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A METHOD FOR CONSTRUCTION OF LIE GROUP INVARIANTS

ABSTRACT. For an adjoint action of a Lie group G (or its subgroup)
on Lie algebra Lie (G) we suggest a method for construction of
invariants. The method is easy in implementation and may shed the
light on algebraical independence of invariants. The main idea is to
extent automorphisms of the Cartan subalgebra to automorphisms
of the whole Lie algebra Lie (G). Corresponding matrices in a linear
space $V \cong \text{Lie}(G)$ define a Reynolds operator "gathering" invariants
of torus $\mathcal{T} \subset G$ into special polynomials. A condition for a linear
combination of polynomials to be G-invariant is equivalent to the
existence of a solution for a certain system of linear equations on
the coefficients in the combination.

As an example we consider the adjoint action of the Lie group
SL(3) (and its subgroup SL(2)) on the Lie algebra $\mathfrak{sl}(3)$.

§1. INTRODUCTION

Algorithms in invariant theory [1, 2] become inefficient for Lie groups
interesting for physics. To overcome this problem, we try to use the follow-
ing well known fact. Every automorphism of a Lie algebra root system (in
particular, an element of the Weyl group $\mathcal{W}$) defines an automorphism of
the Cartan subalgebra $\mathfrak{h}$ and, subsequently, can be extended to the auto-
morphism of the whole Lie algebra $\mathfrak{g}$ [3]. For example, let us consider $\mathfrak{sl}(2)$
algebra with the standard basis $X, Y, H = [X, Y]$ and define the adjoint
action $\text{ad}_A B := [A, B]$. The operator

$$S = (\exp \text{ad}_X) (\exp -\text{ad}_Y) (\exp \text{ad}_X)$$

induces a second order inner automorphism of $\mathfrak{sl}(2)$:

$$S(X) = -Y, \quad S(Y) = -X, \quad S(H) = -H.$$

The action of S on to the Cartan subalgebra $\mathfrak{h} = \{H\}$ corresponds to a Weyl group $\mathcal{W} = \mathbb{Z}/2$ reflection of the 1-dimensional $\mathfrak{sl}(2)$-root system:

$$\sigma_\alpha(\alpha) = -\alpha.$$

This paper is organized as follows. Section 2 illustrates a method on the construction of two Casimir invariants of the Lie group SL(3). An adjoint representation for the Lie algebra $\mathfrak{sl}(3)$ is done in Sec. 2.1. In 2.2, using a simple algorithm ([2, §4.3]), we construct a Hilbert basis of invariants for a maximal torus $\mathcal{T} \subset \mathrm{SL}(3)$. Sections 2.3 and 2.4 describe two main steps of our method. In 2.3, we construct a set of the S-operators extending the action of the Weyl group for the $\mathfrak{sl}(3)$ root system and apply corresponding Reynolds operator to the torus invariants. Derived invariants we call *Weyl blocks*. In 2.4, we join them into linear combinations which are invariant relative to the adjoint action of SL(3). In Sec. 3, SL(2)-invariants in the algebra $\mathfrak{sl}(3)$ are found. In Sec. 3.1, an embedding of SL(2) into SL(3) is used to construct all elements necessary for our method. In Sec. 3.2, we list a set of invariants and prove that it is a fundamental system of SL(2)-invariants in the adjoint representation of the algebra $\mathfrak{sl}(3)$.

§2. SL(3) INVARIANTS

2.1. Linear representation of SL(3)**-adjoint action on** $\mathfrak{sl}(3)$**.** The dimension of $\mathfrak{sl}(3)$ is equal to 8 and the rank is 2 and, correspondingly, two $\mathfrak{sl}(2)$-triples generate the algebra:

$$X_1 = E_{12}, \quad Y_1 = E_{21}, \quad H_1 = E_{11} - E_{22}, \tag{2.1}$$

$$X_2 = E_{23}, \quad Y_2 = E_{32}, \quad H_2 = E_{22} - E_{33}, \tag{2.2}$$

where $(E_{ij})_{kl} = \delta_{ik}\delta_{jl}$ are unit matrices. These operators with two additional

$$X_3 := [X_1, X_2], \quad Y_3 := [Y_2, Y_1] \tag{2.3}$$

form the Cartan-Weyl basis of $\mathfrak{sl}(3)$. Below we use the following notations for the basis elements:

Y_1	X_1	Y_2	X_2	Y_3	X_3	H_1	H_2
e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8

$$\tag{2.4}$$

Consider $\mathfrak{sl}(3)$ as a vector space V with the basis $\{e_i\}_{i=1}^8$ corresponding to the generators e_i. The adjoint representation of $\mathfrak{sl}(3)$

$$\mathrm{ad}_{\mathrm{e}_i} : \ \mathrm{e}_j \mapsto [\mathrm{e}_i, \mathrm{e}_j] = c_{ij}{}^k \mathrm{e}_k \tag{2.5}$$

induces a linear transformation $\mathcal{A}_i$ in V by the matrix $(A_i)_j{}^k = -c_{ij}{}^k$:

$$\mathcal{A}_i\,(e_j) := e_k\,(\widetilde{A}_i)^k{}_j, \quad \widetilde{A}_i = \chi^{-1} A_i\,\chi, \tag{2.6}$$

where k numerates rows, j numerates columns of the matrix $(\widetilde{A}_i)^k{}_j$ and χ is the matrix of the Cartan metric on $\mathfrak{sl}(3)$:

$$\chi_{ij} := \operatorname{Tr}(A_i \cdot A_j) = c_{ip}{}^q c_{jq}{}^p. \tag{2.7}$$

The adjoint action of the Lie group SL(3) on its algebra $\mathfrak{sl}(3)$

$$\exp(t\,\mathrm{e}_i)\,\mathrm{e}_j\,\exp(-t\,\mathrm{e}_i) = \exp(t\,\mathrm{ad}_{\mathrm{e}_i})\,\mathrm{e}_j \quad \mathrm{e}_i, \mathrm{e}_j \in \mathfrak{sl}(3), \tag{2.8}$$

can be represented as a linear transformation of V by one-parameter groups:

$$\mathcal{C}_i\,(e_j) := e_k(\exp[t\,\widetilde{A}_i])^k{}_j\,. \tag{2.9}$$

In this section we construct polynomial SL(3)-invariants in variables

e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8
y_1	x_1	y_2	x_2	y_3	x_3	h_1	h_2

(2.10)

2.2. Torus Invariants.
A d-dimensional torus $\mathcal{T} \cong (\mathbb{C}^*)^d$ acting on the ring $\mathbb{C}[V] = \mathbb{C}[x_1, \ldots, x_n]$ is isomorphic to a group of the diagonal matrices

$$T = \operatorname{diag}\left(\prod_{i=1}^{d} t_i^{a_{1i}}, \ldots, \prod_{i=1}^{d} t_i^{a_{ni}} \right), \tag{2.11}$$

where $t_1, \ldots, t_d \in \mathbb{C}^*$ are torus variables and x_i have the weights

$$\omega_i = (a_{i,1}, \ldots, a_{i,d}), \quad a_{ij} \in \mathbb{Z}.$$

$\mathcal{T}$ maps monomials into monomials and its invariant ring $\mathbb{C}[V]^{\mathcal{T}}$ is monomial. Suppose $\mathcal{T} \subset \mathrm{G}$, for the Lie group G. Then the algebra of G-invariants is a subalgebra of torus invariants

$$\mathbb{C}[V]^{\mathrm{G}} \subset \mathbb{C}[V]^{\mathcal{T}},$$

i.e. a G-invariant polynomial is a linear combination of $\mathcal{T}$-invariant monomials.

By definition a weight ω_i of an element X_i is the set of values of the corresponding root α_i on the elements H_j of the Cartan subalgebra $\mathfrak{h}$. The values $\alpha_i(H_j)$ can be directly read from the commutation relations in the Cartan-Weyl basis:

$$[H_j, X_i] = \alpha_i(H_j)X_i. \tag{2.12}$$

Weights of the variables (2.10) are ones of the generators (2.4)

y_1	x_1	y_2	x_2	y_3	x_3	h_1	h_2
$(-2,1)$	$(2,-1)$	$(1,-2)$	$(-1,2)$	$(-1,-1)$	$(1,1)$	$(0,0)$	$(0,0)$

(2.13)

An efficient algorithm 4.3.1 from [2] allows to find the *Hilbert basis* [1], i.e. a minimal generating set, for the torus invariants. We realize it in the following way. Let us construct the d-dimensional set Ω of boxes marked by weights of the variables x_i and 'filled in' with the corresponding x_i considered as initial monomials. After that we produce new monomials degree by degree multiplying on x_i monomials derived at the previous step. If a new monomial $m = x_1^{n_1} \ldots x_k^{n_k}$ with the weight

$$\omega_m = \left(\sum_{i=1}^{k} a_{i,1} n_i, \ldots, \sum_{i=1}^{k} a_{i,d} n_i \right) \tag{2.14}$$

is not divisible on derived early monomials with the same weight ω_m we put m into the corresponding box. We discard a new monomial if its weight is not in Ω. An output of the algorithm is the set of $\mathcal{T}$-invariant monomials with the weight $\underbrace{(0, \ldots, 0)}_{d}$.

