $l \geqslant 2$, assume additionally that any $c \in R$ is contained in the ideal $c^2R + 2cR$.*

Then one has

$$\begin{aligned}&[\![E(\Phi, R, I_1), G(\Phi, R, I_2), \dots, G(\Phi, R, I_m)]\!]\\ &\qquad = [\![E(\Phi, R, I_1), E(\Phi, R, I_2), \dots, E(\Phi, R, I_m)]\!].\end{aligned}$$

These theorems are broad generalisations of the double commutator formulas. Let us explain, why they do not reduce to the double formula. Consider *three* ideals I, J, K of R and form the commutator

$$[[E(R, I), G(R, J)], G(R, K)].$$

The double commutator formula implies that

$$[[E(R, I), G(R, J)], G(R, K)] = [[E(R, I), E(R, J)], G(R, K)].$$

But as we know, the relative commutator subgroup $[E(R, I), E(R, J)]$ may be strictly larger, than $E(R, I \circ J)$, so it is not at all clear, why

$$[[E(R, I), E(R, J)], G(R, K)] = [[E(R, I), E(R, J)], E(R, K)]?$$

This is indeed the key new leap in the proof of Theorem 6, and the commutator calculus developed in [45, 46, 50] is not powerful enough here. Actually, once the above equality is established, Theorem 6 follows by level calculations and elementary group theory. But this step itself requires a new layer of the relative commutator calculus, which we briefly sketch in the next section.

[2]Recently the second author removed the extra condition in this theorem

§9. A further piece of commutator calculus

We prove Theorem 6 by induction on m. The induction base $m = 2$ is precisely the relative standard commutator formula, Theorem 1.

However, the non-trivial part is the induction step, corresponding to the next case, $m = 3$. In fact, the proof of the following special case constitutes bulk of the proof of Theorem 6.

Theorem 7A. *Let $n \geqslant 3$, and let A be a quasi-finite ring. Further, let I, J and K be three two-sided ideals of A. Then*

$$[[E(n,R,I),\mathrm{GL}(n,R,J)],\mathrm{GL}(n,R,K)] = [[E(n,R,I),E(n,R,J)],E(n,R,K)].$$

As we have just observed, the standard commutator formula implies that

$$[[E(n,R,I),\mathrm{GL}(n,R,J)],\mathrm{GL}(n,R,K)] = [[E(n,R,I),E(n,R,J)],\mathrm{GL}(n,R,K)].$$

Thus, to prove Theorem 7A it remains to establish the following equality

$$[[E(n,R,I),E(n,R,J)],\mathrm{GL}(n,R,K)] = [[E(n,R,I),E(n,R,J)],E(n,R,K)].$$

However, this last equality does not follow from the standard commutator formula. Instead, it is established by the same method that was used to prove the standard commutator formula by the first and the fourth authors [50]. Let us reproduce a typical auxiliary result. Morally, it is another version of commutator caluculs. We do not *yet* try to come up with a most general version, and content ourselves with a form is sufficient to prove Theorem 7A. The following result is [51, Lemma 14].

Lemma. *Let $n \geqslant 3$, let A be a quasi-finite R-algebra. Further, let I, J and K be three two-sided ideals of A and $t \in R$. Then for any given $y \in E(n, A_t, K_t)$ and any integer l there exists a sufficiently large integer p such that*

$$[x,y] \in [[E(n,A,t^lI),E(n,A,t^lJ)],E(n,A,t^lK)]$$

for all $x \in [E(n,t^pI),E(n,A,J)]$.

The proof of this result, as also the proofs of similar results for other groups, are mostly prestidigitation and tightrope walking, and similar in spirit to the relative commutator calculus in [49]. However, this piece of commutator calculus operates at a different level of technical sophistication. For instance, now we have to plug in not just the elementary generators, or their conjugates, as in [45, 50] and [46], but also the other two types of generators constructed in Theorem 3.

Similar result for unitary groups is [47, Theorem 7].

Theorem 7B. *Let $n \geqslant 3$, R be a commutative ring, (A,Λ) be a form ring such that A is a quasi-finite R-algebra. Further, let (I,Γ), (J,Δ) and (K,Ω) be three form ideals of a form ring (A,Λ). Then*

$$\big[[\,\mathrm{EU}(2n,I,\Gamma),\mathrm{GU}(2n,J,\Delta)],\mathrm{GU}(2n,K,\Omega)\big] = \big[[\,\mathrm{EU}(2n,I,\Gamma),\mathrm{EU}(2n,J,\Delta)],\mathrm{EU}(2n,K,\Omega)\big].$$

Its prove is even more toilsome, than that of Theorem 7A. In fact, just the proof of the unitary analogue of the above triple commutator lemma, [47, Lemma 13], consists of some 6 solid pages of calculations.

After Theorem 7 is established, Theorem 6 follows by 2–3 pages of artless formal juggling with level calculations and commutator identities. For GL_n and unitary groups, the details of calculations can be found in [51] and [47], for Chevalley groups they are still unpublished.

§10. Multiple ⇝ double

In connection with Theorems 6 and 7 it is natural to ask, whether the equality

$$[[E(R,I),E(R,J)],E(R,K)] = [E(R,I),[E(R,J),E(R,K)]]$$

holds for any three ideals I, J and K of R. If this were the case, one could drop the requirement that the arrangement of brackets on the left hand side and the right hand side of these theorems should coincide.

However, in general this equality fails, as can be shown by easy examples. Let us retreat to the case of GL_n. In fact, setting here $K = R$ we see that

$$\begin{aligned} E(n,R,I\circ J) &= [E(n,R,I\circ J),E(n,R)] \\ &\leqslant [[E(n,R,I),E(n,R,J)],E(n,R)] \leqslant [\mathrm{GL}(n,R,I\circ J),E(n,R)] \\ &= [E(n,R,I\circ J),E(n,R)] = E(n,R,I\circ J), \end{aligned}$$

so that the left hand side equals $E(n,R,I\circ J)$. On the other hand, the right hand side equals $[E(n,R,A),E(n,R,B)]$. Thus, in this case associativity of commutators amounts to the equality

$$[E(n,R,A),E(n,R,B)] = E(n,R,I\circ J).$$

As we know from §4, this equality does not hold, in general.

To motivate the next theorem, let us calculate these triple commutators. Combining Theorems 1 and 2, we see that

$$\begin{aligned}[E(n,R,I\circ J),E(n,R,K)] &\leqslant [[E(n,R,I),E(n,R,J)],E(n,R,K)]\\ &\leqslant [\mathrm{GL}(n,R,I\circ J),E(n,R)] = [E(n,R,I\circ J),E(n,R,K)].\end{aligned}$$

In other words,

$$[[E(n,R,I),E(n,R,J)],E(n,R,K)] = [E(n,R,I\circ J),E(n,R,K)].$$

Similarly, one can verify that

$$[E(n,R,I),[E(n,R,J),E(n,R,K)]] = [E(n,R,I),E(n,R,J\circ K)].$$

Plugging in the above calculation Theorem 6 instead of Theorem 1, we get the following amazing corollary. It asserts that multiple commutators of relative elementary subgroups can always be expressed as *double* commutators of such subgroups, corresponding to some symmetrised product ideals. The following is observed in [41].