For $T \subset \mathrm{SL}(3)$ the two dimensional torus T (2.11) is the following:

$$\mathrm{T} = \mathrm{diag}\, (t_1^{-2}t_2^1,\ t_1^2 t_2^{-1},\ t_1^1 t_2^{-2}\ t_1^{-1} t_2^2,\ t_1^{-1} t_2^{-1},\ t_1^1 t_2^1,\ t_1^0 t_2^0,\ t_1^0 t_2^0), \tag{2.15}$$

and the Hilbert basis consists from the monomials:

$$h_1,\ h_2,\ x_1 y_1,\ x_2 y_2,\ x_3 y_3,\ x_1 x_2 y_3,\ x_3 y_1 y_2. \tag{2.16}$$

2.3. Weyl blocks. The root system $\Phi_{\mathfrak{sl}(3)}$ of $\mathfrak{sl}(3)$ algebra lies in the Euclidean plane and has six roots $\pm\alpha_i$, $i = 1, 2, 3$ (see Fig. 1). A group of the reflections σ_i relative to hyperplanes orthogonal to the roots α_i, i.e. the Weyl group $\mathcal{W}_{\mathfrak{sl}(3)}$, is the permutation group $\mathfrak{S}_3$ [3]
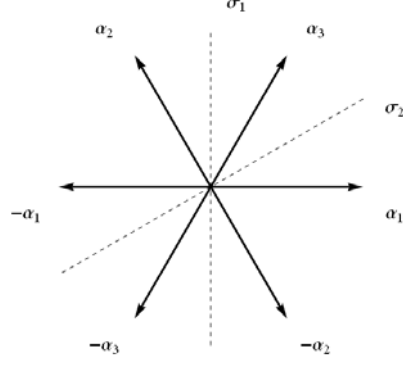
$$\mathfrak{S}_3 = \{\mathbb{I}, (12), (23), (13), (123), (132)\} \tag{2.17}$$

The permutations (12) and (23) correspond to the reflections σ_1 and σ_2 (relative to hyperplanes shown by dashed lines in the figure 1) respectively and generate $\mathfrak{S}_3$ according to formulas:

$$(123) = (12) \circ (23), \tag{2.18}$$

$$(132) = (23) \circ (12), \tag{2.19}$$

$$(13) = (12) \circ (23) \circ (12). \tag{2.20}$$

Fig. 1. $\mathfrak{sl}(3)$ root system.

Suppose we treat $\mathfrak{sl}(3)$ algebra as a linear space V according to Sec. 2.1. The reflections σ_1 and σ_2 can be extended to linear operators in V analogously to $\mathfrak{sl}(2)$ case (see Introduction)

$$S_1 = \exp[\widetilde{A}_{X_1}]\exp[-\widetilde{A}_{Y_1}]\exp[\widetilde{A}_{X_1}] \tag{2.21}$$

$$S_2 = \exp[\widetilde{A}_{X_2}]\exp[-\widetilde{A}_{Y_2}]\exp[\widetilde{A}_{X_2}] \tag{2.22}$$

as we have two $\mathfrak{sl}(2)$-triples (2.1), (2.2) in $\mathfrak{sl}(3)$. Transformations by S_1, S_2 of basis elements of $\mathfrak{sl}(3)$

$$\mathcal{S}_i\,(e_j) := e_k\,(S_i)^k{}_j\,, \tag{2.23}$$

are shown in the following tables:

$$\mathcal{S}_1: \quad \begin{array}{c|c|c|c|c|c|c|c} y_1 & x_1 & y_2 & x_2 & y_3 & x_3 & h_1 & h_2 \\ \hline -x_1 & -y_1 & y_3 & x_3 & -y_2 & -x_2 & -h_1 & h_3 \end{array} \tag{2.24}$$

$$\mathcal{S}_2: \quad \begin{array}{c|c|c|c|c|c|c|c} y_1 & x_1 & y_2 & x_2 & y_3 & x_3 & h_1 & h_2 \\ \hline -y_3 & -x_3 & -x_2 & -y_2 & y_1 & x_1 & h_3 & -h_2 \end{array} \tag{2.25}$$

where $h_3 := h_1 + h_2$. As a proof of correctness of extension we compare the action of $\mathcal{S}_1, \mathcal{S}_2$ on the Cartan subalgebra $\mathfrak{h} = \{h_1, h_2, h_3\}$ and Weyl group reflections of the dual roots $\alpha_i \leftrightarrow H_i$, $i = 1, 2, 3$ in the $\mathfrak{sl}(3)$ root

space $\Phi_{\mathfrak{sl}(3)}$:

$$\sigma_1(\alpha_1) = -\alpha_1 \quad \mathcal{S}_1(h_1) = -h_1$$
$$\sigma_1(\alpha_2) = \alpha_3 \quad \mathcal{S}_1(h_2) = h_3$$
$$\sigma_1(\alpha_3) = \alpha_2 \quad \mathcal{S}_1(h_3) = h_2$$

$$\sigma_2(\alpha_1) = \alpha_3 \quad \mathcal{S}_2(h_1) = h_3$$
$$\sigma_2(\alpha_2) = -\alpha_2 \quad \mathcal{S}_2(h_2) = -h_2$$
$$\sigma_2(\alpha_3) = \alpha_1 \quad \mathcal{S}_2(h_3) = h_1$$

Taking a Cartan decomposition

$$\mathfrak{sl}(3) = \{h_1, h_2\} \oplus \coprod_{\alpha \in \Phi_{\mathfrak{sl}(3)}} L_\alpha \tag{2.26}$$

we see from the tables (2.24), (2.25) that every one-dimensional subspace L_α is transformed according to the reflection σ_i: $\mathcal{S}_i : L_\alpha \to L_{\sigma_i(\alpha)}$.

Let us consider matrices

$$S_0 = \mathbb{I}, \quad S_3 = S_1 S_2 S_1, \tag{2.27}$$
$$S_1, \qquad S_4 = S_1 S_2, \tag{2.28}$$
$$S_2, \qquad S_5 = S_2 S_1, \tag{2.29}$$

where we construct S_3, S_4, S_5 following to (2.20), (2.18), (2.19), respectively. $S_0, \ldots, S_5$ are not a representation of $\mathcal{W}_{\mathfrak{sl}(3)} = \mathfrak{S}_3$ [3]. S_1, S_2 generate a non-abelian matrix group of order 24 but we postpone investigation of this group.

We define a Reynolds operator Rey on a polynomial P in variables (2.10)

$$Rey(P) = \sum_S \mathcal{S}_i(P), \quad S = \{\mathcal{S}_0, \mathcal{S}_1, \ldots, \mathcal{S}_5\} \tag{2.30}$$

where transformations $\mathcal{S}_i$ correspond to the matrices S_i according to (2.23).

Application of Rey to the torus invariants (2.16) yields polynomials which we call *Weyl blocks* $w_{i,j}$ (j numerates Weyl blocks of degree i). Linear combinations of the torus invariants give *initial* Weyl blocks:

$$w_{2,1} = x_1 y_1 + x_2 y_2 + x_3 y_3, \quad w_{3,1} = x_1 x_2 y_3 + x_3 y_1 y_2. \tag{2.31}$$

Taking products of torus invariants we derive additional blocks:

$$w_{2,2} = h_1^2 + h_1 h_2 + h_2^2, \tag{2.32}$$

$$w_{3,2} = 2h_1^3 + 3h_1^2 h_2 - 3h_1 h_2^2 - 2h_2^3, \tag{2.33}$$

$$w_{3,3} = x_1 y_1 (h_1 + 2h_2) - x_2 y_2 (2h_1 + h_2) + x_3 y_3 (h_1 - h_2). \tag{2.34}$$

2.4. SL(3) Invariants. The group SL(3) is generated by eight one-para-meter subgroups $\mathcal{C}^{(k)}$, $k = 1, \ldots, 8$ (2.9) corresponding to the basis elements (2.4). A difference of a Weyl block $w_{i,j}$ under the action of $\mathcal{C}^{(k)}$ has a part which is linear in the parameter t:

$$\Delta_{i,j}^{(k)} := \frac{d}{dt} \mathcal{C}^{(k)}(w_{i,j}) \Big|_{t=0} \tag{2.35}$$

A homogeneous linear combination $\sum_{i,j} \mu_{i,j} w_{i,j}$ of the Weyl blocks with nu-merical coefficients $\mu_{i,j}$ is SL(3)-invariant if and only if the sum $\sum_{i,j} \mu_{i,j} \Delta_{i,j}^{(k)}$ is equal to zero for all $\mathcal{C}^{(k)}$.