Theorem 8A. *Let A be a quasi-finite ring with 1 and let $I_i \trianglelefteq A$, $i = 1,\dots,m$, be ideals of A. Consider an arbitrary configuration of brackets $[\![\dots]\!]$ and assume that the outermost pairs of brackets cuts between positions h and $h+1$. Then one has*

$$\begin{aligned}&[\![E(n,R,I_1),E(n,R,I_2),\dots,E(n,R,I_m)]\!]\\ &\qquad = [E(n,R,I_1\circ\dots\circ I_h),E(n,R,I_{h+1}\circ\dots\circ I_m)].\end{aligned}$$

For the unitary case it is Theorem 7 of [47].

Theorem 8B. *Let (A,Λ) be a quasi-finite ring with 1 and let (I_i,Γ_i), $i = 1,\dots,m$, be form ideals of the form ring (A,Λ). Consider an arbitrary configuration of brackets $[\![\dots]\!]$ and assume that the outermost pairs of brackets cuts between positions h and $h+1$. Then one has*

$$\begin{aligned}&[\![\mathrm{EU}(2n,I_1,\Gamma_1),\mathrm{EU}(2n,I_2,\Gamma_2),\dots,\mathrm{EU}(2n,I_m,\Gamma_m)]\!]\\ &= [\mathrm{EU}(2n,(I_1,\Gamma_1)\circ\dots\circ(I_h,\Gamma_h)),\mathrm{EU}(n,(I_{h+1},\Gamma_{h+1})\circ\dots\circ(I_m,\Gamma_m))].\end{aligned}$$

Of course, similar result also holds in the context of Chevalley groups, once we have Theorem 6C.

Theorem 8C. *Let R be a commutative ring with 1 and let $I_i \trianglelefteq R$, $i = 1,\dots,m$, be ideals of R. Consider an arbitrary configuration of brackets*

$[\![\ldots]\!]$ *and assume that the outermost pairs of brackets cuts between positions* h *and* $h+1$. *Then one has*

$$[\![E(\Phi,R,I_1),E(\Phi,R,I_2),\ldots,E(\Phi,R,I_m)]\!] \\ = [E(\Phi,R,I_1\ldots I_h),E(\Phi,R,I_{h+1}\ldots I_m)].$$

§11. Commutators of congruence subgroups

The multiple commutator formula we stated in §8 depends only on some commutativity conditions. The *general* multiple commutator formula formula we are going to discuss now, only works over *finite-dimensional* rings.

Let, as before, I_i, $i=1,\ldots,m$, be ideals of the ring R. Once we have Theorem 6, it is natural to ask, and this was stated as [120], Problem 2, whether the presence of an elementary factor is essential there? In other words, does the following stronger commutator formula

$$[\![G(R,I_1),G(R,I_2),\ldots,G(R,I_m)]\!] = [\![E(R,I_1),E(R,I_2),\ldots,E(R,I_m)]\!]$$

hold, at least under some assumptions on m and R?

Some special cases of this formula were indeed known before. Let us look at the two first instances, 0-dimensional rings, and 1-dimensional rings. In fact, these results are stated in terms of the stable rank $\mathrm{sr}(A)$ of the ring A, see [15].

• When $\mathrm{sr}(A)=1$, one has $\mathrm{SL}(n,A,I)=E(n,A,I)$. For the commutative case, this follows from a classical result by Hyman Bass [15]. In general, this is essentially Bak's *definition* of $\mathrm{SL}(n,A,I)$, we discuss below. However, Bak proves that in the commutative case his definition does agree with the usual one in terms of determinants, see [9, Lemma 3.7].

• When $\mathrm{sr}(A)=2$ a classical result by Alec Mason and Wilson Stothers asserts that for any two two-sided ideals I and J of A one has

$$[\mathrm{GL}(n,A,I),\mathrm{GL}(n,A,J)] = [E(n,A,I),E(n,A,J)].$$

In fact, this last result is the first non-trivial case of the following theorem.

Theorem. *Let A be any associative ring with* 1, *let* $n\geqslant \mathrm{sr}(A)+1,3$, *and let* $I,J\trianglelefteq A$ *be two-sided ideals of* A. *Then one has*

$$[\mathrm{GL}(n,A,I),\mathrm{GL}(n,A,J)] = [E(n,A,I),E(n,A,J)].$$

For commutative rings, under somewhat stronger assumptions, this is [77, Corollary 3.9]. For non-commutative rings this theorem is stated in [74, Theorem 1.2], with an indication that the proof follows the same lines. In [41] we give a short proof of this result, based on Theorem 4.

The main result we wish to report in this talk is a generalisation of this theorem to *multiple* commutators over an *arbitrary* finite-dimensional ring. We cannot do this in terms of the stable rank $\mathrm{sr}(A)$ itself, since that would be far too much. There is no obvious way to implement induction on $\mathrm{sr}(A)$.

Following Anthony Bak [9] we use Bass–Serre dimension $\delta(R)$ of R instead. One of the [many!] deep external results on which our proof depends, is the following *induction lemma* by Bak.

Lemma. *Let R be a commutative ring of finite Bass–Serre dimension $\delta(R)$. Let $X_1 \cup \ldots \cup X_r$ be a decomposition of* $\mathrm{Max}(R)$ *into irreducible Noetherian subspaces of dimension $\leqslant \delta(R)$. If $s \in R$ is an element such that for each I_k, $1 \leqslant k \leqslant r$, there exists an ideal $\mathfrak{m}_k \in X_k$ such that $s \notin \mathfrak{m}_k$. Then $\delta(\widetilde{R_{(s)}}) < \delta(R)$.*

To state the next result, we have to recall the definition of super special linear groups, introduced by Bak [9]. These groups are another major tool to implement induction in our proofs. Let A be a module finite algebra over a commutative ring R and let I be an ideal in A. Define

$$\mathrm{S}^m\mathrm{L}(n, A, I) = \bigcap_{\phi} \mathrm{Ker}\big(\,\mathrm{GL}(n, A, I) \longrightarrow \mathrm{GL}(n, A', I')/E(n, A', I')\big),$$

where the intersection is taken over all homomorphisms $A \longrightarrow A'$ of rings such that A' is module finite over a commutative ring R' of Bass–Serre dimension $\delta(R') \leqslant m$, I' is the ideal of A' generated by $\phi(I)$.

This definition can be extended to all quasi-finite rings by passage to limits. Namely, let $A = \varinjlim A_i$, where A_i is module finite over a commutative ring R_i. Consider the ideal $I_i = I \cap A_i$. Then

$$\mathrm{S}^m\mathrm{L}(n, A, I) = \varinjlim \mathrm{S}^m\mathrm{L}(n, A_i, I_i).$$

By definition $\mathrm{S}^m\mathrm{L}(n, A, I)$ is functorial. The group $\mathrm{S}^0\mathrm{L}(n, A, I)$ will be denoted simply by $\mathrm{SL}(n, A, I)$. When $A = R$ is itself commutative, it coincides with the usual special congruence subgroup.