Now let us take into account that $\Delta_{i,j}^{(k)}$ are polynomials in the variables e_i, $i = 1, \ldots, 8$ (2.10),

$$\Delta_{i,j}^{(k)}(e_1, \ldots, e_8) = \sum_{d \in \mathrm{Deg}} \phi_{i,j}^{(k)}(d) \, e_1^{d_1} \ldots e_8^{d_8}, \quad d = (d_1, \ldots, d_8).$$

where Deg is a set of multidegrees d of monomials $e_1^{d_1} \ldots e_8^{d_8}$ in $\Delta_{i,j}^{(k)}$. Rewriting the equation

$$\sum_{i,j} \mu_{i,j} \Delta_{i,j}^{(k)} = 0 \tag{2.36}$$

as a sum over d, we get a homogeneous system of linear equations (one equation for every monomial e^d and fixed k)

$$\sum_{i,j} \mu_{i,j} \phi_{i,j}^{(k)}(d) = 0 \tag{2.37}$$

for the coefficients $\mu_{i,j}$.

For the Weyl blocks (2.31), (2.32) we get equations:
second order $w_{2,i}$,

$$\mu_{2,1} - 3\mu_{2,2} = 0, \tag{2.38}$$

third order $w_{3,i}$,

$$\mu_{3,1} - 3\mu_{3,3} = 0, \quad 9\mu_{3,2} - \mu_{3,3} = 0. \tag{2.39}$$

The equations have solutions spanned by vectors

$$\{\mu_{2,1}, \mu_{2,2}\} = \{3, 1\}, \quad \{\mu_{3,1}, \mu_{3,2}, \mu_{3,3}\} = \{27, 1, 9\} \tag{2.40}$$

Thus we derive two Casimir invariants in the algebra $\mathfrak{sl}(3)$ under the adjoint action of the group SL(3):

$$\mathfrak{C}_2 = 3w_{2,1} + w_{2,2}, \tag{2.41}$$

$$\mathfrak{C}_3 = 27w_{3,1} + w_{3,2} + 9w_{3,3}. \tag{2.42}$$

The Weyl blocks $w_{2,2}$ and $w_{3,2}$ represent reductions of the Casimir invariants onto the Cartan subalgebra $\mathfrak{H}$ because the variables h_1, h_2 correspond to the generators $H_1, H_2 \in \mathfrak{H}$. We substitute h_i by dual roots α_i and get invariants of the Weyl group

$$w_{2,2} \rightarrow \mathcal{I}_2^{\mathcal{W}} = \alpha_1^2 + \alpha_1\alpha_2 + \alpha_2^2, \tag{2.43}$$

$$w_{3,2} \rightarrow \mathcal{I}_3^{\mathcal{W}} = 2\alpha_1^3 + 3\alpha_1^2\alpha_2 - 3\alpha_1\alpha_2^2 - 2\alpha_2^3 \tag{2.44}$$

in accordance with the Chevalley theorem about the homomorphism of Lie group invariants and Weyl group invariants [3].

§3. SL(2) INVARIANTS IN $\mathfrak{sl}(3)$ ALGEBRA

3.1. $\mathfrak{sl}(2)$ as a subalgebra of $\mathfrak{sl}(3)$. The generators $\{Y_1, H_1, X_1\}$ (see (2.1)) form the subalgebra $\mathfrak{su}(2)$ in $\mathfrak{su}(3)$. We decompose $\mathfrak{sl}(3)$ into a direct sum of irreducible $\mathfrak{sl}(2)$-modules looking at the commutation relations:

$$[H_1, Y_2] = Y_2 \qquad [Y_1, Y_2] = -Y_3 \quad [X_1, Y_2] = 0, \tag{3.1}$$

$$[H_1, Y_3] = -Y_3 \quad [Y_1, Y_3] = 0 \qquad [X_1, Y_3] = -Y_2. \tag{3.2}$$

$$[H_1, X_3] = X_3 \qquad [Y_1, X_3] = X_2 \quad [X_1, X_3] = 0, \tag{3.3}$$

$$[H_1, X_2] = -X_2 \quad [Y_1, X_2] = 0 \qquad [X_1, X_2] = X_3. \tag{3.4}$$

It is convenient to introduce a new operator H_0,

$$H_0 := \tfrac{1}{2}H_1 + H_2, \tag{3.5}$$

with commutators:

$$[H_1, H_0] = [Y_1, H_0] = [X_1, H_0] = 0. \tag{3.6}$$

According to the above relations, $\mathfrak{su}(3)$ is a direct sum of four $\mathfrak{sl}(2)$-modules:

$$V_0 \oplus V_1' \oplus V_1'' \oplus V_2 \tag{3.7}$$

where each module V_λ is a disjoint union of one-dimensional subspaces

$$V_\lambda = \coprod_\mu V_{\lambda,\mu}, \quad \mu = \lambda, \lambda - 2, \ldots, -(\lambda - 2), -\lambda. \qquad (3.8)$$

The decomposition can be summarized by the following table.

$\mu \backslash V_\lambda$	V_2	V_1'	V_1''	V_0
2	X_1			
1		Y_2	X_3	
0	H_1			H_0
-1		Y_3	X_2	
-2	Y_1			

$$(3.9)$$

Embedding $\mathfrak{sl}(2)$ into $\mathfrak{sl}(3)$ allows us to use many constructions from the previous section. Let us reorder $\mathfrak{su}(3)$-generators (2.4) in accordance with the table (3.9)

$$Y_1, \ X_1, \ H_1, \ Y_2, \ Y_3, \ X_2, \ X_3, \ H_0 \qquad (3.10)$$

The group SL(2) is generated by three one-parameter subgroups

$$\mathcal{C}_{Y_1}, \ \mathcal{C}_{X_1}, \ \mathcal{C}_{H_1}$$

(see (2.9)) and acts on the row

$$v = \{y_1, x_1, h_1, y_2, y_3, x_2, x_3, h_0\} \qquad (3.11)$$

from the right.

The Cartan subalgebra of $\mathfrak{sl}(2)$ is one-dimensional: $\mathfrak{H}_{\mathfrak{sl}(2)} = \{H_1\}$. And the weights of the basis elements (3.10) are given by the first component of their $\mathfrak{sl}(3)$-weights (2.13) except H_0 with the weight 0. From other hand, $\mathfrak{sl}(2)$-weights are given by the value of μ in table (3.9). Accordingly, one-dimensional torus $\mathcal{T}_{\mathrm{SL}(2)}$

$$\mathcal{T}_{\mathrm{SL}(2)} = \mathrm{diag}(t^{-2}, t^2, t^0, t^1, t^{-1}, t^{-1}, t^1, t^0) \qquad (3.12)$$

acts on 8-dimensional vector v (3.11) from the right. The Hilbert basis of the torus invariants includes new monomials

$$y_2 y_3, \quad x_2 x_3,$$
$$y_1 y_2^2, \quad y_1 x_3^2, \quad x_1 x_2^2, \quad x_1 y_3^2 \tag{3.13}$$

in addition to the invariants of $\mathcal{T}_{\mathrm{SL}(3)}$ (2.16) where h_0 must be included instead of h_2.

The root system $\Phi_{\mathfrak{sl}(2)}$ is represented by $\pm\alpha_1$ and has the Weyl group $\mathcal{W}_{\mathfrak{sl}(2)} = \mathbb{Z}_2$. The operator s_1 (2.21) has the block-diagonal form

$$
\begin{pmatrix}
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\tag{3.14}
$$

Now the Reynolds operator (2.30) includes only summation over s_1 and the identity operator.

3.2. SL(2)-Invariant polynomials in $\mathfrak{sl}(3)$ algebra.

Following the described above procedure, we get as a result:

degree	invariants
1	$\mathcal{I}_1 = \underline{h_0}$
2	$\mathcal{I}_2 = h_1^2 + 4\,\underline{x_1 y_1}$
	$\mathcal{I}_3 = \underline{x_2 y_2 + x_3 y_3}$
3	$\mathcal{I}_4 = h_1 y_2 y_3 + \underline{y_1 y_2^2 - x_1 y_3^2}$
	$\mathcal{I}_5 = h_1 x_2 x_3 + \underline{x_1 x_2^2 - x_3^2 y_1}$
	$\mathcal{I}_6 = h_1 (x_2 y_2 - x_3 y_3) - 2\,\underline{(y_1 y_2 x_3 + x_1 x_2 y_3)}$

$$\tag{3.15}$$

where initial Weyl blocks are underlined.

We shall prove that the set of invariants $\mathcal{I}_i$, $i = 1, \ldots, 6$ generates the invariant ring $\mathbb{C}[y_1, x_1, \ldots, h_0]^{\mathrm{SL}(2)}$, i.e. this set is a fundamental system of SL(2)-invariants in $\mathfrak{sl}(3)$ algebra. This is the case if and only if the Hilbert

series $H(R,q)$ of the subalgebra $R = \mathbb{C}[\mathcal{I}_1, \ldots, \mathcal{I}_6]$ is equal to the Molien series [1] of $\mathbb{C}[y_1, x_1, \ldots, h_0]^{\mathrm{SL}(2)}$.

For this purpose we calculate the Molien function [2] for the adjoint action of SU(2) (i.e. the real compact form of SL(2)) on the algebra $\mathfrak{su}(3)$. An injective map of SU(2) into SU(3)

$$\rho = \begin{pmatrix} \mathrm{SU}(2) & 0 \\ 0 & 1 \end{pmatrix} \subset \mathrm{SU}(3) \tag{3.16}$$

corresponds to our choice of representation of $\mathfrak{sl}(2)$ in $\mathfrak{sl}(3)$ (see the previous subsection 3.1). One can instead of the adjoint action

$$\rho \, a \, \rho^{-1} \quad a \in \mathfrak{su}(3) \tag{3.17}$$

consider a linear representation L of ρ on $V = \mathbb{R}^9$

$$V'_A = L_{AB} V_B, \quad L_{AB} \in \pi(\rho) \otimes \overline{\pi(\rho)},$$

where $\overline{\pi(\rho)}$ is the complex conjugation of $\pi(\rho)$. After that we calculate the Molien function

$$M\left(\mathbb{C}[V]^{\mathrm{SU}(2)}, q\right) = \int\limits_{\mathrm{SU}(2)} \frac{d\mu}{\det(\mathbb{I} - q\,\pi(\rho) \otimes \overline{\pi(\rho)})}, \quad |q| < 1, \tag{3.18}$$

where $d\mu$ is a Haar measure for SU(2), $\mathbb{I}$ is an identity operator, and $\pi(\rho)$ is a representation of ρ. Using Weyl integration formula [5], one can reduce the integral over SU(2) to the integral over its maximal torus $\mathcal{T}_{\mathrm{SU}(2)}$. An expression of $d\mu$ via the Haar measure $d\phi$ on $\mathcal{T}_{\mathrm{SU}(2)}$ is the following one:

$$\begin{aligned} d\mu &= \frac{1}{\pi} \sin^2(\phi)\, d\phi \quad 0 \leqslant \phi \leqslant 2\pi \\ &= -\frac{1}{4\pi i} \frac{dz}{z^3} (1 - z^2)^2 \quad \text{where} \quad z = e^{i\phi}. \end{aligned} \tag{3.19}$$

$\pi(\rho)$ has a diagonal form

$$\pi(\rho) = \begin{pmatrix} z & 0 & 0 \\ 0 & z^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{3.20}$$

and, hence,

$$\begin{aligned} \pi(\rho) \otimes \overline{\pi(\rho)} &= \mathrm{diag}(z, z^{-1}, 1) \otimes (z^{-1}, z, 1) \\ &= \mathrm{diag}(1, z^2, z, z^{-2}, 1, z^{-1}, z^{-1}, z, 1), \end{aligned} \tag{3.21}$$

$$\det \|\mathbb{I} - q\pi_G \otimes \overline{\pi}_G\| = (1-q)^3 (1-qz)^2 (1-qz^2)(1-qz^{-1})^2 (1-qz^{-2}). \tag{3.22}$$

Substituting (3.19), (3.22) in the expression (3.18) we derive that the Molien function is equal to the following contour integral over the unit circle:

$$M\left(\mathbb{C}[V]^{\mathrm{SU}(2)}, q\right)$$

$$= -\frac{1}{4\pi i}\frac{1}{(1-q)^3}\oint\frac{zdz(1-z^2)^2}{(1-qz)^2(1-qz^2)(z-q)^2(z^2-q)}. \quad (3.23)$$

There are two simple poles $z = \pm\sqrt{q}$ and a double pole $z = q$ (of order $m = 2$). Application of the residue theorem [2] yields the result:

$$M\left(\mathbb{C}[V]^{\mathrm{SU}(2)}, q\right) = \frac{1+q^3}{(1-q)^2(1-q^2)^2(1-q^3)^2}. \quad (3.24)$$

$M\left(\mathbb{C}[V]^{\mathrm{SU}(2)}, q\right)$ satisfies a functional equation

$$M\left(\mathbb{C}[V]^{\mathrm{SU}(2)}, q^{-1}\right) = q^9 M\left(\mathbb{C}[V]^{\mathrm{SU}(2)}, q\right), \quad (3.25)$$

hence, $\mathbb{C}[V]^{\mathrm{SU}(2)}$ being a graded Cohen-Macaulay algebra [4] is the Gorenstein one according to Stanley [6].