The main result of Bak [9] is the existence of nilpotent filtration on relative K_1, which becomes finite for finite dimensional rings.

Theorem. *Let A be a quasi-finite ring over a commutative ring R, let $n \geqslant 3$, and let I be an ideal of A. Then*

- *Each* $\mathrm{S}^m\mathrm{L}(n, A, I)$ *is a normal subgroup of* $\mathrm{GL}(n, A)$.
- *The sequence*

$$\mathrm{S}^0\mathrm{L}(n, A, I) \geqslant \mathrm{S}^1\mathrm{L}(n, A, I) \geqslant \mathrm{S}^2\mathrm{L}(n, A, I) \geqslant \cdots$$

is a descending $\mathrm{S}^0\mathrm{L}(n, A, I)$*-central series.*

- *The conjugation action of* $\mathrm{GL}(n, A)$ *on* $\mathrm{GL}(n, A, I)/\,\mathrm{S}^0\mathrm{L}(n, A, I)$ *is trivial.*
- *If Bass–Serre dimension of R is finite, $\delta(R) < \infty$, then*

$$\mathrm{S}^m\mathrm{L}(n, A, I) = E(\Phi, R, I),$$

whenever $m \geqslant \delta(R)$.

Actually, in the first part of this paper we have already discussed generalisations of such nilpotent filtrations to relative K_1, for unitary groups, and for Chevalley groups, obtained in [11, 36, 43].

This theorem implies the following result, where we cite a slightly better bound for m, which follows from Bass' result (a special case of the Mason–Stothers theorem above).

Theorem. *Let A be a quasi-finite algebra with* 1 *over a commutative ring R of finite Bass–Serre dimension $\delta(R)$, let $n \geqslant 3$, and further let I be a two-sided ideal of A. Assume that $m \geqslant \max(\delta(R) + 3 - n, 1)$. Then*

$$[\mathrm{SL}(n, A, I), \mathrm{SL}(n, A), \ldots, \mathrm{SL}(n, A)] = E(n, A, I),$$

where the number of $\mathrm{SL}(n, A)$ *equals m.*

§11. General multiple commutator formula

Here we state the *general* multiple relative commutator formula for GL_n, which is the main result of our paper [41]. It is a very powerful result, which, when valid, simultaneously generalises all previously known commutator formulas and nilpotent filtrations of relative K_1.

However, it requires some finiteness conditions, and relies on a host of deep external results. At this time, we only have a complete proof for the case of the general linear group. In this section we limit ourselves with the case $G = \mathrm{GL}_n$. It is not that we lack the machinery to generalise these results to groups, as we shall see in the next sections, all requisite localisation techniques itself is there. What is not there, especially for Chevalley groups, are definitive analogues of such classical results as Whitehead lemma, stability of K_1, etc.

Theorem 9A. *Let A be a quasi-finite algebra with* 1 *over a commutative ring R of finite Bass–Serre dimension $\delta(R)$, let $n \geqslant 3$, and further let $I_i \trianglelefteq R$, $i = 1, \dots, m$, be two-sided ideals of A. Assume that $m \geqslant \max(\delta(R) + 3 - n, 1)$. Then*

$$[\![\mathrm{SL}(n, A, I_1), \mathrm{SL}(n, A, I_2), \dots, \mathrm{SL}(n, A, I_m)]\!] = [\![E(n, A, I_1), E(n, A, I_2), \dots, E(n, A, I_m]\!].$$

The strategy of the proof is induction on $\delta(R)$, using Bak's induction lemma. Eventually, by induction we show that

$$[\![\mathrm{SL}(n, A, I_1), \mathrm{SL}(n, A, I_2), \dots, \mathrm{SL}(n, A, I_m)]\!] = [E(n, R, I_1 \circ \ldots \circ I_h), E(n, R, I_{h+1} \circ \ldots \circ I_m)],$$

the right hand side of this equality being, as we know from Theorem 8A, just another expression for $[\![E(n, A, I_1), E(n, A, I_2), \dots, E(n, A, I_m]\!]$.

The following result serves as the base of induction. It is essentially a combination of known results.

Theorem 10A. *Let A be a quasi-finite algebra with* 1 *over a commutative ring R of finite Bass–Serre dimension $\delta(R)$, let $n \geqslant 3$, and let $I, J \trianglelefteq A$ be two-sided ideals of A. Then one has*

$$[\mathrm{GL}(n, R, A), \mathrm{GL}(n, R, B)] = \mathrm{S}^{\delta(R)-n+3}\mathrm{L}(n, R, A \circ B).$$

In fact, this is an immediate corollary of the above Mason–Stothers theorem, Bass' estimate of the stable rank of A in terms of Bass–Serre dimension $\delta(R)$, and Bass–Vaserstein theorem on injective stability of relative K_1.

On the other hand, induction step now looks as follows.

Theorem 11A. *Let A be a quasi-finite algebra with* 1 *over a commutative ring R, $n \geqslant 3$, and let $I, J, K \trianglelefteq A$ be three two-sided ideals of A. Then for any m one has*

$$[[\mathrm{S}^m\mathrm{L}(n, A, I), \mathrm{S}^m\mathrm{L}(n, A, J)], \mathrm{S}^0\mathrm{L}(n, A, K)] \leqslant [\mathrm{S}^{m+1}\mathrm{L}(n, A, I \circ J), \mathrm{S}^{m+1}\mathrm{L}(n, A, K)].$$

Ideologically, the proof of this result – as the proofs in [36,43] and [11] – is still modeled on Bak's paper [9]. However, in most important technical aspects the proof is *completely* new, as we explain in the next sections. The most important innovation can be described as follows. In Bak's paper [9], as also in [36,43], the whole interplay between localisation and completion was implemented at the *global* level. After that relative results were derived from the corresponding absolute results by a version of relativisation, which affords some form of splitting.

This is how the proof of relative results was carried through in Bak's paper [9], in the case of GL_n. Our joint paper with Bak [11], where similar relative results were obtained for unitary groups, and for Chevalley groups, followed the same general strategy. Of course, considerable additional technical strain was due to the fact that relativisation with respect to *form* ideals was harder to implement.

However, in the above papers Bak and ourselves always considered *one* ideal. Now, we have to relativise with respect to *several* ideals – well, at least with respect to two ideals – and use some form of *relative* splitting principle, rather than just the absolute one. There is no need to persuade anyone, who has herself ever tried to fool around with relativisation with *two* parameters, that this is a gruesome task. An attempt to prove Theorem 10 along these lines, by splitting several ideals simultaneously, immediately lead to rather awkward technical impediments.