Required Molien series of $\mathrm{SU}(2)$-invariant ring $\mathbb{C}[V]^{\mathrm{SU}(2)}$ is nothing then an expansion of $M\left(\mathbb{C}[V]^{\mathrm{SU}(2)}, q\right)$ in series of powers of q. Now we turn out to the calculation of the Hilbert series $H(R, q)$ for the subalgebra $R = \mathbb{C}[\mathcal{I}_1, \ldots, \mathcal{I}_6]$. Using slack-variables method [1], we find one syzygy between the invariants $\mathcal{I}_i$ (3.15),

$$\mathcal{I}_2\mathcal{I}_3^2 - 4\mathcal{I}_4\mathcal{I}_5 - \mathcal{I}_6^2 \quad (3.26)$$

thus $\{\mathcal{I}_1, \ldots, \mathcal{I}_5\}$ are algebraically independent while $\mathcal{I}_6$ can be considered as algebraically dependent. This implies that R is decomposed as the direct sum of graded $\mathbb{C}$-vector space

$$\mathbb{C}[\mathcal{I}_1, \ldots, \mathcal{I}_6] = \mathbb{C}[\mathcal{I}_1, \ldots, \mathcal{I}_5] \oplus \mathcal{I}_6\,\mathbb{C}[\mathcal{I}_1, \ldots, \mathcal{I}_5] \quad (3.27)$$

with the Hilbert series

$$H(R, q) = \frac{1+q^3}{(1-q)(1-q^2)^2(1-q^3)^2}. \quad (3.28)$$

We see that the denominator of the Molien function (3.24) has an extra term $1 - q$ in comparison with $H(R, q)$. The reason is that we have calculated $M\left(\mathbb{C}[V]^{\mathrm{SU}(2)}, q\right)$ in 9-dimensional vector space but $\mathfrak{sl}(3)$ has dimension 8. This completes the proof.

As a corollary we get that a Hironaka decomposition [1] of R is done by the formula (3.27). $\mathcal{I}_1, \ldots, \mathcal{I}_5$ are primary invariants and $\mathcal{I}_6$ is a secondary

invariant. For example, a decomposition of the Casimir invariants (2.41) is the following:

$$\mathfrak{C}_2 = \mathcal{I}_1^2 + \tfrac{3}{4}\,\mathcal{I}_2 + 3\,\mathcal{I}_3, \qquad (3.29)$$

$$\mathfrak{C}_3 = -2\,\mathcal{I}_1^3 + \tfrac{9}{2}\,\mathcal{I}_1\mathcal{I}_2 - 9\,\mathcal{I}_1\mathcal{I}_3 - \tfrac{27}{2}\,\mathcal{I}_6. \qquad (3.30)$$

§4. Summary

For construction of Lie group invariants we propose a method based on the Weyl block structure of invariants. To produce $\mathcal{W}$-blocks, we extend the Weyl group action on the Cartan subalgebra to the automorphism of the whole Lie algebra. Corresponding set of operators defines a Reynolds operator Rey. Being applied to Hilbert basis of torus invariants operator Rey produces initial $\mathcal{W}$-blocks. Beside that we construct $\mathcal{W}$-blocks from all possible products of torus invariants up to the higher degree of invariants from the Hilbert basis. A homogeneous linear combination of $\mathcal{W}$-blocks is invariant relative to the adjoint action of a Lie group if a difference of the combination under a transformation is zero. This condition gives a system of linear equations. If there is a solution of the system, then we get an invariant.

Using the described above method, we construct fundamental sets of invariants of the adjoint actions of the Lie groups $SL(3)$ and $SL(2)$ on the Lie algebra $\mathfrak{sl}(3)$. We see that the Weyl block structure of invariants reflects very deep properties of Lie algebras and Lie groups such as generating relations, properties of root systems, irreducible representations and so on. All this information can be useful for studying rings of invariants. For example, let us mention that the initial $\mathcal{W}$-blocks of the invariants (3.15) taken with their coefficients satisfy the same syzygy (3.26) as invariants themselves.

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