Our idea was then to prove *multirelative* versions of Bak's localisation completion theorem itself. Recall that localisation completion theorem is another LOCAL GLOBAL PRINCIPLE. Essentially, it asserts that the commutator of something that becomes elementary under principal s-localisation with something else that becomes elementary under s-adic completion is indeed *globally* elementary. The proof is an exemplary manifestation of the combined force of continuity and density, in s-adic topology.

Morally, the advantage of our new approach is that it consists in moving all relativisation to the *local* level, where congruence subgroups coincide with relative elementary subgroups, so that splitting is not an issue at all. This would be possible if we could rely on the the full force of *relative* commutator calculus.

In fact, as reported in [39] and in §9, we worked out *some* form of relative commutator calculus in [45, 50] and [46], and then a slightly fancier one in [51] and [47]. Regretfully, in all these papers except [46] we implemented only *first localisation*, whereas now we need *much* stronger versions of

Theorem 1, with two denominators. To get that, we have to implement what is called *second loclaisation*.

This means that we had to turn the crank again, to redo all relative commutator calculus *from scratch*, allowing *two* denominators. The target results of this version of commutator calculus look like a blend of Theorem 1 or Theorem 7 with the commutator calculus lemmas used in their proofs, and will be reproduced in the next section.

§12. Relative commutator calculus, revisited

To state the target results of the relative commutator calculus, we need to introduce some notation, similar to the notation we used in the first part of this work, [39, §5]. Namely, for two additive subgroups B, C of a ring R we denote by $E^L(C,B)$ the *set* of all products of $\leqslant L$ relative elementary generators $z(\xi,\eta)$ of the group $E(R,RBR)$, such that $\xi\in B$, $\eta\in C$.

Thus, for instance, $E^L(\Phi,\frac{1}{s^k}R,\frac{1}{s^k}I)$ is the set of products of $\leqslant L$ elements of the form

$$x_\alpha\big(\frac{\xi}{s^k},\frac{\eta}{s^k}\big)={}^{x_{-\alpha}\left(\frac{\eta}{s^k}\right)}x_\alpha\big(\frac{\xi}{s^k}\big),\qquad \xi\in I,\ \eta\in R.$$

In other words,

$$E^L\big(\Phi,\frac{1}{s^k}R,\frac{1}{s^k}I\big)=\left({}^{E^1\left(\Phi,\frac{t^l}{s^k}R\right)}E^1\big(\Phi,\frac{t^l}{s^k}I\big)\right)^L.$$

Clearly, for any $x\in E(\Phi,R_s,I_s)$ there exist positive integers k and L such that $x\in E^L(\Phi,\frac{1}{s^k}R,\frac{1}{s^k}I)$.

The following result is [46, Theorem 2]. We only established this result under somewhat stronger assumption in rank 2, than those in theorem 1C, to spare some 2–3 further pages of calculations. Anyway, for groups of types C_2 and G_2 relativisation has to be considered separately, in the more general setting of admissible pairs, or even radices.

Theorem 12C. *Let Φ be a reduced irreducible root system,* $\mathrm{rk}(\Phi)\geqslant 2$. *In the cases $\Phi=\mathrm{C}_2,\mathrm{G}_2$ assume additionally that $2\in R^*$. Then for any $s\in R$, $s\neq 0$, any p,k and L, there exists an r such that for any two ideals I and J of a commutative ring R, one has*

$$\begin{aligned}\Big[E^L\big(\Phi,\frac{1}{s^k}R,\frac{1}{s^k}I\big),F_s(G(\Phi,R,s^rJ))\Big]&\\ \leqslant\big[E(\Phi,F_s(s^pR),F_s(s^pI)),E(\Phi,F_s(s^pR),F_s(s^pJ))\big].&\end{aligned}$$

Regrettably, in [50] and [45] we only implemented the *first* localisation and patching, whereas theorems of that type require also the *second* localisation. To prove analogues of the above theorem for GL_n and for unitary groups, we were forced to replay the whole relative commutator calculus from the very start once again, allowing *two* denominators. The following results are established in [41] and in [42], respectively.

Theorem 12A. *Let $n \geqslant 3$, R be a commutative ring, and let A be a quasi-finite R-algebra. Then for any $s \in R$, $s \neq 0$, any p, k and L, there exists an r such that for any two ideals I and J of A, one has*

$$\Big[E^L\big(n, \frac{1}{s^k}A, \frac{1}{s^k}I\big), F_s(\,\mathrm{GL}(n, A, s^r J))\Big] \leqslant \big[E(n, F_s(s^p A), F_s(s^p I)), E(n, F_s(s^p A), F_s(s^p J))\big].$$

Theorem 12B. *Let $n \geqslant 3$, R be a commutative ring, (A, Λ) be a form ring such that A is a quasi-finite R-algebra. Then for any $s \in R_0$, $s \neq 0$, any p, k and L, there exists an r such that for any two form ideals (I, Γ) and (J, Δ) of (A, Λ), one has*

$$\Big[\,\mathrm{EU}^L\,\big(n, \frac{1}{s^k}I, \frac{1}{s^k}\Gamma\big), F_s(\,\mathrm{GU}(n, J, s^r\Delta))\Big] \leqslant \big[\,\mathrm{EU}\,(n, F_s(s^p I), F_s(s^p\Gamma)), \mathrm{EU}\,(n, F_s(s^p J), F_s(s^p\Delta))\big].$$

§13. Relative localisation completion

In the first part of the present paper we have already discussed Bak's localisation completion theorem [9], and its analogues for unitary groups, due to the first author [36, 37], and for Chevalley groups, due to the first and the third authors [43], see [39, Theorem 8].

We start with explaining this idea in the simplest case of *one* ideal. This is not yet sufficient to prove Theorem 10, but it allows to give easier proofs of nilpotency of K_1.

With this end, recall the necessary notation concerning completion, see [9] or [39, §11]. Let $s \in A$. Usually, the s-completion $\widehat{A}_s$ of the ring A is defined as the following inverse limit:

$$\widehat{R}_s = \varprojlim R/s^n R, \quad n \in \mathbb{N}.$$

However, for our purposes one has to modify this definition, by interchanging the order of taking limits. Namely, we set

$$\widetilde{A}_s = \varinjlim (\widehat{A}_i)_s,$$

where the limit is taken over all finitely generated subrings A_i of A which contain s. Let us denote by $\widetilde{F}_s$ the canonical map $A \longrightarrow \widetilde{A}_s$. For the case, where R is Noetherian, $\widetilde{F}_s = \widehat{F}_s$ coincides with the inverse limit of reduction homomorphisms $\pi_{s^n} : A \longrightarrow A/s^nA$

In [39], we discussed definition of the groups $G(R, s^{-1})$ and $G(R, \widehat{s})$. First of all, we have to introduce their relative analogues.

First, let A be an algebra over a commutative ring R, $n \geqslant 3$, $s \in R$, and further let I be an ideal of A. In the case of GL_n the relative analogues of the above groups are defined as follows

$$\mathrm{GL}(n, A, I, s^{-1}) = \mathrm{Ker}(\mathrm{GL}(n, A, I) \longrightarrow \mathrm{GL}(n, A_s, I_s)/E(n, A_s, I_s)),$$

$$\mathrm{GL}(n, A, I, \widehat{s}) = \mathrm{Ker}(\mathrm{GL}(n, R, I) \longrightarrow \mathrm{GL}(n, \widetilde{A}_{(s)}, \widetilde{I}_{(s)})/E(n, \widetilde{A}_{(s)}, \widetilde{I}_{(s)})).$$

Now, let A_i be the inductive system of all finite R_i-subalgebras, where R_i ranges over all finitely generated subrings of R containing s. We set $I_i = I \cap A_i$. Then

$$\mathrm{GL}(n, A, I, s^{-1}) = \varinjlim \mathrm{GL}(n, A_i, I_i, s^{-1}),\ \mathrm{GL}(n, A, I, \widehat{s}) = \varinjlim \mathrm{GL}(n, A_i, I_i, \widehat{s}),$$

which reduces most problems about these groups to the case, where A is finite over a Noetherian ring R. The same argument works for other groups and below we usually assume that the ground ring is Noetherian.

The following result is [41, Theorem 9].

Theorem 13A. *Let A be a quasi-finite algebra over a commutative ring R, $n \geqslant 3$, let I be an ideal of R, and let $s \in R$. Then*

$$[\mathrm{GL}(n, A, I, s^{-1}), \mathrm{GL}(n, A, \widehat{s})] \leqslant E(n, A, I).$$

Similar result holds also in the unitary setting. Let (A, Λ) be a form ring, which is module finite over a commutative ring R. Further, let (I, Γ) be a form ideal of (A, Λ). Take $s \in R_0$ and define

$$U(2n, I, \Gamma, s^{-1}) = \mathrm{Ker}\big(U(2n, I, \Gamma) \longrightarrow U(2n, I_s, \Lambda_s)/\,\mathrm{EU}(2n, I_s, \Gamma_s)\big),$$

$$U(2n, I, \Gamma, \hat{s}) = \mathrm{Ker}\big(U(2n, I, \Gamma) \longrightarrow U(2n, \widetilde{(I, \Gamma)}_{(s)})/E(2n, \widetilde{(I, \Gamma)}_{(s)})\big),$$

A similar result for unitary groups can be stated as follows. It is a generalisation of one of the main results of the Thesis by the first named author [36, 37].

Theorem 13B. *Let (A,Λ) be a module finite form ring over a commutative ring R, let (I,Γ) be a form ideal of (A,Λ), and let $s\in R_0$. Then*
$$[U(2n,I,\Gamma,s^{-1}),U(2n,A,\Lambda,\hat{s})]\leqslant \mathrm{EU}(2n,I,\Gamma).$$

Finally, let Φ be an irreducible root system of rank ≥ 2, let R be a commutative ring, let I be an ideal of R, and let $s\in R$. Define
$$G(\Phi,R,I,s^{-1})=\mathrm{Ker}(G(\Phi,R,I)\longrightarrow G(\Phi,R_s,I_s)/E(\Phi,R_s,I_s)),$$
$$G(\Phi,R,I,\widehat{s})=\mathrm{Ker}(G(\Phi,R,I)\longrightarrow G(\Phi,\widetilde{R}_{(s)},\widetilde{I}_{(s)})/E(\Phi,\widetilde{R}_{(s)},\widetilde{I}_{(s)})).$$

The following result for Chevalley groups is unpublished, and below we sketch its proof.

Theorem 13C. *Let A be a quasi-finite algebra over a commutative ring R, $n\geqslant 3$, let I be an ideal of R, and let $s\in R$. Then*
$$[G(\Phi,R,I,s^{-1}),G(\Phi,R,\widehat{s})]\leqslant E(\Phi,R,I).$$

The following proof is essentially an enhancement of the proof of [43, Theorem 6.1], where we simply plug in a more powerful version of the commutator calculus, *relative*, instead of absolute. Of course, the proof in [43] was itself just a streamlined adaptation of the original Bak's argument [9].

Denote by $E^K(\Phi,\frac{1}{s^k}R,\frac{1}{s^k}I)$ the subset of $E(\Phi,R_s,I_s)$, consisting of products of $\leqslant K$ elements of the form $z_\alpha(\xi,\zeta)$, where $\alpha\in\Phi$, $\xi\leqslant\frac{1}{s^k}I$, $\zeta\in\frac{1}{s^k}R$.

The usual argument allows us to reduce the proof to the case, where R is Noetherian. Let $x\in G(\Phi,R,I,s^{-1})$ and $y\in G(\Phi,R,\widehat{s})$. By definition, the condition on x means that $F_s(x)\in E^K(\Phi,\frac{1}{s^k}R,\frac{1}{s^k}I)$ for some k and some K. On the other hand, the condition on y means that $\pi_{s^m}(y)\in E(\Phi,R/s^mR)$ for *all* m, or, what is the same, that $y=uz$, where $u\in\in E(\Phi,R)$ and $z\in\mathrm{GL}(n,R,s^mR)$.

Thus, $[x,y]=[x,uz]=[x,u]\cdot{}^u[x,z]$. The first factor belongs to $E(\Phi,R)$ simply because it is normal in $G(\Phi,R)$. As for the second factor, a typical target result of the relative commutator calculus – in this situation [a special case of] [46, Theorem 2] – can be stated as follows: for any q there exists a sufficiently large m such that $F_s([x,z])\in E(\Phi,F_s(s^qR),F_s(s^qI))$. On the other hand, since $G(\Phi,R,s^qR)$ is normal in $G(\Phi,R)$, one has $[x,z]\in G(\Phi,R,s^qR)$. Now, since R is assumed to be Noetherian, the usual argument based on injectivity of localisation homomorphisms on small neighbourhoods of e in s-adic topology convinces us that $[x,z]\in E(\Phi,R,s^qI)$.

This shows that both $[x,u]$ and ${}^{u}[x,z]$, and thus also $[x,y]$ are elementary, as claimed.

Observe that Theorems 3A–3C easily imply by induction nilpotency of *relative* K_1, in other words, main results of [9] and [11]. From the very start, this proof works at the relative level, without any need to relativise results on nilpotent filtrations of absolute K_1. We believe that this proof is both better conceptually and (once the target results of relative commutator calculus are established!) technically easier than the original proof.

§14. Birelative localisation completion

However, to prove the general commutator formula we need stronger results.

Theorem 14A. *Let A be a quasi-finite algebra over a commutative ring R, $n \geqslant 3$, $s \in R$, and let $I, J \trianglelefteq A$ be two-sided ideals of A. Then*

$$[\mathrm{GL}(n,A,I,s^{-1}), \mathrm{GL}(n,A,J,\widehat{s})] \leqslant [E(n,A,I), E(n,A,J)].$$

Theorem 14B. *Let (A,Λ) be a module finite form ring over a commutative ring R, let (I,Γ) and (J,Δ) be two form ideals of (A,Λ), and let $s \in R_0$. Then*

$$[U(2n,I,\Gamma,s^{-1}), U(2n,J,\Delta,\hat{s})] \leqslant [\mathrm{EU}(2n,I,\Gamma), \mathrm{EU}(2n,J,\Delta).$$

Theorem 14C. *Let A be a quasi-finite algebra over a commutative ring R, $n \geqslant 3$, let I be an ideal of R, and let $s \in R$. Then*

$$[G(\Phi,R,I,s^{-1}), G(\Phi,R,J,\widehat{s})] \leqslant [E(\Phi,R,I), E(\Phi,R,J)].$$

Lemma. *Let A be a module finite algebra over Noetherian ring R, I a two-sided ideal of A. Then for any given l and $s \in R$ there is a sufficiently large m such that $I \cap s^m A \subseteq s^l I$.*

Indeed, consider the following sequence of ideals

$$J_m = (I : s^m) = \{a \in A \mid s^m A \in I\}, \qquad m = 0,1,2,\ldots$$

Clearly, $J_0 \leqslant J_1 \leqslant J_2 \leqslant \ldots$. Since R is Noetherian and A is module finite over R, there exists such an k that $J_k = J_{k+1} = J_{k+2} = \ldots$. The equality $J_m = J_k$ for any $m > k$, amounts to the following: for any $a \in R$ inclusion $s^m a \in I$ implies that already $s^k a \in I$. In particular, taking $m = k + l$, and an $a \in R$ such that $s^m b \in I \cap s^m A$, we can conclude that $s^k b \in I$, and thus $s^m a = s^{k+l} a = s^l(s^k a) \in s^l I$.

Theorem 15A. *Let A be a quasi-finite algebra over a commutative ring R, $n \geqslant 3$, $s \in R$, and let $I, J, K \trianglelefteq A$ be two-sided ideals of A. Then*

$$[[\mathrm{GL}(n, A, I, \widehat{s}), \mathrm{GL}(n, A, J, \widehat{s})], \mathrm{GL}(n, A, K, s^{-1})] \leqslant [E(n, A, I \circ J), E(n, A, K)].$$

For unitary groups, part of the requisite relative commutator calculus with two denominators was developed in our paper [42]. We are convinced that in this case we could prove an analogue of Theorem 11 by the same strategy. In this case, the external equipment, on which our prove depends, such as appropriate versions of Whitehead lemma, stability theorems, etc., can be mostly found in the existing literature.

For Chevalley groups, the situation is different. On the one hand, in [46] we have already developed a stronger version of relative commutator calculus, and [46, Theorem 2], is precisely what we need to prove the birelative localisation completion theorem. On the other hand, many of the background results are not available in

The second author proved an analogue of Theorem 8A for Chevalley groups with the use of his universal localisation method. With this approach all problems related to splitting are shifted the affine ring of G and other universal rings.

§15. Relative commutator length

In the first part of this paper we have already discussed results on the commutator width in the absolute case, what we called the "anti-re" behavious of groups of points of algebraic groups and their elementary subgroups over rings of dimension $\geqslant 2$. Namely, in [86] and [94] we proved that commutators have bounded length in elementary generators in GL_n and Chevalley groups respectively, in the absolute case.

Here, we state the relative versions of these results, recently obtained by the second author. We limit ourselves with the results themselves and a minimum of comments. One should consult our Porto Cesareo conference paper [40] for a much broader picture, including background, history, motivation, ideas of proof, analogues, and unsolved problems.

First, we replace one occurence of R by an ideal. Of course, in this case the elementary generators, but rather their conjugates $z_{ij}(\xi, \eta)$, $Z_{ij}(\xi, \eta)$, and $z_\alpha(\xi, \eta)$, we introduced in § 5. These results are new even in the case of the general linear group.

Theorem 16A. *Let R be a commutative ring and let $I \trianglelefteq R$ be an ideal of R. Then there exists an L such that any commutator $[x,y]$, where*

$$x \in \mathrm{SL}(n,R,I), \quad y \in E(n,R) \quad \textit{or} \quad x \in \mathrm{SL}(n,R), \quad y \in E(n,R,I)$$

is a product of not more than L elementary generators $z_{ij}(\xi,\zeta)$, where $1 \leqslant i \neq j \leqslant n$, $\xi \in I$, $\zeta \in R$.

Theorem 16C. *Let R be a commutative ring and let $I \trianglelefteq R$, be an ideal of R. Then there exists an L such that any commutator $[x,y]$, where*

$$x \in G(\Phi,R,I), \quad y \in E(\Phi,R) \quad \textit{or} \quad x \in G(\Phi,R), \quad y \in E(\Phi,R,I)$$

is a product of not more than L elementary generators $z_\alpha(\xi,\zeta)$, where $\alpha \in \Phi$, $\xi \in I$, $\zeta \in R$.

Now we state the ultimate *birelative* versions of the results on commutator length. These results use the elementary generating systems of relative commutator subgroups $[E(R,I),E(R,J)]$, constructed in §6, and thus could not even be stated this way before [47, 48, 51]. In these results *both* occurences of R are replaced by its ideals.

Theorem 17A. *Let R be a commutative ring and let $I,J \trianglelefteq R$ be two ideals of R. Then there exists an L such that any commutator*

$$[x,y], \qquad x \in \mathrm{SL}(n,R,I), \quad y \in E(n,R,J)$$

is a product of not more than L elementary generators listed in Theorem 5A.

Theorem 17C. *Let R be a commutative ring and let $I,J \trianglelefteq R$ be two ideals of R. Then there exists an L such that any commutator*

$$[x,y], \qquad x \in G(\Phi,R,I), \quad y \in E(\Phi,R,J)$$

is a product of not more than L elementary generators listed in Theorem 5C.

Quite remarkably, the bound L in these theorems does not depend either on the ring R, or on the choice of the ideals I, J. The proofs of these theorems are paradigmatic applications of the method of UNIVERSAL LOCALISATION, introduced by the second author [92], specifically to eliminate any dependence upon dimension of R. Actually, we sketched the main ideas of this method in [39, §10].

These proofs are not particularly long, but rely on a whole bunch of universal constructions. From the proofs, it becomes apparent that similar results hold also in other such situations: for any other functorial generating set, for multiple relative commutators [40, 92], etc., etc.

It is natural to ask, whether similar results hold for unitary groups? Well, this is an open question. The point is that Bak's unitary groups are not always algebraic, and it is not immediate, how to generalise the universal constructions of [92] from groups defined by equations to groups defined by congruences.

As a result, at this time we only have *weaker* versions of the above results, with bounds depending on the Jacobson dimension of the centre. The proofs of the following results use localisation in the same style as [45, 47], they will be published in [42].

Theorem 16B. *Let $n \geqslant 3$, and let (A, Λ) be a quasi finite form algebra over a commutative ring R with $\dim \operatorname{Max}(R) = d < \infty$. Further, let (I, Γ) be a form ideal of (A, Λ). Then there exists an L such that any commutator $[x, y]$, where*

$x \in \mathrm{SU}(2n, I, \Gamma),\ y \in \mathrm{EU}(2n, R, \Lambda)$ *or* $x \in \mathrm{SU}(2n, R, \Lambda),\ y \in \mathrm{EU}(2n, I, \Gamma)$

is a product of not more than L elementary generators

$$Z_{ij}(\xi, \zeta) \in \mathrm{EU}(2n, I, \Gamma).$$

Theorem 17B. *Let $n \geqslant 3$, and let (A, Λ) be a quasi finite form algebra over a commutative ring R with $\dim \operatorname{Max}(R) = d < \infty$. Further, let (I, Γ) and (J, Δ) be two form ideals of (A, Λ). Then there exists an L such that any commutator*

$$[x, y], \qquad x \in \mathrm{SU}(n, I, \Gamma), \quad y \in \mathrm{EU}(n, J, \Delta)$$

is a product of not more than L elementary generators listed in Theorem 5B.

Thus, the following problem naturally suggests itself.

Problem 3. *Develop versions of universal localisation in the non-algebraic setting, in particular, for unitary groups.*

Since the publication of [39] the second author had some progress in this direction, but a definitive solution is still missing.

§16. Relative Quillen–Suslin lemma

As another important recent advance towards developing *relative* versions of localisation, let us state the main result of the paper by Himanee

Apte and the second author [7, Theorem 7.1]. This is the following relative version of the Quillen–Suslin localisation principle for Chevalley groups.

Theorem 18C. *Let Φ be a root system of rank $\geqslant 2$, $I \trianglelefteq R$ be an ideal of a commutative ring R. In the cases $\Phi = \mathrm{C}_2, \mathrm{G}_2$ assume additionally that R does not have residue field $\mathbb{F}_2$ of 2 elements. Further, let $g \in G(\Phi, R[x], xI[x])$ be such that $F_{\mathfrak{m}} \in E(\Phi, R_{\mathfrak{m}}[x], xI_{\mathfrak{m}}[x])$. Then $g \in E(\Phi, R[x], xI[x])$.*

For classical groups such relative versions of the Quillen–Suslin principle were obtained by Ravi Rao and his students, see, for instance, [6].

Observe, that the condition on the ring here is somewhat weaker than the one we assumed throughout this paper. This is due to the fact that the proof in [7] only relies on the *absolute* commutator formulae (actually, for relative level calculations the authors of [7] use the same assumption as we do).

In [7] we observed that for the proof of local-global principle only requires a *weaker* form of commutator calculus, called **dilation principle** by the Indian school. The following version of this principle is [7, Theorem 6.3].

Theorem 19C. *Let Φ be a root system of rank $\geqslant 2$, $I \trianglelefteq R$ be an ideal of a commutative ring R. In the cases $\Phi = \mathrm{C}_2, \mathrm{G}_2$ assume additionally that R does not have residue field $\mathbb{F}_2$ of 2 elements. Further, let $s \in R$ and $g(x) \in G(\Phi, R[x], xI[x])$ be such that $F_{\mathfrak{m}} \in E(\Phi, R_s[x], xI_s[x])$. Then there exists an $m \in \mathbb{N}$ such that $g(s^m x) \in E(\Phi, R[x], xI[x])$.*

Let us cite another interesting result of [7], which establishes a remarkable connection of splitting with the relative commutator subgroups.

Recall that a two-sided ideal $I \trianglelefteq A$ is called a *splitting ideal*, if $R = I \oplus B$ as an additive group, and at that $B \leqslant R$ is a subring of R. Clearly, in this case $B \cong R/I$ and the projection $R \longrightarrow B \leqslant R$ is a retraction. The following [absolute] splitting principle is classically known and widely used

Lemma. *Let I be a splitting ideal of an associative ring A. Then*

$$G(\Phi, R, I) \cap E(\Phi, R) = E(\Phi, R, I).$$

However, in the relative case this principle implies only that if I be a splitting ideal of an associative ring R and $J \trianglelefteq R$ be any two-sided ideal. Then

$$G(\Phi, R, I) \cap E(\Phi, R, J) = E(\Phi, A, I) \cap E(\Phi, R, J).$$

The following result, which is essentially [7], Lemma 2.2, is a proper relative version of the splitting principle.

Lemma. *Assume as in Theorem* 3C, *let I be a splitting ideal of R, $R = I \oplus B$, and let $J \trianglelefteq R$ be a two-sided ideal of R generated by an ideal $K \trianglelefteq B$. Then*

$$G(\Phi, R, I) \cap E(\Phi, R, J) = [E(\Phi, R, I), E(\Phi, R, J)].$$

Actually, [7] states a slighly more general result, without extra-assumptions in the case $\Phi = \mathrm{C}_l$. But then, of course, on has to multiply the group in the right hand side by $E(\Phi, R, IJ)$.

§17. Where next?

> Let us hear the suspicions. I will look after the proofs.
> Arthur Conan Doyle

In conclusion, we attach an updated list of unsolved problems related to the results of [39] and the present paper. The results of the present paper solve some of the problems stated in [39], whereas some others need corrections. We keep working on these problems and hope to be able to address them in subsequent publications.

Problem 1. *Obtain explicit length estimates in the relative conjugation calculus and commutator calculus.*

Problem 2. *Obtain explicit length estimates in the universal localisation.*

Notice that in the vast majority of results on bounded width of commutators we had to assume that both factors have determinant 1. So far we were unable to fight the toral factor. The following problem is not fully solved even in the absolute case for the general linear group. In [86] there is a partial result for rings of geometric origin.

Problem 4. *Prove the bounded width of commutators of the form $[x, y]$, where $x \in \mathrm{GL}(n, R)$, $y \in E(n, R)$.*

After this is done, one could certainly improve the corresponding results also at the relative level.

Problem 5. *Replace in Theorems* 7A, 7B, 9A, 9B *commutativity of the ground ring by a weaker commutativity assumption.*

Proposition 2.7 of [86] is a stable version of the main result on the bounded width of commutators. It asserts that under condition $n \geqslant \mathrm{sr}(R)+1$ the length of any commutator $[x,y]$, where $x,y \in \mathrm{GL}(n,R)$ in elementary generators is bounded. It would be natural to generalise this result to unitary groups, and to Chevalley groups. However, it would require very precise forms of injective stability for K_1, with efficient proofs, if one wishes to obtain explicit bounds.

Problem 6. *Prove that under condition* $n \geqslant \Lambda\,\mathrm{sr}(R)+1$ *the width of commutators of the form* $[x,y]$*, where* $x,y \in \mathrm{GU}(2n,R,\Lambda)$*, in the elementary generators of the group* $\mathrm{EU}(2n,R,\Lambda)$ *is bounded.*

It seems that for Chevalley groups one still has to do some work to find appropriate stability conditions.

Problem 7. *Find stability conditions, under which the width of commutators of the form* $[x,y]$*, where* $x,y \in G(\Phi,R)$*, in the elementary generators of the group* $E(\Phi,R)$ *is bounded.*

Let us mention yet another variation of the finite width of commutators in elementary generators. It seems that such a result would require centrality of the extension $\mathrm{St}(n,R) \longrightarrow E(n,R)$, hence the restriction $n \geqslant 4$. Thus, with our present state of knowledge, there is little hope to generalise it to other groups. Even for the linear case it is only proven for rings of geometric origin, [86, Theorem 2.1].

Problem 8. *Let* R *be a commutative ring. Prove the bounded width of commutators of the form* $[x,y]$*, where* $x,y \in \mathrm{St}(n,R)$*,* $n \geqslant 4$*, with respect to the elementary generators.*

Another important challenge is to improve rank bounds in the commutator calculus for the unitary groups. In the absolute case it is the usual condition stated in [13, 14].

Problem 9. *Develop conjugation calculus and commutator calculus in the group* $\mathrm{GU}(4,R,\Lambda)$*, provided* $\Lambda R + R\Lambda = \Lambda$*.*

In [39] without much thinking we conjectured that the same condition would suffice to develop relative commutator calculus. However, a closer look shows that in the relative case the above condition is far too weak. The correct conditions should be stated in terms of the *relative* form parameters, and seem to be extremely restrictive, so that we are not sure we would be inclined to work out all details under such ridiculous confinements.

Problem 10. *Prove relative commutator formulae for the group* $\mathrm{GU}(4, R, \Lambda)$*, provided* $\Gamma J + J\Gamma = \Gamma$*,* $\Delta I + I\Delta = \Delta$*.*

Another important problem is the description of *subnormal* subgroups of $G(R)$. For the case of $\mathrm{GL}(n, R)$ this problem has a fully satisfactory answer, due to the works by John Wilson, Leonid Vaserstein, and others, see in particular [8, 68, 105, 109, 112, 126, 130].

For unitary groups, there are works by Günter Habdank, the fourth author, and You Hong, see, in particular, [33, 34, 129–132]. Presently You Hong, the third and the fourth authors are working towards an improvement of bounds. But there are still a number of loose ends so that an alternative approach based on the methods of [11, 36, 37, 42, 45, 47] would be very much desirable.

Problem 11. *Give localisation proofs for the description of subgroups of the unitary group* $\mathrm{GU}(2n, R, \Lambda)$*, normalised by the relative elementary subgroup* $\mathrm{EU}(\Phi, I, \Gamma)$*, for a form ideal* (I, Γ)*.*

Similar problem for Chevalley groups is still largely unsolved. Again, a localisation approach based on the methods [43, 46, 94] would be most welcome.

Problem 12. *Using relative localisation, describe subgroups of a Chevalley group* $G(\Phi, R)$*, normalised by the relative elementary subgroup* $E(\Phi, R, I)$*, for an ideal* $I \trianglelefteq R$*.*

Actually, the third and the fourth author classified such subgroups in Chevalley groups of types E_6 and E_7, but we used a *completely different* geometric method, the proof from the Book.

Now that we have relativised results on nilpotent filtration for the general linear group, it is natural to obtain similar results for unitary groups. This will require improvement and relative versions of many known results, for instance, those related to stabilisation.

Problem 13. *Obtain the analogues of the general relative multiple commutator formula for unitary groups.*

In [39] and the present paper we work at the level of K_1. It is natural to ask, to which extent these methods carcy over to the level of K_2.

Problem 14. *Develop localisation methods at the level of* K_2*. In particular, devise a localisation proof of the centrality of* K_2*.*

The fact that higher mixed commutators of relative elementary subgroups can always be expressed as double commutators, seems to be very surprising. It strongly suggests that there is close connection with some finer aspects of the structure of K_1, such as homotopy stability, exact sequences and excision. Especially striking is the analogy with the works of Susan Geller and Charles Weibel [27–30] on doubly relative K_1 and excision.

Problem 15. *Establish connection of* $[E(\Phi,R,A),E(\Phi,R,B)]$ *with the excision kernel.*

Morally, the results of [39,40,86,92,94] assert that $G(R)$ has *very* few commutators when $\dim(R)\geqslant 2$. Positive results on commutator width only hold under some very strong finiteness conditions on R, and require completely different techniques. See [87] for some specific conjectures.

It is very challenging to understand, to which extent such behaviour is typical for more general classes of group words. There are a lot of recent results showing that the verbal length of the finite simple groups is strikingly small [32,64,65,87,88]. In fact, under some natural assumptions for large finite simple groups this verbal length is 2. We do not expect similar results to hold for rings other than the zero-dimensional ones, and some arithmetic rings of dimension 1.

Powers are a class of words in a certain sense opposite to commutators. Alireza Abdollahi suggested that before passing to more general words, we should first look at powers. An answer – in fact, *any* answer! – to the following problem would be amazing. However, we would be less surprised if for rings of dimension $\geqslant 2$ the verbal maps in $G(R)$ would have very small images.

Problem 16. *Establish finite width of powers in elementary generators, or lack thereof.*

In this connection let us mention the following purely group theoretical problem. It is well known that a commutator is a product of [not more than] three squares $[x,y]=x^2(x^{-1}y)^2y^2$. Similarly, a commutator of commutators $[[x,y],[u,v]]$ is a product of not more than 60 cubes [4], whereas the Engelian commutator $[x,y,y]$ is a product of 3 cubes [5].

Problem 17. *When a higher commutator of commutators can be expressed as a bounded product of* m*-th powers?*

The following two problems are in fact not individual clear cut problems, but rather huge research projects.

Problem 18. *Generalise results of* [39] *and the present paper to odd unitary groups.*

Problem 19. *Obtain results similar to those of* [39] *and the present paper for* [*groups of points of*] *isotropic reductive groups.*

In the first one of these settings there are foundational works by Victor Petrov [80–82], while in the second one there are papers by Victor Petrov, Anastasia Stavrova, Alexander Luzgarev and Ekaterina Kulikova [61, 73, 83, 90] with versions of Quillen–Suslin lemma. But that's about it. Most of the conjugation calculus and the commutator calculus, including relative results, explicit estimates, etc., have to be developed from scratch.

In particular, this applies to the relative results. If one is not willing to stipulate invertibility of 2 and 3, these results should be stated not in terms of ideals of the ground ring, but rather in terms of much fancier non-associative structures.

Let us mention two much less ambitious subprojects. First, all isotropic reductive groups are *inner* forms of quasi-split groups = twisted Chevalley groups. The following problem seems to be much more immediate than Problem 19. It can be easily approached by the methods developed in [43, 48, 92, 94]. On the other hand, it is not clear, how much helpful it might be to treat it separately, outside of the general context of isotropic reductive groups.

Problem 20. *Obtain results similar to those of* [39] *and the present paper for twisted Chevalley groups.*

Another much tamer common piece of Problems 18 and 19 can be stated as follows.

Problem 21. *Generalise results of* [39] *and the present paper to orthogonal and symplectic groups of Witt index* $\geqslant 3$.

It seems that most necessary tools are contained already in the works by Victor Petrov, Tony Bak, Rabeya Basu, Ravi Rao and Reema Khanna [10, 17–20, 80–82].

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